3rd Emléktábla Workshop

Even more combinatorics 06. 27-30. 2011.

Preliminary Schedule

Day 1:
10:14 Welcome
10:15 - 11:00 Gábor Sárközy
11:15 - 12:00 József Balogh
Lunch Break
14:00 - 14:45 Gábor Tardos
15:00 - 15:45 Miklós Bóna
16:30 from in front of Rényi: Traveling together to Balatonalmádi by private bus.

Other Days:
9:29 Waking up
8:30 - 9:30 Breakfast
9:30 Partitioning to Groups of 3-5 for the day
9:30 - 12:30 Work in Groups of 3-5
11:00 Coffee Break
12:30 - 14:00 Lunch Break
14:00 Optional Repartitioning for the afternoon
14:00 - 17:00 Work in Groups of 3-5
15:45 Coffee Break
17:00 - 18:30 Discussion of Results
18:30 - Dinner and other activities

Last Day:
Discussion from 16:00 and then Return to Budapest.

List of Participants

József Balogh, University of Illinois Urbana-Champaign
Miklós Bóna, Univeristy of Florida
Gábor Sárközy, Worcester Polytechnic Institute and Computer and Automation Research Institute of the Hungarian Academy of Sciences
Gábor Tardos, Rényi Institute
János Barát, University of Pannónia
Péter Csikvári, Eötvös University/Rényi Institute
Endre Csóka, Eötvös University
Péter Csorba, Rényi Institute
Jane Gao, Max-Planck-Institut für Informatik, Saarbrücken
Dániel Gerbner, Rényi Institute
Balázs Keszegh, Rényi Institute
Ida Kantor, Charles University, Prague
Younjin Kim, University of Illinois Urbana-Champaign
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Bernard Lidicky, Charles University, Prague
Alessandro Maddaloni, University of Southern Denmark
Tamás Mészáros, Central European University
Padmini Mukkamala, Rutgers University
Zoltán Nagy, Eötvös University
Lale Ozkahya, Iowa State University
Cory Palmer, Rényi Institute
Dömötör Pálvölgyi, Eötvös University
Balázs Patkós, Rényi Institute
Rados Radoicic, City University New York
Ago-Erik Riet, Univeristy of Memphis
Blerina Sinaimeri, Sapienzia University of Rome
Tamás Terpai, Université de Genčve
György Venter, London School of Economics
Máté Vizer, Central European University

Invited Problems

Coverings by monochromatic pieces

by Gábor N. Sárközy

Abstract

The typical problem in (generalized) Ramsey theory is to find the order of the largest monochromatic member of a family $\mathcal{F}$ (for example matchings, paths, cycles, connected subgraphs) that must be present in any edge coloring of a complete graph K_n with t colors. Another area is to find the minimum number of monochromatic members of $\mathcal{F}$ that partition or cover the vertex set of every edge colored complete graph. Here we propose a problem that connects these areas: for a fixed positive integers $s \leq t$, at least how many vertices can be covered by the vertices of no more than s monochromatic members of $\mathcal{F}$ in every edge coloring of K_n with t colors. Several problems and conjectures are presented, among them a possible extension of a well-known result of Cockayne and Lorimer on monochromatic matchings for which we prove an initial step: every t-coloring of K_n contains a $(t-1)$-colored matching of size k provided that

$$n \geq 2k + \left\lfloor \frac{k-1}{2^{t-1}-1} \right\rfloor .$$

This problem was proposed at last year's Emléktábla workshop by András Gyárfás.

1 Introduction

I present a problem that came out of recent investigations with András Gyárfás and Stanley Selkow [18].

The typical problem in (generalized) Ramsey theory is to find the order of the largest monochromatic member of a family $\mathcal{F}$ (for example matchings, paths, cycles, connected subgraphs) that must be present in any edge coloring of a complete graph K_n with t colors. For easier reference these problems are called Ramsey problems in this paper. Another well studied area, we call them *cover problems*, is to find the minimum number of monochromatic members of $\mathcal{F}$ that partition or cover the vertex set of every edge colored complete graph.

Here we propose a common generalization of Ramsey and cover problems. For a fixed positive integer s, at least how many vertices can be covered by the vertices of no more than s monochromatic members of $\mathcal{F}$ in every edge coloring of K_n with t colors? A somewhat related problem was proposed by Chung and Liu [4]: for a given graph G and for fixed s, t, find the smallest n such that in every t-coloring of the edges of K_n there is a copy of G colored with at most s colors.

Several problems and conjectures are formulated, among them a possible extension of a well-known result of Cockayne and Lorimer on monochromatic matchings. Our main result (Theorem 17) is that every t-coloring of K_n contains a $(t-1)$-colored matching of size k provided that

$$n \geq 2k + \left\lfloor \frac{k-1}{2^{t-1}-1} \right\rfloor .$$

This result is sharp. A simple consequence (Corollary 18) is that every t-colored K_{2^t-2} has a perfect matching missing at least one color. This is a special case of a conjecture proposed at last year's Emléktábla workshop by András Gyárfás [13].

1.1 Path and cycle covers

As far as path covers in infinite graphs are concerned, Rado [28] has a "perfect" result stated here in an abridged form with its simplified original proof.

Theorem 1. *The vertex set of any t-colored countable complete graph can be partitioned into finite or one-way infinite monochromatic paths, each of a different color.*

Proof. Call a set $C \subseteq \{1,\dots,t\}$ of k colors, $1 \le k \le t$, *perfect* if there exists a set $\mathcal{P} = \{P_1,\dots,P_k\}$ of k vertex disjoint finite paths $P_1 = \dots x_1,\dots,P_k = \dots,x_k$ with the following property: P_i is monochromatic in color c_i and there is an infinite set Y of vertices such that Y is disjoint from the paths of $\mathcal{P}$ and for each $i \in \{1,\dots,k\}$ and for all $y \in Y$, $x_i y$ is colored with c_i. A perfect color set exists since any color c_1 present on infinitely many edges of a star incident to vertex x forms such a (one-element) set. Select a perfect set C of k colors so that k is as large as possible ($k \le t$); this is witnessed by $\mathcal{P}$ and Y. Let u be an arbitrary vertex not covered by $\mathcal{P}$. Consider a color c such that uy has color c for every $y \in Y^*$ where $Y^* \subseteq Y$, is infinite. It follows from the choice of k that $c \in C$. Now u can be added to the end of the c-colored path of $\mathcal{P}$, either directly if $u \in Y$, or through a vertex $v \in Y^*$ if $u \notin Y$. The infinite set witnessing the extension is either Y^* or $Y^* \setminus \{v\}$. Clearly the extensions can be continued to place all vertices of the countable complete graph so that all paths of $\mathcal{P}$ are finite or one-way infinite. $\square$

There are several possibilities to "finitize" Theorem 1. The 2-color version works perfectly as noted in a footnote in [8].

Proposition 2. *The vertex set of any 2-colored finite complete graph can be partitioned into monochromatic paths, each of a different color.*

Proof. If $P_1 = ..., x_1, P_2 = ..., x_2$ are red and blue paths and v is uncovered then either v can be placed as the last vertex of one of the paths P_i or one of the bypasses P_1, x_2, v or P_2, x_1, v extends one monochromatic path (and shortens the other). $\square$

Lehel conjectured that Proposition 2 remains true if paths are replaced by cycles (where the empty set, one vertex and one edge are accepted as a cycle). Although the existence of a 'near partition' (where the two monochromatic cycles intersect in at most one vertex) follows easily, see [10], it took a long time until this was proved for large n in [24], [1]. Recently an elementary proof was found by Bessy and Thomassé [3] that works for all n.

Theorem 3. *([3]) The vertex set of any 2-colored complete graph can be partitioned into two monochromatic cycles of different colors.*

In [11] and [6] several possible extensions of Proposition 2 were suggested. It turned out that for 3 colors one can not expect full partition by distinct colors, the first example of this phenomenon is from Heinrich [20]. Recently the asymptotic ratios of monochromatic path and cycle partitions with three distinct colors was obtained in [17].

Theorem 4. *([17]) In every 3-colored K_n at least $(\frac{3}{4} - o(1))n$ vertices can be partitioned into monochromatic cycles of distinct colors.*

We note that here the asymptotic ratio $\frac{3}{4}$ is best possible. Nevertheless in [6] the authors formulated the striking conjecture that Proposition 2 and Theorem 3 carry over to any number of colors if repetitions of colors are possible.

Conjecture 5. *([6]) The vertex set of every t-colored complete graph can be partitioned into t monochromatic cycles.*

The current best result in the direction of this beautiful conjecture is the following (this improved an earlier estimate from [6]).

Theorem 6. *([14]) For every integer $t \geq 2$ there exists a constant $n_0 = n_0(t)$ such that if $n \geq n_0$ and the edges of the complete graph K_n are colored with t colors then the vertex set of K_n can be partitioned into at most $100t \log t$ vertex disjoint monochromatic cycles.*

The case of three colors was recently solved in an asymptotic sense.

Theorem 7. *([17]) In every 3-colored K_n at least $(1 - o(1))n$ vertices can be partitioned into three monochromatic cycles.*

The proofs of Theorems 4, 6 and 7 rely on the Regularity Lemma and relaxations of cycles to connected matchings (see Subsection 1.2).

In [29] we generalized Conjecture 5 to non-complete graphs which have a given maximum independent set size $\alpha(G) = \alpha$. We formulated the following conjecture.

Conjecture 8. *([29]) The vertex set of every t-colored graph G with $\alpha(G) = \alpha$ can be partitioned into αt monochromatic cycles.*

The conjecture is known to be true for the special case $t = 1$ (a theorem of Pósa [27], see also Exercise 8.3 in [23]). Perhaps the case $t = 2$ is not hopelessly difficult, especially in light of the above mentioned Bessy-Thomassé result [3]. In general the current best bound is the following.

Theorem 9. *([29]) If the edges of a graph G with $\alpha(G) = \alpha$ are colored with t colors then the vertex set of G can be partitioned into at most $25(\alpha t)^2 \log(\alpha t)$ vertex disjoint monochromatic cycles.*

Returning to complete graphs the new problem we propose here is the following.

Problem 10. *Suppose $1 \leq s \leq t$. What is the maximum number of vertices that can be covered by s monochromatic cycles (paths) in every t-coloring of the edges of K_n?*

We do not have a general conjecture here, not even for the asymptotics (for fixed s, t and large n). The case $s = 1$ is the path Ramsey number which is wide open. The case $t = 3$ ([16]) is the only evidence that perhaps $\frac{n}{t-1}$ is the true asymptotic value ($\frac{n}{t}$ is an easy lower bound). But already the case $t = 4$ is open. The case $s = t$ is in Conjecture 5.

The first interesting special case is $t = 3, s = 2$.

Conjecture 11. *In any 3-colored K_n there are two vertex disjoint monochromatic paths (cycles) covering at least $\frac{6n}{7}$ vertices.*

A weaker form (for matchings instead of paths) of Conjecture 11 follows from Theorem 17 below (when $t = 3$).

1.2 Connected matchings.

One technique used recently in many papers (for example [7], [14] and [16]) in Ramsey and in covering problems related to paths or cycles is to replace the paths or cycles by a simpler structure, *monochromatic connected matchings,* and rely on the Regularity and Blow-up Lemmas to create paths or cycles from them. A connected monochromatic matching means that all edges of the matching are in the same component of the subgraph induced by the edges in the color of the matching.

Thus a natural step towards proving an asymptotic (or sharp for large enough n) version of Conjecture 11 would be to prove it for connected matchings. However, in the problem mentioned above we are a step behind, we could only prove Conjecture 11 for matchings (without the connectivity condition). A logical plan is to treat connected pieces and matchings separately, this is done in Subsection 1.3 and in Section 2.

1.3 Covers by connected components

Since every connected component contains a spanning tree of the component, we use here the somewhat simpler tree language. A special case of a conjecture attributed to Ryser, (appearing in his student, Henderson's thesis [21]) states that every intersecting t-partite hypergraph has a transversal of at most $t-1$ elements. Using the dual of the hypergraph of monochromatic components in a t-coloring of complete graphs, one can easily see that the following form of the conjecture (introduced in [9]) is equivalent.

Conjecture 12. *In every t-coloring of the edges of a complete graph, the vertex set can be covered by the vertices of at most $t-1$ monochromatic trees.*

If Conjecture 12 is true then an easy averaging argument would easily extend it as follows.

Conjecture 13. *For every $1 \leq s \leq t-1$ and for every t-coloring of K_n at least $\frac{ns}{t-1}$ vertices can be covered by the vertices of at most s monochromatic trees.*

Since Conjecture 12 is known to be true for $2 \leq t \leq 5$, Conjecture 13 is true for $1 \leq s \leq t \leq 5$. Also, the case $s = 1$ is known for arbitrary t (originally in [9], [12] is a recent survey). Perhaps a good test case is to try to prove Conjecture 13 for $s = 2$ (and for general t).

Since Ryser's conjecture is extended further in [6] by changing cover to partition in Conjecture 12, one may perhaps even require partition in Conjecture 13 as well.

1.4 Covers by copies of a fixed graph

It seems that to find the percentage of vertices that can be covered by monochromatic copies of a fixed graph H having at most s colors is a difficult problem. Indeed, even the case when H is a single edge seems difficult. However, somewhat surprisingly, for any fixed *connected non-bipartite graph* H and for any fixed $t \geq 3$ and fixed $s \leq t$, the percentage of vertices of K_n that can be covered by vertex disjoint monochromatic copies having at most s colors can be rather well approximated. In fact, the following theorem can be easily obtained from the results of [25]. Let $R_t(H)$ denote the smallest integer n such that in every t-coloring of the edges of K_n there is a monochromatic copy of H.

Theorem 14. *Suppose that $t \geq 3, 1 \leq s \leq t$ and H is a connected non-bipartite graph. Then in every t-coloring of the edges of K_n, at least $\frac{s(n-R_t(H))}{t}$ vertices can be covered by vertex disjoint monochromatic copies of H using at most s colors. On the other hand, for any n that is divisible by t, the edges of K_n can be t-colored so that at most $\frac{sn}{t}$ vertices can be covered by vertex disjoint monochromatic copies of H having at most s colors.*

Proof. The first part follows by selecting successively monochromatic copies of H, removing after each step the part covered so far. Clearly, the process stops only when fewer than $R_t(H)$ vertices remain. Then, an obvious averaging argument gives that the copies in s suitable colors cover the claimed quantity.

The second part follows from the following construction. Partition $V(K_n)$ into t equal parts and color the edges within the parts with t different colors, say within part i every edge gets color i. The crossing edges (going from one part to another) are all colored with the same color between any fixed pair of parts. There are two rules. On one hand, crossing edges of color i cannot be incident to part i. On the other hand, the union of crossing edges of color i should span a bipartite graph. It is easy to see that these rules can be easily guaranteed for $t \geq 3$ (and impossible to meet for $t = 2$). Because H is connected, not bipartite and crossing edges of color i are not adjacent to part i, each monochromatic copy of H must be completely within a part. Therefore copies of H having at most s colors are covered by the at most s parts, proving the second statement of the theorem. $\square$

2 Covers by matchings - how to generalize Cockayne - Lorimer theorem?

Here we return to the most basic case, when we want to cover by copies of an edge (the simplest bipartite graph), i.e. by matchings. A matching in a t-colored complete graph is called an s-colored matching if at most s colors are used on its edges. To describe easily certain t-colorings of K_n we need, consider partition vectors with t positive integer coordinates whose sum is equal to n. Assume that $V(K_n) = \{1, 2, \ldots, n\}$. Then $[p_1, p_2, \ldots, p_t]$ represents the coloring obtained by partitioning $V(K_n)$ into parts A_i so that $|A_i| = p_i$ for $i = 1, 2, \ldots, t$ and the color of any edge $e = (x, y)$ is the minimum j for which $\{x, y\}$ has non-empty intersection with A_j.

Problem 15. *Suppose $1 \le s \le t$. What is the size of the largest s-colored matching that can be found in every t-coloring of the edges of K_n?*

The Ramsey problem, the case $s = 1$ in Problem 15, was completely answered by Cockayne and Lorimer [5]. Here we state its diagonal case only.

Theorem 16. *([5]) Assume $n \ge (t+1)p + 2$ and K_n is arbitrarily t-colored. Then there is a monochromatic matching of size $p + 1$.*

Observe that Theorem 16 is sharp, the coloring $[p, p, \ldots, p, 2p+1]$ of the complete graph $K_{(t+1)p+1}$ does not contain a monochromatic matching with $p + 1$ edges.

Notice that the case $s = t$ of Problem 15 is trivial, any perfect (or near-perfect if n is odd) matching is obviously optimal. In this paper we settle the case $s = t - 1$, by showing that the extremal coloring is close to the coloring $[p, 2p, 4p, \ldots, 2^{t-1}p]$. More precisely we prove the following.

Theorem 17. *Every t-coloring of K_n contains a $(t-1)$-colored matching of size k provided that*

$$n \ge 2k + \left\lfloor \frac{(k-1)}{2^{t-1} - 1} \right\rfloor .$$

This is sharp for every $t \ge 2, k \ge 1$.

This problem was proposed at last year's Emléktábla workshop by András Gyárfás. In case of $t = 3$ Theorem 17 gives Conjecture 11 in a weaker form. Noting that for $k < 2^{t-1}$ the second term is zero in Theorem 17, we get the following.

Corollary 18. *Every t-colored $K_{2^t - 2}$ has a perfect matching missing at least one color.*

We note here that for $t = 2, 3, 4$ there are results stronger than Theorem 17. Namely, not only a $(t-1)$-colored matching of size k can be guaranteed, but a $(t-1)$-colored path on $2k$ vertices. For $t = 2$ this is a well-known result [8], for $t = 3$ it was proved in [26] and for $t = 4$ in [22]. In fact, it was conjectured in [22] that Theorem 17 holds also if the matching of size k is replaced by P_{2k}.

For the case $t = 4, s = 2$ we suspect that the extremal coloring is essentially $[p, p, 2p, 4p]$. That leads to

Conjecture 19. *If $n \ge \lfloor \frac{8k-2}{3} \rfloor$ then every 4-coloring of K_n contains a 2-colored matching of size k*
.

For the case $s = 2, t = 5$, the coloring $[p, p, p, 2p, 4p]$ and the coloring $[p, p, p, p, 2p]$ corresponding to Theorem 16 give essentially the same parameters so we do not risk a conjecture here. Moreover, for $s = 2, t = 6$ the latter coloring $[p, p, p, p, p, 2p]$ is better than $[p, p, p, p, 2p, 4p]$. This leads to the dilemma whether there are better colorings in this case or $[p, p, p, p, p, 2p]$ is the extremal one? The latter possibility would be similar to the phenomenon discussed in Subsection 1.4, saying vaguely that in a 6-colored complete graph the size of the largest 2-matching is twice the size of the largest monochromatic matching.

3 Large $(t-1)$-colored matchings in t-colored complete graphs.

Here we prove Theorem 17. To show that it is sharp, set $N = 2k - 1 + \left\lfloor \frac{(k-1)}{2^{t-1}-1} \right\rfloor = 2k - 1 + p$, where $p = \left\lfloor \frac{N}{2^t-1} \right\rfloor = \left\lfloor \frac{k-1}{2^{t-1}-1} \right\rfloor$. Consider the coloring $[p, 2p, 4p, \dots, 2^{t-2}p, q]$ of K_N with $q = N - (2^{t-1} - 1)p$.

If a matching in this coloring misses color $j \neq t$ then it misses at least $2^{j-1}p - \sum_{i<j} 2^{i-1}p = p$ vertices from the vertex set to which color class j is incident to. Thus at most $N - p = 2k - 1$ vertices are covered by this matching so its size is smaller than k. A matching that misses color t has at most $\sum_{i<t} 2^{i-1}p = (2^{t-1} - 1)p \leq k - 1$ edges.

To prove the upper bound, consider a t-colored K_n where

$$n = 2k + \left\lfloor \frac{(k-1)}{2^{t-1}-1} \right\rfloor = 2k + p.$$

Set $V = V(K_n)$, let G_i denote the subgraph of K_n with vertex set V and containing edges of colors different from color i, $1 \leq i \leq t$. We are going to show that for at least one i, G_i has a matching of size k. The proof is indirect: if the maximum matching of G_i, $\nu(G_i)$, is at most $k - 1$ for each i, then for the *deficiency* of G_i, $def(G_i)$, defined as the the number of vertices uncovered by any maximum matching of G_i, we have

$$def(G_i) \geq n - 2\nu(G_i) \geq 2k + p - 2(k-1) = p + 2.$$

We apply the following well-known result, where $c_o(G)$ is the number of odd components of G.

Theorem 20. *(Berge formula)* $def(G) = max\{c_o(V(G) \setminus X) - |X| : X \subset V(G)\}$.

Thus, for each i, there is a set $X_i \subset V$ such that

$$c_o(V \setminus X_i) \geq |X_i| + p + 2. \tag{1}$$

Assume w.l.o.g. that $|X_1| \leq \dots, \leq |X_t|$ and observe that the edges between connected components of G_i in $V \setminus X_i$ are all colored with color i. Let $C_1^1, \dots, C_{m_1}^1$ be the vertex sets of the connected components of G_1 in $V \setminus X_1$, from (1) we have $m_1 \geq |X_1| + p + 2$.

Lemma 21. *There is an index $l \in \{1, 2, \dots, m_1\}$, say $l = 1$, such that for every $j > 1$, $\cup_{i \neq 1} C_i^1 \subset X_j$.*

Proof. Suppose that $v, w \notin X_2$ where $v \in C_q^1, w \in C_r^1$ and $q \neq r$. This implies that the edges of color 1 form a complete multipartite graph M on $V \setminus (X_1 \cup X_2)$ with at least two partite classes. Therefore all vertices of M must be in the same connected component of G_2 in $V \setminus X_2$. Thus G_2 has at most $1 + |X_1| \leq 1 + |X_2| < |X_2| + p + 2$ odd components in $V \setminus X_2$, contradicting (1). Thus X_2 must cover all but at most one among the C_i^1-s, say C_1^1 can be uncovered.

Next we show that for all $j \geq 2$, we have $\cup_{i \neq 1} C_i^1 \subset X_j$. We have seen this for $j = 2$, so assume $j > 2$. The argument of the previous paragraph gives that X_j covers all but one C_i^1, say the exceptional one is C_l^1. Suppose $l \neq 1$, say $l = 2$. The inequality $|X_2| \leq |X_j|$ implies $|X_2 \setminus X_j| \leq |X_j \setminus X_2|$ and from this

$$|X_1 \cap (X_2 \setminus X_j)| + |C_2^1 \setminus X_j| \leq |X_1 \cap (X_j \setminus X_2)| + |C_1^1 \setminus X_2|. \tag{2}$$

On the other hand, using (1),

$$|V \setminus X_j| = |X_1 \cap (X_2 \setminus X_j)| + |X_1 \setminus (X_2 \cup X_j)| + |C_2^1 \setminus X_j| \geq c_o(V \setminus X_j) \geq |X_j| + p + 2$$

$$\geq |X_1 \cap (X_j \setminus X_2)| + |C_1^1 \setminus X_2| + |X_j \cap C_2^1| + |\cup_{i \geq 3} C_i^1| + p + 2$$

which can be rearranged as

$$(|X_1 \cap (X_2 \setminus X_j)| + |C_2^1 \setminus X_j|) - (|X_1 \cap (X_j \setminus X_2)| + |C_1^1 \setminus X_2|) + |X_1 \setminus (X_2 \cup X_j)| \geq$$
$$\geq |X_j \cap C_2^1| + |\cup_{i\geq 3} C_i^1| + p + 2. \tag{3}$$

Note that from (2) the left hand side of (3) is at most $|X_1 \setminus (X_2 \cup X_j)|$. Thus from (3) we get

$$|X_1| \geq |X_1 \setminus (X_2 \cup X_j)| \geq |X_j \cap C_2^1| + |\cup_{i\geq 3} C_i^1| + p + 2 \geq m_1 - 2 + p + 2 = m_1 + p$$

and this contradicts $m_1 \geq |X_1| + p + 2$ and finishes the proof of the lemma. □

Call $Ker_1 = C_2^1 \cup \ldots \cup C_{m_1}^1$ the first kernel. With this notation Lemma 21 claims that each X_j with $j > 1$ contains Ker_1. We may iterate Lemma 21 to define the set $Ker_i = C_2^i \cup \ldots \cup C_{m_i}^i$, the i-th kernel, so that each X_j with $j > i$ contains Ker_i. Furthermore, these kernels are disjoint, since X_{i+1} contains Ker_i, but Ker_{i+1} is contained in $V \setminus X_{i+1}$. This implies that we have the following recursion on the sizes of the X_i's.

Claim 1. *For every $2 \leq i \leq t$ we have*

$$|X_i| \geq |X_1| + \ldots + |X_{i-1}| + (i-1)(p+1).$$

Indeed, X_i contains all the disjoint kernels $Ker_1, \ldots, Ker_{i-1}$ and thus using (1) we get

$$|X_i| \geq \sum_{j=1}^{i-1} |Ker_j| \geq \sum_{j=1}^{i-1} (c_0(V \setminus X_j) - 1) \geq \sum_{j=1}^{i-1} (|X_j| + p + 1),$$

as desired.

Claim 1 implies easily by induction the following

$$|X_i| \geq (2^{i-1} - 1)(p+1). \tag{4}$$

But then, since the kernels are disjoint, using (1) again we get the following contradiction

$$n \geq \sum_{i=1}^{t} |Ker_i| \geq \sum_{i=1}^{t} (c_0(V \setminus X_i) - 1) \geq \sum_{i=1}^{t} (|X_i| + p + 1) =$$

$$= \sum_{i=1}^{t} |X_i| + t(p+1) \geq \sum_{i=1}^{t} 2^{i-1}(p+1) = (2^t - 1)(p+1) > n.$$

Here for the last inequality we have to check

$$(2^t - 1)(p+1) \geq n + 1 = 2k + p + 1.$$

This is equivalent to

$$p + 1 \geq \frac{k}{2^{t-1} - 1}, \tag{5}$$

which is always true for our choice $p = \lfloor \frac{k-1}{2^{t-1}-1} \rfloor$. Indeed,

$$\left\lfloor \frac{k-1}{2^{t-1} - 1} \right\rfloor = \left\lfloor \frac{k}{2^{t-1} - 1} \right\rfloor$$

(and so (5) is trivially true) for all cases except when $\frac{k}{2^{t-1}-1}$ is an integer, but (5) is true in this case as well, finishing the proof of Theorem 17. □

References

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Chromatic threshold of graphs

by József Balogh

This is based on a paper of Balogh, Butterfield, Hu, Lenz, and Mubayi.

Abstract

Let $\mathcal{F}$ be a family of r-uniform hypergraphs. The *chromatic threshold* of $\mathcal{F}$ is the infimum of all non-negative reals c such that the subfamily of $\mathcal{F}$ comprising hypergraphs H with minimum degree at least $c\binom{|V(H)|}{r-1}$ has bounded chromatic number. This parameter has a long history for graphs ($r = 2$), and in this paper we begin its systematic study for hypergraphs.

Łuczak and Thomassé recently proved that the chromatic threshold of near bipartite graphs is zero, and our main contribution is to generalize this result to r-uniform hypergraphs. For this class of hypergraphs, we also show that the exact Turán number is achieved uniquely by the complete $(r+1)$-partite hypergraph with nearly equal part sizes. This is one of very few infinite families of nondegenerate hypergraphs whose Turán number is determined exactly. In an attempt to generalize Thomassen's result that the chromatic threshold of triangle-free graphs is 1/3, we prove bounds for the chromatic threshold of the family of 3-uniform hypergraphs not containing $\{abc, abd, cde\}$, the so-called generalized triangle.

In order to prove upper bounds we introduce the concept of *fiber bundles*, which can be thought of as a hypergraph analogue of directed graphs. This leads to the notion of *fiber bundle dimension*, a structural property of fiber bundles which is based on the idea of Vapnik-Chervonenkis dimension in hypergraphs. Our lower bounds follow from explicit constructions, many of which use a generalized Kneser hypergraph. Using methods from extremal set theory, we prove that these generalized Kneser hypergraphs have unbounded chromatic number. This generalizes a result of Szemerédi for graphs and might be of independent interest. Many open problems remain.

4 Introduction

An r**-uniform hypergraph on** n **vertices** is a collection of r-subsets of V, where V is a set of n elements. If $r = 2$ then we call it a graph. The r-sets in a hypergraph are called **edges**, and the n elements of V are called **vertices**. For a hypergraph H let $V(H)$ denote the set of vertices. We denote the set of edges by either $E(H)$ or simply H. The **chromatic number** of a hypergraph H, denoted $\chi(H)$, is the least integer k for which there exists a map $f : V(H) \to [k]$ such that if E is an edge in the hypergraph then there exist $v, u \in E$ for which $f(v) \neq f(u)$. For a vertex v in a hypergraph H we let $d(v)$ denote the number of edges in H that contain v. We let $\delta(H) = \min\{d(v) : v \in V(H)\}$, called the **minimum degree** of H.

Definition. Let $\mathcal{F}$ be a family of r-uniform hypergraphs. The **chromatic threshold** of $\mathcal{F}$, is the infimum of the values $c \geq 0$ such that the subfamily of $\mathcal{F}$ consisting of hypergraphs H with minimum degree at least $c\binom{|V(H)|}{r-1}$ has bounded chromatic number.

We say that F is a subhypergraph of H if there is an injection from $V(F)$ to $V(H)$ such that every edge in F gets mapped to an edge of H. Notice that this is only possible if both H and F are r-uniform for some r. If H is an r-uniform hypergraph, then the family of H**-free** hypergraphs is the family of r-uniform hypergraphs that do not contain H as a (not necessarily induced) subgraph.

The study of the chromatic thresholds of graphs was motivated by a question of Erdős and Simonovits [6]: "If G is non-bipartite, what bound on $\delta(G)$ forces G to contain a triangle?" This question was answered by Andrásfai, Erdős, and Sós [3], who showed that the answer is $2/5\,|V(G)|$, achieved by the blowup of C_5. Andrásfai, Erdős, and Sós's [3] result can be generalized to construct triangle-free graphs with chromatic number at least k and large minimum degree. As k increases, these constructions have minimum degree approaching $1/3$. This led to the following conjecture: if $\delta(G) > (1/3 + \epsilon)\,|V(G)|$ and G is triangle-free, then $\chi(G) < k_\epsilon$, where k_ϵ is a constant depending only on ϵ.

Note that the conjecture is equivalent to the statement that the family of triangle-free graphs has chromatic threshold $1/3$. The conjecture was proven by Thomassen [19]. Subsequently, there have been three more proofs of the conjecture: one by Łuczak [13] using the Regularity Lemma, a result of Brandt and Thomassé [4] proving that one can take $k_\epsilon = 4$, and a recent proof by Łuczak and Thomassé [14] using the concept of Vapnik-Chervonenkis dimension (which is defined later in this paper).

For other graphs, Goddard and Lyle [9] proved that the chromatic threshold of the family of K_r-free graphs is $(2r-5)/(2r-3)$ while Thomassen [20] showed that the chromatic threshold of the family of C_{2k+1}-free graphs is zero for $k \geq 2$. Recently, Łuczak and Thomassé [14] gave another proof that the class of C_{2k+1}-free graphs has chromatic threshold zero for $k \geq 2$, as well as several other results about related families, such as Petersen-free graphs. The main result of Allen, Böttcher, Griffiths, Kohayakawa and Morris [1] is to determine the chromatic threshold of the family of H-free graphs for all H.

We finish this section with some definitions. For an r-uniform hypegraph H and a set of vertices $S \subseteq V(H)$, let $H[S]$ denote the r-uniform hypergraph consisting of exactly those edges of H that are completely contained in S. We call this the hypergraph **induced by** S. A set of vertices $S \subseteq V(H)$ is called **independent** if $H[S]$ contains no edges and **strongly independent** if there is no edge of H containing at least two vertices of S. A hypergraph is s-partite if its vertex set can be partitioned into s parts, each of which is strongly independent.

If $\mathcal{H}$ is a family of r-uniform hypergraphs, then the family of **$\mathcal{H}$-free** hypergraphs is the family of r-uniform hypergraphs that contain no member of $\mathcal{H}$ as a (not necessarily induced) subgraph. For an r-uniform hypergraph H and an integer n, let $ex(n,H)$ be the maximum number of edges an r-uniform hypergraph on n vertices can have while being H-free and let

$$\pi(H) = \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{r}}.$$

We call $\pi(H)$ the **Turán density** of H.

Let $T_{r,s}(n)$ be the complete n-vertex, r-uniform, s-partite hypergraph with part sizes as equal as possible. When $s = r$, we write $T_r(n)$ for $T_{r,r}(n)$. Let $t_r(n)$ be the number of edges in $T_r(n)$. We say that an r-uniform hypergraph H is **stable** with respect to $T_r(n)$ if $\pi(H) = r!/r^r$ and for any $\epsilon > 0$ there exists $\delta > 0$ such that if G is an H-free r-uniform hypergraph with at least $(1-\delta)t_r(n)$ edges, then there is a partition of $V(G)$ into $U_1, U_2, \ldots, U_r$ such that all but at most ϵn^r edges of G have exactly one vertex in each part.

Let $\mathrm{TK}^r(s)$ be the r-uniform hypergraph obtained from the complete graph K_s by enlarging each edge with $r-2$ new vertices. The **core vertices** of $\mathrm{TK}^r(s)$ are the s vertices of degree larger than one. For $s > r$, let $\mathcal{TK}^r(s)$ be the family of r-uniform hypergraphs such that there exists a set S of s vertices where each pair of vertices from S are contained together in some edge. The set S is called the set of **core vertices** of the hypergraph. For $s \leq r$, let $\mathcal{TK}^r(s)$ be the family of r-uniform hypergraphs such that there exists a set S of s vertices where for each pair of vertices $x \neq y \in S$, there exists an edge E with $E \cap S = \{x,y\}$ (the definition is different when $s \leq r$ so that a hypergraph which is just a single edge is not in $\mathcal{TK}^r(s)$). It is obvious that $\mathrm{TK}^r(s) \in \mathcal{TK}^r(s)$.

5 Chromatic threshold: Results and Problems

5.1 F_5

A classical 3-uniform hypergraph is the so-called "generalized triangle", which is sometimes such called F_5 is isomorphic to $\{abc, abd, cde\}$. Its extremal hypergraph is the complete 3-partite 3-graph (proved by Frankl and Füredi).

Theorem 22. *The chromatic threshold of the family of F_5-free 3-uniform hypergraphs is between $6/49$ and $(\sqrt{41}-5)/8 \approx 7/40$.*

Construction. Our construction is inspired by a construction by Hajnal [6] of a dense triangle-free graph with high chromatic number. Hajnal's key idea was to use the Kneser graph to obtain large chromatic number. The Kneser graph $\mathrm{KN}(n,k)$ has vertex set $\binom{[n]}{k}$, and two vertices F_1, F_2 form an edge if and only if $F_1 \cap F_2 = \emptyset$. We use an extension of Kneser graphs to hypergraphs. Alon, Frankl, and Lovász [2] considered the Kneser hypergraph $\mathrm{KN}^r(n,k)$, which is the r-uniform hypergraph with vertex set $\binom{[n]}{k}$, and r vertices $F_1, \ldots, F_r$ form an edge if and only if $F_i \cap F_j = \emptyset$ for $i \neq j$. They gave a lower bound on the chromatic number of $\mathrm{KN}^r(n,k)$ as follows.

Theorem 1. *If $n \geq (t-1)(r-1) + rk$, then $\chi(\mathrm{KN}^r(n,k)) \geq t$.*

Fix $t \geq 2$ and $\epsilon > 0$. Pick $k \geq 2t$ and $n = 3k + 2(t-1)$ and note that $n < 4k$. By Theorem 1, $\mathrm{KN}^3(n,k)$ has chromatic number at least t and it is easy to check that it is F_5-free. For integers u, v, and w where n divides u, let U, V and W be disjoint vertex sets of size u, v, and w respectively. Partition U into $U_1, \ldots, U_n$ such that $|U_i| = \frac{u}{n}$ for each i. Let H be the hypergraph with vertex set $V(\mathrm{KN}^3(n,k)) \cup U \cup V \cup W$ and the following edges.

- For $\{S_1, S_2, S_3\} \in \mathrm{KN}^3(n,k)$, make $\{S_1, S_2, S_3\}$ an edge of H.
- For $S \in V(\mathrm{KN}^3(n,k))$, $x \in U_i$ with $1 \leq i \leq n$, and $y \in V$, make $\{S, x, y\}$ an edge of H if $i \in S$.
- For $x \in U$, $y \in V$, and $z \in W$, make $\{x, y, z\}$ an edge of H.

Notice that H has chromatic number at least t because $\mathrm{KN}^3(n,k)$ is a subhypergraph.

Conjecture. The lower bound, i.e. the construction is sharp for F_5.

5.2 0-chromatic thresholds: Cycles

Definition. *Let C_m^r be the r-uniform hypergraph with m edges on n vertices $v_1, \ldots, v_n$ for which*

1. *the n vertices are arranged consecutively in a circle,*
2. *each edge contains r consecutive vertices,*
3. *if $m = 2k+1$ for some integer $k > 0$ then $n = rk + (r-1)$, and if $m = 2k$ then $n = rk$,*
4. *edges E_i and E_j share vertices if and only if $i \in \{j-1, j+1\}$ or $i = 1$ and $j = m$,*
5. *for $i \leq m-1$, if i is odd then $|E_i \cap E_{i+1}| = 1$; if i is even then $|E_i \cap E_{i+1}| = r-1$, and*
6. *if m is even then $|E_1 \cap E_m| = 1$; if m is odd then $|E_1 \cap E_m| = r-1$.*

Theorem 23. *The chromatic threshold of the family of C_{2k+1}^r-free hypergraphs is zero for $k \geq 2$.*

Theorem 24. *The extremal hypergraph of C_{2k+1}^r is the complete r-partite graph for $k \geq 2$ and $r \in \{3,4\}$.*

Problem What is the extremal hypergraph of C_{2k+1}^r for $k \geq 2$ and $r \geq 5$?

5.3 Chromatic thresholds of 3-uniform hypergraphs

Many open problems remain; for most 3-uniform hypergraphs A the chromatic threshold for the family of A-free hypergraphs is unknown. Interesting hypergraphs to study are those for which we know the extremal number, $ex(n, A)$, and we will examine a few of those here along with partial results and conjectures. We conjecture that most of the lower bounds given by the constructions in this section are tight.

Mubayi [16] showed that if $s > r$ then $\mathrm{ex}(n, \mathcal{TK}^r(s)) = |T_{r,s-1}(n)|$ and $\mathrm{ex}(n, \mathrm{TK}^r(s)) = (1 + o(1))\,|T_{r,s}(n)|$. Recently, Pikhurko [17] has shown that for large n and $s > r$, $\mathrm{ex}(n, \mathrm{TK}^r(s)) = |T_{r,s-1}(n)|$ and that $T_{r,s-1}(n)$ is the unique extremal example. Because F_5 is a member of $\mathcal{TK}^3(4)$ it follows that the chromatic threshold of $\mathcal{TK}^3(4)$-free hypergraphs is at most $(\sqrt{41} - 5)/8$. The following simple variation of another construction provides a lower bound of $18/361$ for both $\mathrm{TK}^3(4)$-free and $\mathcal{TK}^3(4)$-free hypergraphs.

Proposition 2. *The chromatic threshold of* $\mathcal{TK}^3(4)$*-free hypergraphs is at least* $\frac{18}{361}$.

Construction. It is very similar to the one of F_5. Choose k, n, u, v, w, U, V, W as in the construction for F_5; that is k, n, u, v, w are integers and U, V, W are disjoint sets of vertices of size u, v, w respectively. Divide U into $U_1, \ldots, U_n$ so that $|U_i| = u/n$ and divide V into $V_1, \ldots, V_n$ such that $|V_i| = v/n$. Let H be the hypergraph formed by taking $\mathrm{KN}^3(n, k)$ and adding the complete 3-partite hypergraph on U, V, W and the following edges. For $S \in V(\mathrm{KN}^3(n, k))$ and $x \in U_i$ and $y \in V_j$, make $\{S, x, y\}$ an edge if $i, j \in S$. The minimum degree is maximized when $a = b$ and $c = a/9$, which gives minimum degree approximately $a^2N^2/9 \approx \frac{18}{361} \cdot \binom{19a/9}{2} N^2$, where $N = u + v + w + \binom{n}{k}$ is the number of vertices in the hypergraphs.

This gives lower bounds on the chromatic thresholds of $\mathrm{TK}^3(4)$-free and $\mathcal{TK}^3(4)$-free hypergraphs and leads to the following questions.

Question 3. *What is the chromatic threshold for* $\mathrm{TK}^3(4)$*-free hypergraphs? It is between* $18/361$ *and* $2/9$*. What is the chromatic threshold for* $\mathcal{TK}^3(4)$*-free hypergraphs? It has the same lower bound as for* $\mathrm{TK}^3(4)$*-free hypergraphs, and because* $F_5 \in \mathcal{TK}^3(4)$ *the upper bound is* $(\sqrt{41} - 5)/8$.

A similar construction provides a $\mathcal{TK}^3(s)$-free hypergraph for any $s \geq 5$. We have not optimized the values.

Lemma 4. *When* $s \geq 5$*, the chromatic threshold of* $\mathcal{TK}^3(s)$*-free hypergraphs is at least* $\frac{(s-2)(s-3)(s-4)^2}{(s^2-13)^2} = 1 - \frac{13}{s} + O(\frac{1}{s^2})$.

5.4 $S(7)$-free hypergraphs

Next, consider the Fano plane $S(7)$. de Caen and Füredi [5] showed that $\mathrm{ex}(n, S(7)) = (\frac{3}{4} + o(1))\binom{n}{3}$. The extremal hypergraph for $S(7)$, proven to be extremal by Füredi and Simonovits [8] and also by Keevash and Sudakov [10], is the hypergraph formed by taking two almost equal vertex sets U and V and taking all edges which have at least one vertex in each of U and V. We can modify the hypergraph from an earlier Section to obtain a lower bound on the chromatic threshold of $S(7)$-free hypergraphs.

Proposition 5. *The chromatic threshold of* $S(7)$*-free hypergraphs is at least* $9/17$.

Construction. Fix $t \geq 2$ and $0 < \epsilon \ll 1$. Then by Lemma 1 there exists k large enough that if $n = (3 + \epsilon)k$ then $\mathrm{KN}^3(n, k)$ has chromatic number at least t. Fix some such k, and fix $N \gg \binom{n}{k}$.

Partition N vertices into two sets, U and V, with $|U| = 9N/17$ and $|V| = 8N/17$. Further partition U into n parts, $U_1, \ldots, U_n$, each of size $|U|/n$. Include as an edge each triple that has at least one vertex in each of U, V. Let H be the hypergraph formed by taking the disjoint union of this hypergraph and $\mathrm{KN}^3(n,k)$ and adding the following edges. For $u \in U_i$, $u' \in U_j$, and $X \in V(\mathrm{KN}^3(n,k))$ include $\{X, u, u'\}$ as an edge if $i, j \in X$ (recall that vertices in $\mathrm{KN}^3(n,k)$ are subsets of $[n]$). Notice that H has chromatic number at least t, and that $V(H) = N + \binom{n}{k}$.

Question 6. *What is the chromatic threshold of $S(7)$-free hypergraphs? It is at least $9/17$ and at most $3/4$, where the upper bound is from the extremal hypergraph of $S(7)$.*

5.5 T_5-free hypergraphs

Recall that the 3-uniform hypergraph T_5 has vertices A, B, C, D, E and edges $\{A, B, C\}$, $\{A, D, E\}$, $\{B, D, E\}$, and $\{C, D, E\}$.

Let $B^3(n)$ be the 3-uniform hypergraph with the most edges among all n-vertex 3-graphs whose vertex set can be partitioned into X_1, X_2 such that each edge contains exactly one vertex from X_2. Füredi, Pikhurko, and Simonovits [7] proved that for n sufficiently large the extremal T_5-free hypergraph is $B^3(n)$. It follows that the chromatic threshold for the family of T_5-free hypergraphs is at most $4/9$.

Proposition 7. *The chromatic threshold of T_5-free hypergraphs is at least $16/49$.*

Construction Fix $t \geq 2$ and $0 < \epsilon \ll 1$. Then by Lemma 1 there exists k large enough that if $n = (3/2 + \epsilon)k$ then $\mathrm{KN}^3_2(n,k)$ has chromatic number at least t. Fix some such k, and fix $N \gg \binom{n}{k}$.

Partition N vertices into two parts, U and V, with $|U| = 4N/7$ and $|V| = 3N/7$. Further partition U into n parts, $U_1, \ldots, U_n$, each of size $|U|/n$. Include as an edge any triple with two vertices in U and one in V. Let H be the hypergraph formed by taking the disjoint union of this graph and $\mathrm{KN}^3_2(n,k)$ and including the following edges. If $X \in V(\mathrm{KN}^3_2(n,k))$ and $u \in U_i$ and $v \in V$ then let $\{u, v, X\}$ be an edge if $i \in X$ (recall that vertices of $\mathrm{KN}^3_2(n,k)$ are subsets of $[n]$). Let $K = V(\mathrm{KN}^3_2(n,k))$. Notice that H has chromatic number at least t, and that $V(H) = N + \binom{n}{k}$.

The minimum degree of H is at least

$$\min\left\{\frac{2|U||V|}{3}, |U||V|, \binom{|U|}{2}\right\} = \frac{8}{49}N^2 - \frac{2}{7}N.$$

5.6 Co-chromatic thresholds

There is another possibility when generalizing the definition of chromatic threshold from graphs to hypergraphs: we can use the co-degree instead of the degree. Recall that if H is an r-uniform hypergraph and $\{x_1, \ldots, x_{r-1}\} \subseteq V(H)$, then the **co-degree** $d(x_1, \ldots, x_{r-1})$ **of** $x_1, \ldots, x_{r-1}$ is $|\{z : \{x_1, \ldots, x_{r_1}, z\} \in H\}|$. Let F be a family of r-uniform hypergraphs. The **co-chromatic threshold** of F is the infimum of the values $c \geq 0$ such that the subfamily of F consisting of hypergraphs H with minimum co-degree at least $c\,|V(H)|$ has bounded chromatic number. More generally, the k**-degree** $d(x_1, \ldots, x_k)$ **of** $x_1, \ldots, x_k$ is $|\{\{z_{k+1}, \ldots, z_r\} : \{x_1, \ldots, x_k, z_{k+1}, \ldots, z_r\} \in H\}|$ and we can define the k-chromatic threshold similarly. Given a hypergraph H and subsets U, V, W of $V(H)$, we say that an edge $\{u, v, w\}$ is of type UVW if $u \in U, v \in V$ and $w \in W$.

The co-chromatic thresholds of F_5-free hypergraphs and $\mathrm{TK}^3(4)$-free hypergraphs are trivially zero because if the minimum co-degree of H is at least 10 then H contains a copy of $\mathrm{TK}^3(4)$ and a copy of F_5. For the Fano plane, the last author proved [15] that for every $\epsilon > 0$ there exists n_0 such that any 3-uniform hypergraph with $n > n_0$ vertices and minimum co-degree greater than $(1/2 + \epsilon)n$ contains a copy of $S(7)$.

Notice that the lower bound construction for the chromatic threshold described above has non-zero minimum co-degree but the co-degree depends on the parameter t. We can modify the construction to prove a better lower bound on the co-chromatic threshold of $S(7)$-free hypergraphs.

Proposition 8. *The co-chromatic threshold of $S(7)$-free hypergraphs is at least $2/5$.*

Construction Fix $t \geq 2$ and $0 < \epsilon \ll 1$. Then by our Lemma there exists k large enough that if $n = (3/2 + \epsilon)k$ then $\mathrm{KN}_2^3(n,k)$ has chromatic number at least t. Fix $N \gg \binom{n}{k}$.

Partition N vertices into two parts, U and V, of size $\frac{3N}{5}$ and $\frac{2N}{5}$ respectively. Include as an edge any triple with at least one vertex in each part. Further partition U into n sets, $U_1, \ldots, U_n$, each of size $|U|/n$. Let H be the hypergraph formed by taking the disjoint union of this hypergraph with $\mathrm{KN}_2^3(n,k)$ and including the following edges. Include any edge of type KUV, where $K = V(\mathrm{KN}_2^3(n,k))$. For any $X, Y \in K$, if $|X \cap Y| < k - 4\epsilon k$ then include every edge of the form $\{X, Y, u\}$ where $u \in U_i$ for some $i \in X \cup Y$. If $|X \cap Y| \geq k - 4\epsilon k$ then include every edge of the form $\{X, Y, u\}$ where $u \in U_i$ for some $i \in X \cap Y$. Notice that H has chromatic number at least t and that $V(H) = N + \binom{n}{k}$.

It remains only to compute the minimum degree of H. Vertices $S_1, S_2 \in K$ have co-degree at least $\frac{k-4\epsilon k}{n}|U|$ if $|S_1 \cap S_2| \geq k - 4\epsilon k$ and at least $\frac{k+4\epsilon k}{n}|U|$ otherwise. Vertices $u_1, u_2 \in U$ have co-degree at least $|V|$ and vertices $v_1, v_2 \in V$ have co-degree at least $|U|$. All other pairs of vertices have co-degree at least $|U|$ or $|V|$. The minimum co-degree is therefore at least

$$\min\left\{\frac{k(1-4\epsilon)}{k(3/2+\epsilon)}|U|, |U|, |V|\right\} = \left\{\frac{2-8\epsilon}{3+2\epsilon} \cdot \frac{3}{5}N, \frac{3}{5}N, \frac{2}{5}N\right\}.$$

For some choice of ϵ, this is approximately $\frac{2}{5}|V(H)|$.

Question 9. *What is the co-chromatic threshold of the Fano-free hypergraphs? It is between $2/5$ and $1/2$.*

In [1] it was proved that if a family $\mathcal{F}$ of graphs has positive chromatic threshold then the chromatic threshold of $\mathcal{F}$ is in fact at least $1/3$. We think that a similar statement holds for hypergraphs. For 3-uniform hypergraphs, we believe that the least positive chromatic threshold is achieved by the family of $\mathrm{TK}^3(4)$-free hypergraphs.

Conjecture 10. *If a family $\mathcal{F}$ of 3-uniform hypergraphs has positive chromatic threshold then the chromatic threshold of $\mathcal{F}$ is at least $18/361$.*

5.7 Kneser hypergraphs

Sarkaria [18] considered the generalized Kneser hypergraph $KN_s^r(n,k)$, which is the r-uniform hypergraph with vertex set $\binom{[n]}{k}$, in which r vertices $F_1, \ldots, F_r$ form an edge if and only if no element of $[n]$ is contained in more than s of them. Note that the Kneser hypergraph $KN^r(n,k)$ is $KN_1^r(n,k)$. Sarkaria [18] and Ziegler [21] gave lower bounds on the chromatic number of $KN_s^r(n,k)$, but Lange and Ziegler [12] showed that the lower bounds obtained by Sarkaria and Ziegler apply only if one allow the edges of $KN_s^r(n,k)$ to have repeated vertices. We conjecture that for $KN_s^r(n,k)$, a statement similar to Theorem 1 is true.

Conjecture There exists $T(r,s,t)$ such that if $n \geq T(r,s,t) + rk/s$, then $\chi\left(KN_s^r(n,k)\right) \geq t$.

The following much weaker statement is sufficient for our purposes. The proof is similar to an argument of Szemerédi which appears in a paper of Erdős and Simonovits [6], and the proof of Claim 1 is motivated by an argument of Kleitman [11].

Theorem 25. *Let $c > 0$; then for any integers r, t, there exists $K_0 = K_0(c, r, t)$ such that if $k \geq K_0$, $s = r - 1$, and $n = (r/s + c)k$, then $\chi\left(KN_s^r(n, k)\right) > t$.*

6 Bipartite v.r.to trianglee-free

Remark. We start by describing the structure of triangle-free graphs with high minimal degrees. For $d \geq 1$ we define a graph F_d as follows. The vertex set $V(F_d)$ consists of the integers modulo $3d-1$, which we denote by $\mathbb{Z}_{3d-1}$. The vertex $v \in \mathbb{Z}_{3d-1}$ is adjacent to the vertices $v+1, v+4, v+7, \cdots, v-1$. Thus F_d is a d-regular graph on $3d-1$ vertices. For example, $F_1 = K_2$ consists of a single edge, and $F_2 = C_5$ is a 5-cycle.

For which graphs do the largest bipartite subgraph and largest triangle-free subgraph have the same number of edges? This question was raised by Erdős [25], who noted that there is equality for the complete graph K_n (by Turán's theorem). Babai, Simonovits and Spencer [22] showed that equality holds almost surely for the random graph where edges are chosen with probability 1/2. A general condition implying equality was given by Bondy, Shen, Thomassé and Thomassen [24], who showed that a minimum degree condition is sufficient.

For a graph G we write $b(G)$ for the number of edges in its largest bipartite subgraph, and $t(G)$ for the number of edges in its largest triangle-free subgraph. Clearly $t(G) \geq b(G)$. Write δ_c for the least number so that, for n sufficiently large, any graph G on n vertices with minimum degree $\delta(G) \geq (\delta_c + o(1))n$ has $t(G) = b(G)$. Bondy et al. [24] showed that $0.675 \leq \delta_c \leq 0.85$. J. Balogh, B. Sudakov and P. Keevash [23] strengthened this as follows.

Theorem 26. $0.75 \leq \delta_c < 0.791$.

Moreover, we believe that the lower bound is tight and propose the following conjecture.

Conjecture 27. *In any graph on n vertices with minimum degree at least $(3/4 + o(1))n$ the largest triangle-free and largest bipartite subgraphs have equal size.*

Theorem 28. *For any $\delta < 3/4$ there is n and a graph G on n vertices with minimum degree at least δn in which the largest triangle-free subgraph has more edges than the largest bipartite subgraph. Therefore $\delta_c \geq 3/4$.*

Construction. The vertex set $V = V(G)$ of our graph will be divided into parts V_i, $i \in \mathbb{Z}_5$ each of size $n/5$. All pairs uv with $u, v \in V_i$ or $u \in V_i$, $v \in V_{i+1}$ for some i are edges of G. Also, for every i each pair uv with $u \in V_i$, $v \in V_{i+2}$ is chosen to be an edge randomly and independently with probability θ, for some $\theta < 3/8$.

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Simple questions on axis-parallel rectangles in the plane.

by Gábor Tardos

1. Oldest: packing versus piercing. Given a collection H of axis-parallel rectangles in the plane $\nu = \nu(H)$ is its packing number: the largest cardinality of a pairwise disjoint subfamily of H, while $\tau = \tau(H)$ is its piercing number: the smallest cardinality of a point set meeting all members of H. Clearly $\tau \geq \nu$ (separate points are needed to pierce the disjoint rectangles), but already a system of five rectangles can show they don't have to be equal. It's a very old open problem to determine if $\tau = O(\nu)$ holds in general. The best bound known is $\tau = O(\nu \log \nu)$.

2. Glass cutting: Given a collection H of n pairwise disjoint convex compact sets on an infinite sheet of glass (i.e., the plane) cut out as many as you can. The glass is brittle, so you have to cut along a straight line. It must be the full line at first and then each piece can be further cut along the intersection of a line with the piece of glass in question. You are done if you identified a subset of H that lies each on separate pieces of the glass and each are intact (not cut through). How big a subset can you always find?

For $n = 2$ one can separate the two pieces, but for some collections with $n = 3$ sets the first cut has to break one set in order to separate the two others.

The best bounds for the largest number $f(n)$ that can always be separated from among n sets are $f(n) = O(n^{\log 2/\log 3})$ and $f(n) = \Omega(n^{1/3})$.

One can consider the restriction of this problem to line segments or (to get closer to the title) to axis parallel rectangles, where the situation is very similar to problem 1 above. Among n pairwise disjoint axis parallel rectangles one can separate $\Omega(n/\log n)$ even if restricted to use horizontal and vertical cuts only, but it is not ruled out that a linear number can be separated this way and I propose this as a problem.

3. Newest (but was still proposed last year on the same workshop): weak epsilon-nets. Is the following statement true for any set H of n points in the plane: There exists a set H' of $n/2$ points in the plane that "represents" H in the sense, that for every axis parallel rectangle containing 1000 points from H also contains at least one point of H'.

I believe the answer is negative and (assuming it remains negative for any constant in place of 1000) this would imply non-trivial lower bounds for so called weak epsilon nets with respect to axis parallel rectangles. A strong negation of the above statement was proved when H' must be a subset of of H. Here it is:

For a uniform random set H of n points in the unit square the following holds with high probability: For any SUBSET H' of H of cardinality $n/2$ one can find an axis parallel rectangle containing $\Omega(\log \log n)$ points of H and none of H'.

The $\log \log n$ is tight in this bound for any point set. The proof is based on estimating the probability that any fixed H' would work, and realizing that this is small even if multiplied by the total number of subsets of H (namely 2^n). This approach have obvious problems when applying to the original setting with an infinite number of possibilities for H', still I believe that something like this must be true.

Background and references in the talk.

The Permutation Pattern Avoidance Problem
Classic Questions and New Directions

by Miklós Bóna

7 Classic Problems

The classic definition of pattern avoidance for permutations is as follows. Let $p = p_1p_2 \cdots p_n$ be a permutation, let $k < n$, and let $q = q_1q_2 \cdots q_k$ be another permutation. We say that p *contains* q as a pattern if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ so that for all indices j and r, the inequality $q_j < q_r$ holds if and only if the inequality $p_{i_j} < p_{i_r}$ holds. If p does not contain q, then we say that p *avoids* q. In other words, p contains q if p has a subsequence of entries, not necessarily in consecutive positions, which relate to each other the same way as the entries of q do.

Example 29. *The permutation 3174625 contains the pattern 1324. Indeed, consider the second, fourth, sixth and seventh entries.*

The enumeration of permutations avoiding a given pattern is a fascinating subject. Let $S_n(q)$ denote the number of permutations of length n (or, in what follows, n-permutations) that avoid the pattern q.

7.1 Patterns of Length Three

Among patterns of length three, there is no difference between the monotone pattern and other patterns as far as $S_n(q)$ is concerned. This is the content of our first theorem.

Theorem 30. *Let q be any pattern of length three, and let n be any positive integer. Then $S_n(q) = C_n = \binom{2n}{n}/(n+1)$. In other words, $S_n(q)$ is the nth Catalan number.*

7.2 Patterns of Length Four

When we move to longer patterns, the situation becomes much more complicated and less well understood. In his doctoral thesis Julian West published the following numerical evidence.

- for $S_n(1342)$, and $n = 1, 2, \cdots, 8$, we have 1, 2, 6, 23, 103, 512, 2740, 15485
- for $S_n(1234)$, and $n = 1, 2, \cdots, 8$, we have 1, 2, 6, 23, 103, 513, 2761, 15767
- for $S_n(1324)$, and $n = 1, 2, \cdots, 8$, we have 1, 2, 6, 23, 103, 513, 2762, 15793.

These data are startling for at least two reasons. First, the numbers $S_n(q)$ are no longer independent of q; there are some patterns of length four that are easier to avoid than others. Second, the monotone pattern 1234, special as it is, does not provide the minimum or the maximum value for $S_n(q)$. We point out that for each q of the other 21 patterns of length four, it is known that the sequence $S_n(q)$ is identical to one of the three sequences $S_n(1342)$, $S_n(1234)$, and $S_n(1324)$.

Exact formulas are known for two of the above three sequences. For the monotone pattern, Ira Gessel gave a formula using symmetric functions.

Theorem 31. *(Gessel) For all positive integers n, the identity*

$$S_n(1234) = 2 \cdot \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k}^2 \frac{3k^2 + 2k + 1 - n - 2nk}{(k+1)^2(k+2)(n-k+1)} \tag{6}$$

$$= \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}. \tag{7}$$

The formula for $S_n(1342)$ is due to the present author and is quite surprising.

Theorem 32. *For all positive integers n, we have*

$$S_n(1342) = (-1)^{n-1} \cdot \frac{(7n^2 - 3n - 2)}{2} + 3 \sum_{i=2}^{n} (-1)^{n-i} \cdot 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \cdot \binom{n-i+2}{2}.$$

This result is unexpected for two reasons. First, it shows that $S_n(1342)$ is not simply less than $S_n(1234)$ for every $n \geq 6$; it is *much less*, in a sense that we will explain in Subsection 7.4. For now, we simply state that while $S_n(1234)$ is "roughly" 9^n, the value of $S_n(1342)$ is "roughly" 8^n. Second, the formula is, in some sense, simpler than that for $S_n(1234)$. Indeed, it follows from Theorem 32 that the ordinary generating function of the sequence $S_n(1342)$ is

$$H(x) = \sum_{i \geq 0} F^i(x) = \frac{1}{1 - F(x)} = \frac{32x}{-8x^2 + 20x + 1 - (1-8x)^{3/2}}.$$

This is an *algebraic* power series. On the other hand, it is known that the ordinary generating function of the sequence $S_n(1234)$ is *not* algebraic. So permutations avoiding the monotone pattern are not even the *nicest* among permutations avoiding a given pattern, in terms of the generating functions that count them.

There is no known formula for the third sequence, that of the numbers $S_n(1324)$. However, the following inequality is known

Theorem 33. *(Bóna) For all integers $n \geq 7$, the inequality*

$$S_n(1234) < S_n(1324)$$

holds.

7.3 Monotone Patterns of Any Length

For general k, there are some good estimates known for the value of $S_n(\alpha_k)$. The first one can be proved by an elementary method.

Theorem 34. *For all positive integers n and $k > 2$, we have*

$$S_n(123 \cdots k) \leq (k-1)^{2n}.$$

7.4 Stanley-Wilf Limits

The following celebrated result of Adam Marcus and Gábor Tardos shows that in general, it is very difficult to avoid any given pattern q.

Theorem 35. *For all patterns q, there exists a constant c_q so that*

$$S_n(q) \leq c_q^n. \tag{8}$$

It is not difficult to show using Fekete's lemma that the sequence $(S_n(q))^{1/n}$ is monotone increasing. The previous theorem shows that it is bounded from above, leading to the following.

Corollary 36. *For all patterns q, the limit*

$$L(q) = \lim_{n\to\infty} (S_n(q))^{1/n}$$

exists.

The real number $L(q)$ is called the *Stanley-Wilf limit*, or *growth rate* of the pattern q. In this terminology, it can be proved that $L(\alpha_k) = (k-1)^2$. In particular, $L(1234) = 9$, while Theorem 32 implies that $L(1342) = 8$. So it is not simply easier to avoid 1234 than 1342, it is *exponentially* easier to do so.

Numerical evidence suggests that in the multiset of $k!$ real numbers $S_n(q)$, the numbers $S_n(\alpha_k)$ are much closer to the maximum than to the minimum. This led to the plausible conjecture that for any pattern q of length k, the inequality $L(q) \leq (k-1)^2$ holds. This would mean that while there are patterns of length k that are easier to avoid than α_k, there are none that are much easier to avoid, in the sense of Stanley-Wilf limits. However, this conjecture has been disproved by the following result of Michael Albert and al.

Theorem 37. *The inequality $L(1324) \geq 11.35$ holds.*

In other words, it is not simply harder to avoid 1234 than 1324, it is *exponentially* harder to do so.

7.5 Questions

1. What is the smallest constant c_k so that $S_n(q) < c_k^n$ for all patterns of length k?

2. What patterns of length k are the easiest and hardest to avoid? Why?

3. What kind of numbers can occur as limits $L(q)$? So far, in all cases when $L(q)$ is known, it is of the form $a + b\sqrt{2}$, with a and b non-negative integers.

4. Find a formula for $S_n(1324)$.

5. Find more patterns q for which $L(q)$ can be computed.

8 New Directions

8.1 Superpatterns

A *k-superpattern* is a permutation that contains all $k!$ patterns of length k. Let $sp(k)$ be the length of the shortest k-superpattern. For instance, $sp(2) = 3$, as 132 is a 2-superpattern of length three, and obviously, there is no shorter 2-superpattern. It is easy to see that $sp(3) = 5$, and a little bit harder to see that $sp(4) = 9$.

In general, a recent construction of Alison Miller shows that $sp(k) \leq \binom{k+1}{2}$. As far as lower bounds go, we only have the trivial lower bound $sp(k) \geq k^2/e^2$.

The questions here are obvious. Improve the lower and upper bounds on $sp(k)$, or prove that they are optimal.

8.2 Supersequences

Find the shortest sequence whose elements are from the set $\{1, 2, \cdots, n\}$ that contains all $n!$ permutations of length n. It is known that if $m(n)$ is the length of the shortest such sequence, then

$$n^2 - cn^{7/4} \leq m(n) \leq n^2 - 2n + 4.$$

8.3 Tight Pattern Avoidance

Same as pattern avoidance, but the entries forming a pattern must be in consecutive positions. Let $T_n(q)$ be the number of n-permutations avoiding q. Let q be of length k. Then a ten-year old conjecture of Elizalde and Noy states that

$$T_n(q) \leq T_n(\alpha_k).$$

Contributed Problems

Ratio of max and min degree in maximal intersection family

by Balázs Patkós

Let $\mathcal{F} \subseteq \binom{[n]}{r}$ be a maximal intersecting family such that $\bigcup \mathcal{F} = [n]$. Try to maximize and minimize $R(\mathcal{F}) = \frac{\Delta(\mathcal{F})}{\delta(\mathcal{F})}$, the ratio of the maximum and the minimum degree. It is not hard to see that if r is fixed, then

$$\frac{1}{r^r}n \leq R(\mathcal{F}) \leq (1+o(1))n,$$

where the upper bound is assymptotically tight but the lower bound seems to be very weak. I conjecture that

$$\frac{1}{r}n \leq R(\mathcal{F}) \leq (1+o(1))n,$$

provided $r = o(n^{1/2})$.

Red-blue alternating paths

by Balázs Keszegh

Theorem 38. *[Gyárfás and Lehel] For $2 \le i \le k$, let T_i be a path or a star on i vertices. $T_2, \dots, T_k$ can be packed into K_k.*

Conjecture 39. *[Gerbner Daniel, Cory Palmer, Balázs Keszegh] For $2 \le i \le k$, let T_i be a path or a star on i vertices. If G is a k-chromatic graph, then $T_2, \dots, T_k$ can be packed into G.*

The following conjecture implies Conjecture 39 and is an analogue of a key lemma used in the proof of Theorem 38 by Gyárfás and Lehel.

Conjecture 40. *Let G and H be two graphs on the same vertex set where the edges of G are red and the edges of H are blue. Suppose there is a subset of the vertices $V_k = \{v_1, v_2, \dots, v_k\}$ such that for $i = 1, 2, \dots, k$ we have degree $d(v_i) \ge i$ in both G and H. Then it is always possible to find a family of vertex-disjoint paths of alternating edge colors (among the edges of G and H together) such that*

1 for every $i = 1, 2, \dots, k$, the vertex v_i is contained in a path,

2 each path consists of vertices only from V_k except for exactly one of its endpoints, which must be outside of V_k.

When $G = H$, Conjecture 40 is exactly the same as the lemma in the paper of Gyárfás and Lehel.

Homomorphism-free coloring of graphs

by János Barát

A vertex coloring of a graph G is homomorphism-free if and only if the identity is the only color-preserving homomorphism of G. The minimum k, for which there exists a homomorphism-free coloring of G with k colors, is denoted by $hf(G)$.

Observe that $hf(G) = 1$ if and only if G is a rigid graph [3].

Problem 1. Which graphs satisfy $hf(G) = 2$?

I have collected some results in [2]. For instance, P. Varjú and myself proved the following

Lemma If G is a graph with maximum degree Δ, then $hf(G) \leq \Delta + 1$.

I believe there is a Brooks-type theorem here. I think it follows from the result for the distinguishing number in [4].

On the other hand, I proved that the distinguishing number [1] and hf can be far apart for some graphs [2].

There are numerous results for trees.

Problem 2. What are the trees, for which hf is large? There is an explicit construction by X. Zhu for such candidates.

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Choosability with separation

by Mohit

Given a graph G, a list L is called a (k,c)-list if $|L(v)| = k, \forall v$ and $|L(v) \cap L(u)| \leq c, \forall (u,v) \in E(G)$. Kratochvil, Tuza and Voigt [1] introduced $\chi_l(G,c)$ to be the minimum k so that G is L colorable for each (k,c) list.

Among other things they showed the following:
1. $\chi_l(G,c) \leq \sqrt{2ec(\Delta - 1)}$
2. $\chi_l(G,1) \leq 4$, G planar
3. $\sqrt{cn/2} \leq \chi_l(K_n,c) \leq \sqrt{2ecn}$

It is now known [2] that $\lim_{n\to\infty} \frac{\chi_l(K_n,c)}{\sqrt{cn}} = 1$

Question 1: Is it true that $\chi_l(H,c) \leq \chi_l(G,c)$ when H is a non-induced subgraph of G ?
Question 2: Is it true that $\chi_l(G,c) \leq \chi_l(K_n,c)$ when G is an n-vertex graph ?

Remark: There are easy examples of (non-uniform) hypergraphs for which Question 1 is not true.

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The sensitivity of 2-colorings of the d-dimensional integer lattice

by Scott Aaronson

This question is stolen from mathoverflow.

Consider the d-dimensional integer lattice, Z^d. Call two points in Z^d neighbors if their Euclidean distance is 1 (i.e., if they differ by 1 on exactly one coordinate).

Let C be a two-coloring of Z^d, which makes each point either red or blue. We'll assume C has the following property: the origin is colored red, but on each of the d axes through the origin, there's a point on that axis that's colored blue.

Let the "sensitivity" of a point x with respect to C, or $s^x(C)$, be the number of x's neighbors that are colored differently from x. Then let $s_d(C) = \min_C \max_{x \in Z^d} s^x(C)$.

QUESTION: How much is $s_d(C)$ in terms of d?

As an example, here I show that the sensitivity might be only two even in six dimensions. The construction is to color everything red except for the following six blue affine subspaces: $(30....), (.30...), (0.3...), (...30.), (....30), (...0.3)$ where the numbers mean the fixed coordinates, the dots the free ones.

More generally, if we suppose that there are exactly d blue axis-aligned affine subspaces, then a simple Turan-type argument shows that $d = 2s^2 - s$ is the biggest dimension where sensitivity s is possible. Can you give any better construction for any dimension?

Can we at least prove that $s_d(C)$ is not bounded by some constant?

Does every polyomino tile $\mathbb{R}^n$ for some n?

by Adam Chalcraft

I have also stolen this problem from mathoverlow.

A *polyomino* is usually defined to be a finite set of unit squares, glued together edge-to-edge. Here I generalize it to mean a finite set of unit hypercubes, glued together facet-to-facet.

Given a polyomino P in $\mathbb{R}^m$, I can $lift$ it to a polyomino in a higher-dimensional Euclidean space $\mathbb{R}^{m+n}$ by crossing it with a unit n-cube: the lifted polyomino is just $P \times [0,1]^n$.

Obviously, not all polyominos tile space.

QUESTION: Is it true that given *any* polyomino P in $\mathbb{R}^m$, there exists some n such that the lifted polyomino $P \times [0,1]^n$ tiles $\mathbb{R}^{m+n}$?

Note that it does not matter whether we consider only connected dominoes or not: If the original polyomino, P, is d dimensional, then we can construct a 2d dimensional connected polyomino, Q, that can be tiled with P. Clearly, this proves the statement, as if it is impossible to tile any space with P, it is also impossible to do so with Q.

Denote a large enough d dimensional brick that contains P by R. Take the 2d dimensional polyomino P x R, so here every original cube of P is replaced by a 2d brick, 1 x R. Note that P x R is contained in an R x R brick. Fill in the missing parts of this R x R brick by 1 x P polyominos. Notice that this means that R x P will be also filled up completely. This polyomino, Q, will be connected, as we can freely move anywhere in the first d coordinates in R x P and in the last d coordinates in P x R.

Note: The complement of the set obtained this way is R\ P x R\ P. If we repeat this, then it can be achieved that our polyomino is arbitrarily dense, i.e. it fills out at least 99% of a brick.

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