Cohomology of certain $\mathbb{N}$-graded Lie algebras

D. V. Millionshchikov and A. Fialowski

In [1] infinite-dimensional $\mathbb{N}$-graded Lie algebras with two generators over a field $\mathbb{K}$ of characteristic zero were classified. In particular, it follows from this classification that an arbitrary $\mathbb{N}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{i\geqslant 1} \mathfrak{g}_i$ with one-dimensional homogeneous components $\mathfrak{g}_i$ such that $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, $i \geqslant 2$, is isomorphic to one (and only one) of the three algebras L_1, $\mathfrak{m}_0$, and $\mathfrak{m}_2$ that are defined by the basis $\{e_i, i \in \mathbb{N}\}$, $\mathfrak{g}_i = \langle e_i \rangle$, and the non-trivial commutation relations

$$L_1\colon \quad [e_i, e_j] = (j-i)e_{i+j} \quad \forall\, i,j \in \mathbb{N};$$
$$\mathfrak{m}_0\colon \quad [e_1, e_i] = e_{i+1} \quad \forall\, i \geqslant 2;$$
$$\mathfrak{m}_2\colon \quad [e_1, e_i] = e_{i+1} \quad \forall\, i \geqslant 2, \quad [e_2, e_j] = e_{j+2} \quad \forall\, j \geqslant 3.$$

The cohomology $H^*(L_1)$ with trivial coefficients was calculated by Goncharova in [2]. The purpose of the present note is to calculate the cohomology $H^*(\mathfrak{m}_0)$ and $H^*(\mathfrak{m}_2)$.

Let $\mathfrak{g} = \bigoplus_i \langle e_i \rangle$ be an arbitrary $\mathbb{N}$-graded Lie algebra with one-dimensional components. We consider the dual basis $\{e^i, e^i(e_j) = \delta^i_j\}$. The space $C^q(\mathfrak{g})$ of q-dimensional cochains of the algebra $\mathfrak{g}$ can be identified with the space $\Lambda^q(\mathfrak{g}^*)$ of formal series $\sum \alpha_{i_1\dots i_q} e^{i_1} \wedge \dots \wedge e^{i_q}$, so that $C^q(\mathfrak{g}) = \widehat{\bigoplus}_k C^q_k(\mathfrak{g})$, where the (finite-dimensional) subspace $C^q_k(\mathfrak{g})$ is generated by the monomials $e^{i_1} \wedge \dots \wedge e^{i_q}$ such that $i_1 + \dots + i_q = k$. The latter decomposition defines a second grading of the cochain complex $C^*(\mathfrak{g})$ and correspondingly of the cohomology $H^*(\mathfrak{g})$.

Let $\mathfrak{b}$ be an ideal of codimension one in some Lie algebra $\mathfrak{g}$. We choose $X \in \mathfrak{g}$, $X \notin \mathfrak{b}$. The conjugate operator $\operatorname{ad} X^*$ of the operator $\operatorname{ad} X$ can be extended to $\Lambda^q(\mathfrak{g}^*)$:

$$\operatorname{ad} X^*(\xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_q) = \operatorname{ad} X^* \xi_1 \wedge \xi_2 \wedge \dots \wedge \xi_q + \xi_1 \wedge \operatorname{ad} X^* \xi_2 \wedge \dots \wedge \xi_q + \dots + \xi_1 \wedge \xi_2 \wedge \dots \wedge \operatorname{ad} X^* \xi_q.$$

The operator $\operatorname{ad} X^*$ commutes with the differential d of the cochain complex and thus defines a map $\operatorname{ad} X^*\colon H^*(\mathfrak{b}) \to H^*(\mathfrak{b})$ in cohomology.

Theorem 1 (Dixmier, [3]). *There is the following long exact sequence in cohomology of Lie algebras*:

$$\dots \to H^{q-1}(\mathfrak{b}) \xrightarrow{\operatorname{ad} X^*} H^{q-1}(\mathfrak{b}) \to H^q(\mathfrak{g}) \to H^q(\mathfrak{b}) \xrightarrow{\operatorname{ad} X^*} H^q(\mathfrak{b}) \to \cdots.$$

We apply Theorem 1 to $\mathfrak{g}_1 = \mathfrak{m}_0$, $\mathfrak{b}_1 = \langle e_2, e_3, \dots \rangle$, $X_1 = e_1$. The cohomology of the Abelian ideal $\mathfrak{b}_1$ is the (completed) exterior algebra $\Lambda^*(\mathfrak{b}_1^*)$. One can show that the operator $\operatorname{ad} X_1^* = \operatorname{ad} e_1^*$ is surjective on $\Lambda^*(\mathfrak{b}_1^*)$, and the problem of calculating the cohomology $H^*(\mathfrak{m}_0)$ reduces to finding its kernel. Let $P_q(k)$ denote the number of partitions of a positive integer k into exactly q summands: $k = x_1 + \dots + x_q$, $1 \leqslant x_1 \leqslant x_2 \leqslant \dots \leqslant x_q$.

Theorem 2. *The cohomology algebra $H^*(\mathfrak{m}_0) = \bigoplus_{k,q} H^q_k(\mathfrak{m}_0)$ is linearly generated by the classes of the following homogeneous cocycles*:

$$e^1, \quad e^2, \quad \omega(e^{i_1} \wedge \dots \wedge e^{i_q} \wedge e^{i_q+1}) = \sum_{l\geqslant 0} (-1)^l (\operatorname{ad} e_1^*)^l (e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_q}) \wedge e^{i_q+1+l}, \qquad (1)$$

where $1 \leqslant q$, $2 \leqslant i_1 < i_2 < \dots < i_q$; *in particular*,

$$\dim H^q_{k+\frac{q(q+1)}{2}}(\mathfrak{m}_0) = P_q(k) - P_q(k-1).$$

This research was carried out with the financial support of the Russian Foundation for Basic Research (grant no. 02-01-00659) and a Hungarian National Science Research Grant (grant nos. T043641 and T043034).

AMS 2000 *Mathematics Subject Classification.* Primary 17B56, 17B70; Secondary 17B65.

DOI 10.1070/RM2004v059n06ABEH000804

The multiplicative structure is defined as follows:

$$[e^1]\wedge\omega(\xi\wedge e^i\wedge e^{i+1})=0,\quad [e^2]\wedge\omega(\xi\wedge e^i\wedge e^{i+1})=\omega(e^2\wedge\xi\wedge e^i\wedge e^{i+1}),\tag{2}$$

$$\omega(\xi\wedge e^i\wedge e^{i+1})\wedge\omega(\eta\wedge e^j\wedge e^{j+1})=\sum_{l=0}^{j-i+1}(-1)^l\omega((\operatorname{ad}e_1^*)^l(\xi\wedge e^i)\wedge e^{i+1+l}\wedge\eta\wedge e^j\wedge e^{j+1})$$
$$+(-1)^{j-i+\deg\eta}\sum_{s\geqslant1}\omega((\operatorname{ad}e_1^*)^{j-i-1+s}(\xi\wedge e^i)\wedge(\operatorname{ad}e_1^*)^s(\eta\wedge e^j)\wedge e^{j+s}\wedge e^{j+s+1})$$
$$+(-1)^{j-i+\deg\eta+1}\sum_{s\geqslant1}\omega((\operatorname{ad}e_1^*)^{j-i+1+s}(\xi\wedge e^i)\wedge(\operatorname{ad}e_1^*)^s(\eta\wedge e^j)\wedge e^{j+s+1}\wedge e^{j+s+2})$$

for $i<j$, *where* ξ *and* η *are arbitrary homogeneous forms in* $\Lambda^*(e^2,\dots,e^{i-1})$ *and* $\Lambda^*(e^2,\dots,e^{j-1})$, *respectively.*

Example. The following cocycles form a basis of $H^2(\mathfrak{m}_0)$:

$$e^2\wedge e^3,\ e^3\wedge e^4-e^2\wedge e^5,\ e^4\wedge e^5-e^3\wedge e^6+e^2\wedge e^7,\ \dots,\ \omega(e^j\wedge e^{j+1})=\sum_{l=0}^{j-2}(-1)^l e^{j-l}\wedge e^{j+1+l},\ \dots.$$

We now apply Theorem 1 to $\mathfrak{g}_2=\mathfrak{m}_2$, $\mathfrak{b}_2=\langle e_1,e_3,e_4,\dots\rangle$, $X_2=e_2$. The ideal $\mathfrak{b}_2$ is in the obvious way isomorphic to $\mathfrak{m}_0$ and therefore its cohomology is already known. The operator $\operatorname{ad}e_2^*$ is 'almost surjective': $\operatorname{Im}(\operatorname{ad}e_2^*)\oplus\langle e^3\rangle=H^*(\mathfrak{b}_2^*)$.

Theorem 3. *The following cocycles form a homogeneous basis of the bigraded cohomology algebra* $H^*(\mathfrak{m}_2)=\bigoplus_{q,k}H^q_k(\mathfrak{m}_2)$:

$$e^1,\quad e^2,\quad e^2\wedge e^3,\quad e^3\wedge e^4-e^2\wedge e^5,$$
$$w_{i_1,\dots,i_q,i_q+1,i_q+2}=\sum_{l\geqslant0}\frac{1}{2^l}\omega\big((\operatorname{ad}e_2^*+(\operatorname{ad}e_1^*)^2)^l(e^{i_1}\wedge\dots\wedge e^{i_q})\wedge e^{i_q+1+l}\wedge e^{i_q+2+l}\big),\tag{4}$$

where $1\leqslant q$, $3\leqslant i_1<i_2<\dots<i_q$; *in particular*, *for* $q\geqslant3$

$$\dim H^q_{k+\frac{q(q+1)}{2}}(\mathfrak{m}_2)=P_q(k)-P_q(k-1)-P_q(k-2)+P_q(k-3).$$

Remark. In fact, the formulae (1)–(4) involve only finite sums, since the operators $\operatorname{ad}e_1^*$ and $\operatorname{ad}e_2^*+(\operatorname{ad}e_1^*)^2$ lower the second grading by one and two, respectively. The operators $\operatorname{ad}e_2^*$ and $(\operatorname{ad}e_1^*)^2$ coincide only on the space of 1-forms. It is worth mentioning that Theorem 2 is valid over a field of any characteristic, and Theorem 3 over an arbitrary field of odd characteristic. This is important, because all the algebras considered above are Lie algebras of maximal class (see [4]), which are widely used in the theory of p-groups.

Bibliography

[1] A. Fialowski, *Vestn. Mosk. Univ. Ser.* I *Mat. Mekh.* **1983**, no. 2, 62–64; English transl., *Mosc. Univ. Math. Bull.* **38**:2 (1983), 76–79.
[2] L. V. Goncharova, *Funkts. Anal. Prilozh.* **7**:2 (1973), 6–14; English transl., *Funct. Anal. Appl.* **7** (1973), 91–97.
[3] J. Dixmier, *Acta Sci. Math. Szeged.* **16** (1955), 246–250.
[4] A. Shalev and E. I. Zelmanov, *J. Algebra* **189** (1997), 294–331.

Moscow State University;
Eötvös Loránd University, Budapest
E-mail: million@mech.math.msu.su, fialowsk@cs.elte.hu
Received 24/NOV/04

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX