8^{th} Emléktábla Workshop

Combinatorial Geometry 06.18. - 06.22.2018.

Preliminary Schedule

Monday

Arrival in the hotel from 14:00.
16:00 Short presentations of problems
18:00 Dinner

Tuesday-Thursday

9:30 Short meeting after breakfast
9:35-17:00 Working in groups of 3-5
12:30 Lunch (except Wednesday)
17:00 Presentations of daily progress
18:00 (or whenever we are done) Dinner

Friday

Check-out at 10:00.

List of Participants

Martin Balko, Charles University
Zoltán Blázsik, MTA-ELTE GAC
Neal Bushaw, Virginia Commonwealth University
Jean Cardinal, Université libre de Bruxelles (ULB)
Gábor Damásdi, Hebrew University
Beka Ergemlidze, Central European University
Nóra Frankl, London School of Economics
Rado Fulek, IST Austria
Andrés Santamaría Galvis, University of Primorska
Dániel Gerbner, Rényi Institute
Tamás Hubai, MTA-ELTE CoGe
Balázs Keszegh, Rényi Institute
Younjin Kim, Ewha Womans University
Dániel Korándi, École Polytechnique Fédérale de Lausanne
István Kovács, Budapest University of Technology and Economics
Andrey Kupavskii, Moscow Institute of Physics and Technology, University of Birmingham
Dániel Lenger, ELTE
Leonardo Martínez, Ben Gurion University of the Negev
Viola Mészáros, University of Szeged
Abhishek Methuku, Central European University
Tamás Róbert Mezei, Rényi Institute
Dániel T. Nagy, Rényi Institute
Zoltán Lóránt Nagy, MTA-ELTE GAC
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János Pach, École Polytechnique Fédérale de Lausanne, Rényi Institute
Cory Palmer, University of Montana
Dömötör Pálvölgyi, MTA-ELTE CoGe
Balázs Patkós, Rényi Institute
Nika Salia, Central European University
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István Tomon, École Polytechnique Fédérale de Lausanne
Casey Tompkins, Rényi Institute
Géza Tóth, Rényi Institute
Tomas Valla, Technical University Prague
Máté Vizer, Rényi Institute
Zhiyu Wang, University of South Carolina
Russ Woodroofe, University of Primorska
Oscar Zamora Luna, Central European University

Empty triangles in x-monotone drawings of K_n

by Martin Balko

Source: Proposed by Jan Kynčl in 2014, personal communication.

Definitions:

- In a *drawing of a graph* the vertices are mapped to distinct points of the plane and every edge is represented by a simple continuous arc connecting the images of its endpoints.
- A drawing of a graph is *semisimple* if no two adjacent edges cross. A drawing is *simple* if it is semisimple and, moreover, no two edges have more than one common crossing.
- A curve γ in $\mathbb{R}^2$ is *x-monotone* if every vertical line intersects γ in at most one point. A drawing of a graph in which every edge is represented by an x-monotone curve is called *x-monotone*.
- In a semisimple drawing of K_n, the edges of three pairwise connected vertices form a Jordan curve which we call a *triangle*. We say that a triangle T is *empty* if there is no vertex in the bounded component of $\mathbb{R}^2 \setminus T$. Let $t(D)$ denote the number of empty triangles in a semisimple drawing D.

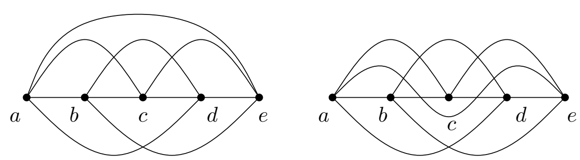


Figure 1: Simple (left) and semisimple (right) x-monotone drawings of K_5 with four empty triangles.

Problem 1. *What is the minimum number of empty triangles in a simple x-monotone drawing of K_n?*

Formally, we are interested in the (asymptotic) growth rate of the function $f(n) = \min t(D)$ where the minimum is taken over all simple x-monotone drawings D of K_n. The same question can be formulated for drawings where we allow independent edges to cross more than once.

Problem 2. *What is the minimum number of empty triangles in a semisimple x-monotone drawing of K_n?*

Related results:

- Obviously, we have $f(n) \geq n - 2$ and the same bound holds for semisimple x-monotone drawings. We should have the bound $f(n) \geq 2n - 6$.
- The following table summarizes the minimum number of empty triangles for small cases:

n	3	4	5	6	7	8	9	10	11
Simple x-monotone	1	2	4	6	10	14	18		
Semisimple x-monotone	1	2	4	6	8	10	12	≤ 14	≤ 16

- There is a characterization of simple and semisimple x-monotone drawings of K_n using small forbidden subconfigurations in [3].

- In simple (not necessarily x-monotone) drawings of K_n the number of empty triangles is known to be at least n [1] and at most $2n-4$ [5]. The upper bound is tight for $n \leq 8$ [1]. However in both these estimates we also count *exterior-empty* triangles, that is, a triangle T is considered *empty* if there is no vertex in at least one of the two components of $\mathbb{R}^2 \setminus T$.

- In *rectilinear drawings* of K_n, that is, drawings where every edge is represented by a line segment, the number of empty triangles is at least $n^2 - \frac{32}{7}n + \frac{22}{7}$ [2] and at most $1.6196n^2 + o(n^2)$ [4].

- Combining the previous results with the fact that every rectilinear drawing of K_n is x-monotone, we have $\Omega(n) \leq f(n) \leq O(n^2)$.

References

[1] O. Aichholzer, T. Hackl, A. Pilz, P. A. Ramos, V. Sacristán, B. Vogtenhuber. Empty triangles in good drawings of the complete graph, *Graphs and Combinatorics* **31**(2) (2015), 335–345.

[2] O. Aichholzer, R. Fabila-Monroy, T. Hackl, C. Huemer, A. Pilz, and B. Vogtenhuber. Lower bounds for the number of small convex k-holes, *Proc. 24th Canadian Conference on Computational Geometry CCCG'12* (2012), 247–252.

[3] M. Balko, J. Kynčl and R. Fulek. Crossing numbers and combinatorial characterization of monotone drawings of K_n, *Discrete and Computational Geometry* **53**(1) (2015), pages 107–143.

[4] I. Bárány and P. Valtr. Planar point sets with a small number of empty convex polygons, *Studia Scientiarum Mathematicarum Hungarica* **41**(2) (2004), 243–266.

[5] H. Harborth. Empty triangles in drawings of the complete graph, *Discrete Mathematics* **191** (1989), 109–111.

The square of a directed graph

by Zoltán L. Blázsik

This problem deals with the *square of an oriented graph.* An oriented graph is a simple graph (no loops or multiple edges) in which each edge is replaced by an arc. Thus you produce a simple directed graph (without multiple edges or "reversed arcs").

To get the square of an oriented graph (or any directed graph) you leave the vertex set the same, keep all the arcs, and for each pair of arcs of the form (u, v), (v, w) you add the arc (u, w) if that arc was not already present.

Problem 1. *Prove that for every oriented graph, D, there exists a vertex whose out-degree at least doubles when you square the oriented graph.*

Remarks. The question was posed by Nate Dean here: `http://dimacs.rutgers.edu/~hochberg/undopen/graphtheory/graphtheory.html`.

Nate Dean learned this problem from Paul Seymour. David Fisher proved this theorem for tournaments (i.e., orientations of complete graphs).

A Fake Geometry Problem

by Neal Bushaw

A graph G is **strongly regular** if there are parameters $k, \lambda, \mu \in \mathbb{N}$ such that G is k-regular, every pair of neighbors have exactly λ common neighbors, and every pair of non-neighbors have exactly μ common neighbors. A considerable generalization of this is the **distance regular graph** – for every pair of vertices u, v at distance i, the number of vertices at both distance j from u and distance k from v is determined only by i, j, k (and not by the specific choices of u, v). Things that involve distance are geometric, so I hereby declare this a geometry related problem.

It is easy to prove that all bipartite strongly regular graphs are hamiltonian. Is the same true for distance regular graphs?

Conjecture 1. *Every bipartite distance regular graph is hamiltonian.*

(This is likely hard to prove, so I propose some simple problems to start.)

Problem 2. *Prove or disprove: every bipartite distance regular graph with diameter three is hamiltonian.*

This has been proven computationally (there are finitely many such graphs). A combinatorial proof ought to be doable, and would give nice insight to proving the higher diameter cases.

Problem 3. *Prove or disprove: every bipartite distance regular graph with diameter four is hamiltonian.*

And then, induction?

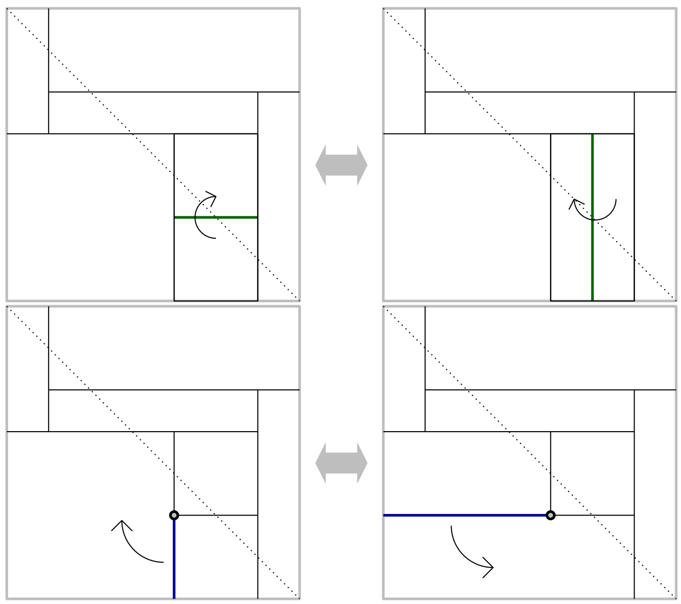
Remarks. The above problems and conjectures are closely related to conjectures of Lovász in the 1970s about vertex transitive and distance transitive graph hamiltonicity. The conjectures were generated independently in May 2018 by an automated conjecturing project at VCU. There are several related conjectures made by the software that may give additional directions for work. (I.e., is it true that every perfect distance regular graph is hamiltonian?)

The flip diameter of diagonal rectangulations

by Jean Cardinal

A diagonal rectangulation is a partition of the unit square into axis-aligned rectangles, such that every rectangle intersects the (upper left to bottom right) diagonal. Combinatorial equivalence classes of such rectangulations are in bijection with Baxter permutations and so-called twin binary trees.

The flip graph on these rectangulations has one vertex per rectangulation, and two are adjacent whenever the corresponding rectangulations differ by a flip, as illustrated below.



Problem 1. *What is the diameter of the flip graph of diagonal rectangulations with n rectangles?*

The best known upper bound is due to Ackerman et al. (LATIN'14), and is $11n$.

Chromatic number of the odd distance graph of the plane

by Gábor Damásdi

Let G be the graph whose vertices are the points of the plane and two point is connected by an edge if they are odd integer distance apart.

Problem 1. *What is the chromatic number of G? Is it even finite?*

Problem 2. *Which graphs are subgraphs of G?*

Remark 1. *$K_{n,n,n}$ is a subgraph, so every 3 colorable graph is a subgraph of G. On the other hand K_4 and the 5-wheel graph are not subgraphs. (The 5-wheel graph is a six vertex graph that you get by adding an extra vertex to C_5 and connecting it to all other vertices)*

On an Generalized Turán problem for trees

by Beka Ergemlidze

Let $ex(n, H, F)$ denote the maximum possible number of copies of H in an F-free graph on n vertices. Alon and Shikhelman started the systematic study of $ex(n, H, F)$ and showed that for any two trees H and F, we have $ex(n, H, F) = \Theta(n^m)$ where m is an integer depending on H and F.

Recently, Shikhelman asked the following question.

Question 1. *Is it true that for any graph H and tree T, $ex(n, H, T) = \Theta(n^m)$ for some integer m?*

In other words, can the exponent be a fraction if a tree is forbidden? As a first step, it would be interesting to at least prove the above statement for a large class of trees or disprove it.

References

[1] Noga Alon and Clara Shikhelman. "Many T copies in H-free graphs." Journal of Combinatorial Theory, Series B 121 (2016): 146–172.

Equilateral sets in ℓ_p^d

by Nóra Frankl

In a normed space X a subset $S \subset X$ is *equilateral* if the distance between any two points of S is one.

Problem 1 (Kusner). *Prove that the cardinality of an equilateral set in ℓ_p^d is at most $d+1$ for $2 < p < \infty$.*

Problem 2 (Kusner). *Prove that the cardinality of an equilateral set in ℓ_1^d is at most $2d$.*

Remarks. Problem 1 is easy for $p = 2$ and solved for $p = 4$ by Swanepoel. The best general upper bounds are due to Alon and Pudlák: An equilateral set in ℓ_p^d has at most $c_p d^{(2p+2)/(2p-1)}$ for some constant c_p, and an equilateral set in ℓ_1^d has at most $cd \log d$ points for some constant c.

Max-min (non) orientable genus of a (complete) bipartite graph

by Radoslav Fulek

Let $G = (V, E)$ be a simple graph. The *orientable genus* of G, denoted by $g(G)$, is the minimum $g \in \mathbb{N}$ such that G can be embedded, i.e., drawn without crossings, on an orientable surface with orientable genus g. Informally, a surface of orientable genus g is a 2-sphere with g handles. It is a folklore result that $g(G)$ exists for every graph, see [1, Section 3.4] for a more precise definition of the orientable genus and its variants.

An embedding of a connected graph on a surface is *cellular* if each of its faces is homeomorphic to an open disc. A minimum genus embedding of a connected graph is always cellular, and a cellular embedding of G is up to an orientation preserving homeomorphism of the surface determined by the set cyclic orders of the edges at the vertices of G, a.k.a. rotations. Thus, computing $g := g(G)$ corresponds to figuring out a set of rotations of its vertices that yields an embedding of G on an orientable surface of genus g.

The *maximum orientable genus* $g_{max}(G)$ of a connected graph G is the maximum g such that G has a cellular embedding on an orientable surface of genus g.

The following questions about a variant of the orientable genus posted recently on mathoverflow `https://mathoverflow.net/questions/295766/max-min-genus-of-a-bipartite-graph` by Greg Bodwin was brought to my attention by Jan Kynčl.

Let G be a connected bipartite graph with bipartition $V = V_1 \uplus V_2$. We define the *max-min genus*, denoted by $g_{max-min}(G)$ as follows. First, Player 1, who is trying to maximize genus, picks a circular ordering of the edges at each of the vertices in V_1. Then Player 2, who is trying to minimize genus, views Player 1's choice and picks a circular ordering of the edges at each of the vertices in V_2. Then $g_{max-min}(G)$ is defined as the orientable genus of G after orderings are fixed under optimal play.

Question 1. *Is there an interesting upper bound on $g_{max-min}(G)$ in terms of $|V_1|$ and $|V_2|$?*

Clearly, $g(G) \leq g_{max-min}(G) \leq g_{max}(G)$, and by Euler's formula we immediately get $g_{max}(G) \leq \left\lfloor \frac{|E|-|V|+1}{2} \right\rfloor$.

Question 2. *Can we asymptotically improve upon $g_{max-min}(G) \leq \left\lfloor \frac{|E|-|V|+1}{2} \right\rfloor$ (if G has sufficiently many edges in terms of the number of vertices)?*

Question 3. *What is the value of $g_{max-min}(K_{n,m})$?*

If this is too hard. One can try to prove non-trivial upper and lower bounds.

The variant of the problem for non-orientable genus $\widetilde{g}(G)$ [1, Section 3.4], in which the second player can also choose the signs on the edges, seems more tractable. One can start with the complete bipartite graph $K_{n,m}$, $n > 2$, $m > 2$, for which the value of the non-orientable genus is $\left\lceil \frac{(n-2)(m-2)}{2} \right\rceil$ [1, Theorem 4.4.7] and the non-orientable maximum genus $mn-n-m+1$ [1, Theorem 4.5.1].

Question 4. *What is the max-min non-orientable genus of $K_{n,m}$?*

References

[1] Mohar, B., and Thomassen, C. *Graphs on surfaces*, vol. 10. JHU Press, 2001.

Sperner families of separable sets of points

by Dániel Gerbner

Suppose we have n points on the plane. We say that a subset of the points is *separable* if it can be strictly separated from the remaining points by a line. We say that a family of sets is *Sperner* if none of the sets contains another set from the family.

Problem 1. *Given n points, what is the maximum size of a Sperner family of separable sets?*

Remarks. A separable set of cardinality k is often called k-set. Géza Tóth showed that there exists a set of n points in the plane with $ne^{\Omega(\sqrt{logk})}$ k-sets. This gives a lower bound $ne^{\Omega(\sqrt{logn})}$. On the other hand it is known that the total number of sparable sets is $O(n^2)$. More info and links about k-sets can be found here: `http://jeffe.cs.illinois.edu/open/ksets.html`.

Colouring unit-distance graphs of lattices

by Tamás Hubai

De Grey's recent breakthrough paper[1] shows that the plane cannot be 4-coloured without unit distances within a single colour class. In particular we have a computer-assisted proof that $\mathbb{Z}[\omega_1, \omega_3, \omega_4]$ is not 4-colourable where $\omega_t = \exp(i \arccos(1 - \frac{1}{2t}))$. We also know that $\mathbb{Z}[\omega_1, \omega_3]$ only has a finite number of 4-colourings, which implies the previous claim.

Problem 1. *Remove computer dependence from claims about the 4-colourings of $\mathbb{Z}[\omega_1, \omega_3]$.*

Remarks. The set of valid colourings is described at [2]. Here is a short summary. Any colouring of the triangular grid $\mathbb{Z}[\omega_1]$ that extends to a colouring of $\mathbb{Z}[\omega_1, \omega_3]$ consists of alternating stripes of two colours each, recurring with a period of 8. Rotation by ω_3 around a vertex yields the same colouring as rotation by either ω_1^2 or ω_1^{-2}, with the same choice for all vertices.

Even if a ring $S = \mathbb{Z}[z_1, \dots, z_t]$ has an infinite number of colourings, we are interested in its asymptotics, so we define the colouring freedom per vertex as $\lambda_k(S) = \limsup \frac{\log c_k(S_i)}{|S_i|}$ where c_k is the number of k-colourings and $S_i \subset S$ denotes the set of vertices with description length $\leq i$. For instance, $\lambda_4(\mathbb{Z}) = 3$, $\lambda_4(\mathbb{Z}[\omega_1]) \approx 1.5$ and $\lambda_4(\mathbb{Z}[\omega_1, \omega_3]) = 0$.

Problem 2. *We know that if S has a finite number of k-colourings then the plane cannot be k-coloured. Does the same hold for $\lambda_k(S) = 0$, i.e. a subexponential number of k-colourings?*

Problem 3. *Can we bound $\lambda_k(S)$ from above using $\lambda_k(\mathbb{Z}[z_1, \dots, z_{t-1}])$?*

References

[1] de Grey, The chromatic number of the plane is at least 5, `arxiv.org/abs/1804.02385`

[2] Polymath16 blog, fourth thread, `dustingmixon.wordpress.com/?p=4902#comment-4366`

[3] Polymath16 wiki, `michaelnielsen.org/polymath1/?Hadwiger-Nelson_problem`

Proper 3-coloring points with respect to disks and related problems

by Balázs Keszegh

Problem 1. *Is it true, that there exists an m such that for any pseudo-disk arrangement any finite set of points admits a 3-coloring such that any pseudo-disk that contains at least m points contains two points with different colors.*

For pseudo-disks $m = 3$ is not enough as observed by Géza Tóth, but $m = 4$ might be enough. For disks $m = 2$ is not enough but $m = 3$ might be enough.

This conjecture also has a natural dual counterpart:

Problem 2. *The members of any pseudo-disk arrangement admit a 3-coloring such that any point that is contained in at least m pseudo-disks is contained in two pseudo-disks with different colors.*

Remarks. These problems are asked in [4], for disks asked already in [2], which case is also still open. **Solving these problems for disks would be already very nice.**

At least for homothets of a convex polygon we know that there is such a 3-coloring (for some m depending on the polygon) [4]. In both the primal [7] and dual cases such 2-colorings do not exist (whatever is m) even if instead of all disks we just take unit disks [5]. If 4 colors can be used then we know that this is possible even for $m = 2$. E.g., in the point coloring case for disks, 4-coloring the (planar) Delaunay-graph is good, for the dual setting see [1] for disks and [6, 3] for pseudo-disks. In [3] a common generalization is posed about coloring intersection hypergraphs of two pseudo-disk families. Even in this general case 4 colors are enough [3], but with 3 colors the problem is open.

References

[1] Guy Even, Zvi Lotker, Dana Ron, and Shakhar Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM Journal on Computing*, 33(1):94–136, jan 2003.

[2] Balázs Keszegh. Coloring half-planes and bottomless rectangles. *Computational geometry*, 45(9):495–507, 2012.

[3] Balázs Keszegh. Coloring intersection hypergraphs of pseudo-disks. In *Symposium on Computational Geometry*, 2018, to appear, `https://arxiv.org/abs/1711.05473`

[4] Balázs Keszegh and Dömötör Pálvölgyi. Proper coloring of geometric hypergraphs. In *Symposium on Computational Geometry*, volume 77 of *LIPIcs*, pages 47:1–47:15. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.

[5] János Pach and Dömötör Pálvölgyi. Unsplittable coverings in the plane. In Ernst W. Mayr, editor, *Graph-Theoretic Concepts in Computer Science - 41st International Workshop, WG 2015, Garching, Germany, June 17-19, 2015, Revised Papers*, volume 9224 of *Lecture Notes in Computer Science*, pages 281–296. Springer, 2015.

[6] Shakhar Smorodinsky. On the chromatic number of geometric hypergraphs. *SIAM Journal on Discrete Mathematics*, 21(3):676–687, 2007.

[7] János Pach, Gábor Tardos, and Géza Tóth. *Indecomposable Coverings*, pages 135–148. Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.

Poset Ramsey numbers

by Dániel Korándi

For a poset P, the *k-color poset Ramsey number* $R_k(P)$ is the smallest integer N such that any k-coloring of the elements of the N-dimensional Boolean lattice Q_N (i.e., the poset on the power set $2^{[N]}$ of $[N] = \{1, \dots, N\}$ with the relation $\subseteq$) contains a monochromatic *induced* copy of P. Axenovich and Walzer [1] showed (in 10 lines) that $2n \leq R_2(Q_n) \leq n^2 + 2n$

Problem 1. *Improve the lower or the upper bound for $R_2(Q_n)$ (or both).*

As for more colors, Cheng et al. [2] proved $R_3(Q_2) = 6$, and asked if $R_k(Q_2) = 2k$ (the 2-colored case $R_2(Q_2) = 4$ was observed in [1]).

Problem 2. *Prove that $R_k(Q_2) > 2k$.*

Of course, it would also be interesting to prove $R_k(Q_2) = 2k$, or any other good bound on this quantity. It is known [1] that $R_k(P) = \Theta(k)$ for every poset P.

References

[1] Axenovich, Maria; Walzer, Stefan, Boolean lattices: Ramsey properties and embeddings. Order 34. 2017, 287-298.

[2] Cheng, Yen-Jen; Li, Wei-Tian; Liu, Chia-An; Wu, Zi-Ying, Ramsey-type of problems on posets in the Boolean lattices. In preparation.

Reversing permutations

by István Kovács

Two permutations π_1, π_2 of $[n]$, as vectors of length n are reversing if there are two coordinates which contain the same elements in π_1 and π_2, but in reversed order. For example: $\{8,7,6,5,4,3,2,1\}$ and $\{8,7,4,5,6,3,2,1\}$ are reversing since in both permutations the third and fifth coordinates contain the same two elements (4 and 6), but the order of these two elements in the two permutations is different.

Conjecture 1. *[J. Körner] There is a constant C such that the maximal number of pairwise reversing permutations of $[n]$ is at most C^n*

It is not hard to construct an exponentially large pairwise reversing family of permutations as the "Cartesian product" of two pairwise reversing constructions is also pairwise reversing. The best upper bound follows from the following result of Cibulka.

Theorem 1 (Cibulka)**.** *The maximal number of pairwise not reversing permutations of $[n]$ is $n^{n/2+o(n)}$.*

Cibulka's result can be utilized as follows. Let G be a graph whose vertices are the permutations of $[n]$ and two vertices are connected by an edge when the corresponding permutations are reversing. The maximal number of pairwise reversing permutations is G's clique number: $\omega(G)$. Cibulka's result says that G's independence number, $\alpha(G) \approx n^{n/2}$. It is easy to see that G is vertex-transitive. It is a folklore result that for vertex-transitive graphs $\omega(G)\alpha(G) \leq |V(G)|$. (For our proof, the lower bound on $\alpha(G)$ is enough which is the easier part of Cibulka's result.)

An equivalent reformulation: Two perfect matchings on $2n$ vertices are C_4-creating if their union (the union of their edges) contains C_4 as a subgraph.

Conjecture 2. *There is a constant D such that the maximal number of pairwise C_4-creating perfect matchings on $2n$ vertices is at most D^{2n}*

It is not hard to prove and it is certainly not hard to feel that Conjecture 1 is equivalent to Conjecture 2 but we will not prove their equivalence now. The advantage of the language of conjecture 2 is that it is easier to see that there is a large number of paiwise not C_4-creating perfect matchings. For many values of $2n$, there are bipartite, C_4-free and (roughly) $\sqrt{2n}$-regular graphs by a theorem or Reiman. Using the van der Warden theorem, it is easy to show that any such graph must contain roughly $\sqrt{2n}^{2n} = (2n)^n$ perfect matchings. These perfect matchings are clearly not C_4-creating since not just every pairwise union, but the union of the whole system is also C_4-free.

It is also unknown whether the maximal number of pairwise C_{2k}-creating perfect matchings is less than an exponential function of the ground set.

Tilings with noncongruent triangles

by Andrey Kupavskii

Problem 1. *Is it true that in any tiling of the plane by triangles of unit area and perimeter*

(i) there are two isometric triangles of the same orientation?

(ii) there are k pairwise isometric triangles for any $k \in \mathbb{N}$?

Together with J. Pach and G. Tardos [1] we showed that in such a tiling there are no two triangles sharing a side, and thus no two isometric triangles (but, potentially, with different orientation). This answered a question of R. Nandakumar [2].

References

[1] A. Kupavskii, J. Pach, G. Tardos, *Tilings with noncongruent triangles*, to appear in European Journal of Combinatorics, arXiv:1711.04504

[2] R. Nandakumar, *Filling the plane with non-congruent pieces*, Blog entries, `http://nandacumar.blogspot.in`, December 2014, January 2015, June 2016

Two games on a square grid

by Dániel Lenger

Square achievement game: Two players alternately write O's (first player) and X's (second player) in the unoccupied cells of an $n \times n$ grid. The first player (if any) to occupy four cells at the vertices of a square with horizontal and vertical sides is the winner. What is the outcome of the game given optimal play?

Problem 1. *What is the outcome of the Square achievement game if both player play with optimal strategy?*

`http://www.openproblemgarden.org/op/a_game_on_an_n_x_n_grid`

Transversal achievement game: Two players alternately write O's (first player) and X's (second player) in the unoccupied cells of an $n \times n$ grid. The first player (if any) to occupy a set of n cells having no two cells in the same row or column is the winner.

Problem 2. *What is the outcome of the Transversal achievement game if both player play with optimal strategy?*

`http://www.openproblemgarden.org/op/a_transversal_achievement_game_on_a_square_grid`

Remarks. Both question were asked by Martin Erickson.

For the Square achievement game there are some known results when n is small, mostly proved by computer: `http://archive.ysjournal.com/article/an-investigation-of-ericksons-square-game-using-the-minimax-algorithm/`

Roland Bacher and Shalom Eliahou proved that every 15×15 binary matrix contains four equal entries (all 0's or all 1's) at the vertices of a square with horizontal and vertical sides. So the game must result in a winner for $n \geq 15$.

Bichromatic intersections

by Leonardo Martínez

This problem is related to the Colorful Helly Theorem, and more precisely, to a recent development that was accepted for SoCG 2018 and can be found on the following link: `https://arxiv.org/abs/1803.06229`. In that paper the authors state a dimensionality conjecture. This proposal concerns the first unknown case of the conjecture.

Let F and G be two finite families of convex sets on $\mathbb{R}^3$. We say that F and G have the *colorful intersection hypothesis* if for every $A \in F$ and $B \in F$ the intersection $A \cap B$ is non-empty.

Problem 1. *Can we find a constant k or a positive real number $\alpha \in (0, 1)$ for which any of the following results is true?*

- *If F and G have the colorful intersection hypothesis, then either F or G can be pierced with k lines.*
- *If F and G have the colorful intersection hypothesis, then either F has a transversal line through at least $\alpha|F|$ sets or G has a transversal line through at least $\alpha|G|$ sets.*
- *If F is pairwise intersecting, then F can be pierced with k lines.*
- *If F is pairwise intersecting, then F has a transversal line through at least $\alpha|F|$ sets.*

Remarks.

Clearly $i)$ is the strongest and implies the rest, but proving any of the results would be interesting.

Under the colorful intersection hypothesis, a projection argument combined with the Colorful Helly Theorem on the line yields that for any vector v, for at least one of F or G we can find a transversal hyperplane through every set.

Luis Montejano proved that if $|F| = |G| = 3$ and they satisfy the colorful intersection hypothesis, then there is either a line through the three sets of F or a line through the three sets of G. This result suggests that the conjecture could be "dimensionally" correct. Unfortunately, this result cannot be applied directly to the problem since there cannot be any (p, q)-like theorem for lines on $\mathbb{R}^3$.

Colored spanning trees

by Viola Mészáros

There are n vertices colored by $2 < k < n$ colors. A *multicolored spanning tree* is a spanning tree where the endpoints of each edge are of distinct colors. A *minimum (or maximum)* spanning tree is a spanning tree with minimum (or maximum) total edge length.

Problem 1. *Compute the minimum (or maximum) multicolored spanning tree with $k > 2$ colors.*

In fact the minimum or maximum spanning tree is to be computed in a complete $k-partite$ geometric graph.

Remarks.

Biniaz, Bose, Eppstein, Maheshwari, Morin and Smid present $\Theta(n \log n)$-time algorithms that solve the minimum and maximum spanning tree problems for $k = 2$, and they give $O(n \log n \log k)$-time algorithms for $k > 2$. A faster algorithm is desired when $k > 2$ or a matching lower bound.

You may find the previous results here: `https://arxiv.org/pdf/1611.01661.pdf`.

On an Extremal Problem for Poset Dimension

by Abhishek Methuku

Let $f(n)$ be the largest integer such that every poset on n elements has a 2-dimensional subposet on $f(n)$ elements. What is the asymptotics of $f(n)$? This question is due to Dorais.

Applying Dilworth's theorem it is easy to show that $f(n) \geq \sqrt{n}$. Guśpiel, Micek and Polak showed the best known upper bound: $f(n) \leq 4n^{2/3} + o(n^{2/3})$. Their main idea was to take a $(k \times k \times k)$-cube with a natural order on its elements and show that it does not contain a large 2-dimensional subposet by using a multidimensional version of Marcus-Tardos theorem first proved by Klazar and Marcus.

In summary the best known bounds on $f(n)$ are the following.

Theorem 1.

$$\sqrt{n} \leq f(n) \leq 4n^{2/3} + o(n^{2/3}).$$

It would be interesting to determine if the lower bound is true.

References

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Lower bound on point guards in orthogonal art galleries

by Tamás Róbert Mezei

An *orthogonal polygon* in the plane is a polygon composed of axis-parallel lines segments, such that the line segments only intersect in their end-vertices. An *orthogonal art gallery* is a region bounded by an orthogonal polygon. Two points in a gallery have *r-vision* of each other, if the minimal axis-parallel rectangle containing both of them is contained in the gallery. A *point guard* is a point in the gallery equipped with r-vision (the region covered by a point guard is called an r-star). A *horizontal mobile guard* (alternatively, *vertical*) is a horizontal line segment contained in the gallery; it covers a point x in the gallery if x is r-visible from a point on its line segment.

Let P be an orthogonal art gallery. Let p be the minimum number of point guards required to cover P. Let m_H (alternatively, m_V) be the minimum number of horizontal (vertical) mobile guards required to cover P. Informally, the goal of the following problem is to understand the relationship between these parameters.

Problem 1. *Find a non-trivial lower bound on p which is a function of m_V, m_H, and/or some other attributes of P.*

Motivation. With Ervin Győri[1], I recently proved that $p \leq \lfloor \frac{4}{3}(m_V + m_H - 1) \rfloor$. The upper bound can be computed in linear time. The result is sharp, as demonstrated on Figure 1. It would be nice to have an estimate complementing this upper bound. Trivially, $\max\{m_H, m_V\} \leq p$. Figure 2 demonstrates that without using other parameters nothing stronger holds. This weak inequality and our previously mentioned result already imply that $\frac{4}{3}(m_V + m_H - 1)$ is an $\frac{8}{3}$-approximation of p. I expect that a non-trivial lower bound will help us come up with an upper bound (preferably one which can still be computed in linear time) which approximates p to a tighter factor than $\frac{8}{3}$.

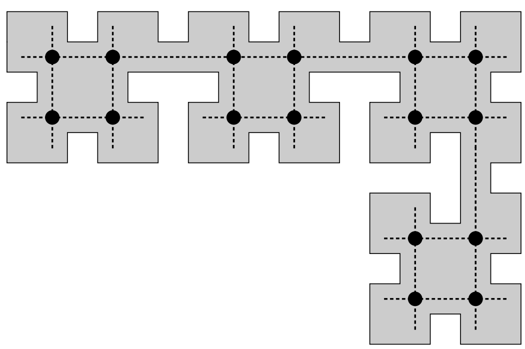

Figure 1: $m_V + m_H = 13$, $p = 16$

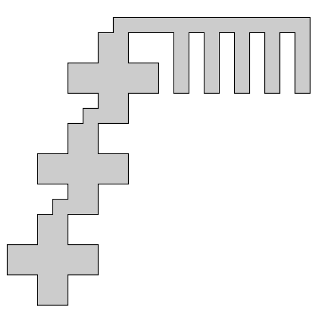

Figure 2: $m_V, m_H \geq 1$ arbitrary, $p = \max\{m_V, m_H\}$

[1] https://trm.hu/publication/mobile-vs-point-guards/

Discrete square boundaries

by Dániel T. Nagy

Let $B \subset \mathbb{Z}^2$ be a finite set and let S be the set of points that are centers of (discrete) square boundaries contained in B.

Problem 1. *How small can $|B|$ be if $|S|$ is given?*

In [1] we showed that $|B| \geq \Omega\left((|S|/\log(|S|))^{7/8}\right)$, and there are constructions for $|B| \leq O(|S|^{7/8})$. We should look for the exact order of magnitude and possibly for strong constants in the bounds.

If this proves to be too easy, there is similar problem in n dimensions about k-skeletons of cubes and their centers. Thornton [2] showed constructions with $|B| \leq O\left(|S|^{1-\frac{n-k}{2n^2}}\right)$ and proved that $|B| \geq \Omega\left(|S|^{\alpha}\right)$ holds for every $\alpha < 1 - \frac{n-k}{2n^2}$.

References

[1] T. Keleti, D.T. Nagy, P. Shmerkin, JAMA (2018) 134 (2) 643-669.

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Crossing families of triangles

by Zoltán Lóránt Nagy

Problem 1. *Consider N points in the 3-D space so that no four points lie on a plane. Each triple of points determines a triangle. How large a family of mutually crossing triangles must there be? We say that two triangles cross if they intersect on their interiors.*

Remarks. The 2-D variant of this problem, concerning line segments of a plane determined by point pairs from a planar point set, is discussed for example in the paper of B. Aronov, P. Erdős, W. Goddard, D. J. Kleitman, M. Klugerman, J. Pach, and L. J. Schulman, namely in Crossing Families, Combinatorica 14 (1994), 127-134. It has been shown that there must always be a family of size $\sqrt{N/12}$, but it is believed that there must always be families of much larger size as well.

A Helly type question on piercing certain set families in $\mathbb{R}^d$

by Márton Naszódi

The *transversal (or piercing) number* of a set family $\mathcal{F}$ is the minimum number t such that there is a t-point set that intersects all members of $\mathcal{F}$.

Problem 1. *Let $\mathcal{F}$ be a finite family of*

(a) half-spaces in $\mathbb{R}^d$,

(b) half-spaces in $\mathbb{R}^d$ not containing the origin,

(c) spherical caps on $\mathbb{S}^{d-1}$,

(d) translates of a convex body/convex polytope/Euclidean ball/... in $\mathbb{R}^d$,

(e) positive homothets of a convex body/convex polytope/Euclidean ball/... in $\mathbb{R}^d$,

(f) your favorite objects.

Is there a Helly–type theorem for piercing $\mathcal{F}$? More formally, is there a $c = c(k, d)$, such that if any subfamily of c members of $\mathcal{F}$ has a transversal of size at most k, then so does $\mathcal{F}$?

Warning 1: If $\mathcal{F}$ is any family of convex sets, then there is no such c, even for $d = k = 2$. See Klee's rosette on p.12 of [1].

Warning 2: A family of translates of a convex polytope in $\mathbb{R}^3$ may have arbitrarily large VC–dimension [2].

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[2] Naszódi, M., and Taschuk, S. On the transversal number and VC-dimension of families of positive homothets of a convex body. *Discrete Math. 310*, 1 (2010), 77–82.

Counting subgraphs

by Cory Palmer

Let H and F be graphs. Denote the maximum number of copies of the graph H in an n-vertex F-free graph by $\mathrm{ex}(n, H, F)$.

When H is the graph of a single edge, K_2, then this is just the ordinary Turán function, $\mathrm{ex}(n, F)$. The systematic study of $\mathrm{ex}(n, H, F)$ (for graphs beyond K_2) was initiated by Alon and Shikhelman. A famous example counts pentagons in triangle-free graphs. Hatami, Hladký, Kráľ, Norine and Razborov and independently Grzesik proved,

$$\mathrm{ex}(n, C_5, K_3) \leq \left(\frac{n}{5}\right)^5.$$

For complete graphs, Erdős showed, that if $t < k$, then

$$\mathrm{ex}(n, K_t, K_k) = \binom{k-1}{t}\left(\frac{n}{k-1}\right)^t + o(n^t).$$

However, many specific cases for H and F remain open. A example is,

Problem 1. *Determine* $\mathrm{ex}(n, C_5, K_4)$.

This is also interesting for other cycles C_k and complete graphs K_t. Another (likely much harder) problem posed by Erdős,

Problem 2. *Determine* $\mathrm{ex}(n, K_3, K_{r,r,r})$.

Bichromatic pencils

by Dömötör Pálvölgyi

A *pencil* is a collection of some lines through a point, called the *center* of the pencil. If the points of the plane are colored, then call a pencil *bichromatic* if there is a color that is present on all the lines of the pencil such that this color is different from the color of the center of the pencil.

Problem 1. *Given any non-monochromatic coloring of the plane with finitely many colors, and m directions, $\alpha_1, \ldots, \alpha_m$, is it true that there is a point p and an angle φ such that the pencil determined by the lines of direction $\alpha_1 + \varphi, \ldots, \alpha_m + \varphi$ through p is bichromatic?*

Remarks. I can only prove the statement for $m = 2$; a natural easiest open case is when there are $m = 3$ lines that close a $60°$ angle.

The question is related to polymath16, you can see here how: `https://dustingmixon.wordpress.com/2018/05/05/polymath16-fourth-thread-applying-the-probabilistic-method/#comment-4306`.

I've already posed this problem on mathoverflow: `https://mathoverflow.net/questions/299616/bichromatic-pencils`.

In fact, it might be even possible to get that all the lines of the pencil are intersected by some another line such that the intersection points are all of the same color (and this color differs from the color of the center of the pencil). When can, in general, ask under what conditions it is possible to find a similar copy of an *almost monochromatic* finite point configuration (S, s_0), where almost monochromatic means that all the points of S have the same color, and the color of s_0 is required to be different. I cannot even answer this if $(S, s_0) \subset \mathbb{R}$. Naturally, I've posed also this problem on mathoverlow: `https://mathoverflow.net/questions/300604/almost-monochromatic-point-sets`.

Independent sets in tangled grids

by Dömötör Pálvölgyi

A poset P is called an $n \times n$ *tangled grid* if it can be partitioned into chains $A_1, \dots, A_n$, and also into chains $B_1, \dots, B_n$, which have the additional property that $|A_i \cap B_j| \le 1$ for any i, j.

Problem 1. *What is the maximum number $f(n)$ of antichains that can occur in an $n \times n$ tangled grid?*

It was observed in [2] that $f(n)$ also gives an upper bound for the maximum possible number of stable matching among n men and n women. Here the A_i correspond to the men and the B_j to the women of the stable matching, and every intersection corresponds to an operation called *rotation*. In fact, since in each rotation there are at least two-two men and women, some elements of this poset should be contracted, but for an upper bound it will do.

It was proved in [2] that $f(n) \le C^n$ for some large enough C. The goal would be to determine the best possible C, which I conjecture to be 4 (with possibly some polynomial multiplicative factor). This is attained in the (untangled) $n \times n$ grid ordered as a diamond (with a unique smallest and largest element), there the answer is $\binom{2n}{n}$.

A possible approach to bound $f(n)$ is to denote the maximum number of antichains among $n \times n$ tangled grids with m elements by $F(m) = F_n(m)$ (note that $f(n) \le F(n^2)$) and apply the counting argument used also for the famous proof of the Crossing-lemma `https://en.wikipedia.org/wiki/Crossing_number_inequality#Proof`. This, however, doesn't give any good bounds on C. Nevertheless, I sketch it below.

We obviously have $F(m) \le 2^m$. Denote by r the number of $(x, y) \in \binom{P}{2}$ that are in *strict relation*, that is, for which $x <_P y$ and there is no i or j for which $x, y \in A_i$ or $\in B_j$ (i.e., they are not contained in the same chain). It is easy to see that $r \ge m - 2n$. If we keep every chain with probability $p = \frac{3n}{m}$, then the new poset will have $n' = pn$ chains[2], $m' = p^2 m$ elements and $r' = p^4 r$ strict relations. The inequality $r' \ge m' - 2n'$ is equivalent to $p^4 r \ge p^2 m - 2pn$, which gives $r \ge \frac{m^3}{27n^2}$. This means that for some element $p \in P$ is in (strict) relation with at least $\frac{m^2}{27n^2}$ other elements. Depending whether p is a part of the antichain of not, we get $F(m) \le F(m-1) + F(m - 1 - \frac{m^2}{27n^2})$ (using the convexity of F). This is practically the same recursion as the one obtained in [1], which finishes the proof. Unfortunately, the exponent is quite bad, and it has been improved very little, so this approach might not give any good bound.

In [2] they obtain the weaker recursion that some element $p \in P$ is in relation with $\Omega\left(\frac{m^{\frac{3}{2}}}{n^{\frac{3}{2}}}\right)$ other elements, but both from above and below, which gives a simpler but weaker recursion.

References

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[2] Anna R. Karlin, Shayan Oveis Gharan, Robbie Weber: A Simply Exponential Upper Bound on the Maximum Number of Stable Matchings. `https://arxiv.org/abs/1711.01032`

[2]Here we cheat a bit, as the number of A_i and B_j chains can differ!

The Chen-Chvátal conjecture for set systems

by Balázs Patkós

A famous theorem of DeBruijn and Erdős states that every set of n non-collinear points in the plane determine at least n lines. One can define lines in metric spaces in the following way: if $(\mathcal{M}, d)$ is a metric space with $x, y \in \mathcal{M}$, then

$$\begin{aligned}\overline{xy} = \{x, y\} \cup \{z \in \mathcal{M} : d(x, y) + d(y, z) = \\ = d(x, z)\} \cup \{z \in \mathcal{M} : d(x, z) + d(y, z) = \\ = d(x, y)\} \cup \{z \in \mathcal{M} : d(x, y) + d(x, z) = d(y, z)\}\end{aligned}$$

The Chen-Chvátal conjecture states that if there is no pair $x, y \in \mathcal{M}$ with $\overline{xy} = \mathcal{M}$, then $\mathcal{M}$ contains at least $|\mathcal{M}|$ distinct lines. I would be pretty happy to see the following special case solved: $\mathcal{M}$ is a family of finite sets and d is the Hamming distance. (More importantly, Vašek Chvátal would also be delighted.)

Tiling $\mathbb{Z}^n$

by István Tomon

A *tile* T is a finite subset of the discrete integer lattice $\mathbb{Z}^n$. A subset T' of $\mathbb{Z}^n$ is a *copy* of T if T' is isometric to T. Recently, it was proved by Leader, Gruslys and Tan [1] that for any tile $T \subset \mathbb{Z}^d$ there exists a positive integer n such that $\mathbb{Z}^n$ can be completely tiled with copies of T. In particular, they proved that if $T \subset [k]^d$, then we can choose $n = \exp(100(d \log k)^2)$.

Let $n(T)$ denote the smallest n for which $\mathbb{Z}^n$ can be tiled with copies of T. What can we say about $n(T)$ in general? Is it true that $n(T)$ can be bounded by a function of $|T|$ and d alone? This question seems already challenging for one dimensional tiles.

Conjecture 1. *(Gruslys,Leader, Tan) For every positive integer t there exists a positive integer $n(t)$ such that if $T \subset \mathbb{Z}$ with $|T| = t$, then $n(T) \leq n(t)$.*

Also, there are no examples (to the best of my knowledge) of tiles for which $n(T)$ is large, that is, for which $n(T)$ is super-polynomial in k (or in $|T|$). It would be interesting to find families of tiles which have particularly bad properties regarding tiling.

References

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Saturation version of the Erdős-Szekeres theorem

by Casey Tompkins

Problem 1. *Let $f(n)$ denote the minimum number of points one can take in the plane in general position and containing no convex n-gon, such that the addition of any further point yields a convex n-gon. Find good bounds (for starters, the order of magnitude) of this function.*

Remarks. For $n \leq 5$ the value of $f(n)$ is equal to that of the Erdős-Szekeres extremal problem. I think (but can no longer reconstruct) that for $n = 6$ the functions differ. My guess is that $f(n)$ is linear in n or at worst polynomial.

One could consider the weaker condition where the point set is allowed to contain a convex n-gon, but any further point must yield a new convex n-gon. However, in a conversation years ago with David Malec, we noticed that here the answer is to simply take $2n - 4$ points in a circle. It is easy to check that such a configuration is (weakly) saturated. Moreover, if we have a set S of only $2n - 5$ points, then it cannot contain both an $n - 1$ cap and an $n - 1$ cup since a cap and a cup can intersect in at most 2 points. Thus, we could add a new point far north or far south of the configuration without creating a convex n-gon.

Edges of multigraphs

by Michael Kaufmann, Torsten Ueckerdt, János Pach, Géza Tóth

A drawing of a graph G, with possible parallel edges, but no loops, is called *nice*, if
1. parallel edges do not cross,
2. both of the two regions determined by two parallel edges contain a vertex of G,
3. any two non-parallel edges cross at most once.

Determine the maximum number of edges of a graph G of n vertices that has a nice drawing. Our best bounds are cn^2 and cn^3.

János Pach, Géza Tóth: A crossing lemma for multigraphs, SoCG 2018.

Geometric Ramsey numbers

by Tomáš Valla

Complete geometric graph K_P on point set $P \subset \mathbb{R}^2$ in general position is a complete graph with vertex set P, whose edges are drawn as straight-line segments. If P is in convex position, then K_P is a *convex complete geometric graph*.

Given a graph G, the *geometric Ramsey number* of G, denoted by $R_g(G)$, is the smallest integer n such that every complete geometric graph K_P on n vertices with edges arbitrarily coloured by two colours contains a monochromatic non-crossing copy of G. The *convex geometric Ramsey number* of G, denoted $R_c(G)$, is defined analogously, only K_P is convex complete geometric graph.

Problem 1. *Does there exist a polynomial $p(n)$, such that for every n and every outerplanar graph G on n vertices, the geometric Ramsey number satisfies $R_g(G) \leq p(n)$?*

Problem 2 (Károlyi)**.** *Does there exists a universal constant c such that $R_g(G) < cn^2$ for every outerplanar graph G with n vertices?*

Problem 3. *For a path P_n it is known that $R_c(P_n) = 2n - 3 \leq R_g(P_n) \leq O(n^{3/2})$. Try to improve the bounds.*

The last (?) paper on the topic is `https://arxiv.org/pdf/1308.5188.pdf`.

Turán numbers of ordered 6-cycles

by Máté Vizer

An *ordered graph* is a simple graph $G = (V, E)$ with a linear ordering on its vertex set. We say that the ordered graph H is an *ordered subgraph* of G if there is an embedding of H in G that respects the ordering of the vertices. The Turán problem for a set of ordered graphs $\mathcal{H}$ asks the following. What is the maximum number $ex_<(n, \mathcal{H})$ of edges that an ordered graph on n vertices can have without containing any $H \in \mathcal{H}$ as an ordered subgraph? When $\mathcal{H}$ contains a single ordered graph H, we simply write $ex_<(n, H)$.

The *interval chromatic number* of an ordered graph H, is the minimum number of intervals the (linearly ordered) vertex set of H can be partitioned into, so that no two vertices belonging to the same interval are adjacent in H.

Pach and Tardos [2] started the systematic study of the Turán numbers of ordered graphs. For example they proved an ordered analogue of the Erdős-Stone-Simonovits theorem. A consequence of this result is that the Turán number of ordered graphs with interval chromatic number larger than 2 is asymptotically determined.

In [1] we investigated the Turán number of some families of ordered 6-cycles with interval chromatic number 2.

Problem 1. *What is the order of magnitude of $ex_<(n, C)$, where C is an ordered 6-cycle with interval chromatic number 2?*

You can find the "conjecture version" of this problem in [1] (Conjecture 2).

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[1] E. Győri, D. Korándi, A. Methuku, I. Tomon, C. Tompkins, M. Vizer. On the Turán number of some ordered even cycles. *European Journal of Combinatorics*, accepted.

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Cover half graph with complete bipartite graph

by Zhiyu Wang

A *difference graph* $H(a, b; f)$ is a bipartite graph on $a + b$ vertices with partite sets $U = \{u_1, \ldots, u_a\}$ and $W = \{w_1, \ldots, w_b\}$, equipped with a non-increasing function $f : [a] \to [b]$ such that $f(1) = b$ and, for all $i \in [a]$, $N(v_i) = \{w_1, \ldots, w_{f(i)}\}$ if $f(i) \geq 1$. The definition of H above is symmetric with respect to the roles of U and W. That is, if $H(a, b; f)$ is a difference graph, then the function $g(j) := \max\{i : f(i) \geq j\}$ witnesses that $H(b, a; g) = H(a, b; f)$.

A *difference graph cover* of a graph G is a family $\mathcal{H}$ of subgraphs of G such that $E(G) = \bigcup_{H \in \mathcal{H}} E(H)$ and each H is a difference graph. For a vertex $v \in G$, we use $mult(v, \mathcal{H})$ to denote the number of difference graphs in $\mathcal{H}$ that contain v. The *local difference graph cover number* of G, denoted by $ldc(G)$ is defined as

$$ldc(G) = \min \left\{ \max_{v \in V(G)} \{mult(v, \mathcal{H})\} : \mathcal{H} \text{ is a difference graph cover of } G \right\}$$

Similarly, the *local bipartite graph cover number* of G, denoted by $lbc(G)$ is defined as

$$lbc(G) = \min\left\{ \max_{v \in V(G)} \{mult(v, \mathcal{H})\} : \mathcal{H} \text{ is a bipartite graph cover of } G \right\}$$

Because every nonempty complete bipartite graph is a difference graph, it is clear that $ldc(G) \leq lbc(G)$ for every graph G. It is not very hard to show that for every difference graph $H = H(m, n; f)$, we have $lbc(H) \leq \lceil \log_2(m+1) \rceil$, noting $ldc(H) = 1$. As a result, for any graph G with v vertices, $lbc(G)/ldc(G) = O(\log v)$.

Proposition 1. *Let $H = H(m, n; f)$ be a difference graph. Then $lbc(H) \leq \lceil \log_2(m+1) \rceil$. Consequently, for all graphs G on v vertices,*

$$ldc(G) \leq lbc(G) \leq ldc(G) \lceil \log_2(v/2+1) \rceil .$$

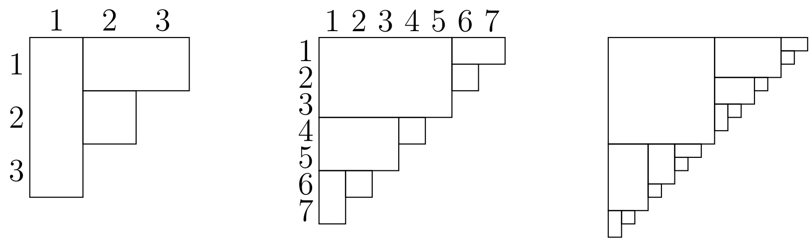

Figure 1: Young diagrams are given which represent complete bipartite graph covers (partitions, in fact) of the edge set of $H_n = H(n, n, f_n)$ with $f_n(i) = n + 1 - i$ for $n = 3, 7, 15$, respectively. The cases for $n = 3, 7$ are labeled. The cover for H_3 corresponds to the graphs $\{1, 2, 3\} \times \{1\}$, $\{1\} \times \{2, 3\}$, and $\{2\} \times \{2\}$. The cover for H_{15} shows $lbc(H_{15}) \leq 3$.

Remark 2. *It may be convenient to visualize $H = H(m, n; f)$ as a Young diagram in which the i^{th} row has length $f(i)$, for $i \in [m]$, so each square corresponds to an edge in the difference graph. (See Figure 1.) A complete bipartite graph cover is equivalent to a cover of the Young diagram with generalized rectangles. That is, a bipartite graph corresponds to the product set $S \times T$ so that $S \subseteq [m]$, $T \subseteq [n]$ and $S \times T$ is contained entirely in the Young diagram. Then $lbc(H)$ is the maximum number of generalized rectangles in any row or column.*

In Proposition 1, we establish an upper bound on $lbc(H)$ for difference graphs H that is logarithmic in the smallest partition class, however it is not clear whether this bound is achieved. We would like to determine the largest value of $lbc(H)$ over all difference graphs H. For difference graph $H_n = H(n, n; f_n)$ with $f_n(i) = n + 1 - i$, the construction in Figure 1 for H_{15} can be extended to show $lbc(H_n) \leq \log(n+1) - 1$ when $n + 1$ is a power of 2 and $n \geq 15$, but the following question remains:

Problem 1. *Let $n + 1$ be a power of 2 and let $H_n = H(n, n; f_n)$ be the difference graph such that $f_n(i) = n + 1 - i$. What is the exact value of $lbc(H_n)$?*

This problem was raised by Heather Smith in our joint paper in [1] .

References

[1] J. Kim, R. Martin, T. Masarik, W. Shull, H. Smith, A. Uzzell and Z. Wang. On Difference graphs and the local dimension of posets. *Arxiv: 1803.08641*

Obstructions to shellability

by Russ Woodroofe and Andrés David Santamaría-Galvis

Before stating some questions, let us recall that...

A *simplicial complex* on a finite vertex set V is a set family $\Delta \subseteq \mathcal{P}(V)$ which is closed under taking subsets. A member σ of Δ is a *face* of Δ, and it has *dimension* $\dim \sigma := |\sigma| - 1$. Notice that the empty set is also a face of Δ with $\dim \varnothing = -1$. A maximal face with respect to inclusion is a *facet* and we say that a complex is *pure* if all the facets have the same dimension. For a face τ of Δ, the *link* of τ in Δ is defined by $\operatorname{link}_\Delta(\tau) := \{\gamma \in \Delta : \gamma \cap \tau = \varnothing\, , \gamma \cup \tau \in \Delta\}$

In a simplicial complex, a sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ of the facets is called a *shelling* if it satisfies that $\left(\bigcup_{i=1}^{j-1} \overline{\sigma_i}\right) \cap \overline{\sigma_j}$ is a pure $(\dim \sigma_j - 1)$-dimensional subcomplex for every $2 \leq j \leq t$, with $\overline{\sigma}$ denoting the set of all faces included in σ. The complex is *shellable* if it has a shelling.

An *obstruction to shellability* (OTS) is a minimal non-shellable complex with the property that every induced subcomplex is shellable. A related notion is of a *cd-obstruction to shellability* (cd-OTS), which is non-shellable but has every proper induced subcomplex and every proper link shellable.

The OTS notion was introduced by Wachs in [1] almost 20 years ago. She showed that for every positive integer d there is an OTS of dimension d. Then, she asked whether there are finitely many OTS in each dimension. Nowadays it is known only for very low dimensions. For instance, there is no obstructions to shellability of dimension 0, a unique obstruction of dimension 1, and OTS of dimension 2 have at most 7 vertices.

Question 1. *From the previous discussion, it is almost natural to ask ourselves in a Ramsey way, what is the number of vertices that a d-dimensional OTS can have?*

Bounds for these numbers are also welcomed and similar questions can be stated for cd-OTS.

On the other hand, OTS has been fully characterized for a complete family of simplicial complexes: Given a finite graph G we can get the *independence complex* $\Delta(G)$ whose facets are the maximal independent sets of G. Woodroofe characterized OTS for those kind of simplicial complexes in [2] proving that the OTS here are exactly the independence complexes of the cycles C_n, where $n = 4$ or $n \geq 6$. With the goal to extend these results to hypergraphs, an interesting starting problem would be:

Problem 2. *Characterize OTS that are independence complexes of* 3*-uniform hypergraphs or non-uniform hypergraphs with hyperedges of cardinality 2 and 3.*

References

[1] M.L. Wachs, Obstructions to shellability, Discrete & Comput. Geom. 22 (1999), 95–103.

[2] R. Woodroofe, Vertex decomposable graphs and obstructions to shellability, Proc. Amer. Math. Soc. 137 (2009), 3235–3246.

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