4^{th} Emléktábla Workshop

Positional Games 08. 06-09. 2012.

Preliminary Schedule

Day 1:
9:29 Welcome
9:30 - 10:15 Michael Krivelevich (Tel-Aviv University): Introduction to Positional Games
10:30 - 11:15 Dan Hefetz (University of Birmingham): Maker-Breaker Games
11:30 - 12:15 Milos Stojakovic (Univeristy of Novi Sad): Avoider-Enforcer Games
Lunch Break
14:00 - 14:45 Asaf Ferber (Tel-Aviv University): Strong Games and Fast Weak Games
15:30 from in front of Rényi: Traveling together to Tihany by private bus.

Other Days:
9:29 Waking up
8:30 - 9:30 Breakfast
9:30 Partitioning to Groups of 3-5 for the day
9:30 - 12:30 Work in Groups of 3-5
12:30 - 14:00 Lunch Break
14:00 Optional Repartitioning for the afternoon
14:00 - 17:00 Work in Groups of 3-5
17:00 - 18:30 Discussion of Results
18:30 - Dinner and other activities

Last Day:
Discussion from after lunch and then Return to Budapest starting 15:30 (ETA 17:30).

List of Participants

Ross Atkins, University of Oxford
János Barát, Monash University
Dennis Clemens, FU Berlin
Péter Csikvári, ELTE and Rényi Institute
Rado Fulek, EPF Lausanne
Dániel Gerbner, Rényi Institute
Roman Glebov, FU Berlin
Andrzej Grzesik, Jagellonian University, Krakow
Rani Hod, Tel-Aviv University
Ida Kantor, Charles University
Balázs Keszegh, Rényi Institute
Younjin Kim, MSRI
Anita Liebenau, FU Berlin
Hong Liu, UIUC
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Fiona Skerman, University of Oxford
Tomas Valla, Charles University
Máté Vizer, Central European University
Dominik Vu, University of Memphis
Kerstin Weller, University of Oxford
Marcin Witkowski, Adam Mickiewicz University
Bartosz Zaleski, Adam Mickiewicz University

Selected topics in positional games

Asaf Ferber Dan Hefetz Michael Krivelevich Miloš Stojaković

1 Positional Games – Introduction

1.1 General framework, examples

Positional games involve two players alternately occupying the elements of a given set X, the *board* of the game. The focus of their attention is a given family $\mathcal{F} = \{A_1, \ldots, A_k\} \subseteq 2^X$ of subsets of X, usually called the *winning sets*; the family $\mathcal{F}$ is called sometimes the *hypergraph of the game*. The players exchange turns occupying previously unoccupied elements (vertices) of X. In the most general version there are two additional parameters – positive integers p and q, the first player in his turn takes p unoccupied vertices, the second player responds by taking q unoccupied vertices (in the most basic version $p = q = 1$ – the so called *unbiased* game). The game is specified completely by defining who wins in every final position, or more generally for every possible game scenario. Of course, the last sentence is not quite mathematically sound, but it is left vague intentionally so as to accommodate a large variety of games, unified under the same roof; later we will be more specific when defining a concrete game type.

Here are some concrete illustrative examples of positional games.

Example. Tic-Tac-Toe. This one certainly does not need a formal introduction – it has been to everyone's childhood. The game of Tic-Tac-Toe (or Crosses & Noughts) is played by two players, alternately claiming one unoccupied cell each from a 3-by-3 board; the player completing a winning line first wins, where winning lines are three horizontal lines, three vertical lines and two diagonals; if none of the lines is claimed in its entirety by either of the players by the end of the game, the game is declared a draw. In this case, the board X is $[3]^2$, and the winning sets $\mathcal{F}$ form a 3-unform hypergraph with eight edges. The phrase "every child knows it is a draw" is right to the point here – this is indeed a draw, and the only known way to prove it is by case analysis (this could well be your first case analysis proof ...).

Example. $\mathbf{n^d}$. This is a far reaching and extremely interesting (and complicated) generalization of Tic-Tac-Toe. Here the board is the d-dimensional cube $X = [n]^d$, and the winning sets are the so called *combinatorial lines* in X. A combinatorial line l is a family of n distinct points $(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)})$ of X such that for each coordinate $1 \leq j \leq d$ the sequence of corresponding coordinates $(a_j^{(1)}, a_j^{(2)}, \ldots, a_j^{(n)})$ is either $1, 2, \ldots, n$ (increasing) or $(n, n-1, \ldots, 1)$ (decreasing) or a constant (and of course, at least one of the coordinates should be non-constant). The winner is the player who occupies a whole line first, otherwise the game ends in a draw. The familiar Tic-Tac-Toe is 3^2 in this notation. Our understanding of this family of games is rather limited, we know in particular (and will explain it later) that for a fixed n, the first player is the one who wins for large enough d.

Exercise 1. *What is the number of winning sets in n^d?*

Example. Hex. This game was apparently invented by the Danish mathematician Piet Hein in 1942 and was played and researched by none other than John Nash in his student years. The game is played on a rhombus of hexagons of size $n \times n$ (in commercial/recreational versions n is usually 11), where two players, White and Black, take the two opposite sides of the board each, and then alternately occupy unoccupied hexagons of the board with their own color. Whoever connects his own opposite sides of the board first, wins the game. Strictly speaking, Hex as we presented it does not fit our general framework (can you see why?), but there is a legitimate way to cast it as a positional game. Apparently thinking of Hex, Nash came up with the idea of Strategy Stealing (see later) and proved that the first player is the winner; one can argue rather easily that at least one player wins; proving that there is *only* one winner is a topological-type statement, see [18].

Example. H-building game. This, somewhat less popular commercially, game is described as follows. The game is played on the edges of the complete graph K_n on n vertices, Player 1 (Maker) takes one unoccupied edge in each round, Player 2 (Breaker) responds by taking q unoccupied edges. Maker wins if he creates a copy of a fixed graph H from his edges, otherwise the win is of Breaker. Here our knowledge is rather satisfactory, see [7].

Exercise 2. *Try to analyze this game for the case of H being a triangle K_3; aim to prove that for $q = c\sqrt{n}$ for small enough $c > 0$ Maker wins, while for $q = C\sqrt{n}$ for large enough $C > 0$ Breaker is the winner. This is a result of Chvátal and Erdős [9].*

The above game represents a very wide and important class of games, whose board is the edge set of the complete graph K_n; sometimes it is generalized further to games played on the edge set of a general (rather than complete) graph. We will see many such games later.

The fundamental monograph [4] of József Beck, the main proponent and contributor of the field for several decades, can serve as a thorough introduction to the subject, covering many of its facets, and posing many interesting problems (beware though that some of them have been solved since the first print!). Beck's recent text [5] has a lot of stuff about games too.

We now proceed to describe and discuss concrete game types.

1.2 Strong games

Strong games is probably the most natural type of games – these are the games played for fun by normal human beings. A **strong game** is played on a hypergraph $(X, \mathcal{F})$ by two players, called Player 1 (Red) and Player 2 (Blue), who take turns in occupying previously unclaimed elements of the board, one element each time (thus this is an unbiased game); Player 1 starts. The winner is the *first* player who completes a winning set $A \in \mathcal{F}$; if this did not happen for the duration of the game, the game is declared a draw. Both Tic-Tac-Toe and n^d are strong games.

Exercise 3. *Prove that the strong connectivity game, played on the edge set of the complete graph K_n (the player completing a spanning tree first wins), is Red's win.*

Since strong games are perfect information games with deterministic moves, assuming the two players play according to their optimal strategies, the game outcome is determined and it can be in principle: win of Red, win of Blue, or a draw.

In reality, there are only two possible outcomes for this kind of games (again, assuming optimal strategies). The most basic fact about strong games is the so called *strategy stealing principle*, asserting formally the advantageous position of the first player.

Theorem 4. *In a strong game played on $(X, \mathcal{F})$, the first player (Red) can guarantee at least a draw.*

Proof. (Sketch) Assume to the contrary that Blue has a winning strategy $\mathcal{S}$. The strategy is a complete recipe, prescribing Blue how to respond to each move of Red, and to reach a win eventually. Now, Red steals (or rather borrows for the duration of the game, in politically correct terms) this strategy $\mathcal{S}$ and adopts it as follows. He starts with an arbitrary move and then pretends to be Blue (by ignoring his first move). After each move of Blue, Red consults the strategy book $\mathcal{S}$ and responds accordingly. If he is told to claim an element of X which is still available, then he does so; if this element has been taken by him as his ignored arbitrary first move, then he takes another arbitrary move instead. The important point to observe is that an extra move can only benefit Red. □

This is a very powerful result, due to its amazing applicability. Yet, it is a pretty useless statement at the same time – it is absolutely inexplicit and provides no clue for Red on how to play for (at least) a draw.

Another general component in the theory of strong games is Ramsey-type results. This is summarized in the following theorem.

Theorem 5. *If in a strong game played on $(X, \mathcal{F})$ there is no final drawing position, then Red has a winning strategy.*

Proof. The game is at least a draw for Red by Theorem 4; it cannot end in a draw by the theorem's assumption. The only remaining possibility is for Red to win. □

The most striking example of the application of this tandem (Strategy Stealing + Ramsey) is probably for the n^d game. Hales and Jewett, in one of the cornerstone papers of modern Ramsey theory [21] (notice its title – the paper was about positional games!), proved that for a given n and a large enough $d \geq d_0(n)$, every 2-coloring of $[n]^d$ contains a monochromatic combinatorial line. Thus, the strong game played on such a board cannot end in a draw (the two colors are marks of the players), and we conclude that Red wins!

Quite disappointingly, the above two statements constitute the whole contents of our bag of general tools available for tackling strong games. Strong games are notoriously hard to analyze, and rather few results are available at present. The difficulty in their analysis is partly due to the fact that they are *not* hypergraph monotone – adding another edge e to the game hypergraph $\mathcal{F}$ is not necessarily beneficial for the first player and can change the outcome of the game. Problems abound here, let us mention one concrete example.

Problem 6. *Show that for every positive integer k there exist t and n_0 such that for every $n \geq n_0$ Red can win in at most t moves the strong game played on the edge set of the complete graph K_n, where the goal is to create a copy of the complete graph K_k.*

In other words, we are basically asking for an *explicit* winning strategy for Red in the K_k-game. Of course, by Strategy Stealing+Ramsey, for $n \geq R(k,k)$ the game cannot end in a draw and is thus Red's win, but this is non-explicit.

1.3 Maker-Breaker games

Given that the first player has at least a draw in every strong game, it makes sense for the second player to play explicitly for a draw – which is to put his element into every winning set of the game hypergraph. This leads us naturally to the concept of Maker-Breaker games which are also known as *weak* games. In a **Maker-Breaker game**, the first player, called Maker, aims to complete a winning set by the end of the game (i.e. not necessarily first) in which case he wins, while the second player, called Breaker, aims to prevent Maker from fulfilling his goal, and the win is his if he succeeds. Consequently here draw is not a possible outcome. For concreteness, let us assume that Maker moves first. Observe the very non-symmetric positions of the two players here. Maker-Breaker games are probably the most accessible and most researched type of games, with a large variety of results obtained over the years; our understanding of this type of games is most advanced with a nice set of tools and methods at hand.

Quite a few unbiased Maker-Breaker games are an easy win for (a way too powerful) Maker.

Example. Hamiltonicity game. Consider the Maker-Breaker Hamiltonicity game $\mathcal{HAM}$ played on the edges of K_n. The winning sets are all graphs on n vertices containing a Hamilton cycle, i.e., Maker wins if he completes a Hamilton cycle by the end of the game. The unbiased version of this game turns out to be a rather easy (and fast) win for Maker – Chvátal and Erdős proved [9] that, for all large enough n, Maker has a strategy to complete a Hamilton cycle in at most $2n$ rounds; this was taken all the way down to the optimal $n+1$ rounds in [26].

The overwhelming power of Maker in a large variety of games leads naturally to consider biased $(1:b)$ Maker-Breaker games, for some integer $b \geq 1$ (possibly depending on the size of the board), to even out the odds of the players. In such a game Breaker responds each time by claiming b elements of the board (or claiming the rest of the board when there are less than b elements available).

Maker-Breaker games are *bias monotone* as given in the following proposition.

Proposition 7. *Let $(X,\mathcal{F})$ be a hypergraph and let p and q be positive integers. If Maker wins the $(p:q)$ game on $\mathcal{F}$, then he also wins the $(p+1:q)$ game and the $(p:q-1)$ game. Similarly, if Breaker wins the $(p:q)$ game on $\mathcal{F}$, then he also wins the $(p-1:q)$ game and the $(p:q+1)$ game .*

Proof. We will prove the statement for Breaker; the analogous statement for Maker can be proved similarly. Let $\mathcal{S}_B$ be a winning strategy of Breaker for the $(p:q)$ game. When playing the $(p-1:q)$ game on $\mathcal{F}$, Breaker plays according to $\mathcal{S}_B$. Whenever Maker claims his $p-1$ board elements, Breaker (in his mind) chooses an arbitrary free board element and "assigns" it to Maker. Whenever Maker claims a free board element that already "belongs" to him in Breaker's mind, Breaker "gives" him another arbitrary free board element. By the end (in Breaker's mind) of the game, he has already blocked every winning set (as he played according to his winning strategy $\mathcal{S}_B$). Breaker will thus win the game no matter how it proceeds.

When playing the $(p:q+1)$ game, Breaker plays according to $\mathcal{S}_B$, where in every move he claims one additional arbitrary free element of X. Whenever he is instructed by $\mathcal{S}_B$ to claim some element which he has already claimed, Breaker claims an arbitrary free element. Since Breaker follows $\mathcal{S}_B$ which is assumed to be a winning strategy, by the end of the game Maker will not have claimed a complete winning set. Since every other element was claimed by Breaker, he has won the game. $\square$

Corollary 8. *Let $(X, \mathcal{F})$ be a hypergraph satisfying $\mathcal{F} \neq \emptyset$ and $\min\{|A| : A \in \mathcal{F}\} \geq 2$. Then there exists a positive integer b^* such that Breaker wins the $(1 : b)$ game on $\mathcal{F}$ if and only if $b \geq b^*$.*

The integer b^* is referred to as the *threshold bias* of $\mathcal{F}$. Determining or estimating the threshold bias of a game is a typical research goal for many Maker-Breaker games. We will see some concrete examples and questions about the threshold bias later.

Another possible type of questions for Maker-Breaker games is not to determine the winner, but rather, say assuming that the winner is Maker, to estimate in how few moves he is guaranteed to claim an entire winning set. This parameter is sometimes called the *move number* of the game. The above mentioned result about the Hamiltonicity game is of this type – the move number of the Hamiltonicity game is $n + 1$, for all large enough n. Finding strategies for fast wins can sometimes prove useful for attacking (usually much more challenging) strong games played on the same game hypergraph, more about it later.

1.4 Avoider-Enforcer games

Similarly to card games, in a **misére game** the goals of the players are reversed, and the player who would win under normal circumstances is declared the loser. For example, an **Avoider-Enforcer game** is the misère version of a Maker-Breaker game: in an Avoider-Enforcer game played on a game hypergraph $(X, \mathcal{F})$, the first player, called Avoider, wins if by the end of the game he does not occupy completely any of the winning sets of $\mathcal{F}$. The second player, Enforcer, wins otherwise, i.e., if he has a strategy to force Avoider to occupy a winning set. We assume that Avoider is the first to make a move. Avoider-Enforcer games are certainly quite interesting for their own sake, they can also be used to analyze perhaps more natural Maker-Breaker games.

Example. Biased Maker-Breaker non-planarity game. This example is taken from [23]. Consider the biased $(1 : b)$ non-planarity Maker-Breaker game played on $E(K_n)$, where Maker wins if in the end he creates a non-planar graph. According to the result of Bednarska and Pikhurko [8] if $b = b(n)$ is such that Maker completes the game with $n - 1$ edges or less, then Breaker has a strategy to force Maker to end up with a forest – which is of course a planar graph. One possible strategy to derive a matching lower bound for the threshold bias is to argue that if $b = (1 - \varepsilon)n/2$, then Maker has a strategy allowing him to avoid claiming cycles of length at most k almost till the end of the game, where $k = k(\varepsilon)$ is a large enough constant. This will show that, if this Maker's strategy is successful, then towards the end of the game his graph M will have about $(1 + \varepsilon/2)n$ edges and girth higher than k. A standard application of Euler's formula gives then that such a graph is necessarily non-planar. Observe the – quite typical – change of roles of players for the analysis of the game, here Maker assumes the role of cycle-Avoider, allowing him to win the original game.

Exercise 9. *Try to complete the details for the above proof sketch.*

One extremely important characteristics of Maker-Breaker games is bias monotonicity. Alas, as noticed in [27], Avoider-Enforcer games are *not* bias monotone under the traditional game rules – claiming less elements in each turn does not necessarily help either side.

Example. Non-monotonicity of Avoider-Enforcer games under traditional rules. Consider the Avoider-Enforcer $(p:q)$ game played on the hypergraph $\mathcal{F}$, whose edge set consists of two disjoint sets of size two. It is easy to see that for $p = q = 2$ Avoider wins, for $p = 1, q = 2$ the win is Enforcer's, and finally for $p = q = 1$ Avoider is the winner again.

This, quite disturbing, feature of the traditional Avoider-Enforcer rules prompted [25] to adjust, in a rather natural way, the game rules so as to ensure bias monotonicity. Under the so called *monotone rules*, for given bias parameters p and q, in a monotone $(p : q)$ Avoider-Enforcer game played on $(X, \mathcal{F})$, Avoider claims *at least* p elements of X per turn and Enforcer claims at *least* q elements of X per turn. These rules can be argued to be bias monotone very similarly to the proof of Proposition 7, and thus the threshold bias becomes a well defined notion. Perhaps somewhat surprisingly, monotone Avoider-Enforces games turn out to be rather different from those played under traditional rules, and in quite a few cases known results about traditional rules provide a rather misleading clue about the location of the threshold bias for the monotone version. We will discuss it in more details later.

1.5 Where does it all belong and how to deal with it?

The term "positional games" can be somewhat misleading mathematically. Classical Game Theory, initiated by John von Neumann is largely based on the notions of uncertainty and lack of perfect information, giving rise to probabilistic arguments and the key concept of a mixed strategy. Positional games in contrast are perfect information games and as such can in principle be solved completely by an all-powerful computer. In reality, this is (luckily – recall your chess games!) not the case, due to the prohibitive complexity of the exhaustive search approach; this only stresses the importance of accessible mathematical criteria for analyzing such games. A probably closer relative is what is sometimes called "Combinatorial Game Theory", popularized by John Conway and others, which includes such games as Nim; the latter is heavily based on algebraic arguments and various notions of decomposition. Positional games are usually quite different and call for combinatorial arguments of various sorts.

An important and rather peculiar feature of positional games is a quite unexpected yet almost ubiquitous presence of probabilistic considerations and arguments in the analysis of positional games. One striking (meta-)example is the so called Erdős paradigm: for quite a few graph properties P, the threshold bias $b^* = b^*(P)$ for the Maker-Breaker P-game played on K_n is approximately equal to the ratio $\binom{n}{2}/(m^*+1)$, where m^* is the threshold edge number m, for which the random graph $G(n, m)$ with n vertices and m edges starts typically having P. Another manifestation of the importance of probability in positional games is the fact that for some games nearly optimal strategies of the players are shown to be random, see, e.g., [7].

2 Maker-Breaker games

2.1 General setting

Let p and q be positive integers, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. A $(p : q)$ *Maker-Breaker* game $(X, \mathcal{F})$ is played by two players, called Maker and Breaker. The players take turns in claiming previously unclaimed elements of X (the *board*). Maker claims exactly p (*Maker's bias*) board elements per turn and Breaker claims exactly q (*Breaker's bias*) board elements per turn (in his last move a player might claim less elements than his bias if not enough free board elements remain). Unless explicitly stated otherwise, we will assume that Maker is the first player. Maker wins the game if he is able to fully claim some element of $\mathcal{F}$; otherwise Breaker wins.

2.2 Tools of the trade

The following theorems describe sufficient conditions for one of the players to have a winning strategy.

Theorem 10 (Biased Erdős - Selfridge Theorem [6])**.** *Let $(X, \mathcal{F})$ be a hypergraph and let p and q be positive integers. If*

$$\sum_{A \in \mathcal{F}} (1+q)^{-|A|/p} < \frac{1}{1+q},$$

then Breaker (as the second player) has a winning strategy for the $(p : q)$ game $\mathcal{F}$. If Breaker is the first player, then $\sum_{A \in \mathcal{F}} (1+q)^{-|A|/p} < 1$ is enough to ensure Breaker's win.

The special case of Theorem 10 where $p = q = 1$ is due to Erdős and Selfridge [13].

Another (simpler but less applicable) sufficient condition for Breaker's win is the following.

Theorem 11 (Degree criterion for pairing [4])**.** *Let $\mathcal{F}$ be an n-uniform hypergraph. If $\Delta(\mathcal{F}) \leq n/2$ (where $\Delta(\mathcal{F})$ is the maximum degree of $\mathcal{F}$), then Breaker has a winning (pairing) strategy for the game $\mathcal{F}$.*

Proof. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$. Breaker would like to use a *pairing strategy*, that is, for every $1 \leq i \leq m$ he would like to find a pair $\{x_i^0, x_i^1\} \subseteq A_i$ such that $\{x_i^0, x_i^1\} \cap \{x_j^0, x_j^1\} = \emptyset$ holds for every $1 \leq i < j \leq m$. Given such sets he would play as follows: whenever Maker claims x_i^k for some $1 \leq i \leq m$ and $k \in \{0, 1\}$, Breaker responds by claiming x_i^{1-k} (if he didn't already claim it). In any other case he makes an arbitrary move. This is clearly a winning strategy for Breaker; it remains to prove that it works, that is, that the required pairs exist. However, this follows from Hall's Theorem. Indeed, to every hyperedge A_i we need to match two of its elements. We build a bipartite graph with two copies of each A_i on one side and with the elements of $\bigcup_{i=1}^m A_i$ on the other. We connect some $x \in \bigcup_{i=1}^m A_i$ and (both copies of) some A_i by an edge if and only if $x \in A_i$. Note that each A_i has n neighbors (since $\mathcal{F}$ is n-uniform) and each $x \in \bigcup_{i=1}^m A_i$ has at most n neighbors (since $\Delta(\mathcal{F}) \leq n/2$ and there are 2 copies of each A_i). Therefore, the required matching exists by Hall's Theorem. $\square$

Remark 12. *In the proof of Theorem 11 the requirement $\{x_i^0, x_i^1\} \cap \{x_j^0, x_j^1\} = \emptyset$ could be relaxed to $|\{x_i^0, x_i^1\} \cap \{x_j^0, x_j^1\}| \neq 1$.*

Exercise 13. 1. *Prove that Breaker* does not *have a pairing winning strategy for the* 4^2 *Maker-Breaker game (note that he does have a winning strategy for this game).*

2. *Find an explicit pairing winning strategy for Breaker in the* 5^2 *Maker-Breaker game.*

3. *Given an explicit pairing winning strategy for Breaker in the* n^2 *Maker-Breaker game, devise an explicit pairing winning strategy for Breaker in the* $(n+2)^2$ *Maker-Breaker game.*

Theorem 14 (Weak win criterion [6])**.** *Let* $(X, \mathcal{F})$ *be a hypergraph and let* p *and* q *be positive integers. Let* $\Delta_2(\mathcal{F})$ *denote the* maximum pair degree of $\mathcal{F}$*, that is,* $\Delta_2(\mathcal{F}) = \max\{|\{A \in \mathcal{F} : \{u,v\} \subseteq A\}| : u, v \in X\}$*. If*

$$\sum_{A \in \mathcal{F}} \left(\frac{p+q}{p}\right)^{-|A|} > \frac{p^2 q^2}{(p+q)^3} \cdot \Delta_2(\mathcal{F}) \cdot |X|$$

then Maker (as the first player) has a winning strategy for the $(p:q)$ *game* $\mathcal{F}$*.*

For the special case of hypergraphs consisting of pairwise disjoint hyperedges, we have a sufficient and necessary condition for Maker's win. Such a Maker-Breaker game was first studied by Chvátal and Erdős [9] who called it the *Box Game*. In the biased $(p:1)$ Box Game $Box(p; a_1, \ldots, a_n)$, there are n pairwise disjoint sets, labeled $1, \ldots, n$, that we call boxes. For every $1 \leq i \leq n$, we denote the size of box i by a_i. In each round, Maker claims p elements of the boxes and then Breaker responds by claiming one element. Maker's goal is to claim a whole box by the end of the game. The following result was proved in [9] (there was in fact an error in the proof, but the statement itself is correct. The proof was corrected in [22]).

Theorem 15. *[9] Let* p *and* $a_1, \ldots, a_n$ *be positive integers such that* $|a_i - a_j| \leq 1$*, for every* $1 \leq i < j \leq n$*. If Breaker is the first player, then Maker has a winning strategy for* $Box(p; a_1, \ldots, a_n)$ *if and only if* $\sum_{i=1}^{n} a_i \leq f(n,p)$*, where* $f(n,p)$ *is defined by the following recursion,* $f(1,p) = 0$ *and* $f(n,p) = \left\lfloor \frac{n(f(n-1,p)+p)}{n-1} \right\rfloor$ *for every* $n \geq 2$*.*

Remark 16. *When using Theorem 15 it is useful to note that*

$$(p-1)n \sum_{i=1}^{n-1} 1/i \leq f(n,p) \leq pn \sum_{i=1}^{n-1} 1/i\,.$$

Note that a more general result was proved in [22]. Their condition applies to a $(p:q)$ game (rather than just $q=1$) and for all values of $a_1, \ldots, a_n$ (without the assumption that they are almost equal). Moreover, the proof describes explicitly the strategies of both players. Roughly speaking, Breaker always claims his elements is the smallest boxes in which he did not previously play whereas Maker ignores some boxes and tries to level the sizes of the other boxes.

2.3 The threshold bias

Determining the threshold bias of a game is a central problem in the theory of Maker-Breaker games. It is very hard to determine exactly and we are therefore aiming to find its asymptotic value. Let $X = E(K_n)$ be the edge set of the complete graph on n vertices and let $\mathcal{F}$ be a family of subsets of X such that every $A \in \mathcal{F}$ corresponds to a subgraph of K_n with positive minimum degree. One way for Breaker to win the $(1:q)$ game $(X, \mathcal{F})$ is by isolating a vertex. This was done in the seminal paper of Chvátal and Erdős which introduced biased Maker-Breaker games [9].

Theorem 17. *For every $\varepsilon > 0$ there exists an integer n_0 such that for every $n \geq n_0$, playing a $(1:q)$ game on K_n, Breaker can isolate a vertex provided that $q \geq (1+\varepsilon)n/\log n$.*

Proof. (sketch) Breaker's strategy is didvided into the following two stages.

Stage I: Breaker builds a clique C of order $\frac{n}{2\log n}$ such that every $v \in V(C)$ is isolated in Maker's graph. This is possible since, for as long as $v(C) < \frac{n}{2\log n}$, Breaker can add at least 2 vertices to his clique and Maker can touch at most 1 vertex of the clique (note that this stage lasts $O(n/\log n)$ turns and so Breaker can find isolated vertices to add to his clique).

Stage II: Let $V(C) = \{u_1, \ldots, u_k\}$. For every $1 \leq i \leq k$ let $A_i = \{u_i w \text{ is a free edge } : w \in V(K_n) \setminus V(C)\}$ and let $\mathcal{F} = \{A_1, \ldots, A_k\}$. Breaker claims all elements of some A_i (and thus isolates u_i). Since the sets A_i are pairwise disjoint, he can do so by assuming the role of Maker in the game $\text{Box}(q; |A_1|, \ldots, |A_k|)$ (to avoid confusion with Maker of the original game, we refer to Maker in the BoxGame as BoxMaker). Applying Theorem 15 with these parameters shows that BoxMaker can win this game and thus Breaker wins the original game. $\square$

Theorem 17 provides an upper bound on the threshold bias of many natural games, played on the edge set of the complete graph. Some natural examples are the Connectivity game $\mathcal{C}$, the Hamilton cycle game $\mathcal{HAM}$, the Perfect matching game $\mathcal{PM}$, the Minimum-degree-k game $\mathcal{D}^k$, the k-edge-onnectivity game $\mathcal{EC}^k$ and the k-vertex-connectivity game $\mathcal{VC}^k$ (in the last three games k is a constant). Obtaining a matching lower bound turned out to be much harder. Nevertheles, following a series of papers spanning about 30 years, the threshold bias of all 6 games mentioned above was proved to be $(1+o(1))n/\log n$ (see [20] and [30] for the final touch). Note that, playing with this bias, at the end of the game Maker will have $(1/2+o(1))n/\log n$ edges. Interestingly, the random graph $G(n,m)$ starts having each of these properties (that is, being connected, Hamiltonian, etc.) for $m = (1/2+o(1))n/\log n$. Hence, all of these games are examples of the Erdős paradigm.

In order to introduce some of the basic methods in this theory, we sketch below the proof of the following weaker result due to Beck [6]:

Theorem 18. *For every $\varepsilon > 0$ there exists an integer n_0 such that for every $n \geq n_0$ we have $b_{\mathcal{C}_n} \geq (\log 2 - \varepsilon)n/\log n$.*

Proof. Consider the auxiliary game $(E(K_n), \mathcal{F})$, where $\mathcal{F} = \{\{uv : u \in S, v \in V(K_n) \setminus S\} : S \in 2^{V(K_n)} \setminus \{\emptyset, V(K_n)\}\}$. We refer to this game as the *Cut Game* and to the players as CutMaker and CutBreaker. It is evident that Maker can win the $(1:q)$ game $\mathcal{C}_n$ if and only if CutBreaker can win the $(q:1)$ game $(E(K_n), \mathcal{F})$ (with CutBreaker making the first move). However, the latter follows from Theorem 10 since

$$\begin{aligned}
\sum_{A \in \mathcal{F}} 2^{-|A|/q} &\leq \sum_{t=1}^{n/2} \binom{n}{t} 2^{-t(n-t)/q} \leq \sum_{t=1}^{\sqrt{n}} n^t 2^{-t(n-\sqrt{n})/q} + \sum_{t=\sqrt{n}}^{n/2} \left(\frac{en}{\sqrt{n}}\right)^t 2^{-t(n-n/2)/q} \\
&\leq \sum_{t=1}^{\sqrt{n}} \left[n 2^{-(1+\varepsilon')\log n/\log 2}\right]^t + \sum_{t=\sqrt{n}}^{n/2} \left[e\sqrt{n} 2^{-(1+\varepsilon')\log n/(2\log 2)}\right]^t \\
&\leq \sum_{t=1}^{\sqrt{n}} \left[n^{-\varepsilon'}\right]^t + \sum_{t=\sqrt{n}}^{n/2} \left[e n^{-\varepsilon'/2}\right]^t = o(1),
\end{aligned}$$

where $\varepsilon' > 0$ is an appropriate constant. □

2.4 Open problems

2.4.1 Beck's Neighborhood Conjecture

In this subsection we will restrict our attention to $(1:1)$ Maker-Breaker games on an n-uniform hypergraph $(X, \mathcal{F})$. From now on we will simply refer to this game as $\mathcal{F}$. If follows by the Erdős-Selfridge Theorem (plug $p = q = 1$ in Theorem 10) that Breaker has a winning strategy in $\mathcal{F}$ if $|\mathcal{F}| < 2^{n-1}$. That is, if the number of winning sets is not too large as a function of the size of a winning set, then Breaker wins (no matter which n-sets are winning sets and which n-sets are not). However, it is not at all clear that the number of winning sets should matter. Indeed, consider for example the hypergraph $(X, \mathcal{F})$ where $X = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ and $\mathcal{F} = \{\{u_i, v_i\} : 1 \leq i \leq n\}$ (so $\mathcal{F}$ is simply a matching). It is obvious that Breaker wins this game (by a simple pairing strategy) regardless of the value of n. Moreover, there are natural games $\mathcal{F}$ which are Breaker's win even though $\mathcal{F}$ contains *infinitely* many finite sets. One such example (see e.g. [4]) is the unrestricted 8-in-a-row. The board of this game is the infinite 2-dimensional grid $\mathbb{Z}^2$ and the winning sets are all diagonal, horizontal and vertical lines of length 8 (that is, $\bigcup_{i \in \mathbb{Z}} \bigcup_{j \in \mathbb{Z}} \{\{(i,j), (i,j+1), (i,j+2), (i,j+3), (i,j+4), (i,j+5), (i,j+6), (i,j+7)\}, \{(i,j), (i+1,j), (i+2,j), (i+3,j), (i+4,j), (i+5,j), (i+6,j), (i+7,j)\}, \{(i,j), (i+1,j+1), (i+2,j+2), (i+3,j+3), (i+4,j+4), (i+5,j+5), (i+6,j+6), (i+7,j+7)\}, \{(i,j), (i-1,j+1), (i-2,j+2), (i-3,j+3), (i-4,j+4), (i-5,j+5), (i-6,j+6), (i-7,j+7)\}\}$).

Assume that some Maker-Breaker game $(X, \mathcal{F})$ is in progress. Whenever Maker claims some board element $x \in X$, he only affects (at least at that moment) the winning sets $A \in \mathcal{F}$ such that $x \in A$. Hence, it seems plausible to believe that $\Delta(\mathcal{F})$ – the maximum degree of $\mathcal{F}$ – can replace the size of $\mathcal{F}$ in the Erdős-Selfridge Theorem.

Conjecture 19. *[4] There exists a real number $c > 1$ such that Breaker has a winning strategy for the game $\mathcal{F}$ for any n-uniform hypergraph $\mathcal{F}$ with maximum degree at most c^n.*

Currently, our knowledge with respect to this conjecture is very limited. It follows by standard examples (e.g. paths from root to leaves in a complete binary tree) that we cannot have $c \geq 2$ in Conjecture 19. The best upper bound we currently know is the following:

Theorem 20. *[19] There exists a positive integer k_0 such that for every $k \geq k_0$ and $n = 2^k$ there exists an n-uniform hypergraph $\mathcal{F}$ such that $\Delta(\mathcal{F}) \leq \frac{2^{n-1}}{n}$ and yet Maker has a winning strategy for the game $\mathcal{F}$.*

For the opposite direction we have the linear upper bound on $\Delta(\mathcal{F})$ given in Theorem 11.

Let us consider small values of $n \geq 2$. If $n = 2$, then either $\mathcal{F}$ is a matching (in which case $\Delta(\mathcal{F}) = 1$) or there exist edges $A, B \in \mathcal{F}$ such that $|A \cap B| = 1$ (in which case $\Delta(\mathcal{F}) \geq 2$). In the former case Breaker has an obvious pairing strategy whereas in the latter Maker wins the game by claiming the vertex of $A \cap B$ in his first move and a vertex of $(A \cup B) \setminus (A \cap B)$ in his second move.

For $n = 3$ Breaker has a pairing strategy if $\Delta(\mathcal{F}) = 1$. Consider the hypergraph $(X, \mathcal{F})$ where $X = \{1, \ldots, 15\}$ and

$$\mathcal{F} = \{\{1,2,3\}, \{2,6,7\}, \{3,8,9\}, \{6,8,10\}, \{7,9,10\}, \{1,4,5\}, \{4,11,12\},$$

$$\{5, 13, 14\}, \{11, 13, 15\}, \{12, 14, 15\}\}$$

(see [33]). Note that $\Delta(\mathcal{F}) = 2$.

Exercise 21. *Prove that Maker has a winning strategy for the game $\mathcal{F}$ (requires a simple case analysis).*

For $n = 4$, it follows by Theorem 11 that Breaker wins if $\Delta(\mathcal{F}) \leq 2$. There is a construction [29] of a 4-uniform hypergraph $\mathcal{F}$ with $\Delta(\mathcal{F}) = 3$ which is Maker's win.

The most challenging open problem is of course to prove or disprove Conjecture 19. However, there are many interesting partial results which would be nice to obtain. We list a few of them below.

Problem 22. *Does there exist a real number $\varepsilon > 0$ and an integer n_0 such that for every $n \geq n_0$ there exists an n-uniform hypergraph $\mathcal{F}$ with maximum degree $\Delta(\mathcal{F}) \leq (2 - \varepsilon)^n$ such that Maker has a winning strategy for the game $\mathcal{F}$?*

Problem 23. *Does there exist an integer n_0 such that for every $n \geq n_0$ Breaker has a winning strategy for the game $\mathcal{F}$ on any n-uniform hypergraph $\mathcal{F}$ satisfying $\Delta(\mathcal{F}) \leq n$?*

Problem 24. *What is the smallest positive integer t for which there exists a 5-uniform hypergraph $\mathcal{F}$ with maximum degree t such that Maker has a winning strategy for the game $\mathcal{F}$?*

2.4.2 The degree game

Let $G = (V, E)$ be a graph on n vertices and let $1 \leq d \leq n - 1$ be an integer. The board of the *minimum-degree-d game* $\mathcal{D}^d(G)$ is the set E and its winning sets are the subgraphs of G with minimum degree at least d. We would like to determine the largest d for which Maker can win the $(1 : 1)$ game $\mathcal{D}^d(G)$. We denote this parameter by $\hat{d}(G)$.

Exercise 25. *Let G be a graph. Prove that $\hat{d}(G) \leq \lceil \delta(G)/2 \rceil$.*

Exercise 26. *Let $G = (V, E)$ be a graph. Prove that, playing a $(1 : 1)$ game on E, Maker can build a graph M such that $d_M(v) \geq \lfloor d_G(v)/4 \rfloor$ holds for every $v \in V$.*

For simplicity we restrict our attention to regular graphs G. The simple upper bound of $\lceil \delta(G)/2 \rceil$, turns out to be essentially tight for large graphs which are not too sparse. In particular, it was proved by Beck [4] that $n/2 - c_1\sqrt{n \log n} \leq \hat{d}(K_n) \leq n/2 - c_2\sqrt{n}$ for appropriate positive constant c_1, c_2 (some related results can be found, e.g., in [2, 4, 17, 28]). However, in general we do not know how to improve the result stated in Exercise 26. We propose the following open problem:

Problem 27. *Prove that for every sufficiently large integer r there exists a real number $\varepsilon > 0$ such that $\hat{d}(G) \geq (1/4 + \varepsilon)r$ holds for every r-regular graph G.*

2.4.3 The fixed graph game

Given a graph H and positive integers n and q, let $\mathcal{F}_H$ be the Maker-Breaker game whose board is $E(K_n)$ and whose winning sets are the copies of H in K_n. We are interested in determining the threshold bias $b_{\mathcal{F}_H}$ for this game for every graph H and sufficiently large n. It was proved by

Chvátal and Erdős [9] that $\sqrt{2n+2} - 5/2 \leq b_{\mathcal{F}_{K_3}} \leq 2\sqrt{n}$. The upper bound was improved to $b_{\mathcal{F}_{K_3}} \leq (2 - 1/24)\sqrt{n}$ by Balogh and Samotij [3].

A far reaching generalization of this result of Chvátal and Erdős, due to Bednarska and Łuczak [7], asserts that for every graph H with at least 3 non-isolated vertices, the threshold bias satisfies $c_1 n^{1/m_2(H)} \leq b_{\mathcal{F}_H} \leq c_2 n^{1/m_2(H)}$, where $m_2(H) = \max\left\{\frac{e(G)-1}{v(G)-2} : G \subseteq H, v(G) \geq 3\right\}$ and $c_2 > c_1 > 0$ are constants depending on H.

We propose the following related open problems:

Problem 28. *Find a constant c such that $(c - \varepsilon)\sqrt{n} \leq b_{\mathcal{F}_{K_3}} \leq (c + \varepsilon)\sqrt{n}$. It follows by the aforementioned results that $\sqrt{2} \leq c \leq 2 - 1/24$. It is believed that in fact $c = \sqrt{2}$.*

Problem 29. *Prove that for every graph H there exists a constant c_H such that $(c_H - \varepsilon)n^{1/m_2(H)} \leq b_{\mathcal{F}_H} \leq (c_H + \varepsilon)n^{1/m_2(H)}$. This was conjectured in [7]. The currently best known constants c_1, c_2 are quite far apart in general. It would thus also be interesting to narrow the gap between them.*

2.4.4 The k-colorability game

Let $k \geq 2$ be an integer. The winning sets of the k-colorability game $\mathcal{N}C^k$ are the edge sets of all graphs G on n vertices such that $\chi(G) > k$. It was proved in [23] that for sufficently large n there exist constants $c_1 > c_2 > 0$ such that

$$c_2 \frac{n}{k \log k} \leq b_{\mathcal{N}C^k} \leq c_1 \frac{n}{k \log k}\,.$$

This result determines the order of magnitude of $b_{\mathcal{N}C^k}$ for every k. As for its asymptotic value, the constants c_1 and c_2 which were obtained in [23] are not too far apart (but not equal either), provided that k is sufficiently large. We have $c_1 \sim 2$ and $c_2 \sim \log 2/2$ as k tends to infinity. For the special case $k = 2$ (that is, Maker's goal is to claim the edges of an odd cycle) a more accurate result was proved by Bednarska and Pikhurko [8]. They proved that

$$(1 - 1/\sqrt{2} - o(1))n \leq b_{\mathcal{N}C^2} \leq \lceil n/2 \rceil - 1\,.$$

We propose the following related open problems:

Problem 30. *Find a constant $1 - 1/\sqrt{2} \leq c \leq 1/2$ such that $b_{\mathcal{N}C^2} = (c + o(1))n$.*

Problem 31. *For every (sufficiently large) k find a constant c such that $b_{\mathcal{N}C^k} = (c + o(1))\frac{n}{k \log k}$.*

2.4.5 The given spanning tree game

Let $T = (V, E)$ be a tree on n vertices. The board of the Maker-Breaker *tree embedding game* $\mathcal{T}_n$ is the edge set of K_n the complete graph on n vertices. The winning sets of $\mathcal{T}_n$ are all (edge sets of) copies of T in K_n. We would like to determine the threshold bias $b_{\mathcal{T}_n}$ for this game. This value could depend on T; in particular, $b_{K_{1,n-1}} = 1$ (use Exercise 26 to prove this), whereas $b_{P_n} = (1+o(1))n/\log n$ (see [30]). Since Breaker can ensure that Maker's graph will be disconnected if his bias is at least $(1+o(1))n/\log n$ (see Theorem 17 above), it follows that $b_{T_n} \leq (1+o(1))n/\log n$ holds for every tree T on n vertices. It might be that this is asymptotically tight for every tree

T of bounded maximum degree (that is, the maximum degree of T is bounded from above by a constant which is independent of n) – see Problem 1 below. It was proved in [16] that there exist real numbers $\alpha, \varepsilon > 0$ such that Maker has a winning strategy for the $(1:q)$ game $\mathcal{T}_n$ for every tree T on n vertices and maximum degree $\Delta(T) \leq n^{\varepsilon}$ provided that $q \leq n^{\alpha}$.

We propose the following related open problems:

Problem 32. *Find the (asymptotic value of the) threshold bias for the game $\mathcal{T}_n$ for every tree T with bounded maximum degree. Is it $(1+o(1))n/\log n$?*

Problem 33. *For every tree T on n vertices, find the threshold bias for the game $\mathcal{T}_n$.*

2.4.6 A colorability game of Duffus, Łuczak and Rödl

In this subsection we will consider a game which is played on the edge set of a graph which is not necessarily the complete graph. Let $q \geq 1$ and $t \geq 3$ be integers and let G be an arbitrary graph. The board of the $(1:q)$ game $(E(G), \mathcal{F}_t^G)$ is the edge set of the graph G. The winning sets are the edge sets of all subgraphs $H \subseteq G$ such that $\chi(H) \geq t$. Clearly, Maker cannot win this game for every graph G. For example, if $\chi(G) \leq t$, then Breaker wins regardless of his strategy. On the other hand, it seems plausible that if $\chi(G)$ is "much larger" than t, then Maker should win. Indeed, this was conjectured in [12]:

Conjecture 34. *For every positive integers t and q, there exists an integer $r = r(t,q)$ such that Maker has a winning strategy for the $(1:q)$ game $(E(G), \mathcal{F}_t^G)$, played on any graph G such that $\chi(G) \geq r$.*

Very little is known about this conjecture. Using a *strategy stealing* argument, it is not hard to see that Conjecture 34 holds for $q = 1$ and any t. Indeed, let G be a graph on n vertices satisfying $\chi(G) > (t-1)^2$. Let G_M and $G_B = G \setminus G_M$ denote the subgraphs of G, built by Maker and Breaker respectively during the game (played according to some strategies). Clearly $\chi(G_M)\chi(G \setminus G_M) \geq \chi(G)$. Hence, either $\chi(G_M) \geq t$ or $\chi(G_B) \geq t$. Assume for the sake of contradiction that no strategy of Maker guarantees $\chi(G_M) \geq t$. It follows from the above, that there exists a strategy $\mathcal{S}_B$ of Breaker, that ensures $\chi(G_B) \geq t$, regardless of Maker's strategy. However, Maker can "steal" $\mathcal{S}_B$, that is, he can claim an arbitrary first edge and then play according to $\mathcal{S}_B$, pretending to be the second player (whenever he is supposed to claim an edge which is already his, he claims an arbitrary free edge). It follows by the definition of $\mathcal{S}_B$ that $\chi(G_M) \geq t$ contrary to our assumption. Note that strategy stealing is a purely existential argument; we do not know of any explicit strategy for Maker, that ensures his win in the game with these parameters (see Problem 1 below).

For any $q \geq 2$ and any $t \geq 3$ the conjecture is open. Two partial results were obtained in [1]. The first shows that $\chi(G) = \Omega(\log(|V(G)|))$ suffices to ensure Maker's win.

Theorem 35. *Let t and q be positive integers. There exists a constant $c = c(t,q)$ such that, if G is a graph on n vertices and $\chi(G) > c \log n$, then Maker has a winning strategy for the $(1:q)$ game $(E(G), \mathcal{F}_t^G)$.*

The second shows that if $\chi(G) \geq r$ holds in some robust way, then Maker has a winning strategy for the game on G.

Theorem 36. *Let t be a positive integer, let $\varepsilon > 0$ and let $G = (V, E)$ be a graph with n vertices and m edges, where n is sufficiently large. Assume that $\chi(H) \geq t$ holds for every $H \subseteq G$ such that $|E(H)| > (1 - \varepsilon)m$. Then Maker has a winning strategy for the $(1 : q)$ game $(E(G), \mathcal{F}_t^G)$, for every $q \leq \frac{c\varepsilon^2 m}{n \log r}$, where $c > 0$ is an appropriate constant.*

We propose the following related open problems:

Problem 37. *For some sufficiently large positive integer r and for every graph G satisfying $\chi(G) \geq r$, find an explicit winning strategy for Maker in the $(1 : 1)$ game $(E(G), \mathcal{F}_3^G)$.*

Problem 38. *Prove that there exists a positive integer r such that for every graph G satisfying $\chi(G) \geq r$, Maker has a winning strategy for the $(1 : 2)$ game $(E(G), \mathcal{F}_3^G)$.*

3 Avoider-Enforcer games

3.1 Two sets of rules

As we already saw, every positional game $\mathcal{F}$ can be played in Avoider-Enforcer setting. Since Avoider *loses* if he claims a set from $\mathcal{F}$, here we refer to the collection $\mathcal{F}$ as the collection of *losing sets*. Each game can be viewed under two different sets of rules – the *strict* game, where in each move players claim *exactly* the number of elements given by their respective biases, and the *monotone* game, where players claim *at least* the number of elements given by their respective biases. In all games, if not explicitly stated otherwise, we will assume that Avoider starts the game.

Given a positional game $\mathcal{F}$, for its *strict* version we define the *lower threshold bias* $f_{\mathcal{F}}^-$ to be the largest integer such that Enforcer can win the $(1:b)$ game $\mathcal{F}$ for every $b \le f_{\mathcal{F}}^-$, and the *upper threshold bias* $f_{\mathcal{F}}^+$ to be the smallest non-negative integer such that Avoider can win the $(1:b)$ game $\mathcal{F}$ for every $b > f_{\mathcal{F}}^+$.

Exercise 39. *Try to construct a game $\mathcal{F}$, that is, construct the hypergraph of the game (you can choose who starts the game), so that you get the two threshold biases $f_{\mathcal{F}}^-$ and $f_{\mathcal{F}}^+$ as far away from each other as you can.*

If we play the game $\mathcal{F}$ under *monotone* rules, the bias monotonicity implies the existence of the unique threshold bias $f_{\mathcal{F}}^{mon}$ as the non-negative integer for which Enforcer has a winning strategy in the $(1:b)$ game if and only if $b \le f_{\mathcal{F}}^{mon}$.

The first natural question to ask is – given a positional game $\mathcal{F}$, what is the relation of the three threshold biases, $f_{\mathcal{F}}^-$, $f_{\mathcal{F}}^+$ and $f_{\mathcal{F}}^{mon}$? Could it perhaps be that the inequalities $f_{\mathcal{F}}^- \le f_{\mathcal{F}}^{mon} \le f_{\mathcal{F}}^+$ hold for every family $\mathcal{F}$? As we will soon see, this is not true in general, not even for such a natural graph game as the connectivity game, which is even bias monotone under the strict rules (i.e., we have $f^- = f^+$).

Exercise 40. *For various example(s) that you created while solving the previous exercise, determine also the threshold $f_{\mathcal{F}}^{mon}$ and observe its position relative to $f_{\mathcal{F}}^-$ and $f_{\mathcal{F}}^+$.*

As the outcome of strict Avoider-Enforcer games can differ substantially from the outcome of the same game played under monotone rules (even when the strict game is bias-monotone), another natural question one may ask is: *Which set of rules is "better"?*

There is no definite answer to this question. The benefit of the strict rules lies in their applicability to Maker-Breaker games (see, e.g., [23]) or to discrepancy type games (see, e.g., [4, 17, 28]). In these applications, in order to provide a strategy for Maker or for Breaker, one defines an auxiliary Avoider-Enforcer game which models the original Maker-Breaker game, and uses the winning strategy of Avoider or Enforcer in the auxiliary game. Clearly, in this situation the monotone rules are useless. On the down side, in some games the outcome heavily depends on how large is the remainder of integer

division of the size of the board $|B|$ with $b+1$.

The advantage of monotone rules is of course the existence of a threshold bias for every game. Moreover, some of the results concerning the threshold bias of the monotone Avoider-Enforcer game tend to show great similarity to their Maker-Breaker analogues. Generally, we find the study of the differences between the two sets of rules to be quite interesting.

All the Avoider-Enforcer games that we will look at are played on the edge set of the complete graph, $E(K_n)$.

3.2 A general criterion

As we saw, for Maker-Breaker games we have several winning criterions, and the Beck's biased Erdős-Selfridge theorem [6] is probably the most important and the most widely used. For Avoider-Enforcer games we have the following result.

Theorem 41. *[27] If*

$$\sum_{A\in\mathcal{F}} \left(1+\frac{1}{a}\right)^{-|A|+a} < 1,$$

then Avoider wins the biased $(a:b)$ game $\mathcal{F}$, under both strict and monotone rules, for every $b \geq 1$.

This criterion turns out to be rather useful, as it is frequently applied. But, its condition does not take b into account and it is not very effective when b is large. A criterion in full generality would be of great importance for the field.

Problem 42. *Can you extend the criterion from Theorem 41 so that it is sensitive to both a and b?*

3.3 Some games whose losing sets are spanning graphs

We move on to look at some concrete games. The leading term of the threshold bias for the monotone version of several well-studied positional games is given by the following two theorems.

Theorem 43. *[25] If $b \geq (1+o(1))\frac{n}{\log n}$, then Avoider has a winning strategy in the monotone $(1:b)$ min-degree-1 game $\mathcal{D}^1$.*

Theorem 44. *[31] If $b \leq (1-o(1))\frac{n}{\log n}$, then Enforcer has a winning strategy in the monotone $(1:b)$ Hamiltonicity game $\mathcal{HAM}$, and also in the k-connectivity game $\mathcal{VC}^k$.*

From these results we can obtain the leading term of the threshold biases for the connectivity game, the Hamiltonicity game, the perfect matching game, the min-degree-k game (for $k \geq 1$), the k-edge-connectivity game (for $k \geq 1$) and the k-connectivity game (for $k \geq 1$). Indeed, as each of these graph properties implies min-degree-1, and each of them is implied either by the Hamiltonicity or the k-connectivity, we have

$$f_{\mathcal{C}}^{mon}, f_{\mathcal{HAM}}^{mon}, f_{\mathcal{PM}}^{mon}, f_{\mathcal{D}^k}^{mon}, f_{\mathcal{EC}^k}^{mon}, f_{\mathcal{VC}^k}^{mon} = (1+o(1))\,\frac{n}{\log n}. \tag{3.1}$$

Note that for all these games we have the same threshold bias in the Maker-Breaker game.

For connectivity game under strict rules we know the exact value of the lower and upper threshold bias, and they are the same.

Theorem 45. *[27] For connectivity game under strict rules, we have*

$$f_{\mathcal{C}}^{-} = f_{\mathcal{C}}^{+} = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Note the substantial difference between these threshold biases and the monotone threshold bias for the connectivity game (3.1).

As for the k-edge-connectivity game under strict rules, we have bounds that are just a factor of two apart.

Theorem 46. *[27] For k-edge-connectivity game, for $k \geq 2$, we have*

$$\frac{n}{2k} \leq f^{-}_{\mathcal{EC}^k} \leq f^{+}_{\mathcal{EC}^k} \leq \frac{n}{k}.$$

Again, the upper and lower threshold biases are far away from the monotone one in (3.1).

Much less is known for the remaining four mentioned games played under strict rules. We know that the statement of Theorem 44 holds also for the strict rules, so we have the lower bound for the lower threshold biases

$$f^{-}_{\mathcal{HAM}}, f^{-}_{\mathcal{PM}}, f^{-}_{\mathcal{D}^k}, f^{-}_{\mathcal{VC}^k} \geq (1 - o(1)) \frac{n}{\log n}.$$

As for the bounds from above, nothing is known apart from the obvious. In particular, we lack any Avoider's winning strategies that are more powerful than the trivial winning strategy[1]. It is not clear how far can we expect to get, as for example in Theorem 45 a trivial Avoider's strategy turns out to be the optimal one.

Problem 47. *For each of the following games – the Hamiltonicity game, the perfect matching game, the min-degree-k game (for $k \geq 1$) and the k-connectivity game (for $k \geq 2$), all played under strict rules, can you produce better bounds for f^- and f^+? How close is f^+ to f^-, are they equal?*

3.4 A few more games

We move on to results for several other games on graphs. For the non-planarity game, in which Avoider wants to keep his graph planar to the end of the game, we know the following.

Theorem 48. *[23]*

$$\frac{n}{2} - o(n) \leq f^{-}_{\mathcal{NP}} \leq f^{+}_{\mathcal{NP}} \leq 2n^{5/4}.$$

One half of proof. We will just show how to get the lower bound. Let us fix an $\varepsilon > 0$, and assume that $b \leq \frac{n}{2}(1 - \varepsilon)$. We will provide Enforcer with a strategy which guarantees that Avoider will occupy the edges of a non-planar graph. Similarly to the Maker-Breaker version of the game that we already saw, we will rely on Euler's formula to show non-planarity of Avoider's graph.

If $b \leq n/7$, then Avoider surely loses the game as he will claim more edges than a planar graph can have, so from now on we can assume that $b > n/7$.

Let $k = k(\varepsilon)$ be the smallest positive integer such that $\frac{1}{1-\varepsilon/2} > \frac{k}{k-2}$. Enforcer's strategy will be to prevent Avoider from claiming a cycle of length smaller than k, which we will call a "short cycle". If he succeeds, then at the end of the game Avoider's graph will have at least

$$\left\lfloor \frac{\binom{n}{2}}{b+1} \right\rfloor \geq \frac{n}{1 - \varepsilon/2} > \frac{k}{k-2} n$$

[1] We say that Avoider has a *trivial strategy* when the Enforcer's bias is that large that the total number of edges Avoider will claim in the whole game is less than the size of the smallest losing set, so he can win no matter how he plays.

edges, and girth at least k. A graph with such properties cannot be planar, thus Enforcer wins.

It remains to show that Enforcer can indeed prevent Avoider from claiming a short cycle. The result from [7] for the fixed graph Maker-Breaker game (that we already saw) can be applied to the cycle C_i, $i \geq 3$, giving us that in the $(1 : q)$ game played on edges of K_n Breaker can prevent Maker from claiming a copy of C_i if $q > c_i n^{\frac{i-1}{i-2}}$, for a constant $c_i > 0$.

Since for sufficiently large n we have

$$\sum_{i=3}^{k-1} c_i n^{\frac{i-2}{i-1}} \leq \frac{n}{7} < b,$$

Enforcer can simultaneously prevent Avoider from claiming any short cycle C_i, $3 \leq i < k$, by simply playing all $k-3$ games in parallel, pretending to be Breaker in each of them. That is, after Avoider claims an edge, Enforcer responds by claiming $c_3 n^{\frac{1}{2}}$ edges according to the strategy in the "triangle avoidance game", then he claims $c_4 n^{\frac{2}{3}}$ edges according to the strategy in the "4-cycle avoidance game", and so on. His different strategies, for the different cycle-games, might call for claiming the same edge more than once, in which case he just claims an arbitrary unclaimed edge instead. $\square$

Adding an additional twist to the proof of this theorem, it can be shown that for the monotone threshold bias we have the same bounds, $\frac{n}{2} - o(n) \leq f_{\mathcal{NP}}^{mon} \leq 2n^{5/4}$.

Exercise 49. *Go through the above proof and see which parts do not hold if we play under monotone rules. Try fixing it, showing that* $\frac{n}{2} - o(n) \leq f_{\mathcal{NP}}^{mon}$.

Next, we look at the Avoider-Enforcer non-k-colorability game.

Theorem 50. *[23] For every $k \geq 3$ there exists a constant s'_k, such that*

$$s'_k n \leq f^-_{\mathcal{NC}^k} \leq f^+_{\mathcal{NC}^k} \leq 2kn^{1+\frac{1}{2k-3}}.$$

Moreover, $s'_k \sim \frac{\log 2}{2k \log k}$ as $k \to \infty$.

Both upper and lower bound in this theorem hold also for the monotone threshold bias. As we will later see, the monotone threshold bias for avoiding a triangle is of order $n^{3/2}$, which may suggest that the bias for the monotone non-k-colorability is super-linear.

Given $t \geq 3$, the Avoider's goal in the minor game is to avoid a K_t minor. In [23], two linear lower bounds for the lower threshold bias for the minor game are given, depending on how large t is compared to n. On the other side, the best upper bound for the upper threshold bias is of order $n^{5/4}$.

Problem 51. *Improve the bounds on the threshold biases, both monotone or strict, for the non-planarity game, the non-k-colorability game and the minor game.*

3.5 Games with losing sets of constant size

All the games mentioned in (3.1) have one common property – the size of the losing sets grows linearly with n. The extreme opposite are games with losing sets of constant size, and in particular the game $\mathcal{F}_H$ in which Enforcer wants to make Avoider claim a copy of some fixed graph H.

For the 2-path game, we know the order of all three threshold biases, and the monotone one is between the other two.

Theorem 52. *[25]* $f^+_{\mathcal{F}_{P_3}} = \binom{n}{2} - 2$, $f^-_{\mathcal{F}_{P_3}} = \Theta\left(n^{3/2}\right)$, *and* $f^{mon}_{\mathcal{F}_{P_3}} = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor - 1$.

Exercise 53. *There are three statements in the last theorem, how many of them can you verify?*

For the triangle game $\mathcal{F}_{K_3}$ we know the monotone threshold bias, and it turns out to be strikingly far from the Maker-Breaker threshold bias for the triangle game, which is of order $\sqrt{n}$.

Theorem 54. *[25]*

$$f^{mon}_{\mathcal{F}_{K_3}} = \Theta\left(n^{3/2}\right) \ .$$

For larger graphs H we do not know much, and the results for the 2-path and the triangle game are not enough to see a more general picture.

Problem 55. *Obtain some nontrivial bounds for the monotone (or strict) thresholds for some more graphs H (other than 2-path and 3-clique). What is the monotone threshold for avoiding the clique K_k, for $k > 3$?*

It seems that in the strict games with small winning sets the remainder r in the integer division of the size of the board with $b + 1$ plays an important role – it can be anywhere between 0 and b, it changes considerably when the size of the board changes, and when it is large it makes Avoider's job much easier. For that reason, we believe that in any strict H-game the distance between $f^-_{\mathcal{F}_H}$ and $f^+_{\mathcal{F}_H}$ will be substantial.

Problem 56. *How large is the gap between $f^-_{\mathcal{F}_H}$ and $f^+_{\mathcal{F}_H}$, for an arbitrary graph H? Can it be proved that $f^-_{\mathcal{F}_H}$ and $f^+_{\mathcal{F}_H}$ are not of the same order?*

Another interesting observation can be made when comparing the same game played under strict and under monotone rules. Namely, Avoider surely loses the monotone game if he claims a copy of H^- (a copy of H with one edge missing) for which the missing edge is still unclaimed, and there are "many", that is, at least b, additional unclaimed edges.[2] In most cases (unless the remainder r is equal to one) this is not the case when playing under the strict rules, especially when r is large. Hence, it may be reasonable to compare the outcomes and strategies in the strict H^- game and the monotone H game. We are curious whether the fact that the thresholds $f^-_{\mathcal{F}_{P_3}}$ and $f^{mon}_{\mathcal{F}_{K_3}}$ are of the same order is merely a coincidence, since P_3 is exactly one edge short of K_3.

[2] As soon as this happens, Enforcer can claim all the remaining edges except the missing edge, thus winning.

4 Strong win through fast weak win

4.1 General setting and notation

As we saw, in a positional game there is a *board* X (usually a finite set), a family of *winning* sets $\mathcal{F} \subseteq 2^X$ and two players who alternately claim previously unclaimed elements of the board X.

In a *strong* game, the players are called Red (the first player to move) and Blue. The player who *first* completes a winning set wins the game. If no one wins by the time all board elements are fully claimed, then the game ends in a draw.

In a *weak* game, the players are called Maker and Breaker. Maker wins the game as soon as he claims all the elements of some winning set $F \in \mathcal{F}$. If Maker does not win by the time all the board elements are claimed, then Breaker wins the game.

In this chapter we examine a connection between fast winning strategies for Maker in weak games to winning strategies of Red in the analogous strong games. We give some examples, sketch the proofs of some known results and present some open problems.

Unless stated otherwise, most of the games considered in this lecture are *unbiased*. Here, we will mostly deal with games on the board $X = E(G)$, where $E(G)$ is the edge set of some given graph G, and the winning sets are all the subgraphs of G which possess some given monotone increasing graph property $\mathcal{P}$, such as being Hamiltonian $\mathcal{HAM}$, admitting a perfect matching $\mathcal{PM}$, being k-connected $\mathcal{VC}^k$, containing a fixed graph H as a subgraph $\mathcal{F}_H$, in particular, containing a k-clique $\mathcal{F}_{K_k}$, containing a fixed spanning tree $\mathcal{T}$ etc. Since the graph G on whose edges we play the game will not always be the complete graph, whenever it necessary we add G to the notation and denote each of these games by $\mathcal{P}(G)$, where $\mathcal{P}$ is the relevant property.

The *length* (*move number*) of a game $\mathcal{P}(G)$ is defined as:

$$\ell_{\mathcal{P}}(G) = \min\{k \in \mathbb{N} : \text{ Maker has a strategy to win within } k \text{ moves }\}$$

If the game is not Maker's win, then we define $\ell_{\mathcal{P}}(G) = \infty$.

4.2 How fast can Maker win a certain game $\mathcal{F}$?

In this section we discuss winning strategies for Maker in various games while limiting his number of moves. One motivation for that is that most of the natural weak unbiased games played on $E(K_n)$ are drastically in favor of Maker, therefore, one way to even out the odds is to limit the length of the game. In later sections we will also show that fast strategies for Maker in a weak game might help Red to win the analogous strong game. The main problem which we are interested at in this section is to find the fastest possible winning strategy for Maker in a certain game $\mathcal{F}$. There are quite a few known results of this type for some natural games. Here we list a few of them and sketch some proofs.

(1) Maker wins the *connectivity* game $\mathcal{C}$ within $n-1$ moves.

Remark: It follows that $\ell_{\mathcal{C}} = n-1$.

The proof for that is easy (see Exercise 3). We sketch a proof for the following much stronger result: Given a multigraph G, Maker (as the second player) wins the game $\mathcal{C}(G)$ if and only if G contains two edge disjoint spanning trees.

Proof. (Lehman [34]) One direction is quite trivial. Assume that Maker has a winning strategy. Breaker (as the first player) uses strategy stealing to build a spanning tree. Since Maker has a winning strategy, it follows that in the end of the game both of the players have spanning trees and they are disjoint.

The other direction is by induction on $|V(G)|$. Let T_1 and T_2 be two edge disjoint spanning trees in G. Whenever Breaker claims an edge $e \in E(T_1)$ (without loss of generality), he divides T_1 into exactly two components C_1 and C_2. Adding e to T_2 forms a cycle and Maker claims an edge f on this cycle connecting the two components C_1 and C_2 (there must be such an edge). Now, contracting f, we get a new multigraph which contains two edge disjoint spanning trees. By the induction hypothesis Maker gets a spanning tree on this multigraph. Adding the edge f yields a spanning tree of G. □

(2) Maker wins the game $\mathcal{PM}$ within $n/2+1$ moves [24], provided that n is even.

Sketch of proof [24]: In his first few moves, Maker creates a "V" (i.e, claims two edges xy and xz) and makes sure that Breaker does not have edges of the form wy or wz where w is unmatched in Maker's graph. Now, if Breaker claims an edge wy or wz, then Maker makes sure that the vertex w will be covered by his (partial) matching. In his last move, to complete his perfect matching, Maker can attach the remaining vertex to either y or z. □

Exercise 57. *Prove that for every even n, Breaker has a strategy to prevent Maker from winning the game $\mathcal{PM}$ in his $(n/2)$th move. Conclude that $\ell_{\mathcal{PM}} = n/2+1$.*

(3) Maker wins the game $\mathcal{HAM}$ within $n+1$ moves [26].

(4) Maker wins the game $\mathcal{VC}^k$ within $kn/2+1$ moves [15].

We prove the weaker result that Maker can win within $kn/2+\Theta(k^2)$ moves. The strategy for proving that Maker can win within $kn/2+1$ moves involves some more careful arguments and can be found at [15]. The main idea of the proof is that, at the beginning of the game, Maker can partition the board into a few disjoint boards, on each of them Maker wants to play a separate game. If Maker has a winning strategy for each such game, then Maker also has a winning strategy for the union of all these games. Indeed, whenever Breaker plays on a board on which Maker has not won yet, Maker responds according to a fixed winning strategy for this game. Otherwise, Maker chooses a board for which he has not yet won, fakes a move of Breaker (by marking an edge in his mind) and responds according to a winning strategy for this board.

Proof. ([15]) Before we present Maker's strategy, notice that, in a similar way to the proof of (2), we can prove that Maker has a winning strategy for the game $\mathcal{PM}(K_{n,n})$ within $n+1$ moves (convince yourself that it works!). Now, for winning the k-connectivity game, Maker can play as follows:

(i) Maker partitions $V(K_n) = V_1 \cup \ldots \cup V_{k-1} \cup U$ in such a way that $|V_i| = \lfloor n/(k-1) \rfloor$ for every $i \le k-1$ (U might be the empty set).

(ii) On every board $E(V_i, V_j)$, $i \neq j$, Maker plays the perfect matching game.

(iii) On every board $E(V_i)$ Maker plays the Hamiltonicity game.

(iv) For every vertex $u \in U$, Maker claims k edges from u to $V(K_n) \setminus U$.

The resulting graph is clearly k-connected (prove it!) and Maker has wasted at most one move on every board. □

(5) Given a spanning tree T with $\Delta(T) = n^{\varepsilon}$, Maker wins the $(1 : n^{\delta})$ $\mathcal{T}$ game within $n + o(n)$ moves for $\varepsilon > 0$ and $\delta > 0$ relatively small (see [16]).

(6) When $\Delta(T) = \Theta(1)$, Maker wins the $(1 : 1)$ game $\mathcal{T}$ within n moves [11].

Exercise 58. *Find a tree T of constant bounded degree with $|V(T)| = n$ so that Breaker has a strategy to prevent Maker from winning the game $\mathcal{T}$ in $n-1$ moves.*

(7) For every $p \geq \text{polylog}n/n$, the random graph $G(n,p)$ is typically such that Maker wins the games $\mathcal{PM}(G(n,p))$, $\mathcal{HAM}(G(n,p))$ and $\mathcal{VC}^k(G(n,p))$ within $n/2+o(n)$, $n+o(n)$ and $kn/2+o(n)$ moves, respectively [10].

4.3 Finding winning strategies for Red is hard

Unlike Maker-Breaker games, in strong games there are essentially no tools to work with. Therefore, they are much harder to analyze and not too much is known about them.

In order to understand the behavior of these games, it is natural to ask "What does a Red's win hypergraph look like?". For example, given a 2-uniform hypergraph $(X, \mathcal{F})$ (which is actually a graph), Red wins the game $\mathcal{F}$ if and only if $\mathcal{F}$ is not a matching. Moreover, notice that in this case (the 2-uniform case), Red's win is equivalent to Maker's win, and if $\mathcal{F}$ is not a matching, then Red can win in exactly two moves.

It turns out that playing on a 3-uniform hypergraph is completely different (see [32]). There are examples of 3-uniform hypergraphs $\mathcal{F}$ for which Maker wins the game $\mathcal{F}$ and yet, Blue has a drawing strategy (for example, Tic-Tac-Toe is such a game). There are also examples of 3-uniform hypergraphs for which Red wins, but his fastest winning strategy grows with the size of the board X.

Problem 59. *Describe a non-trivial family of hypergraphs $(X, \mathcal{F})$ for which Red has a winning strategy in the game $\mathcal{F}$.*

4.3.1 Non 2-colorable hypergraphs

One non-trivial family of such hypergraphs is the family of all non 2-colorable hypergraphs. Indeed, since a drawing position is impossible when playing on a non 2-colorable hypergraph $\mathcal{F}$, by the strategy stealing argument we conclude that Red wins. For example, the hypergraph of the game $\mathcal{F}_{K_k}$, where $n \geq R(k)$ is not 2-colorable. Another example, is the *k-colorability* game $\mathcal{NC}^k(G)$. The game $\mathcal{NC}^k(G)$ is played on G, the winning sets are all the subgraphs of G which are not k-colorable. Since $\chi(G) \leq \chi(H)\chi(H^c)$ for every subgraph $H \subset G$, it follows that if $\chi(G) > k^2$, then the hypergraph of the game is not 2-colorable.

4.3.2 A perfectly fast Maker wins as Red

Another example of a non-trivial family of hypergraphs $\mathcal{F}$ for which Red has a winning strategy in the game $\mathcal{F}$ is the family of all n-uniform hypergraphs for which Maker has a winning strategy in the analogous weak game in **exactly** n moves. Indeed, by following Maker's strategy, Red wins before Blue has a chance to complete a winning set of his own.

Examples.

(1) The connectivity game $\mathcal{C}$.

(2) The *Hamilton path* game $\mathcal{HP}$ (see [24]).

Problem 60. *Describe a large family of graphs H of fixed size, for which, playing on $E(K_n)$, Maker can build a copy of H in $|E(H)|$ moves, provided that n is large enough.*

4.4 Can almost perfect Maker help Red to win?

Naturally, one can think that given an n-uniform hypergraph $(X, \mathcal{F})$, if Maker can win the game $\mathcal{F}$ within $n+1$ moves, then Red can also win the analogous strong game. Unfortunately, this is not the case as the following example illustrates:

Examples.

(1) The classic $[3]^2$ Tic-Tac-Toe is such a game (prove that!).

(2) A more general example is the following: For $n \geq 3$, let $X = [3]^2 \cup X_0$ where X_0 is an arbitrary set of size $|X_0| = 2n - 6$. A winning set $F \in \mathcal{F}$ is the union of a winning set in the classic Tic-Tac-Toe played on $[3]^2$ and **any** $(n-3)$-element subset of X_0. Maker trivially wins this game within $n+1$ moves by winning the Tic-Tac-Toe game in 4 moves and claiming half of the elements of X_0 (convince yourself that he can do so), and yet, Blue can guarantee at least a draw.

Although the last examples illustrate that a fast winning strategy for Maker does not necessarily imply Red's win, this does not mean that there is no use in finding such a strategy for strong games. Since in a strong game Red has to concentrate on building and blocking, it would be much easier to analyze his possibilities if the game does not last too long. For example, given an n-uniform hypergraph $\mathcal{F}$, if Maker can win within $n+1$ moves, then using his strategy, all Red needs to do is enforce Blue to waste one move during the game.

Examples.

(1) The perfect matching game.

Proof (sketch) A fast winning strategy for Maker is known (see [24]). To turn it into Red's win we should just find a way to enforce Blue to waste enough moves until Red completes his goal. A wasted move, for example, is a move where Blue has claimed an edge which is incident with a vertex of degree at least 1 in his graph (and making it of degree at least 2). Informally, Red plays as follows: In his first move, Red claims an arbitrary edge. After Blue's first move, let x be a vertex which is isolated in Red's graph and is not isolated in Blue's graph (there must be such a vertex). From now on, unless Blue wastes a move, Red claims independent edges without touching x. If Blue wastes a move then Red starts playing as Maker. At the end, by blocking Red from matching x, Blue wastes a move (increases the degree of x to at least 2) and Red can build a trap for Blue, ensuring that every time that Blue blocks him, he must touch x.

(2) The Hamiltonicity game. It is a little bit more tricky and you can read it in [14].

(3) The k-connectivity game. Red builds a k-connected graph before Blue's minimum degree is k [15].

(4) For every constant p, $G(n,p)$ is typically such that Red wins the game $\mathcal{PM}(G(n,p))$ (the main idea of the proof is that Maker can find a partition of the graph into cliques of relatively large constant sizes and then play these games in parallel).

4.5 More open problems

We also suggest the following problems:

(1) Can Red win the fixed spanning tree game $\mathcal{T}$? For which trees?

(2) Find a non-trivial family of graphs H of fixed size, so that Red can win the game $\mathcal{F}$ played on $E(K_n)$ for a sufficiently large n.

(3) Can Red still win the games $\mathcal{PM}$, $\mathcal{HAM}$ and $\mathcal{VC}^k$ when each player claims $a > 1$ elements per move? For which values of a?

(4) Is Red the typical winner of the games $\mathcal{HAM}(G(n,p))$ and $\mathcal{VC}^k(G(n,p))$ for a constant $p > 0$?

(5) For which values of p, and for which trees, can Maker win the fixed spanning tree game played on $G(n,p)$ within $n + o(n)$ moves?

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Contributed Problems

Voronoi games on graphs

by Miloš Stojaković

Let a be a positive integer and G a graph (with at least $2a$ vertices). Two players, Mr White and Mr Black, play a game by alternately claiming unclaimed vertices of G, for a rounds (until *each* of them has claimed a vertices). When they are done, we denote by $\mathcal{W}$ and $\mathcal{B}$ the sets of vertices claimed by Mr White and Mr Black, respectively. Now for each vertex $v \in V(G)$ we find the set N_v of (one or, in case of ties, more of) its nearest neighbors from $\mathcal{W} \cup \mathcal{B}$. If $N_v \subseteq \mathcal{W}$, the vertex v is owned by Mr White, if $N_v \subseteq \mathcal{B}$, the vertex v is owned by Mr Black, and otherwise the vertex v is not owned. The game is won by the player that owns more vertices, and if there is no such player, the game is a tie.

Figure 1 gives a possible course of the game played on the 11×11 grid graph, for two rounds. This particular game is won by Mr White, as in the end (4th picture) he owns 50 vertices, while Mr Black owns only 49.

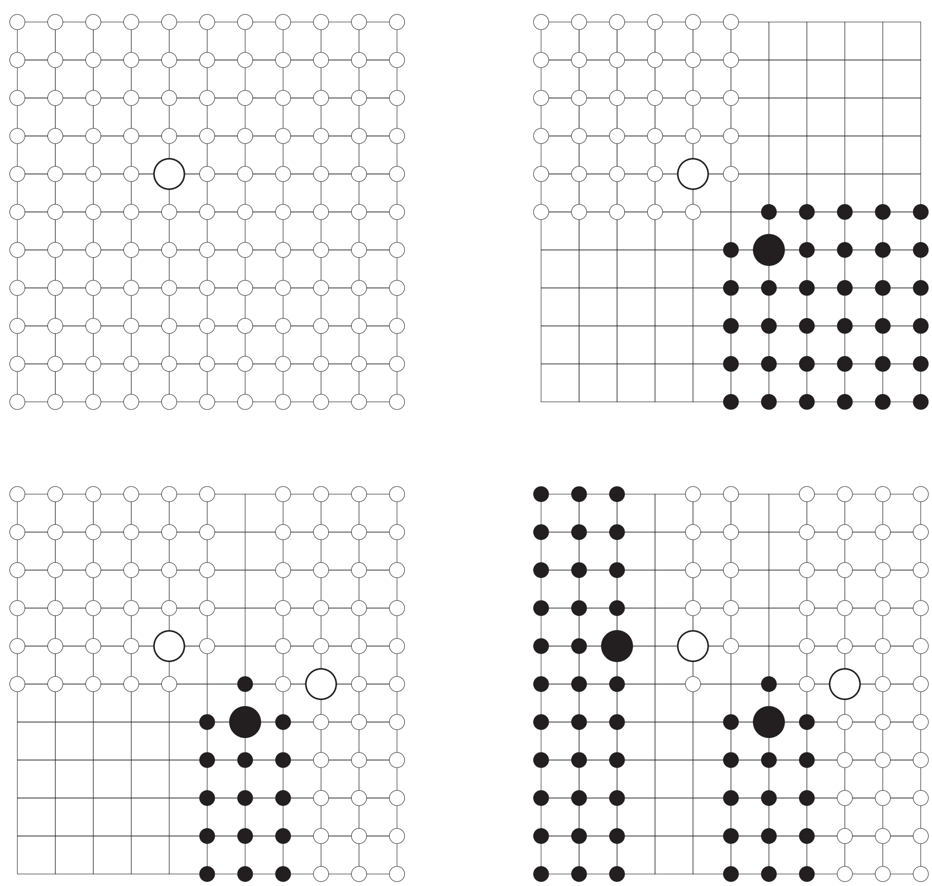

Figure 1: Example of 4 moves (2 rounds) of the game played on the 11×11 grid graph – big circles mark claimed vertices, small circles mark owned vertices.

Although the definition of the game is simple and natural, determining the winner for general G turns out to be rather difficult. Teramoto, Demaine and Uehara [2] determined the outcome of the game for G a (large) complete k-ary tree. They conjectured that the game is always a tie when played on a path, which was confirmed as the main result of [1].

Question 1. Determine the outcome of the game for G being a tree!

As we currently know the outcome only for few special classes of trees, partial results may also be of interest.

In [2] it is shown that determining the winner of the game in full generality is NP-complete, but for G a tree nothing is known.

Question 2. Is there a polynomial time algorithm for determining the outcome of the game for G being a tree?

The list of problems does not end here, as one can ask the same questions for any other class of graphs, e.g. the grid graphs.

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Pairing strategy draws via the combinatorial nullstellansatz

by Dömötör Pálvölgyi

In a Maker-Breaker game we say that Breaker can achieve a *pairing strategy draw* if there is a matching among the points of the board such that every winning set contains at least one pair. It is easy to see that the second player can now force a draw by putting his mark always on the point which is matched to the point occupied by the first player in the previous step (or anywhere, if the point in unmatched). This question deals with five-in-a-row game (aka. Amőba) type games, where Maker has to collect m marks in a row.

It was shown by Hales and Jewett (see [2]), that for the game played on the two dimensional grid where Maker has to collect $m = 9$ marks vertically, horizontally or diagonally, Breaker can achieve a pairing strategy draw. A more general but somewhat weaker result by Kruczek and Sundberg [5] shows that if there are n winning directions (eg. $n = 4$ for the traditional game), then if $m \geq 3n$, Breaker has a pairing strategy (in any dimension). They conjectured that there is always a pairing strategy for $m \geq 2n + 1$, generalizing the result of Hales and Jewett. (It is not hard to show that if $m = 2n$, then such a strategy cannot exist.) An asymptotic version of this conjecture was proved.

Theorem 61. *[6] If $p = m - 1 \geq 2n + 1$ is a prime, then in the Maker-Breaker game played on $\mathbb{Z}^d$, where Maker needs to put at least m of his marks consecutively in one of n given winning directions, Breaker can force a draw using a pairing strategy.*

The proof is by projecting the board onto a general line and using the following simple lemma proved by several pairs of authors independently using the Combinatorial Nullstellensatz ([1]).

Lemma 62. *[3, 4, 7] Given $d_1, \ldots, d_n$ and $p \geq 2n + 1$ prime, we can select $2n$ numbers, $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ all different modulo p such that $x_i + d_i \equiv y_i \mod p$.*

Any improvement of the above lemma would lead to an improvement of the above theorem as well. Unfortunately there is no hope to solve the conjecture this way, as for $n = 3$ if the directions are $(1, 0), (0, 1), (1, 1)$, then there can be no pairing strategy where the pair of a grid point depends only on its projection (although it is not hard to find a pairing strategy). What is the best we can get in one and in more dimensions?

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Saturation games

by Balázs Patkós

The *saturation game* $(X, \mathcal{D})$ is played by two players: former Wales international, FC Liverpool legend, Ian Rush and former Washington State Cougar running back nowadays Mt San Antonio baseball player Dwight Tardy. The players pick distinct elements of the board X alternatingly $r_1, t_1, r_2, t_2, ...$ (or $t_1, r_1, t_2, r_2, ...$) such that after each step the sets $D_j = \{r_1, t_1, r_2, t_2, ..., r_j\}$ and $D'_j = \{r_1, t_1, r_2, t_2, ...r_i, t_j\}$ formed by the elements picked until that point belong to the downwards closed family $\mathcal{D} \subseteq 2^X$. The game stops when the set D of already picked elements is maximal in $\mathcal{D}$, i.e. for every $x \in X \setminus D$ we have $D \cup \{x\} \notin \mathcal{D}$. The *score* of the game is the total number of elements picked by the two players. Rush's aim is to keep the score as low as possible (that is to finish the game as early as possible) while Tardy's aim is obtain a score as high as possible. The problem is to determine $s(X, \mathcal{D})$, the score when both Rush and Tardy play according to their optimal strategy. More precisely, $s_T(X, \mathcal{D})$ denotes the score in a game played according to the optimal strategies and started by Tardy, while $s_R(X, \mathcal{D})$ denotes the score of the game with optimal strategies and started by Rush.

In most examples the board X is either the power set $2^{[n]}$ of the set $[n]$ of the first n positive integers or the set $\binom{[n]}{k}$ of all k-subsetes of $[n]$. In particular, if k equals 2, then the two players pick edges of the complete graph or order n. The only studied instance of the problem is the triangle-free game when $X = \binom{[n]}{2}$ and $\mathcal{D} = \mathcal{D}_\Delta$ is the set of all triangle-free subgraphs of the complete graph K_n on n veretices. Füredi, Reiner and Seress [2] showed that

$$\frac{1}{2} n \log n \leq s(\binom{[n]}{2}, \mathcal{D}_\Delta) \leq \frac{n^2}{5}$$

holds and Biró, Horn and Wildstrom [1] announced some improvements on the constant of the upper bound.

Open Problem 1 Let $2k \leq n$, $X = \binom{[n]}{k}$ and $\mathbb{I}_{n,k} \subset 2^{\binom{[n]}{k}}$ be the set of all k-uniform intersecting families. Try to determine the asymptotics of $s(X, \mathbb{I}_{n,k})$ when k is fixed and n tends to infinity. (Note that the non-uniform intersecting game is not interesting as all maximal non-uniform intersecting subfamily of $2^{[n]}$ has size 2^{n-1}.) It is easy to see that $s_R(X, \mathbb{I}_{n,2}) = 3$ while $s_T(X, \mathbb{I}_{n,2}) = n - 1$

Open Problem 2 Let $X = 2^{[n]}$ and $\mathbb{S}_n \subset 2^{2^{[n]}}$ be the set of all Sperner subfamilies of $2^{[n]}$. Try to give lower and upper bounds on $s_T(X, \mathbb{S}_n)$. (Clearly, $s_R(X, \mathbb{S}_n)$ equals 1.)

References

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The path is more important than the goal, is it?

by Balázs Keszegh

The following problem is in fact a search theoretic problem but we phrase it in a way which looks more like a problem about games.

The *pyramid graph* $Py(n)$ is a directed graph defined in the following way. $Py(n)$ has $N = n(n+1)/2$ vertices on $n+1$ levels, for $1 \leq i \leq n+1$ the ith level having i vertices $v_{i,1}, v_{i,2} \ldots v_{i,i}$, and from every vertex $v_{i,j}$ where $1 \leq i \leq n$ and $1 \leq j \leq i$, there is a *left outgoing edge* going to $v_{i+1,j}$ (its *left child*) and a *right outgoing edge* going to $v_{i+1,j+1}$ (its *right child*). $Py(n)$ has one *root* at the top, $v_{1,1}$ and $n+1$ *sinks* at the bottom, the vertices on the $(n+1)$th level.

The (asymmetric) biased game we regard is the following: A chooses a vertex set S of size k and B chooses for each vertex in S an outgoing edge. A wins if there is only one possible directed path starting at the root and ending in a leaf which is compatible with the choices of B (i.e., for every asked vertex on the path the path goes along the chosen edge). Of course A wins at some point but what is the minimal number of rounds in which A can always win? For $k = 1, 2, 3$ we know the exact answer [1] but for bigger k the problem is open.

Consider the setting when A wins already if all paths that are compatible with the answers of B end in the same leaf. Can A finish the game faster in this case?

The game can of course be played on other directed (acyclic) graphs, for example for complete rooted d-ary trees we know the minimal number of rounds for every k.

This problem was originally proposed by Soren Riis.

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