7th Emléktábla Workshop

Discrete Geometry 06.29. - 07.02. 2015.

Preliminary Schedule

Day 1:
9:29 Welcome
9:30 - 10:15 János Pach, EPFL and Rényi Institute
10:30 - 11:15 Günter Rote, FU Berlin
11:30 - 12:15 Géza Tóth, Rényi Institute
Lunch Break
14:00 - 14:45 Eyal Ackerman, University of Haifa
15:30 from in front of Rényi: Traveling together to Balatonalmádi by private bus.

Other Days:
9:29 Waking up
8:30 - 9:30 Breakfast
9:30 Partitioning to Groups of 3-5 for the day
9:30 - 12:30 Work in Groups of 3-5
12:30 - 14:00 Lunch Break
14:00 Optional Repartitioning for the afternoon
14:00 - 17:00 Work in Groups of 3-5
17:00 - 18:30 Discussion of Results
18:30 - Dinner and other activities

Last Day:
Discussion from after lunch and then Return to Budapest starting 16:00 (ETA 17:30).

List of Participants

Chidambaram Annamalai, EPFL
János Barát, MTA-ELTE GAC
Luis Barba, Université Libre de Bruxelles and Carleton University
Stefan David, Cambridge University
Rado Fulek, Columbia University / IST Austria
Dániel Gerbner, MTA Rényi Institute
Vytautas Gruslys, Cambridge University
Balázs Keszegh, MTA Rényi Institute
Dániel Korándi, ETH Zürich
Andrey Kupavskii, EPFL
Shoham Letzter, Cambridge Univerity
Abhishek Methuku, Central European University
Mirjana Mikalački, University of Novi Sad
Till Miltzow, FU Berlin
Luis Montejano, UNAM
Zoltán Lóránt Nagy, MTA-ELTE GAC
Dömötör Pálvölgyi, Eötvös University
Cory Palmer, University of Montana
Balázs Patkós, MTA-ELTE GAC / MTA Rényi Institute
Alexey Pokrovskiy, FU Berlin
Alexandr Polyanskii, MIPT
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Ago-Erik Riet, University of Tartu
Edgardo Roldán-Pensado, UNAM
Miloš Stojaković, University of Novi Sad
Marko Savić, University of Novi Sad
May Szedlák, ETH Zürich
István Tomon, University of Cambridge
Casey Tompkins, MTA Rényi Institute
Nikola Trkulja, University of Novi Sad
Tomaš Valla, Czech Technical University in Prague
Claudiu Valculescu, EPFL
Máté Vizer, MTA Rényi Institute
Russ Woodroofe, Mississippi State Unversity

Beyond planarity of graphs

by Eyal Ackerman

A *topological graph* is a graph drawn in the plane with its vertices as distinct points and its edges as Jordan arcs that connect the corresponding points and do not contain any other vertex as an interior point. Every pair of edges in a topological graph has a finite number of intersection points, each of which is either a common endpoint or a crossing point. It is usually assumed that exactly two edges cross at every crossing point. A topological graph is called *simple*, if every pair of its edges intersect at most once. If every edge is a straight-line segment, then the graph is *geometric*.

A *plane* graph is a topological graph with no crossings. There are several ways to generalize the notion of planarity. Call a topological graph *k-quasi-planar* if it has no k pairwise crossing edges. Hence, a plane graph is 2-quasi-planar. The following is a well-known and rather old conjecture (see e.g., [9, 15]).

Conjecture 1. *For any integer $k \geq 2$, every n-vertex k-quasi-planar graph has $O_k(n)$ edges.*

This conjecture was verified for $k = 3$ [4, 6, 17], for $k = 4$ [3], and (for any k) for *convex* geometric graphs [10]. For $k \geq 5$ the currently best upper bounds on the size of n-vertex k-quasi-planar graphs are $n(\log n)^{O(\log k)}$ by Fox and Pach [11], and, for simple topological graphs, $O_k(n \log n)$ due to Suk and Walczak [20].

It is likely that the following is easier to prove than Conjecture 1.

Conjecture 2. *Call two edges in a geometric graph* virtually crossing*, if a line that contains one edge crosses the other edge. Then an n-vertex geometric graph with no k pairwise virtually crossing edges has $O_k(n)$ edges.*

Another possible generalization of a plane graph is a *k-planar* topological graph, which is a topological graph in which every edge is crossed at most k times. The notion of 1-planarity was introduced in 1965 by Ringel [19], and since then many properties of k-planar graphs have been studied. Let $e_k(n)$ denote the maximum number of edges in a k-planar topological graph with $n > 2$ vertices, and let $e^*_k(n)$ denote the same quantity for simple topological graphs. Clearly, $e_0(n) = 3n-6$. It was shown by Pach and Tóth [18] that $e^*_k(n) \leq 4.108\sqrt{k}n$ and that $e^*_k(n) \leq (k+3)(n-2)$ for $0 \leq k \leq 4$ (equality holds for $0 \leq k \leq 2$ for infinitely many values of n). Pach et al. [16] observed that the latter bound applies also for not necessarily simple topological graphs when $k \leq 3$, and proved a better bound for $k = 3$, namely, $e_3(n) \leq 5.5n - 11$. The author recently proved that $e^*_4(n) \leq 6n - 12$ [2]. The last two bounds are tight up to an additive constant.

Determining $e_k(n)$ or $e^*_k(n)$ for small values of k, apart from being an interesting problem on its own, leads to better lower bounds in the famous *Crossing Lemma* that states that every topological graph with n vertices and $m \geq 4n$ edges has at least $c\frac{m^3}{n^2}$ crossings [5, 13]. This lemma is tight, apart from the multiplicative constant c. The recent bound on $e^*_4(n)$ implies that that $c > 0.0345$, whereas the best upper bound is $c < 0.09$ [18].

Problem 3. *Improve the bounds on the multiplicative constant in the Crossing Lemma, for example, by determining $e_5(n)$.*

Any improved lower bound for the Crossing Lemma, immediately implies improved bounds for its many applications, such as the maximum number of incidences between points and lines in the plane. Another example is the following nice conjecture due to Albertson.

Conjecture 4. *The* crossing number *of a graph G is at least the crossing number of $K_{\chi(G)}$.*

The Albertson conjecture is known to hold for small values of $r = \chi(G)$: For $r = 5$ it is equivalent to the Four Color Theorem, whereas for $r = 6$, $r \leq 12$, and $r \leq 16$, it was verified by Oporowskia and Zhao [14], Albertson, Cranston, and Fox [7], and Barát and Tóth [8], respectively. Barát and Tóth [8] have also verified the conjecture for n-vertex graphs such that $n \leq r + 4$ or $n \geq 3.57r$, while the recent bound on $e_4^*(n)$ implies that the Albertson conjecture holds also when $r \leq 18$, and when $n \geq 3.03r$.

Returning to k-planar graphs, it is easy to see that a k-planar topological graph G can be decomposed into $k+1$ plane graphs. Thus, a 1-planar graph can be decomposed into two plane graphs. However, recall that an n-vertex 1-planar graph has at most $4n - 8$ edges, which suggests that a 1-planar graph might be decomposed into sparser graphs. Indeed, it was shown in [1] that a 1-planar graph can be decomposed into a plane graph and a plane forest. Since we know that the size of a 2-planar graph is at most $5n - 10$ [18], the following is also possible.

Problem 5. *Is it true that a 2-planar graph can be decomposed into a plane graph and two plane forests?*

Obviously, it would also be interesting to consider decompositions of k-planar graphs for $k > 2$.

Another way to allow crossings but still get a graph that is not 'far' from begin planar, is to require the crossings to be connected in a certain way. For example, call a simple topological graph *fan-planar* if for every edge e every two edges that cross e share a common vertex. Note that every fan-planar graph is 3-quasi-planar and therefore has $O(n)$ edges. Hence, the interesting question is to determine exactly the maximum size of a fan-planar graph.

Problem 6. *Is it true that an n-vertex fan-planar topological graph has at most $5n - 10$ edges?*

This bound, if true, is tight. It is only known to hold if for each edge e every two edges that cross e share a common vertex *on the same side* of e [12] (it would also be interesting to simplify this rather long and technical proof).

Here is another way that crossing edges might be regarded as 'close' to each other.

Problem 7. *Determine the maximum number of edges in an n-vertex (simple) topological graph in which for every pair of crossing edges e and e' there is a crossing-free edge that connects an endpoint of e and an endpoint of e'.*

References

[1] E. Ackerman. A note on 1-planar graphs. *Discrete Applied Mathematics*, 175:104–108, 2014.

[2] E. Ackerman, On topological graphs with at most four crossings per edge, manuscript.

[3] E. Ackerman, On the maximum number of edges in topological graphs with no four pairwise crossing edges, *Disc. Compu. Geometry*, 41 (2009), 365–375.

[4] E. Ackerman and G. Tardos, On the maximum number of edges in quasi-planar graphs, *J. Combinatorial Theory, Ser. A.*, 114:3 (2007), 563–571.

[5] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi, Crossing-free subgraphs, *Theory and Practice of Combinatorics*, North-Holland Math. Stud. 60, North-Holland, Amsterdam, 1982, 9–12.

[6] P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir. Quasi-planar graphs have a linear number of edges. *Combinatorica*, 17(1):1–9, 1997.

[7] M. O. Albertson, D. W. Cranston, and J. Fox. Crossings, colorings, and cliques. *Electr. J. Comb.*, 16(1), 2009.

[8] J. Barát and G. Tóth. Towards the albertson conjecture. *Electr. J. Comb.*, 17(1), 2010.

[9] P. Brass, W. Moser, J. Pach, *Research Problems in Discrete Geometry*, Springer, 2005.

[10] V. Capoyleas and J. Pach. A Turán-type theorem on chords of a convex polygon. *J. Comb. Theory, Ser. B*, 56(1):9–15, 1992.

[11] J. Fox and J. Pach. Coloring K_k-free intersection graphs of geometric objects in the plane. *Eur. J. Comb.*, 33(5):853–866, 2012.

[12] M. Kaufmann and T. Ueckerdt. The density of fan-planar graphs. *CoRR*, abs/1403.6184, 2014.

[13] F. T. Leighton. New lower bound techniques for VLSI. *Mathematical systems theory*, 17(1):47–70, 1984.

[14] B. Oporowski and D. Zhao. Coloring graphs with crossings, *Discrete Mathematics*, 309(9):2948–2951, 2009.

[15] J. Pach, Notes on geometric graph theory, In J. E. Goodman, R. Pollack and W. Steiger, editors, *Discrete and Computational Geometry: Papers from DIMACS special year*, volume 6 of *DIMACS series*, 273–285, AMS, Providence, RI, 1991.

[16] J. Pach, R. Radoičić, G. Tardos, G. Tóth, Improving the crossing lemma by finding more crossings in sparse graphs, *Disc. Compu. Geometry*, 36:4 (2006), 527–552.

[17] J. Pach, R. Radoičić, and G. Tóth. Relaxing planarity for topological graphs. In J. Akiyama and M. Kano, editors, *Discrete and Computational Geometry*, volume 2866 of *Lecture Notes in Computer Science*, pages 221–232. Springer Berlin Heidelberg, 2003.

[18] J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica*, 17:3 (1997), 427–439.

[19] G. Ringel. Ein Sechsfarbenproblem auf der Kugel. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 29(1-2):107–117, 1965.

[20] A. Suk and B. Walczak. New bounds on the maximum number of edges in k-quasi-planar graphs. In S. Wismath and A. Wolff, editors, *Graph Drawing*, volume 8242 of *Lecture Notes in Computer Science*, pages 95–106. Springer International Publishing, 2013.

Erdős-Hajnal and their descendents

by János Pach

1. Erdős-Hajnal conjecture. A classic result of Erdős and Szekeres [8] in Ramsey theory states that every graph on n vertices contains a clique or an independent set of size at least $\frac{1}{2}\log n$. In [6], Erdős showed that this bound is tight up to a constant factor by showing that there exists a graph on n vertices, for every integer $n > 1$, with no clique or independent set of more than $2\log n$ vertices.

In [7], Erdős and Hajnal showed that given a graph F, the family $\mathcal{G}(F)$ of all graphs that do not contain F as an induced subgraph have much stronger Ramsey-type properties. They showed that every graph on n vertices in $\mathcal{G}(F)$ contains a clique or independent set of size $e^{c_1\sqrt{\log n}}$, where $c_1 = c_1(F)$ is a suitable constant. They "conjectured" that one can always find a clique or independent set of size n^{c_2}, where $c_2 = c_2(F)$. Despite much attention by various researchers in the area, the Erdős-Hajnal conjecture remains open [4]. It was shown in [3] that the Erdős-Hajnal conjecture is equivalent to the following.

Conjecture 1. *For any tournament T, every T-free tournament of n vertices has a transitive subtournament of size at least n^ϵ, where $\epsilon = \epsilon(T)$ is a positive constant.*

This conjecture is known to be true for all tournaments T with at most 5 vertices. In contrast, the original Erdős-Hajnal conjecture is open already for $F = C_5$.

Problem 2. *Verify the Erdős-Hajnal conjecture for special forbidden graphs or tournaments.*

One can obtain interesting instances of the above problem using geometric constructions. A *string graph* is the intersection graph of a set of continuous curves in the plane. That is, we assign a vertex to each curve, and connect two vertices by an edge if and only if the corresponding curves intersect. It is well known and easy to see that no string graph contains an induced subgraph isomorphic to the 15-vertex graph obtained from K_5 by subdividing each of its edges by a vertex. Thus, if the Erdős-Hajnal conjecture is true, then so is the following statement.

Conjecture 3. *There is an absolute constant $\epsilon > 0$ such that every string graph of n vertices has a clique or an independent set of size at least n^ϵ.*

This conjecture is true for intersection graphs of segments [18] and, more generally, for the intersection graphs of collections of curves, any two of which have at most k points in common, where k is a fixed integer [11]. In particular, it was shown by Larman et al. [15] that the intersection graph of any collection of n segments in the plane has a clique or an independent set of size at least $n^{1/4}$.

Conjecture 4. *There exists $\epsilon > 0$ such that the intersection graph of n segments in the plane has a clique or an independent set of size at least $n^{1/4+\epsilon}$.*

Let l_1 and l_2 be two non-vertical lines in 3-space. If l_1 passes over l_2, we write $l_1 \succ l_2$.

Problem 5. *Estimate the largest $\epsilon > 0$ such that from any collection of n lines in general position in 3-space we can select n^ϵ members that are totally ordered by the relation $\succ$.*

Why do geometrically defined graphs and hypergraphs have nicer Ramsey-type properties than the generic ones? On a "philosophical" level, we can say that these graphs have lots of interesting structural properties that an "average" graph does not have. It seems that from the point of view of Ramsey theory, random or random-like graphs show the worst behavior, so excluding them significantly modifies the answers. In particular, in many cases we can introduce some natural *partial orders* on a family of n geometric objects, which enable us to use Dilworth's theorem, stating that for every partial order we can find a chain or an antichain of size at least $\sqrt{n}$. Ramsey's theorem implies the existence of chains or antichains of size only $\log n$.

2. Perfect regularity lemmas and Ramsey-type theorems for semi-algebraic graphs. Szemerédi's regularity lemma [22] is one of the most powerful tools in modern combinatorics. It was introduced by Szemerédi in his proof of the Erdős-Turán conjecture on long arithmetic progressions in dense subsets of the integers.

In its simplest version, the regularity lemma gives a rough structural characterization of all graphs. A partition is called *equitable* if any two parts differ in size by at most one. According to the lemma, for every $\varepsilon > 0$ there is $K = K(\varepsilon)$ such that every graph has an equitable partition of its vertex set into at most K parts such that all but at most an ε fraction of the pairs of parts are ε-regular. For a pair (V_i, V_j) of vertex subsets, $e(V_i, V_j)$ denotes the number of edges in the graph running between V_i and V_j. The density $d(V_i, V_j)$ is defined as $\frac{e(V_i,V_j)}{|V_i||V_j|}$. The pair (V_i, V_j) is called ε-regular if for all $V_i' \subset V_i$ and $V_j' \subset V_j$ with $|V_i'| \geq \varepsilon|V_i|$ and $|V_j'| \geq \varepsilon|V_j|$, we have $|d(V_i', V_j') - d(V_i, V_j)| \leq \varepsilon$. The number of parts K grows extremely fast as function of $1/\varepsilon$. It follows from the proof that $K(\varepsilon)$ may be taken to be of an exponential tower of twos of height $\varepsilon^{-O(1)}$. Gowers [12] used a probabilistic construction to show that such an enormous bound is indeed necessary. Consult [9] for other proofs that improve on various aspects of the result.

Alon *et al.* [2] (see also Fox *et al.* [13]) established a strengthening of the regularity lemma for point sets in $\mathbb{R}^d$ equipped with a semi-algebraic relation E. To be more precise, let V be an ordered point set in $\mathbb{R}^d$, and let $E \subset \binom{V}{2}$. We say that E is a *semi-algebraic* relation on V with *complexity* at most t if there are at most t polynomials $g_1, \ldots, g_s \in \mathbb{R}[x_1, \ldots, x_{2d}]$, $s \leq t$, of degree at most t and a Boolean formula Φ such that for vertices $u, v \in V$ such that u comes before v in the ordering,

$$(u, v) \in E \quad \Leftrightarrow \quad \Phi(g_1(u, v) \geq 0; \ldots ; g_s(u, v) \geq 0) = 1.$$

At the evaluation of $g_\ell(u, v)$, we substitute the variables $x_1, \ldots, x_d$ with the coordinates of u, the variables $x_{d+1}, \ldots, x_{2d}$ with the coordinates of v. We may assume that the semi-algebraic relation E is symmetric, i.e., for all points $u, v \in \mathbb{R}^d$, $(u, v) \in E$ if and only if $(v, u) \in E$. We assume that the dimension d and complexity t are fixed parameters, and $n = |V|$ tends to infinity.

It was shown in [2] that any point set $V \subset \mathbb{R}^d$ equipped with a semi-algebraic relation $E \subset \binom{V}{2}$ has an equitable partition into a bounded number of parts such that all but at most an ε-fraction of the pairs of parts (V_1, V_2) behave not only regularly, but *homogeneously* in the sense that either $V_1 \times V_2 \subseteq E$ or $V_1 \times V_2 \cap E = \emptyset$. Their proof is essentially qualitative: it gives a poor estimate for the number of parts in such a partition. Fox, Pach, and Suk [10] gave a much stronger quantitative form of this result, showing that the number of parts can be taken to be polynomial in $1/\varepsilon$.

Consequently, the intersection graph $G = (V, E)$ of any collection of n semi-algebraic sets in $\mathbb{R}^d$ of complexity at most t has two subsets $V_1, V_2 \subset V$ with $|V_1|, |V_2| \geq c(d, t)n$ such that either $V_1 \times V_2 \subseteq E$ or $V_1 \times V_2 \cap E = \emptyset$. In such cases, we say that the family of intersection graphs has the *strong Erdős-Hajnal property.* Indeed, this property implies the existence of a positive constant $\epsilon = \epsilon(d, t) > 0$ such that all n-vertex subgraphs of the family have a clique or an independent set of size at least n^ϵ. This provides another reason why the family of intersection graphs of segments satisfy this condition, that is, has the *(weak) Erdős-Hajnal property*. Notice that string graphs are not semi-algebraic with bounded complexity, therefore, we cannot apply these techniques to them. Moreover, they do not have the strong Erdős-Hajnal property [19]. However, they may well have the weak one.

One can also attempt to estimate *off-diagonal* Ramsey numbers in the semi-algebraic setting. Let $R(s,n)$ denote the smallest positive integer R such that every K_s-free graph of R vertices has an independent set of size n. With a little abuse of notation, hiding the dependence on the parameters d and t, we denote by $R^*(s,n)$ the same quantity restricted to (d-dimensional) semi-algebraic graphs of bounded complexity (t). It is known [1], [14] that $R(3,n) = \Theta(n^2/\log n)$ and, for fixed $s > 3$, $R(s,n) = n^{\Theta(1)}$. For a long time, it was conjectured that $R^*(s,n) = O(n)$, but Pawlik et al. [20] exhibited triangle-free intersection graphs of $\Omega(n \log\log n)$ segments with no independent set of size n.

Conjecture 6. *If a system of $n \log\log n$ segments has no three pairwise intersecting members, then it has $\Omega(n)$ disjoint ones.*

Problem 7. *What is the smallest $c \geq 1$ such that $R^*(3,n) = n^c$?*

There are 4-dimensional triangle-free semi-algebraic graphs with bounded complexity showing that $c \geq 4/3$. Note that each segment can be described by 4 real coordinates. Thus, segment intersection graphs "live" in $4+4=8$-space. It is likely that for $d = 2, 3$, the smallest value of c is close to 1.

Analogously, one can define *semi-algebraic k-uniform hypergraphs* (k-ary relations) of bounded complexity, and define the off-diagonal Ramsey functions $R_k(s,n)$ and $R_k^*(s,n)$. For 3-uniform hypergraphs, we know [5], [21] that $R_3^*(s,n) = 2^{n^{o(1)}}$, for every fixed $s \geq 3$.

Conjecture 8. *For every $s \geq 3$, we have $R_3^*(s,n) = n^c$, with a suitable constant $c = c(s) > 0$.*

Here is another unusual Ramsey-type geometric question, addressed first by Mubayi and Suk [17].

Problem 9. *What is the smallest number $S = S(n)$ such that no matter how we 2-color all $\binom{S}{2}$ segments induced by S points in general position in the plane, we can always find n points that form the vertex set of a convex n-gon and all segments between them are of the same color?*

It was shown in [17] that $2^{2n-2} < S(n) < 2^{O(n^2 \log n)}$.

3. Graphs and hypergraphs of bounded VC-dimension. Let $\mathcal{F}$ be a set system on a ground set V. The *Vapnik-Chervonenkis dimension* (VC-dimension) of $\mathcal{F}$ is the *largest* integer D for which there exists a D-element set $S \subset V$ such that for every subset $B \subset S$, one can find a member $A \in \mathcal{F}$ with $A \cap S = B$. Given a graph $G = (V,E)$, for any vertex $v \in V$, let $N(v)$ denote the neighborhood of v in G, that is, the set of vertices in V that are connected to v. We note that v itself is not in $N(v)$. Then we say that G has VC-dimension D, if the set system induced by the neighborhoods in G, i.e., $\mathcal{F} = \{N(v) \subset V : v \in V\}$, has VC-dimension D.

It can be easily deduced from the Thom-Milnor theorem that for every d and t, there exists an integer $D = D(d,t)$ such that the VC-dimension of every d-dimensional semi-algebraic graph of complexity t is at most D. The question arises: which favorable properties of semi-algebraic graphs remain true for all graphs with bounded VC-dimension.

Consider first the Erdős-Hajnal conjecture. Given a bipartite graph F, the *closure* of a bipartite graph F is the set of all graphs that can be obtained from F by adding edges between two vertices in the same part. It is known (see [16]) that a graph has bounded VC-dimension if and only if it does not contain any induced subgraph that belongs to the closure of some fixed bipartite graph F. By the result of Erdős and Hajnal mentioned earlier, all n-vertex graphs with bounded VC-dimension contain a clique or an independent set of size $e^{\Omega(\sqrt{\log n})}$. For graphs of bounded VC-dimension, Fox, Pach, and Suk improved this bound to $e^{(\log n)^{1-o(1)}}$. In semi-algebraic graphs of bounded complexity, one can always find a clique or independent set of size n^{ϵ}. Can this be extended to all graphs of bounded VC-dimension?

Problem 10. *Is it true that for any positive integer D, there exists $\epsilon = \epsilon(D) > 0$ such that every n-vertex graph with VC-dimension at most D contains a clique or an independent set of size at least n^{ϵ}?*

For graphs of bounded VC-dimension, no "perfect regularity lemma," similar to the one established for semi-algebraic graphs, holds. (Verify this!) However, one can prove a somewhat weaker, "almost perfect" version. For $V_1, V_2 \subset V$, we say that the pair (V_1, V_2) is ϵ-homogenous if the edge density $d(V_1, V_2) = \frac{e(V_1,V_2)}{|V_1||V_2|}$ between them is less than ϵ or greater than $1-\epsilon$. The Lovász and Szegedy [16] "ultra-strong" regularity lemma states that for any $\epsilon > 0$, there is a (least) $K = K(\epsilon)$ such that the vertex set V of any graph with VC-dimension D has an equitable partition into at most $K \leq (1/\epsilon)^{O(d^2)}$ parts such that all but at most an ϵ-fraction of all pairs of parts are ϵ-homogeneous. Fox, Suk, and I improved the upper bound on $K(\epsilon)$ to $c(1/\epsilon)^{2d+1}$, where $c = c(d)$.

References

[1] M. Ajtai, J. Komlós, and E. Szemerédi, A note on Ramsey numbers, *J. Combin. Theory Ser. A* **29** (1980), 354–360.

[2] N. Alon, J. Pach, R. Pinchasi, R. Radoičić, and M. Sharir, Crossing patterns of semi-algebraic sets, *J. Combin. Theory Ser. A* **111** (2005), 310–326.

[3] N. Alon, J. Pach, and J. Solymosi, Ramsey-type theorems with forbidden subgraphs, *Combinatorica* **21** (2001), no. 2, 155–170.

[4] M. Chudnovsky, The Erdős-Hajnal conjecture—a survey, *J. Graph Theory* **75** (2014), no. 2, 178–190.

[5] D. Conlon, J. Fox, J. Pach, B. Sudakov, A. Suk, *Transactions of the American Mathematical Society* **366** (2014), 5043–5065.

[6] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[7] P. Erdős, A. Hajnal, Ramsey-type theorems, *Discrete Appl. Math.* **25** (1989), 37-52.

[8] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compos. Math.* **2** (1935), 463–470.

[9] J. Fox and L. M. Lovász, A tight lower bound for Szemerédi's regularity lemma, arXiv preprint arXiv:1403.1768, 2014.

[10] J. Fox, J. Pach, and A. Suk, Density and regularity theorems for semi-algebraic hypergraphs, in: *Proc. 26th Ann. ACM-SIAM Symp. on Discrete Algorithms(SODA '15). San Diego CA*, 2015, 1517–1530.

[11] J, Fox, J. Pach, and Cs. Tóth, Intersection patterns of curves, *J. Lond. Math. Soc. (2)* **83** (2011), no. 2, 389–406.

[12] W. T. Gowers, Lower bounds of tower type for Szemerédi's uniformity lemma, *Geom. Funct. Anal.* **7** (1997), 322–337.

[13] J. Fox, M. Gromov, V. Lafforgue, A. Naor, and J. Pach, Overlap properties of geometric expanders, *J. Reine Angew. Math.* (Crelle's Journal), **671** (2012), 49–83.

[14] J. H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Structures Algorithms* **7** (1995), 173–207.

[15] D. Larman, J. Matoušek, J. Pach, and J. Törőcsik, A Ramsey-type result for convex sets, *Bull. London Math. Soc.* **26** (1994), no. 2, 132–136.

[16] L. Lovász, B. Szegedy, Regularity Partitions and The Topology of Graphons, An Irregular Mind, I. Bárány, J. Solymosi, and G. Sági eds., *Bolyai Society Mathematical Studies* **21** (2010), 415–446.

[17] D. Mubayi and A. Suk, A Ramsey-type result for geometric ℓ-hypergraphs, *European J. Combin.* **41** (2014), 232–241.

[18] J. Pach and J. Solymosi, Crossing patterns of segments, *J. Combin. Theory Ser. A* **96** (2001), no. 2, 316–325.

[19] J. Pach and G. Tóth, Comment on Fox news. Geombinatorics 15 (2006), no. 3, 150–154.

[20] A. Pawlik, J. Kozik, T. Krawczyk, M. Lasoń, P. Miczek, W. Trotter, and B. Walczak, Triangle-free intersection graphs of line segments with large chromatic number, *J. Combin. Theory, Ser. B* **105** (2014), 6–10.

[21] A. Suk, Semi-algebraic Ramsey numbers, *Symposium of Computation Geometry* (2015), to appear.

[22] E. Szemerédi, Regular partitions of graphs, *Colloques Internationaux CNRS* **260** *Problémes Combinatoires et Théorie des Graphes, Orsay (1976)*, CNRS, 1976, 399–401.

Title?

by Günter Rote

1 Bimonotone Drawings

An x-monotone curve C is the graph $C = \{ (x, f(x)) \mid a \le x \le b \}$ of some continuous function f over some interval; a y-monotone curve is written similarly as $C = \{ (f(x), x) \mid a \le x \le b \}$.

Research Question 1. *What is the complexity of the following problem?*

Given a graph whose vertices have given positions in the plane with distinct x- and y-coordinates, can we draw the edges as non-crossing curves that are both x-monotone and y-monotone?

2 Windrose Planarity

A *quadrant-constrained* or *q-constrained* plane graph assigns to each directed edge a direction SW→NE, SE→NW, NW→SE, or NE→SW. Such a graph is *windrose-planar* if it can be redrawn such that each edge is both x-monotone and y-monotone in the given direction. This is a strengthening of the well-known concept of upward-planarity to two directions: vertical and horizontal.

Exercise 1. *Let G be a triangulated q-constrained graph with the following properties.*

1. *The graph obtained by directing all edges from W to E is acyclic.*
2. *The graph obtained by directing all edges from S to N is acyclic.*
3. *Each interior vertex u has at least one neighbor in each of the 4 quadrants.*

Then choosing the x-coordinates of the vertices consistently with the order implied by the first condition, and choosing the y-coordinates independently consistently with the order of the first condition, is sufficient to produce a straight-line windrose-planar drawing, provided that some trivial necessary conditions are met, see Angelini et al. [1].

Consider a face cycle $v_1 v_2 \dots v_k$ in a 2-connected q-constrained graph G. To each vertex of this cycle, we can assign a *category* from the range $\{0°, 90°, 180°, 270°, 360°\}$. The category is the interior angle between the two incident edges under the assumption that all edges are drawn in $\pm 45°$ directions (slope ± 1) in their respective quadrants.

Exercise 2.
1. *Formulate a rule to decide between the angle categories $0°$ and $360°$ when two adjacent edges point into the same quadrant.*
2. *Prove that, if G is windrose-planar, then the sum of angle categories must be $k\pi - 360°$.*
3. *How can a triangular face have sum of angle categories different from $180°$?*
4. *Prove that every face cycle with correct angle sum can be triangulated into triangles with angle sum $180°$ by inserting $k-3$ diagonals.*

It has been proved [1] that every q-constrained graph that satisfies certain obvious necessary conditions has a windrose-planar drawing. Moreover, we can find such a drawing with at most one bend per edge.

Research Question 2. *Can we* completely *get rid of bends?*

3 Slope Number of Balanced 3-Trees

Consider the plane graph T_k that is obtained from K_4 by repeating the following operation $k-1$ times: Add a vertex into every bounded triangular face and connect it to the three corners of the triangle. T_k has maximum degree $\Delta = 3 \cdot 2^{k-1}$.

Research Question 3. *How many different slopes are needed for a straight-line drawing of T_k?*

Jelínek et al. [2] showed that every subgraph of T_k with maximum degree Δ can be drawn with $O(\Delta^5)$ slopes.

The following relations between the slopes in a drawing of K_k might be handy:

Proposition 3. *Let P_1, P_2, P_3, P_4 be four points in the plane with distinct x-coordinates, and let s_{ij} denote the slope of the line through P_i and P_j. Then*

$$\sum_{\pi \in S_4} \operatorname{sign} \pi \cdot s_{\pi(1)\pi(2)} s_{\pi(2)\pi(3)} s_{\pi(3)\pi(4)} = 0, \tag{3.1}$$

where the summation is over the set S_4 of permutations of $\{1, 2, 3, 4\}$.

Exercise 4. *Show that condition* (3.1) *is invariant under*

1. *scaling the x- or y-axis,*
2. *exchanging the x- and y-axis by a reflection along the diagonal,*
3. *a shearing operation leaving the y-axis fixed.*

Exercise 5. *Use a combination of the operations of Exercise 4 to specialize the point set P_1, P_2, P_3, P_4 to such a position that Proposition 3 can be conveniently proved.*

4 Homotopy of 1-Simple Drawings

We consider a given drawing G of a graph in the plane up to homotopy: that is, each edge can be continuously deformed as long as it does not move over another vertex.

Research Question 4. *If each pair of edges can be homotopically deformed that they cross at most once, does it follow that all edges can be simultaneously deformed such that they cross pairwise at most once?*

For the purposes of this problem, we can assume that the vertices are aligned on the x-axis in the order $v_1, \ldots, v_n$ and the homotopy type of an edge is recorded by the sequence of straight edges (v_i, v_{i+1}) and rays $(-\infty, v_1)$ and (v_n, ∞) that it crosses from top to bottom or from bottom to top.

5 Maximum Cut in a Special Graph

Prove that, for any $n \geq 2$, the maximum of

$$\sum_{i \in A} \sum_{j \in \{0,1,2,\ldots,n-1\} \setminus A} \max\{n - i - j, 0\}$$

over all subsets $A \subseteq \{0, 1, 2, \ldots, n-1\}$ is achieved for some interval A of the form $\{0, 1, \ldots, k\}$. More generally, prove that the maximum over all subsets A of fixed size $|A| = \ell$ is achieved when A is an interval or the complement of an interval.

References

[1] P. Angelini, G. Da Lozzo, G. Di Battista, V. Di Donato, P. Kindermann, G. Rote, and I. Rutter. Windrose planarity. In preparation, 2015.

[2] V. Jelínek, E. Jelínková, J. Kratochvíl, B. Lidický, M. Tesař, and T. Vyskočil. The planar slope number of planar partial 3-trees of bounded degree. *Graphs and Combinatorics*, 29(4):981–1005, 2013.

Saturated topological graphs

by Géza Tóth

A *simple topological graph* G is a graph drawn in the plane so that any pair of edges have at most one point in common, which is either an endpoint or a proper crossing. G is called *saturated* if no further edge can be added so that it remains a simple topological graph. Obviously, if G is a *complete* simple topological graph, then it is saturated.

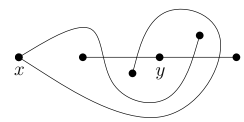

Figure 1: Edge $\{x, y\}$ cannot be added.

The simple topological graph G_1 on Figure 1, found by Kynčl, [7], has six vertices and if we connect x and y by any curve as an edge, two edges with a common endpoint will cross each other. So the resulting topological graph is not simple anymore. All other egdes can be added, so we obtain a saturated simple topological graph of 6 vertices and 14 edges. From this we can construct a saturated simple topological graph of n vertices and $\binom{n}{2} - \lfloor n/6 \rfloor$ edges.

It is a natural question to ask, whether every saturated simple topological graph with n vertices must have $\Omega(n^2)$ edges. It turned out, that there are examples with only a linear number of edges.

Theorem 1. (Kynčl, Pach, Radoičić, Tóth, [6]) *For any $n \geq 4$, let $s(n)$ be the minimum number of edges that a saturated simple topological graph on n vertices can have. Then*

$$1.5n \leq s_1(n) \leq 17.5n.$$

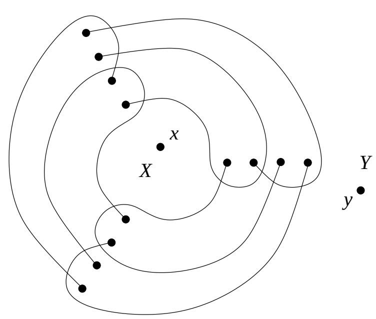

Figure 2: Edge $\{x, y\}$ cannot be added.

The upper bound construction is an iterated version of the topological graph G_2 on Figure 2. It is a simple topological graph, but if we connect vertex x in region X, and vertex y in Y by a curve, it will cross one of the edges of G_2 at least twice.

For the lower bound, it is proved that in a saturated simple topological graph each vertex has degree at least three. Therefore, the number of edges is at least $1.5n$.

The upper bound has been improved recently by Hajnal, Igamberdiev, Rote, and Schulz [5]. For the lower bound, a natural way to improve it is to show that in a saturated simple topological graph each vertex has degree at least four, or five, or even more. In [5] it is also shown, that we

k	1	2	3	4	5	6	7	8	9	10	≥ 11
upper bound in [6]	$17.5n$	$16n$	$14.5n$	$13.5n$	$13n$	$9.5n$	$10n$	$9.5n$	$7n$	$9.5n$	$7n$
upper bound in [5]	$7n$	$14.5n$									

Table 1: Upper bounds on the minimum number of edges in saturated k-simple topological graphs.

cannot expect too much improvement from this simple approach, there could be a vertex of degree four, or many vertices of degree five.

Theorem 2. (Hajnal, Igamberdiev, Rote, and Schulz [5]) (i)

$$s(n) \leq 7n.$$

(ii) *For every $n \geq 6$ there is a saturated simple topological graph on n vertices with a vertex of degree 4.*

(iii) *For every $m \geq 1$ there is a saturated simple topological graph on $10m$ vertices with m vertices of degree 5.*

Problem 1. *Is there a saturated simple topological graph with a vertex of degree three?*

Problem 2. *Construct a saturated simple topological graph with many vertices of degree four.*

Problem 3. *Improve the bounds for $s(n)$.*

In general, for any positive integer k, a topological graph is called a *k-simple* if any two edges have at most k points in common. We also assume that in a k-simple topological graph no edge crosses itself. A 1-simple topological graph is exactly a simple topological graph. It is not obvious at all to construct *non-complete* saturated k-simple topological graphs for $k > 1$.

Theorem 3. (Kynčl, Pach, Radoičić, Tóth, [6]) *For any positive integers k and $n \geq 4$, let $s_k(n)$ be the minimum number of edges that a saturated k-simple topological graph on n vertices can have. Then for $k > 1$ we have*

$$n \leq s_k(n) \leq 16n.$$

For $k = 2$, the upper bound was improved by Hajnal, Igamberdiev, Rote, and Schulz [5].

Theorem 4. (Hajnal, Igamberdiev, Rote, and Schulz [5])

$$s_2(n) \leq 14.5n.$$

For the best upper bounds see Table 1.

In a graph G, an *isolated triangle* is a triangle (K_3) which is not connected to any other vertices. In the proof of the lower bound $s(n) \geq 1.5n$, an essential step is that we prove that there is no isolated triangle in a saturated simple topological graph. The proof does not work for saturated k-simple topological graphs, for $k > 1$, therefore, in this case we can prove only that every vertex has degree at least 2, which implies $s_k(n) \geq n$.

Problem 4. *For $k > 1$, can a saturated k-simple topological graph contain an isolated triangle?*

But unlike in the case of simple topological graphs, even if we knew that a saturated k-simple topological graph cannot contain an isolated triangle, we still cannot prove that in a saturated k-simple topological graph all vertices have degree at least 3.

Problem 5. *Is there a saturated k-simple topological graph for some $k > 1$ with a vertex of degree two?*

Probably the most natural and exciting problem in this topic is the following.

Problem 6. *Is it true that every saturated k-simple topological graph is connected?*

The answer might depend on the value of k, and we do not know the answer for any k.

We assumed that in a k-simple topological graph, no edge can cross itself. For any k, a graph drawn in the plane is called a *k-complicated topological graph* if any two edges have at most k points in common, and an edge is allowed to cross itself, at most k times. Somewhat surprisingly, for saturated k-complicated topological graphs we cannot even prove that every vertex has degree at least two! We can only prove that a saturated k-complicated topological graph does not have isolated vertices. Therefore, the best lower bound we have for the minimum number of edges of a saturated k-complicated topological graph is $c_k(n) \geq n/2$. On the other hand, for $k \geq 6$, using self-crossings, we can improve our upper bound constructions, to obtain that $c_k(n) \leq 5n/2$.

Problem 7. *Is there a saturated k-complicated topological graph with a vertex of degree one?*

Now we study a slightly different problem. It is an easy consequence of Euler's Formula, that every planar graph of n vertices has at most $3n-6$ edges. If it has exactly $3n-6$ edges, then it is a triangulation. If it has less edges and it is drawn in the plane without crossings, than we can extend it to a triangulation.

A topological graph is *1-plane*, if each edge is crossed at most once. A graph is *1-planar*, if it has a 1-plane drawing. It is known that the maximum number of edges of a 1-plane or 1-planar graph is $4n-8$. Brandenburg et al. [3] and independently Eades et al. [4] observed a very interesting phenomenon. They noticed that maximal 1-plane or maximal 1-planar graphs can have much fewer edges.

Theorem 5. (Brandenburg, Eppstein, Gleissner, Goodrich, Hanauer, Reislhuber [3]) *Let $e_1(n)$ (resp. $e'_1(n)$) denote the minimum number of edges of a maximal 1-plane (resp. 1-planar) graph of n vertices. Then we have*

$$2.1n \leq e_1(n) \leq 2.33n,$$

$$2.15n \leq e'_1(n) \leq 2.64n.$$

Both lower bounds were recently improved to $2.22n$ [2].

Problem 8. *Improve the bounds for $e_1(n)$ and $e'_1(n)$.*

For any n, $e_1(n) \leq e'_1(n)$ since any maximal 1-planar graph has a maximal 1-plane drawing. Now the best known lower bounds are the same.

Problem 9. *Is it true that for every n $e_1(n) = e'_1(n)$?*

In general, for every $k \geq 1$, a topological graph is *k-plane*, if each edge is crossed at most once. A graph is *k-planar*, if it has a k-plane drawing. Let $e_k(n)$ (resp. $e'_k(n)$) denote the minimum number of edges of a maximal k-plane (resp. k-planar) graph of n vertices.

Auer et al. [1] proved that $e_2(n) \leq 1.33n$ and $e'_2(n) \leq 2.63n$. It is not hard to see, that $e_k(n) \leq cn/k$ for some $c > 0$.

Problem 10. *Establish some nontrivial bounds for $e_k(n)$ and $e'_k(n)$.*

References

[1] C. Auer, F. J. Brandenburg, A. Gleissner, K. Hanauer: On Sparse Maximal 2-Planar Graphs, Graph Drawing 2012, Lecture Notes in Computer Science **7704** (2013), 555-556.

[2] J. Barát, G. Tóth: Improvements on the density of maximal 1-planar graphs, manuscript.

[3] F. J. Brandenburg, D. Eppstein, A. Gleissner, M. T. Goodrich, K. Hanauer, J. Reislhuber: On the Density of Maximal 1-Planar Graphs, Graph Drawing 2012, Lecture Notes in Computer Science **7704** (2013), 327-338.

[4] P. Eades, S.-H. Hong, N. Katoh, G. Liotta, P. Schweitzer, Y. Suzuki: Testing Maximal 1-Planarity of Graphs with a Rotation System in Linear Time, Graph Drawing 2012, Lecture Notes in Computer Science **7704** (2013), 339-345.

[5] P. Hajnal, A. Igamberdiev, G. Rote, A. Schulz: Saturated simple and 2-simple topological graphs with few edges, 41st International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2015, accepted.

[6] J. Kynčl, J. Pach, R. Radoičić, G. Tóth: Saturated simple and k-simple topological graphs, Computational Geometry: Theory and Applications **48** (2015), 295-310.

[7] J. Kynčl, Improved enumeration of simple topological graphs, *Discrete Comput. Geom.* **50** (2013), 727-770.

Contributed Problems

Planarity testing of cycles and paths with four clusters

by Radoslav Fulek

Let $G = (V, E)$ be a cycle or a path. Let $V = V_1 \cup V_2 \cup V_3 \cup V_4$ be a partition of the vertex set of G, i.e., $V_i \cap V_j = \emptyset$ if $i \neq j$.

Can we decide in sub-exponential time, if G with the given partition of its vertex set is c-planar [3], or in other words, if there exists a planar graph $G' = (V, E')$ such that $E \subseteq E'$ and the induced subgraph $G[V_i]$ is connected for $i = 1, 2, 3, 4$? Is the problem NP-complete ?

This problem was "almost" solved by Cortese et al. [2]. In particular, it is shown that the problem is solvable in polynomial time if one of the V_i's is empty, or G is a cycle and there exists an edge in G between every pair V_i and V_j, $i \neq j$, i.e., $E[V_i, V_j] \neq \emptyset$ if $i \neq j$.

Thrackle conjecture on a cylinder

by Radoslav Fulek

A graph G drawn in the plane is a *thrackle* if every pair of edges in G meet exactly once, either at a common vertex or in their proper crossing. John Conway conjectured [1] that a thrackle cannot have more edges than vertices.

The *cylinder* $\mathcal{C}$ is $S^1 \times I$, where S^1 is a circle and I is an interval. The *projection* to I in $\mathcal{C}$ is the map that maps $(s, i) \in \mathcal{C}$ to i. Does Conway's conjecture hold for graphs drawn on $\mathcal{C}$ such that the projection of every edge to I is injective? It might be helpful first to figure out whether an even cycle can be drawn as a thrackle in that way.

A related case of x-monotone drawings was settled by Pach and Sterling [4]. An x-monotone drawing of a graph is a drawing in the plane in which every vertical line intersects every edge at most once.

References

[1] `https://en.wikipedia.org/wiki/Conway's_thrackle_conjecture`.

[2] Pier Francesco Cortese, Giuseppe Di Battista, Maurizio Patrignani, and Maurizio Pizzonia. On embedding a cycle in a plane graph. *Discrete Mathematics*, 309(7):1856 – 1869, 2009.

[3] Qing-Wen Feng, Robert F. Cohen, and Peter Eades. Planarity for clustered graphs. In Paul Spirakis, editor, *Algorithms - ESA '95*, volume 979 of *Lecture Notes in Computer Science*, pages 213–226. Springer Berlin Heidelberg, 1995.

[4] János Pach and Ethan Sterling. Conway's conjecture for monotone thrackles. *The American Mathematical Monthly*, 118(6):pp. 544–548, 2011.

An extremal problem on crossing vectors

by Till Miltzow

(This problem is originally from Édouard Bonnet and we pose it here with his permission.)

How many flips to non-crossing A *perfect geometric matching* on a set of points P in the plane is a set of non-crossing line segments such that each point is an endpoint of exactly one line segment. Given $2n$ points in general position in the plane it is well known that there always exists a perfect geometric matching on these points.

One elegant argument to see this is to start with any perfect matching, potentially self-intersecting, and remove each crossing by a flip, see figure below. Although the total number of crossings might increase by a flip the total length of all segments decreases. Thus the process will eventually end with a perfect geometric matching.

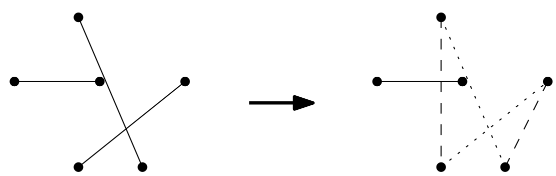

Figure 1: two segments cross in the left matching. We can remove this crossing by removing the segments involved and adding two other segments. This is called a flip.

We say matching M is a successor of matching M' if we can construct M from M' by a flip as above. We say that $\mathcal{M} = (M_1, \ldots, M_k)$ is a valid sequence of matchings, if each matching M_{i+1} is a successor of M_i. The number k denotes the length of $\mathcal{M}$. Given a set P of $2n$ points in the plane we define

$$f(P) = \max\{k : \exists \mathcal{M} \text{ of length } k\}.$$

Consequently the function $g(n)$ is defined as

$$g(n) = \max\{f(P) : P \text{ has } 2n \text{ points}\}.$$

The best upper and lower bounds we currently know are

$$\Omega(n^2) \le g(n) \le O(n^3).$$

The lower bound is straight forward. The upper bound uses the following non-trivial idea: Consider two crossing line segments A and B and let ℓ_1 and ℓ_2 be lines intersecting A and B as in Figure 2. Then any flip of A and B reduces the total number of line-segment intersections by at least two. For any other line ℓ the number of line-segment intersections cannot increase after a flip. Thus the total number of line-segment intersections decreases by at least two after any flip, under the condition that we started with an appropriate set of lines. We can create such a set of lines of size $O(n^2)$. Thus the total number of intersections in the beginning is $O(n^3)$ and this gives the upper bound.

We ask to give better upper or lower bounds!

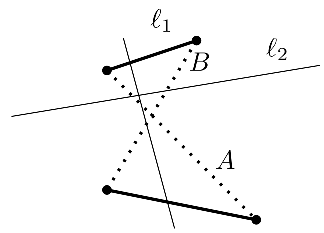

Figure 2: Flipping A and B yields fewer line-segment crossings.

Hadwiger-Debrunner type (p, q, τ) relations

by Dömötör Pálvölgyi

Say that $\mathcal{F}$ satisfies the (p, q, τ) property if for any p sets $F_1, \ldots F_p \in \mathcal{F}$ there are q of them, $F_{i_1}, \ldots, F_{i_q}$, whose union can be stabbed with τ points, $x_1, \ldots x_\tau$, i.e., for every j we have $F_{i_j} \cap \{x_1, \ldots x_\tau\} \neq \emptyset$. The celebrated result of Alon and Kleitman says that if $\mathcal{F}$ is a finite convex family from $\mathbb{R}^d$, which has the $(p, q, 1)$ property for $p \geq q \geq d+1$, then it also has the (n, n, C) property for some $C = C(p, q, d)$. In short, since we only study finite convex families in $\mathbb{R}^d$, we write this as $(p, q, 1) \to (n, n, C)$ (where we should in fact write $\to_d$, but this should be always clear from the context). Helly's theorem translates in this notation to $(d+1, d+1, 1) \to (n, n, 1)$.

We can make some trivial observations among these quantities, like $(p, q, t) \to (p+1, q, t)$, $(p, q, t) \to (p, q-1, t)$, $(p, q, t) \to (p, q, t+1)$, and $(p, q, t) \to (p+q, 2q, 2t)$.

Another well-known lemma, by Hadwiger and Debrunner, is $(d+1+kd, d+1+k(d-1), 1) \to (n, n, k+1)$ and this is sharp. From this it follows, e.g., that in the plane $(5, 4, 1) = 2$.

In general, the bounds are growing quite fast, the only result, which is for the first non-trivial case, is $(4, 3, 1) \to (n, n, 13)$ by Kleitman, Gyárfás and Tóth, while the conjecture is that $(4, 3, 1) \to (n, n, 3)$ with a 100\$ reward by the authors.

Notice that all results are about the $(p, q, 1)$ case, so I would propose to study the general version of the problem.

Problem 1. *When does* $(p, q, t) \to (p', q', t')$*?*

Notice that it follows from the Alon-Kleitman (p, q)-theorem that $(p, td+1, t) \to (n, n, C)$ for some C, while taking hyperplanes in general position shows that $(p, td, t) \not\to (n, n, C)$. (Nabil and Shakhar called my attention to this, if I remember well.) It is also easy to make an example that shows $(n, n, 2) \not\to (n+1, n+1, 2)$.

It would be also interesting to study what we get from supposing multiple things, like we can conclude $(2, 2, 1) \wedge (4, 3, 1) \to (5, 5, 2)$. Can we get some non-trivial implications?

Problem 2. *Is it true that* $(2, 2, 1) \wedge (5, 5, 2) \to (n, n, 2)$*?*

The crossing number of Hanoi graphs

by Cory Palmer

The Tower of Hanoi puzzle consists of d many discs of different sizes distributed among $p \geq 3$ many pegs with the restriction that no disc may sit on top of a disc of smaller size (the divine rule). We may move a single disc from the top of a stack to another (possibly empty) stack as long as we obey the divine rule.

We define the *Hanoi graph* H_p^d as follows. The vertices of H_p^d are the possible arrangements of d discs on $p \geq 3$ pegs obeying the divine rule. Two vertices of H_p^d are connected by an edge if they can be reached from each other by a move of a single disc from the top of a stack to another.

The Hanoi graph H_p^d has p^d vertices and $\frac{1}{2}\binom{p}{2}(p^d - (p-2)^d)$ edges and the degree of a vertex is $\binom{p}{2} - \binom{p-k}{2}$ where k is the number of pegs with at least one disc. Several graph parameters have been determined for Hanoi graphs, e.g.,

1. H_p^d has connectivity $(p-1)$.
2. H_p^d has independence number p^{d-1}.
3. H_p^d has chromatic number p (Arett and Dorée [1]).
4. H_p^d has a Hamiltonian cycle (Hinz and Parisse [3]).

Planarity of Hanoi graphs was characterized by Hinz and Parisse [3]. It is not difficult to show that for the three-peg game that H_3^d is planar. The remaining planar Hanoi graphs H_p^0, H_4^1, and H_4^2. A nice exercise is to show that H_4^3 is not planar. For $d \geq 4$, the Hanoi graph H_4^d contains H_4^3 as a subgraph and is therefore not planar. Furthermore, for $p \geq 5$ the Hanoi graph H_p^d contains a K_5 and is therefore not planar.

The problem to determine the crossing number of Hanoi graphs is posed in the monograph of Hinz, Klavžar, Milutinović, and Petr [2], but the only known bound is by Schmid [4] who showed that the crossing number of H_4^3 is at most 72. Because H_p^d contains many copies of K_p it would be interesting to determine the crossing number of H_p^d in terms of the crossing number of K_p.

References

[1] D. Arett and S. Dorée, Coloring and counting on the tower of Hanoi graphs, *Mathematics Magazine* **83** (2010) 200–209.

[2] A.M. Hinz, S. Klavžar, U. Milutinović, and C. Petr, The tower of Hanoi—myths and maths, Birkhäuser/Springer Basel AG, Basel, 2013.

[3] A.M. Hinz and D. Parisse, On the Planarity of Hanoi Graphs, *Expositiones Mathematicae* **20** (2002) 263–268.

[4] R.S. Schmid, Überschneidungszahlen und Geschlecht bei Hanoi-und Sierpiński-Graphen, Diploma thesis, Ludwig-Maximilians-Universität, München, 2010.

The Chen-Chvatal conjecture

by Balázs Patkós

One of the classical De Bruijn-Erdős theorems states that for any n non-collinear points P_1, P_2, $\ldots, P_n$ in the plane, the number of lines defined by $P_1, P_2, \ldots, P_n$ is at least n. One can generalize the notion of lines in metric spaces with the help of *betweenness*. In a metric space (M, d), we say that y is between x and z if $d(x, y) + d(y, z) = d(x, z)$ holds. This is denoted by $\overline{xyz}$. The line $\ell_{x,y}$ in M defined by the points x and y is the set of points:

$$\ell_{x,y} = \{x, y\} \cup \{z : \overline{zxy}\} \cup \{z : \overline{xzy}\} \cup \{z : \overline{xyz}\}.$$

Conjecture (Chen - Chvatal)**.** *Let (M, d) be a finite metric space. Then either there are points $P_1, P_2 \in M$ with $\ell_{P_1,P_2} = M$ or the number of lines defined by the pairs of points of M is at least $|M|$.*

Lots of special cases of the conjecture are known. Also, there are some general bounds. An incomplete list of references is below.

References

[1] Pierre Aboulker, Adrian Bondy, Xiaomin Chen, Ehsan Chiniforooshan, Vašek Chvátal, Peihan Miao, Number of lines in hypergraphs, Discrete Applied Mathematics 171 (2014), Pages 137–140

[2] Pierre Aboulker, Rohan Kapadia, The Chen-Chvátal conjecture for metric spaces induced by distance-hereditary graphs, European Journal of Combinatorics 43 (2015), Pages 1–7

[3] L. Beaudou, A. Bondy, X. Chen, E. Chiniforooshan, M. Chudnovsky, V. Chvatal, N. Fraiman, and Y. Zwols, A de Bruijn-Erdős theorem for chordal graphs, ELECTRONIC JOURNAL OF COMBINATORICS Volume: 22 Issue: 1

[4] X. Chen, V Chvatal, Problems related to a de Bruijn-Erdős theorem, Discrete Appl. Math., 156 (2008), pp. 2101–2108

[5] Xiaomin Chen, Guangda Huzhangb, Peihan Miaoc, Kuan Yang, Graph metric with no proper inclusion between lines, Discrete Applied Mathematics 185 (2015), Pages 59–70

[6] E. Chiniforooshan and V. Chvatal, A de Bruijn-Erdős theorem and metric spaces, Discrete Math. Theor. Comput. Sci., 13 (2011), pp. 67–74.

[7] V. Chvatal, A de Bruijn-Erdős theorem for 1-2 metric spaces

[8] I. Kantor, B. Patkós, Towards a de Bruijn-Erdős Theorem in the L_1-Metric, Discrete and Computational Geometry 49 (2013), pp 659–670.