2nd Emléktábla Workshop

More combinatorics 01. 24-27. 2011.

Preliminary Schedule

Day 1:
10:14 Welcome
10:15 - 11:00 Pach János
11:15 - 12:00 Tóth Géza
Lunch Break
14:00 - 14:45 Károlyi Gyula
15:00 - 15:45 Győri Ervin
16:30 from in front of Rényi: Traveling together to Gyöngyöstarján by private bus.

Other Days:
9:29 Waking up
8:30 - 9:30 Breakfast
9:30 Partitioning to Groups of 3-5 for the day
9:30 - 12:30 Work in Groups of 3-5
11:00 Coffee Break
12:30 - 14:00 Lunch Break
14:00 Optional Repartitioning for the afternoon
14:00 - 17:00 Work in Groups of 3-5
15:45 Coffee Break
17:00 - 18:30 Discussion of Results
18:30 - Dinner and other activities

Day 2 afternoon: Optional hiking led by Ida

Day 3 evening: Obligatory Csocsó tournament

Last Day:
Discussion from 15:00,
Return to Budapest at 16:30, arrival to Rényi at 18:00

Invited Problems

Boxes and the art of ϵ-net maintenance

by János Pach

All new results mentioned below are based on joint work with Gábor Tardos.

Let X be a *finite* set and let $\mathcal{R}$ be a system of subsets of an underlying set which contains X. In computational geometry, the pair $(X, \mathcal{R})$ is usually called a *range space*. The elements of X and $\mathcal{R}$ are said to be the *points* and the *ranges* of the range space, respectively. Consider a subset $A \subseteq X$. It is called *shattered* if for every subset $B \subseteq A$, one can find a range $R_B \in \mathcal{R}$ with $R_B \cap A = B$. The size of the largest shattered subset of points, $A \subseteq X$, is said to be the *Vapnik-Chervonenkis dimension* (or *VC-dimension*) of the range space $(X, \mathcal{R})$.

In [VaC71], Vapnik and Chervonenkis proved that, from the point of view of random sampling, all range spaces whose VC-dimensions are bounded by a constant behave very nicely. In particular, for any $\varepsilon > 0$, a randomly selected "small" subset of X, whose number of elements depends only on the VC-dimension d and ε, will "hit" every range containing at least $\varepsilon|X|$ points of X, with large probability. A set of points in X with the property that every range $R \in \mathcal{R}$ with $|R \cap X| \geq \varepsilon|X|$ contains at least one of its elements is called an *ε-net* for the range space $(X, \mathcal{R})$. Note that these sets are often called *strong* ε-nets in the literature, to distinguish them from the so-called *weak ε-nets*, which may also contain points from $\cup\mathcal{R} \setminus X$, but must still hit all ranges that contain at least $\varepsilon|X|$ elements of X.

The ideas of Vapnik and Chervonenkis have been adapted by Haussler and Welzl [HaW87] to show that the minimum number $f = f_d(\varepsilon)$ such that every range space of VC-dimension d admits an ε-net of size at most f satisfies $f_d(\varepsilon) = O\left(\frac{d}{\varepsilon}\log\frac{d}{\varepsilon}\right)$. They asked whether the logarithmic factor can be removed in this formula. Pach and Woeginger [PaW90] proved that while $f_1(\varepsilon) = \max(2, \lceil\frac{1}{\varepsilon}\rceil - 1)$, the logarithmic factor is needed for every $d \geq 2$. Moreover, it was shown by Komlós et al. [KoPW92, PaA95]) that for any $d \geq 2$,

$$(d - 2 + \frac{1}{d+2} + o(1))\frac{1}{\varepsilon}\log\frac{1}{\varepsilon} \leq f_d(\varepsilon) \leq (d + o(1))\frac{1}{\varepsilon}\log\frac{1}{\varepsilon},$$

as ε tends to 0. (Here log denotes the natural logarithm.)

Haussler and Welzl discovered that the above results apply to many geometrically defined range spaces. Roughly speaking, the VC-dimension is bounded by a constant for any set of ranges with bounded *description complexity*, that is if the ranges can be described in terms of a bounded number of parameters. In a number of basic geometric scenarios it was possible to improve on the above bounds. For instance, for any finite set of points in the plane, one can find an ϵ-net of size linear in $1/\varepsilon$, where the ranges are half-planes, translates of a convex polygon, disks or certain kind of pseudo-disks. Similar results hold in three-dimensional space for half-space ranges [PaW90, MaSW90, Ma92, PyR08].

Theorem 1. (Matoušek, Seidel, Welzl [MaSW90, Ma92]) *All range spaces $(X, \mathcal{R})$, where X is a finite set of points in $\mathbb{R}^3$ and $\mathcal{R}$ consists of half-spaces, admit ε-nets of size $O(1/\varepsilon)$.*

Theorem 2. (Aronov, Ezra, Sharir [ArES10]) *All range spaces $(X, \mathcal{R})$, where X is a finite set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and $\mathcal{R}$ consists of axis-parallel rectangles (boxes), admit ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$.*

Aronov et al. have also established a similar result for "fat" triangular ranges in the place of axis-parallel rectangles. For weak ε-nets, Ezra [Ez10] extended Theorem B to higher dimensions.

In algorithmic applications, it is often natural to consider the dual range space, in which the roles of points and ranges are swapped [BrG95, PaA95]. Given a finite family $\mathcal{R}$ of ranges in $\mathbb{R}^m$, the *dual range space induced* by them is defined as a set system (hypergraph) on the underlying set $\mathcal{R}$, consisting of the sets $\mathcal{R}_x := \{R \mid x \in R \in \mathcal{R}\}$, for all $x \in \mathbb{R}^m$. (Note that $\mathcal{R}_x$ and$\mathcal{R}_y$ may coincide for $x \neq y$.) It is easy to see that if the VC-dimension of the range space $(X, \mathcal{R})$ is less than d for every $X \subset \mathbb{R}^m$, then the VC-dimension of the dual range space induced by any subset of $\mathcal{R}$ is less than 2^d.

Clarkson and Varadarajan [ClV07] found a simple and beautiful connection between the complexity of the boundary of the union of n members of $\mathcal{R}$ and the size of the smallest epsilon-net in the dual range space. If the complexity of the boundary is $o(n \log n)$, then the dual range space admits ε-nets of size $o\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. This connection has been further explored and improved in [Va09, ArES10]. In particular, it was shown that dual range spaces of "fat" triangles in the plane admit ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\log\frac{1}{\varepsilon}\right)$.

In most range spaces $(X, \mathcal{R})$, one can find roughly $1/\varepsilon$ pairwise disjoint ranges $R \in \mathcal{R}$ such that the sets $R \cap X$ are of size at least $\varepsilon|X|$. In these cases, the size of any ε-net is $\Omega(1/\varepsilon)$. For the last two decades, "the prevailing conjecture" was that in "geometric scenarios," this bound is essentially tight: there always exists an ε-net of size $O(1/\varepsilon)$ (see, e.g., [MaSW90, ArES10]. This conjecture had to be revised after Alon [Al10] discovered some geometric range spaces of small VC-dimension, in which the ranges are straight lines, rectangles or infinite strips in the plane, and which do not admit ε-nets of size $O(1/\varepsilon)$. Alon's construction is based on the density version of the Hales-Jewett theorem [HaJ63], due to Furstenberg and Katznelson [FuK89, FuK91], and recently improved in [Po09]. However, his lower bound is only barely superlinear: $\Omega\left(\frac{1}{\varepsilon}g(\frac{1}{\varepsilon})\right)$, where g is an extremely slowly growing function, closely related to the inverse Ackermann function.

Theorem 3. (P., Tardos, 2011) *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a dual range space Σ^* of VC-dimension 2, induced by n axis-parallel rectangles in $\mathbb{R}^2$, in which the minimum size of an ε-net is at least $C\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$. Here $C > 0$ is an absolute constant.*

From Theorem 1 it is not hard to deduce the following results for primal range spaces.

Theorem 4. (P., Tardos) *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a (primal) range space $\Sigma = (X, \mathcal{R})$ of VC-dimension 2, where X is a set of n points in $\mathbb{R}^4$, $\mathcal{R}$ consists of axis-parallel boxes with one of their vertices at the origin, and in which the size of the smallest ε-net is at least $C\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$. Here $C > 0$ is an absolute constant.*

Theorem 5. (P., Tardos) *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a (primal) range space $\Sigma = (X, \mathcal{R})$ of VC-dimension 2, where X is a set of n points in $\mathbb{R}^4$, $\mathcal{R}$ consists of half-spaces, and in which the size of the smallest ε-net is at least $C\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$. Here $C > 0$ is an absolute constant.*

Theorems 4 and 5 show that Theorems 2 and 1 cannot be generalized to 4-dimensional space. It also follows, by a standard duality argument, that there exist *dual* range spaces induced by half-spaces in $\mathbb{R}^4$, for which the size of the smallest ε-net is $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.

The next result shows that Theorem 2 is tight.

Theorem 6. (P., Tardos) *For any $\varepsilon > 0$ and for any sufficiently large integer $n > n_0(\varepsilon)$, there exists a (primal) range space $\Sigma = (X, \mathcal{R})$, where X is a set of n points in the plane, $\mathcal{R}$ consists of axis-parallel rectangles, and in which the size of the smallest ε-net is at least $C\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}$. Here $C > 0$ is an absolute constant.*

The proofs of Theorems 3 and 5 are based on two constructions from [PaT10] and [ChPS09], related to hypergraph coloring problems.

In [Ez10], Ezra proved that if X is any finite set of points in $\mathbb{R}^d$ and $\mathcal{R}$ consists of all axis-parallel boxes, then $(X, \mathcal{R})$ admits a weak ε-net of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$. This implies that Theorem 4 cannot be strengthened by requiring that the constructed range spaces do not admit *weak* ε-nets of size smaller than $\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$, provided that $\varepsilon > 0$ is sufficiently small.

It is easy to see that the analogue of Theorem 5 is also false for *weak* ε-nets instead of strong ones. Indeed, any finite system of half-spaces in $\mathbb{R}^d$ can be hit by $d+1$ points, so that in (primal or dual) half-space range spaces there always exist weak ε-nets of size $O(1)$.

Problem 1. (P., Tardos) *Does the analogue of Theorem 6 hold for weak ε-nets in place of strong ones?*

Given a point set X in the plane, the *Delaunay graph* with respect to axis-parallel rectangles is a graph defined on the vertex set X, whose two points $x, y \in X$ are connected by an edge if and only if there is a rectangle parallel to the coordinate axes that contains x and y, but no other elements of X.

Problem 2. (Chen, P., Szegedy, Tardos)) *Is it true that the Delaunay graph (with respect to axis-parallel rectangles) of any n-element point set in the plane has an independent set of size at least $n^{1-o(1)}$?*

The best known lower bound, due to Ajwani, Elbassioni, Govindarajan, and Ray [AjEG07] is roughly $\Omega(n^{0.617})$.

Problem 3. (Chen, P., Szegedy, Tardos)) *Give a nontrivial lower bound on the smallest possible size of the maximum independent set in the Delaunay graph of an n-element point set with respect to axis-parallel boxes in d-dimensional space, for a fixed d.*

A trivial bound that follows by repeated application of the Erdős-Szekeres lemma on monotone subsequences is $\Omega(n^{1/2^{d-1}})$.

Consider any family $\mathcal{R}$ of axis-parallel rectangles in the plane, and construct a graph $G = (V, E)$ on the vertex set $V = \mathcal{R}$ by connecting two rectangles if one contains at least one vertex of the other.

Problem 4. (P., Tardos) *Is it true that G is Δ-degenerate for an appropriate absolute constant Δ, that is, every subgraph of G_2 has a vertex of degree at most Δ?*

To prove this, it would be sufficient to show that $|E| \leq \Delta|V|/2$.

Problem 5. (P., Tardos) *Does there exist a constant C with the following property: For any set X of n points in the plane, one can find an $n/2$-element set Y such that every axis-parallel rectangle that is disjoint from S contains at most C elements of X?*

References

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Crossing Numbers

by Géza Tóth

In a *drawing* of a graph G vertices are represented by points and edges are represented by curves connecting the corresponding points. We assume that the edges do not pass through vertices, any two edges have finitely many common points and each of them is either a common endpoint, or a proper crossing. We also assume that no three edges cross at the same point. The *crossing number* $\text{CR}(G)$ is the minimum number of edge-crossings (i. e. crossing points) over all drawings of G.

The following general lower bound on crossing numbers was discovered by Ajtai–Chvátal–Newborn–Szemerédi [1] and, independently, by Leighton [5].

Theorem 1. For any graph G with n vertices and $e \geq 17n$ edges, we have

$$\text{CR}(G) \geq \frac{1}{31.1}\frac{e^3}{n^2}. \tag{1}$$

This estimate has many applications, and it is tight up to a constant factor. The best known constant, $1/31.1$, in (1) is due to Pach, Radoičić, Tardos, and Tóth [7] (see also [9]) who also showed that the result does not remain true if we replace $1/31.1 \approx 0.0321$ by roughly 0.09.

It was shown in [8] that there is a "best constant" in the following sense. Let $\kappa(n, e)$ denote the minimum crossing number of a graph G with n vertices and at least e edges. That is,

$$\kappa(n, e) = \min_{\substack{n(G)=n \\ e(G)\geq e}} \text{CR}(G).$$

Theorem 2. *If $n \ll e \ll n^2$, then*

$$\lim_{n\to\infty} \kappa(n, e)\frac{n^2}{e^3} = C > 0$$

exists.

We have $0.0321 \leq C \leq 0.09$.

Problem 1. *Improve the bounds on C.*

We define three variants of the notion of crossing number.

(1) The *rectilinear crossing number*, $\text{LIN-CR}(G)$, of a graph G is the minimum number of crossings in a drawing of G, in which every edge is represented by a straight-line segment.

(2) The *pairwise crossing number* of G, $\text{PAIR-CR}(G)$, is the minimum number of crossing pairs of edges over all drawings of G. (Here the edges can be represented by arbitrary continuous curves, so that two edges may cross more than once, but every pair of edges can contribute at most one to $\text{PAIR-CR}(G)$.)

(3) The *odd-crossing number* of G, $\text{ODD-CR}(G)$, is the minimum number of those pairs of edges which cross an odd number of times, over all drawings of G.

It readily follows from the definitions that

$$\text{LIN-CR}(G) \geq \text{CR}(G) \geq \text{PAIR-CR}(G) \geq \text{ODD-CR}(G).$$

The simplest proof of Theorem 1, with a weaker constant, easily generalizes to these other crossing numbers.

Theorem 3. For any graph G with n vertices and $e \geq 4.5n$ edges, we have

$$\text{LIN-CR}(G) \geq \text{CR}(G) \geq \text{PAIR-CR}(G) \geq \text{ODD-CR}(G) \geq \frac{1}{60.75}\frac{e^3}{n^2}. \tag{2}$$

Most likely the arguments in [7] and [9] can be (partly) extended to the pair-crossing number and the odd-crossing number, but we were unable to do so.

Problem 2. *Improve the constant* $\frac{1}{60.75}$ *in 2.*

By the Hanani-Tutte theorem [3], [6], if $\text{ODD-CR}(G) = 0$, then the graph is planar, so $\text{CR}(G) = \text{PAIR-CR}(G) = \text{ODD-CR}(G) = 0$. According to the Fáry-Wagner theorem [4], in this case $\text{LIN-CR}(G) = 0$ too. Bienstock and Dean [2] proved that if $\text{CR}(G) \leq 3$ then $\text{CR}(G) = \text{LIN-CR}(G)$. On the other hand, they constructed graphs with crossing number 4, whose rectilinear crossing numbers are arbitrarily large. What about the relationships between the values of the other three crossing numbers? It was shown in [10] that these are related to each other.

Theorem 4. For any graph G,

$$\text{CR}(G) \leq 2\text{ODD-CR}(G)^2.$$

This implies the following two weaker bounds.
(1) $\text{PAIR-CR}(G) \leq \text{CR}(G) \leq 2\text{PAIR-CR}(G)^2$, (2) $\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq 2\text{ODD-CR}(G)^2$.

It was shown by Pelsmajer, Schaefer, and Štefankovič [12] that if $\text{ODD-CR}(G) \leq 3$ then $\text{ODD-CR}(G) = \text{PAIR-CR}(G) = \text{CR}(G)$. (And by the above mentioned result of Bienstock and Dean, $\text{LIN-CR}(G)$ also has the same value.) The bound $\text{CR}(G) \leq 2\text{PAIR-CR}(G)^2$ has been improved several times. Let $k = \text{PAIR-CR}(G)$. It was shown by Valtr [19] that $\text{CR}(G) \leq 2k^2/\log k$, then in [16] that $\text{CR}(G) \leq 9k^2/\log^2 k$, and finally in [17] that $\text{CR}(G) \leq 2k^{7/4}/\log^{3/2} k$. On the other hand, we can not rule out the possibility, that $\text{CR}(G) = \text{PAIR-CR}(G)$ for every graph G! This is probably the most exciting problem in the area.

Problem 3. a. *Is there a constant c such that* $\text{CR}(G) \leq c\text{PAIR-CR}(G)$ *for every graph G?*
b.(*) *Is it true that* $\text{CR}(G) = \text{PAIR-CR}(G)$ *for every graph G?*

In the case of the other inequalities, between PAIR-CR and ODD-CR, the situation is quite different. The bound $\text{PAIR-CR}(G) \leq 2\text{ODD-CR}(G)^2$ has not been improved so far, although it does not look hard at all, and everybody belives that it is very far from the truth.

Problem 4. *Improve the inequality* $\text{PAIR-CR}(G) \leq 2\text{ODD-CR}(G)^2$.

Form the other direction, Pelsmajer, Schaefer, and Štefankovič [11] constructed a series of graphs with

$$\text{ODD-CR}(G) < \left(\frac{\sqrt{3}}{2} + o(1)\right) \cdot \text{PAIR-CR}(G).$$

It was improved in [16] to

$$\text{ODD-CR}(G) < \left(\frac{3\sqrt{5}}{2} - \frac{5}{2} + o(1)\right) \cdot \text{PAIR-CR}(G).$$

by a completely different construction. Note that $\frac{\sqrt{3}}{2} \approx 0.866$ and $\frac{3\sqrt{5}}{2} - \frac{5}{2} \approx 0.855$.

Problem 5. *Is there a constant c such that* $\text{PAIR-CR}(G) \leq c\text{ODD-CR}(G)$ *for every graph G?*

The second construction gives the "smallest" known example where $\textsc{pair-cr}(G)$ and $\textsc{odd-cr}(G)$ are different. More precisely, using the ideas of the second construction, one can construct a graph G with $\textsc{odd-cr}(G) = 9$ and $\textsc{pair-cr}(G) = \textsc{cr}(G) = 10$.

Problem 6. *What is the smallest number k with the property that there is a graph G with $\textsc{odd-cr}(G) = k$ and $\textsc{pair-cr}(G) > k$?*

By the results mentioned before, $4 \leq k \leq 9$.

We can further modify each of the above crossing numbers, by applying one of the following rules:

Rule + **:** Consider only those drawings where two edges with a common endpoint do not cross each other.

Rule 0 **:** Two edges with a common endpoint are allowed to cross and their crossing counts.

Rule $-$: Two edges with a common endpoint are allowed to cross, but their crossing does not count.

In the previous definitions we have always used Rule 0. If we apply Rule + (Rule $-$) in the definition of the crossing numbers, then we indicate it by using the corresponding subscript, as shown in the table below. This gives us an array of nine different crossing numbers. It is easy to see that in a drawing of a graph, which minimizes the number of crossing points, any two edges have at most one point in common. Therefore, $\textsc{cr}_+(G) = \textsc{cr}(G)$, which slightly simplifies the picture.

Rule +	$\textsc{odd-cr}_+(G)$	$\textsc{pair-cr}_+(G)$	$\textsc{cr}(G)$
Rule 0	$\textsc{odd-cr}(G)$	$\textsc{pair-cr}(G)$	
Rule $-$	$\textsc{odd-cr}_-(G)$	$\textsc{pair-cr}_-(G)$	$\textsc{cr}_-(G)$

Moving from left to right or from bottom to top in this array, the numbers do not decrease. It is not hard to generalize Theorem 3 to each of these crossing numbers.

Tutte [18] wrote that "We are taking the view that crossings of adjacent edges are trivial, and easily got rid of.". This is true for the standard crossing number, but not at all obvious for PAIR-CR and ODD-CR.

Pelsmajer, Schaefer, and Štefankovič [13] generalized Theorem 4 as follows.

Theorem 5. For any graph G,

$$\textsc{cr}(G) \leq 2\textsc{odd-cr}_-(G)^2.$$

This is the best known bound so far, moreover, this implies the best known bound between $\textsc{cr}_-(G)$ and $\textsc{cr}(G)$. It is hard to imagine that $\textsc{cr}_-(G)$ and $\textsc{cr}(G)$ can be diferent for any graph.

Problem 7. a. *Is there a constant c such that $\textsc{cr}(G) \leq c\textsc{cr}_-(G)$ for every graph G?*
b.(*) *Is it true that $\textsc{cr}(G) = \textsc{cr}_-(G)$ for every graph G?*

Return to the Hanani-Tutte theorem [3].

Theorem 6. (Weak Hanani-Tutte theorem) If G can be drawn in the plane such that any two edges cross an even number of times, then G is planar.

That is, if $\textsc{odd-cr}(G) = 0$ then $\textsc{cr}(G) = 0$. This result has many proofs, the simplest one is due to Pelsmajer, Schaefer, and Štefankovič [12] Analogous statement holds on any surface instead of the plane [14].

Theorem 7. (Strong Hanani-Tutte theorem) If G can be drawn in the plane such that any two independent edges cross an even number of times, then G is planar.

This statement follows from Kuratowski's theorem, and there is an elementary proof in [12]. In contrast to the weak version, the strong wersion is not known to hold on other surfaces, the only exception is the projective plane (Pelsmajer, Schaefer, Stasi, [15]).

Problem 8. *Prove the strong Hanani-Tutte theorem on the torus.*

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Ramsey-Type Problems for Geometric Graphs

by Gyula Károlyi

A *geometric graph* is a graph drawn in the plane so that every vertex corresponds to a point, and every edge is a closed straight-line segment connecting two vertices but not passing through a third. The $\binom{n}{2}$ segments determined by n points in the plane, no three of which are collinear, form a *complete* geometric graph with n vertices. A geometric graph is *convex* if its vertices correspond to those of a convex polygon. Further, we say that a subgraph of a geometric graph is *non-crossing*, if no two of its edges have an interior point in common.

For any finite graph G, either G or its complement $\overline{G}$ is connected. That is, either G or $\overline{G}$ contains a spanning tree. This observation extends to a geometric setting as follows, see [10].

Theorem A. *If the edges of a finite complete geometric graph are coloured by two colours, there exists a non-crossing spanning tree, all whose edges are of the same colour.*

For *convex* geometric graphs this follows by simple induction on the number of vertices. If all edges along the boundary of the convex hull are of the same colour, then there is a monochromatic non-crossing spanning path. Otherwise let ab and bc be two edges of the convex hull having different colour; omit the vertex b and apply the induction hypothesis. One cannot in general expect a monochromatic non-crossing spanning path (see below), but probably Theorem 1 can still be strengthened. A *caterpillar* is a tree obtained from a path and a set of isolated vertices, connecting each isolated vertex to the path with a new edge. It was suggested by Micha Perles, that the following may be true, at least for convex geometric graphs.

Problem 1. *The edges of a finite complete geometric graph are coloured by two colours. Is it always true that there exists a non-crossing monochromatic spanning caterpillar?*

For any finite sequence $G_1, G_2, \ldots, G_t$ of simple graphs, $R(G_1, G_2, \ldots, G_t)$ denotes the smallest integer r with the property that whenever the *edges* of a complete graph on at least r vertices are partitioned into t colour classes, there is an integer $1 \leq i \leq t$ such that the ith colour class contains a subgraph isomorphic to G_i. Such a subgraph will be referred to as a monochromatic subgraph in the ith colour. In the special case, when each $G_i = K_{k_i}$ is a complete graph on k_i vertices, we will simply write $R(k_1, k_2, \ldots, k_t)$ for $R(G_1, G_2, \ldots, G_t)$. In general, if G_i has k_i vertices, then the existence of $R(G_1, G_2, \ldots, G_t)$ follows directly from that of $R(k_1, k_2, \ldots, k_t)$.

For a sequence of graphs $G_1, G_2, \ldots, G_t$, the *geometric* Ramsey number $R_g(G_1, G_2, \ldots, G_t)$ is defined as the smallest integer r with the property that whenever the edges of a complete geometric graph on at least r vertices are partitioned into t colour classes, the ith colour class contains a non-crossing copy of G_i, for some $1 \leq i \leq t$. The number $R_c(G_1, G_2, \ldots, G_t)$ denotes the corresponding number if we restrict our attention to *convex* geometric graphs only. These numbers exist if and only if each graph G_i is *outerplanar*, that is, can be obtained as a subgraph of a triangulated cycle (convex n-gon triangulated by non-crossing diagonals). The necessity of the condition is obvious, whereas the 'if part' is implied by the following result of Gritzmann et al. [7].

Theorem B. *Let P be an arbitrary set of n points in the plane in general position. For any outerplanar graph H on n vertices, there is a straight-line embedding f of H into the plane such that the vertex set of $f(H)$ is P and no two edges of $f(H)$ cross each other.*

Corollary C. $R(G_1, \ldots, G_t) \leq R_c(G_1, \ldots, G_t) \leq R_g(G_1, \ldots, G_t) \leq R(k_1, \ldots, k_t)$ *holds for arbitrary outerplanar graphs $G_1, \ldots, G_t$ with $k_1, \ldots, k_t$ vertices, respectively.*

Most known results concern the diagonal bi-coloured case, that is, when $t = 2$ and $G_1 = G_2$. For simplicity, write $R(G)$, $R_g(G)$ and $R_c(G)$ for $R(G, G)$, $R_g(G, G)$ and $R_c(G, G)$, respectively. Due to the inequality $R_c(G) \leq R_g(G)$ and the cyclically ordered structure of convex complete geometric graphs, it is generally easier to obtain/prove upper bounds for R_c than to R_g. On the other hand, the largest number of crossing edges in a complete geometric graph occurs when the vertices are in convex position, suggesting that R_g should not be much larger than R_c. I recall once proving $R_g(G) \neq R_c(G)$ (probably) for long enough paths, but I forgot the details.

Problem 2. *Is there an infinite sequence (G_n) of outerplanar graphs G_n on n vertices such that*

(a) $\limsup_{n\to\infty} \{R_g(G_n) - R_c(G_n)\} > 0$;

(b) $\limsup_{n\to\infty} \dfrac{R_g(G_n)}{R_c(G_n)} > 1$;

(x) $\limsup_{n\to\infty} \dfrac{R_g(G_n)}{R_c(G_n)} = \infty$?

Denote by C_k a cycle of k vertices, D_k a cycle of k vertices triangulated from a vertex, P_k a path of k vertices (that is, of length $k - 1$), and S_k a star of k vertices. In addition, $M_{2k} = kP_2$ will stand for any perfect matching on $2k$ vertices. Regarding paths, the following results are known [11].

Theorem D. *If $k \geq 3$, then $2k - 3 = R_c(P_k) \leq R_g(P_k) = O(k^{3/2})$.*

The lower bound is implied by a simple construction. The upper bound concerning the convex case is a consequence of the following result due to Perles (see [11] or [3] for a proof).

Theorem E. *If a convex geometric graph of $n \geq k + 1$ vertices has more then $\lfloor (k-1)n/2 \rfloor$ edges, then it contains a non-crossing path of length k.*

Most likely the general upper bound is very far from the truth, but it may be very difficult to find the right order of magnitude.

Problem 3. *Improve upon the upper bound $R_g(P_k) = O(k^{3/2})$.*

The weaker, but somewhat more general bound $R_g(P_{k+1}, P_{l+1}) \leq kl + 1$ can be proved by the following argument [10]. Let p_i $(0 \leq i \leq kl)$ denote the vertices of a complete geometric graph. Suppose that they are listed in increasing order of their x-coordinates, which are all distinct. Define a partial ordering of the vertices, as follows. Let $p_i < p_j$ if $i < j$ and there is an x-monotone red path connecting p_i to p_j. By Dilworth's theorem [6], one can find either $k + 1$ elements that form a totally ordered subset $Q \subset P$, or $l + 1$ elements that are pairwise incomparable. In the first case, there is an x-monotone red path visiting every vertex of Q. In the second case, there is an x-monotone blue path of length l, because any two incomparable elements are connected by a blue edge. The bound follows noting that an x-monotone path cannot intersect itself. The same idea leads to the following result [11], [13].

Theorem F. $R_g(D_k, D_\ell) \leq (k-2)(\ell-1) + (k-1)(\ell-2) + 2$ *holds for arbitrary integers $k, \ell \geq 3$.*

Since $R_c(C_4) = 14$ [2], this bound is tight for $k = \ell = 4$. Given that $R_c(C_3, C_\ell) = 3\ell - 3$ holds for every $\ell \geq 3$ [13], it is also tight for the case $k = 3$. Besides these cases, nothing better than the general estimate $R_c(C_k, P_\ell) \geq (k-1)(\ell-1) + 1$ [12] is known. Let $X, Y \in \{C, D, P\}$ (except $X = Y = P$).

Problem 4. *Find the exact values of any of the functions $R_c(X_k, Y_\ell)$, $R_g(X_k, Y_\ell)$.*

The estimate $R_c(C_k, G_\ell) \geq (k-1)(\ell-1)+1$ is known for any connected outerplanar graph G_ℓ on ℓ vertices [12]. The following old result of Chvatal [4] can be used to obtain a matching upper bound in certain cases.

Theorem G. $R(K_k, T_\ell) = (k-1)(\ell-1)+1$ *holds for any tree T_ℓ on ℓ vertices.*

This, together with Theorem B implies $R_g(H_k, S_\ell) \leq (k-1)(\ell-1)+1$ for every outerplanar graph H_k on k vertices. Since $R_g(H_k, P_\ell) \leq (k-1)(\ell-1)+1$ can be also proved using the above explained argument, it is quite plausible that similar estimates hold for any tree T_ℓ on ℓ vertices.

Problem 5. *Is it true, that $R_c(H_k, T_\ell) = R_g(H_k, T_\ell) \leq (k-1)(\ell-1)+1$ holds for any tree T_ℓ on ℓ vertices and any outerplanar graph H_k on k vertices, which contains a Hamiltonian cycle?*

Even though the Ramsey function $R(n)$ is exponentially large, it may well be that all geometric Ramsey numbers are relatively small.

Problem 6. *Is there a universal constant c such that $R_g(G_n) < cn^2$ holds for every outerplanar graph G_n with n vertices?*

If not, the sequence L_{2n} of 'ladder' graphs (vertex disjoint paths $p_1 \ldots p_n$ and $q_1 \ldots q_n$ together with the edges $p_i q_i$) seems to be a likely candidate for a counterexample.

The multicolour Ramsey number of matchings was determined by Cockayne and Lorimer [5] as

$$R(M_{2k_1}, M_{2k_2}, \ldots, M_{2k_t}) = \sum_{i=1}^{t} k_i + \max_{1\leq i\leq t} k_i - t + 1.$$

The same result is true also in the geometric setting if $t = 2$; $R_g(M_{2k}, M_{2\ell}) = k + 2\ell - 1$ holds for $k \leq \ell$ [9, 10]. For the diagonal Ramsey number $R_g^{(t)}(M_{2k}) = R_g(\underbrace{M_{2k}, \ldots, M_{2k}}_{t \text{ times}})$ it implies the general upper bound [13]

$$R_c^{(t)}(M_{2k}) \leq R_g^{(t)}(M_{2k}) \leq \begin{cases} \frac{3t}{2}k - \frac{3t}{2} + 2 & \text{for } t \text{ even,} \\ \frac{3t+1}{2}k - \frac{3t+1}{2} + 2 & \text{for } t \text{ odd.} \end{cases}$$

This upper bound is sharp also for $t = 4$ [13], but most likely it is not for larger values of t. If $t \geq 2$ and $k \geq 6t - 10$, then $R_c^{(t)}(M_{2k}) \geq (6/5)tk$.

Problem 7. *Is there a constant $c' < 3/2$ such that $R_c^{(t)}(M_{2k}) < c'tk$ holds for $t \geq t_0$, $k \geq k_0$?*

The existence of such a constant would yield to an improved lower bound on the chromatic number of certain geometric Kneser graphs [1]. The only result pointing in this direction is a minute improvement for the case $k = 2$; for $t \geq t_0$ the additive constant 2 in the above general upper estimate for $R_c^{(t)}(M_4)$ can be replaced by 1 [8].

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Coloring Vertices and Edges of a Graph by Nonempty Subsets of a Set

by Ervin Győri

All the results are joint work with P.N. Balister and R.H. Schelp.

1 Introduction

In a recent article by Hedge [1], he considered coloring certain graphs G, where $|V(G)|+|E(G)| = 2^n - 1$ for some integer n, by nonempty subsets of an n-element set. One primary assignment considered was to assign distinct subsets to the vertices and edges such that each edge is assigned the symmetric difference of its end vertices. Since $|V(G)|+|E(G)| = 2^n-1$ this means all nonempty subsets are used in the assignment. When such an assignment exists it is called a *strong set coloring* of the graph. This assignment is similar to ones frequently studied in coding theory.

One interesting conjecture made in this article was that paths of order 2^{n-1} where $n > 2$ are not strongly set colorable. We proved the following theorem, but we learned after submission of this paper, that A.R. Mehta and G.R. Vijaykumar proved it earlier.

Theorem 1 (Mehta, Vijaykumar [2]). *The paths P_4 and P_8 are not strongly set colorable while all other paths of the form $P_{2^{n-1}}$ are strongly set colorable.*

An equivalent formulation for a graph to be strongly set colorable and one used throughout this note is the following one. Let G be a connected graph such that $|V(G)| + |E(G)| = 2^n - 1$ for some n. We say that G is *strongly set colorable* if there is a bijection f from $V(G) \cup E(G)$ to the set of nonzero vectors in $\mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field with two elements, such that for every edge $xy \in E(G)$, $f(xy)+f(x)+f(y) = 0$. We regard vectors in $\mathbb{F}_2^n$ as 0-1 sequences of length n, addition in $\mathbb{F}_2^n$ corresponding to componentwise addition modulo 2. The vector v then corresponds to the subset $S_v \subseteq \{1, 2, \ldots, n\}$, where S_v consists of all i such that the ith coordinate of v is 1. In this language the Hegde conjecture says there is no permutation $v_1, v_2, v_3, \ldots, v_{2^n-1}$ of the n-dimensional nonzero vectors of $\mathbb{F}_2^n$ such that $v_i + v_{i+1} + v_{i+2} = 0$ for $i = 1, 3, 5, \ldots, 2^n - 3$. Here v_i represents the set coloring of a vertex when i is odd and of an edge when i is even. These conditions will be referred to as *sum conditions*. When a permutation satisfying the sum conditions exists we will refer to it as *representing vectors in a good permutation* or more briefly as a *good permutation.*

2 Results on strong set colorings

We do not show Theorem 1 but we present the short proof of the first half of it.

Theorem 2. *The paths P_4 and P_8 of orders 4 and 8 are not strongly set colorable.*

Proof: First consider the path P_4 and suppose there is a good permutation of its representing vectors $v_1, \ldots, v_7$. Since for all $n > 1$ the sum of all the non-zero vectors in $\mathbb{F}_2^n$ is zero, $v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 = 0$. By the sum condition both $v_1 + v_2 + v_3 = 0$ and $v_5 + v_6 + v_7 = 0$, so we have $v_4 = 0$, a contradiction. Therefore P_4 is not strongly set colorable.

Next consider the path P_8. Suppose that there is a good permutation $v_1, \ldots, v_{15}$ of the representing vectors satisfying the sum conditions. First notice that

$$\begin{aligned} v_4 + v_8 + v_{12} &= \sum_{i=1}^{15} v_i - (v_1+v_2+v_3) - (v_5+v_6+v_7) - (v_9+v_{10}+v_{11}) - (v_{13}+v_{14}+v_{15}) \\ &= 0 - 0 - 0 - 0 = 0. \end{aligned}$$

Claim: The vectors $v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{13}, v_{14}, v_{15}$ can be written in the form $v_i + v_{4j}$ where $i, j = 1, 2, 3$. We call the forms of these 9 vectors canonical.

Proof: [Proof of Claim.] First we show that these 9 vectors are different from $v_1, v_2, v_3, v_4, v_8, v_{12}$. Suppose that $v_{i_1} = v_{i_2} + v_{4j_2}$. Then adding v_{i_2} to this vector, we obtain that either $v_{4j_2} = 0$ if $i_1 = i_2$ or v_{4j_2} is the third vector out of v_1, v_2, v_3 if $i_1 \neq i_2$, a contradiction in both cases. Likewise $v_{4i_1} \neq v_{i_2} + v_{4j_2}$, otherwise $v_{i_2} = 0$ or the third vector of v_4, v_8, v_{12}. Then, by pigeonhole principle, it is sufficient to prove that these 9 vectors are pairwise distinct. Suppose that $v_{i_1} + v_{4j_1} = v_{i_2} + v_{4j_2}$. If $i_1 = i_2$, $j_1 \neq j_2$, then it follows that $v_{4j_1} = v_{4j_2}$, a contradiction. If $i_1 \neq i_2$ then let i_3 be the third index out of $1, 2, 3$. Then $v_{i_3} = v_{i_1} + v_{i_2} = v_{4j_1} + v_{4j_2}$, which is either 0 if $j_1 = j_2$ or the third vector out of v_4, v_8, v_{12} if $j_1 \neq j_2$, a contradiction, completing the proof of this claim.

Define the sets $B_1 = \{v_5, v_6, v_7\}, B_2 = \{v_9, v_{10}, v_{11}\}, B_3 = \{v_{13}, v_{14}, v_{15}\}$ as *blocks*.

Claim: For each block, the vectors $v_1, v_2, v_3, v_4, v_8, v_{12}$ appear exactly once in canonical form of the vectors in the block.

Proof: [Proof of Claim.] Let w_1, w_2, w_3 be the vectors in the block, and suppose that, say, v_{i_1} appears in the canonical form of w_1 and w_2, i.e., $w_1 = v_{i_1} + v_{4j_1}$ and $w_2 = v_{i_1} + v_{4j_2}$ where $j_1 \neq j_2$ since $w_1 \neq w_2$. Then $w_3 = w_1 + w_2 = v_{4j_1} + v_{4j_2} = v_{4j_3}$ where v_{4j_3} is the third vector in $\{v_4, v_8, v_{12}\}$, a contradiction. The case when v_{4j_1} appears twice can be settled similarly.

Having established the above claims we complete the proof of the theorem.

Observe that $v_5 = v_3 + v_4$ by the sum condition. If the canonical form of v_7 contains v_8 then the sum condition implies that v_9 is an element of v_1, v_2, v_3, a contradiction. So v_7 has v_{12} in its canonical form, and since we have symmetry in v_1 and v_2, we may assume that $v_7 = v_1 + v_{12}$ (and so $v_6 = v_2 + v_8$). Then $v_9 = v_1 + v_4$ by the sum condition. As above, v_{11} cannot have v_{12} in its canonical form, and of course, cannot be $v_6 = v_2 + v_8$. Thus $v_{11} = v_3 + v_8$, since if $v_{11} = v_1 + v_8$ then $v_{12} + v_{11} = v_1 + v_4 = v_9 = v_{13}$, a contradiction. Then by the sum condition, $v_{13} = v_3 + v_4 = v_5$, a final contradiction, completing the proof of the theorem.

We noticed that the strategy used in the proof of Theorem [2] can be applied beginning with a strongly set colorable bipartite graph in place of a path.

Theorem 3. *Let G be a strongly set colorable bipartite graph with color classes X, Y and edge set E. Let G_1, G_2, G_3, G_4 be four disjoint copies of G with color classes X_1, Y_1, X_2, Y_2, X_3, Y_3, X_4, Y_4 and edge sets E_1, E_2, E_3, E_3, respectively. Let G_0 denote the graph obtained from the disjoint union of the graphs G_1, G_2, G_3, G_4 by adding edges e_1, e_2, e_2 with the following properties:*

1. *each e_i joins two copies of the same vertex;*
2. *one of the following three possibilities occurs:*
   - *(a) the edges join X_1 and X_2, X_1 and X_3, X_1 and X_4, respectively; or*
   - *(b) the edges join X_1 and X_2, X_2 and X_3, Y_3 and Y_4, respectively; or*
   - *(c) the edges join X_1 and X_2, Y_2 and Y_4, Y_1 and Y_3, respectively.*

Then G_0 is strongly set colorable.

Proof: Consider the same vector coloring of G_1, G_2, G_3 and G_4. Then extend the vectors in X_1, Y_1, and E_1 with 00, the vectors in X_3, Y_4, and E_2 with 01, the vectors in X_2, Y_3, and E_4 with 10, the vectors in X_4, Y_2, and E_3 with 11. If we color the edges e_1, e_2, e_3 according to the sum condition, then it is easy to verify that we obtain a desired vector coloring of G_0, completing the proof.

Symmetric possibilities of those listed in the last theorem are clearly strongly set colorable as well. It is noted in [1] that graphs which are strongly set colorable cannot have it even degree vertices covered by two edges. However this condition is not sufficient for graphs to be strongly set colorable. For example, P_8 is not strongly set colorable but does not have its six even degree vertices covered by two edges. It seems to be difficult to even characterize which trees are strongly set colorable.

3 Strong set colorings of other trees

The binary tree has a particularly nice set coloring. Another interesting possible construction is to take a strong set coloring of G and extend it to a new graph G' obtained from G by adding 2^n pending edges. The color of each of the original edges and vertices of G is obtained by appending 0 to the original color. New edges e_i get color $1p_i$ and new vertices u_i get color $1q_i$ where the pendent edge $e_i = u_iv_i$ is joined to a vertex v_i of color $0v_i$. We need $v_i = p_i - q_i$ for this to give a strong coloring of G', and the set of all p_i and q_i needs to exhaust $\mathbb{F}_2^n$. Note that some of the v_i's may be the same since we can attach several pendent edges to the same vertex. We need $\sum v_i = 0$ for such p_i and q_i to exist (since $0 = \sum p_i + \sum q_i = \sum v_i$). It appears that this is sufficient, so we make the following conjecture.

Conjecture 1. *Given 2^{n-1} non-zero (not necessarily distinct) vectors $v_1, \ldots, v_{2^{n-1}} \in \mathbb{F}_2^n$, $n \geq 2$, with $\sum_{i=1}^{2^{n-1}} v_i = 0$, there exists a partition of $\mathbb{F}_2^n$ into pairs of vectors $\{p_i, q_i\}$, $i = 1, \ldots, 2^{n-1}$ such that for all i, $v_i = p_i - q_i$.*

This conjecture, which appears to be of interest in its own right, is true for $n \leq 5$. It is also true if at least half of all the vectors v_i are the same and each vector occurs an even number of times.

Theorem 4. *Given 2^{n-1} non-zero vectors $v_1, \ldots, v_{2^{n-1}} \in \mathbb{F}_2^n$, $n \geq 2$, with $v_1 = v_2 = \cdots = v_{2^{n-2}}$ and $v_{2i+1} = v_{2i+2}$ for all $i = 0, \ldots, 2^{n-2} - 1$, there exists a partition of $\mathbb{F}_2^n$ into pairs of vectors $\{p_i, q_i\}$, $i = 1, \ldots, 2^{n-1}$ such that for all i, $v_i = p_i - q_i$.*

Proof: Without loss of generality assume $v_1 = 00\ldots01 \in \mathbb{F}_2^n$. Suppose we have chosen p_j and q_j for some pairs $j = 2i+1, 2i+2$ where $v_j \neq v_1$ so that the vectors that have been used form a union of pairs $\{u0, u1\}$, $u \in S \subseteq \mathbb{F}_2^{n-1}$. Suppose $v_{2i+1} = v_{2i+2} \neq v_1$ is another pair of vectors for which p_j and q_j have not yet been assigned. Write $v_{2i+1} = v_{2i+2} = va$ where $v \in \mathbb{F}_2^{n-1}$ and $a \in \{0, 1\}$. Since we have chosen fewer than half of all the p_j and q_j, $|S| < 2^{n-2}$. Thus by the pigeonhole principle, there exists a pair $p, q \notin S$ with $p - q = v$. Now choose $p_{2i+1} = pa$, $q_{2i+1} = q0$, $p_{2i+2} = p\bar{a}$, and $q_{2i+2} = q1$, where $\bar{a} = 1 - a$. Then $v_{2i+1} = p_{2i+1} - q_{2i+1}$ and $v_{2i+2} = p_{2i+2} - q_{2i+2}$, and the chosen vectors form a union of pairs $\{u0, u1\}$ with $u \in S' = S \cup \{p, q\}$. Repeat this process until we have assigned vectors p_j and q_j for all $v_j \neq v_1$. Finally assign for each $v_j = v_1$ one of the remaining $p \notin S$ and set $p_j = p1$ and $q_j = p0$ so that $v_j = p_j - q_j$ for these values of j as well. This gives the required partition of $\mathbb{F}_2^n$ into pairs.

4 Proper set colorings of complete bipartite graphs

In [1], a *proper set coloring* of a graph G is defined as an assignment of subsets of an n-element set to the vertices and edges of G so that

1. each edge receives the symmetric difference of the sets assigned to its end vertices;
2. distinct vertices receive distinct sets;
3. distinct edges receive distinct non-empty sets; and

4. every non-empty set appears on some edge.

Note that the same set can occur on both an edge and a vertex. Clearly, if a proper set coloring of G is to exists we need $|E(G)| = 2^n - 1$.

The following theorem appears as a conjecture in [1].

Theorem 5. *If the complete bipartite graph $K_{s,t}$ has a proper set coloring then either $s = 1$ or $t = 1$.*

Writing $S_1, \ldots, S_s$ and $T_1, \ldots, T_t$ as the sets assigned to the vertices of $K_{s,t}$, this theorem is an immediate consequence of the following.

Theorem 6. *Assume $S_1, \ldots, S_s$ and $T_1, \ldots, T_t$ are subsets of $\{1, \ldots, n\}$ and the symmetric differences $S_i \oplus T_j$, $i = 1, \ldots, s$, $j = 1, \ldots, t$, are non-empty and represent every non-empty subset of $\{1, \ldots, n\}$ exactly once (so in particular $st = 2^n - 1$). Then either $s = 1$ or $t = 1$.*

Some of a lot of open problems:

1. Do we have a strong set coloring of trees if the number of vertices is a power of 2?

2. Find necessary (and sufficient???) conditions for the existence of strong set coloring of graphs. (With nice or less nice edge and vertex numbers.)

3. Similar questions about proper set colorings.

References

[1] S.M. Hegde, Set Colorings of Graphs, *preprint*.

[2] Mehta, Vijaykumar

Contributed Problems

Edge-connectivity and claw-decomposition of graphs

by János Barát

Problem Let G be a bipartite graph of edge-connectivity 1000, and $|E(G)|$ divisible by 3. Is it possible to remove some $K_{1,3}$'s such that the vertex degrees of the remaining graph are all divisible by 3?

A positive answer would imply the following

Conjecture [1] There exists a smallest natural number k_c such that every simple k_c-edge-connected graph G, whose size is divisible by 3, has a $K_{1,3}$-decomposition.

If we replace $K_{1,3}$ by P_4, that is the path with three edges, then Thomassen [3] proved the existence of such a constant:

Theorem Every 171-edge-connected graph admits a decomposition into paths with three edges.

It is a long-standing problem, whether 2-edge-connectedness is sufficient for planar triangle-free graphs, and 3-edge-connectedness for graphs in general [2].

Some claw-decomposition results of [1] can be generalized to $K_{1,2k+1}$-decompositions. It seems to be untested whether even claws behave differently:

Problem Let t be a large enough positive integer and $k \geq 2$. Does every t-edge-connected graph admit a $K_{1,2k}$-decomposition?

References

[1] J. Barát, C. Thomassen, Claw-decompositions and Tutte-orientations. *J. Graph Theory* **52** (2006), 135–146.

[2] M. Jünger, G. Reinelt and W. R. Pulleyblank, On partitioning the edges of graphs into connected subgraphs. *J. Graph Theory* **9** (1985) 539–549.

[3] C. Thomassen, Decompositions of highly connected graphs into paths of length 3. *J. Graph Theory* **58** (2008), 286–292.

Edge Cover Scheduling

by Dave

For this problem, the input is a graph $G = (V, E)$ and a duration $d_e \in \mathbf{Z}_+$ for each edge e. A *schedule* of the edges is a vector $\{x_e\}_{e \in E}$, and in a schedule each edge e is *active* for all times in the interval $[x_e, x_e + d_e]$. The *coverage* of a schedule is the largest T such that, for all $0 \leq t \leq T$ and all $v \in V$, at least one edge incident to v is active at time t. We want to find a schedule with large coverage.

Let δ be the minimum degree weighted according to duration, i.e. $\delta = \min_{v \in V} \sum_{u:uv \in E} d_{uv}$. Clearly no schedule has coverage greater than δ. Is there always a schedule of coverage $\Omega(\delta)$? If not, is it still possible to approximate the maximum coverage within $O(1)$ by some polytime algorithm?

If all durations are unit, a schedule of coverage $\lfloor (3\delta + 1)/4 \rfloor$ is always possible — this is an old result of Gupta, but there is a nice new proof by Alon et al. In the general case, the LLL can show that a schedule of coverage $\Omega(\delta / \log \delta)$ is always possible.

References

[1] N. Alon et al. Polychromatic colorings of plane graphs, Discrete & Computational Geometry 42(3):421-422, 2009. http://dx.doi.org/10.1007/s00454-009-9171-5

[2] B. Bollobás, D. Pritchard, T. Rothvoß, A. Scott. Cover-decomposition and polychromatic numbers in combinatorial settings. http://sma.epfl.ch/ pritchar/math/2010/hypergraph-colouring.pdf

[3] R. P. Gupta. On the chromatic index and the cover index of a multigraph. Proc. Theory and Applications of Graphs, Kalamazoo, 1976. http://dx.doi.org/10.1007/BFb0070378

Cover Decomposition into Multiple Coverings

by Dömötör

Suppose we have a set system $\mathcal{F}$ over some base set S. (E.g. the unit discs in the plane.) Denote by m_k the smallest constant (if exists) such that any finite multiset $X \subset S$ can be colored with k colors such that any $F \in \mathcal{F}$ that contains at least m_k elements of X contains all k colors. (E.g. any finite set of points can be colored such that any unit disc contains all colors.)

Is it true that if m_2 exists, then m_k also exists?

(E.g. m_2 is known to exist for unit discs but m_3 is not.)

Is it true that $m_k = O(m_2)$?

(This is the case for all geometric shapes that were examined so far.)

Another problems related to cover-decomposition is to extend established results about open sets to closed sets. E.g. it is known that any 12-fold covering of the plane with the translates of an open triangle is decomposable into two coverings but the same is not known about closed triangles, not even with a weaker constant.

A few related papers:

References

[1] M. Gibson and K. Varadarajan: Decomposing Coverings and the Planar Sensor Cover Problem, arXiv:0905.1093v1.

[2] P. Mani-Levitska, J. Pach: Decomposition problems for multiple coverings with unit balls, manuscript, 1986.

[3] D. Pálvölgyi, Decomposition of Geometric Set Systems and Graphs, PhD thesis, http://arxiv.org/abs/1009.4641

Coloring points with respect to discs with three colors

by Keszegh

Is there a k for which every finite set of points P can be colored by 3 colors such that if a disc contains at least k points of P, then not all of them have the same color? If yes, find the smallest such k.
With 4 colors, already for $k = 2$ we can give such a coloring. Indeed, a proper coloring of the Delaunay-triangulation of a point set P gives such a coloring.

Collinear Clique Graphs

by Péter Maga

Characterize the graphs whose vertices can be mapped into different points of the plane in such a way that any $k > 2$ points are collinear if and only if the corresponding vertices form a clique.

Universal set of lines for trees

by Rado

Does there exist for any $n > 0$ an arrangement of n lines $\mathcal{L}$ in the plane such that for any tree $T = (V, E)$ on n vertices and any bijection $b : V \to \mathcal{L}$ there exists a non-crossing, straight-line embedding of T in which each $v \in V$ is represented by a point on the line $b(v)$?
The expected answer to this problem is, of course, negative. For general planar graphs it was shown to be false, by Dujmovic and Langerman (http://arxiv.org/abs/1012.0548), whose proof is a bit overkill I think.

Multicolour Ramsey

by Zoli Nagy

For a given p, we want to color the edges of K_n with two colors such that every edge has at least one color and at most a p fraction of the edges can have both colors. What is the biggest monochromatic subgraph, $f_p(n)$, that is guaranteed to exist? (So for $p = 0$ we get back the original Ramsey problem.)
The followings are known:
* If $p < 1$, then $f_p(n) < \frac{2}{\log_2 \frac{2}{1+p}} \log_2 n + 1$ holds.
* Let t be a positive integer, and $p := 1 - \frac{1}{t}$. Then $f_p(n) \leq t f_0(\lceil \frac{n}{t}) \rceil) \leq 2t \log_2(\frac{n}{t})$.
* $f_{\frac{1+p}{2}}(2n) \leq 2 f_p(n)$.

Results

First place: Rado - Nagy Zoli
Second place: Cory - Dani
Third place: Gergő - Viktor
Fourth place: Viola - Dömötör

To Collinear Clique Graphs

by János Barát, Filip Morić, Zoltán Nagy, David Pritchard

We made the following conjecture.

Conjecture 1. *A graph G has an embedding of the desired form if and only if it has no induced diamond.*

Notice that diamond-free graphs have a nice equivalent description in terms of their maximal cliques:

Lemma 2. *Let V be a set of vertices and S a collection of subsets of V, each of size at least 3. Then S is the family of maximal cliques for a diamond-free graph if and only if*

(1) there is no pair of sets from S intersecting in more than one vertex, and

(2) there is no triple of sets such that each two contain a vertex not contained by the third. (I.e., iff there is no hypercycle of length ≤ 3.)

Just to stay on the safe side, we also conjecture that if GQ is a self-dual generalized quadrangle on more than 15 points, then the graph obtained by replacing each set in the GQ by a clique is a counterexample to the original Conjecture above.

You can read more at `http://daveagp.wordpress.com/2011/01/31/721/`

To Cover Decomposition of Closed Sets

by Dave, Dömötör, Máté, Miloš, Viktor, Viola

We tried to extend known results about cover-decomposition of a finite collection of sets to infinite collections. In this case (unlike in the finite case) it matters whether the underlying set is open or closed. We proved that the set of intervals of a line is cover-decomposable, i.e. if we have a 100-fold covering of the line with any intervals, then it decomposes into two coverings. (In fact this was already known with a better constant, it is not hard to prove.) We also proved using a standard compactness argument that if we cover any closed subset of the plane sufficiently many-fold with the open copies of a finite-cover-decomposable, bounded set, then we can decompose this covering.

To Geometric Ramsey-type problems

by Balázs Keszegh, Rado, Filip, Máté

Problem 3. *The edges of a finite complete geometric graph are coloured by two colours. Is it always true that there exists a non-crossing monochromatic spanning caterpillar?*

One could relax this problem:

Problem 4. *The edges of a finite complete geometric graph are coloured by two colours. Is it always true that there exists a red non-crossing spanning tree or a blue non-crossing spanning caterpillar?*

We proved an even more relaxed version:

Theorem 5. *The edges of a finite complete geometric graph are coloured by two colours. There exists a red (maybe crossing) spanning tree or a blue non-crossing spanning caterpillar.*

In other words if the red edges form a non-connected graph then there is a blue non-crossing spanning caterpillar. This holds for arbitrary (not necessarily convex) geometric graphs.

Proof: Suppose that the red edges form a non-connected graph, then there is a complete bipartite subgraph with blue edges. We prove that in this graph there is a spanning caterpillar. Denote by A and B the two classes of this bipartite graph, wlog. suppose A is not bigger then B. If $conv(A)$ and $conv(B)$ are disjoint then there is a separating line and then using a lemma of ? we can see that there is an alternating (between the parts) path covering A. The rest of the vertices of B can be easily joined to the path forming a spanning caterpillar.

If $conv(A)$ and $conv(B)$ have a common point then fix one such point O which is not among the vertices of G. Now we order the vertices of G according to the slopes of the lines between O and the vertices (in clockwise order). This defines a cyclic order of the vertices around O. Vertices from A and B form intervals of type A and B. In each interval take the last vertex and connect it to every vertex of the next interval. It is easy to see that this way we only used edges of the complete bipartite graph and we defined a non-crossing self-closed caterpillar (=cycle+hanging leaves).

It is well known (due to Gyárfás?) and easy to see that an abstract complete graph with edges two-colored has a hamiltonian cycle which is a union of one red path and one blue path (or it is monochromatic). We pose the following geometric generalization as an open problem.

Problem 6. *The edges of a finite complete (convex) geometric graph are coloured by two colours. Is it always true that its vertices can be covered by the union of a non-crossing red path and a non-crossing blue path? Perhaps they can be pairwise crossing-free as well? Perhaps they can have one(two) common endpoint(s) (like in the abstract case)?*

In the construction showing that in a convex geometric graph the maximal monochromatic path cannot be always bigger then $n/2$, it is trivial to find such pair of covering paths. Also, in this construction there is exactly one covering non-crossing tree, which is a caterpillar.

To Edge Cover Scheduling

by Dave

Somewhat after the workshop I was able to solve this problem affirmatively: there is always a schedule of coverage at least $\delta/8$. The proof is given in the appendix of the paper at `http://arxiv.org/abs/1009.6144` and relies on some ad-hoc properties of graphs. We would now like to generalize it to hypergraphs (say of maximum edge size 3), but there are counterexamples showing the current proof does not extend.

To Ramsey-type for geometric graphs

by Marek Krcal, Tomas Valla, Josef Cibulka and Jane Gao

(a) Let M_{2k} be a matching of size k and P_ℓ a path on ℓ vertices. Then $R_c(M_{2k}, P_\ell) \leq \ell + 2(k-1)$.

(b) Let T_ℓ be a tree with ℓ vertices and let $\Delta(T_\ell)$ denote its maximum degree. Then $R_c(M_{2k}, T_\ell) \geq \max\{\ell, 2k + 2\lceil \Delta(T_\ell)/2 \rceil - 3\}$.

(c) Let H_ℓ be any outerplanar graph on ℓ vertices. $R_c(M_{2k}, H_\ell) \leq R_g(M_{2k}, H_\ell) \leq \ell + (k-1)(\ell+1)$.

(d) Let T_n be any caterpillar on n vertices and H_m any outerplanar graph on m vertices. Then $R_c(H_m, T_n) \leq (m-1)(2n-1)+1$.

(e) Let L_{2n} be a ladder graph on $2n$ vertices, then $R_c(L_{2n}) \leq 32n^3$, and $R_g(L_{2n}) = O(n^{10})$.

To Strong Set Coloring

by Patkós Balázs, Ida, Terpai, Younjin, Anita

Definition 7. *A strong set coloring of a graph G is an injective mapping $c : E(G) \cup V(G) \to 2^{[m]} \setminus \{\emptyset\}$ such that for any edge $e = (u, v) \in E(G)$ the equality $c(e) = c(u) \triangle c(v)$ holds, where $A \triangle B$ denotes the symmetric difference $(A \setminus B) \cup (B \setminus A)$. The strong set coloring number $ssc(G)$ of a graph G is the minimum integer m such that a strong set coloring $c : E(G) \cup V(G) \to 2^{[m]}$ exists.*

As all colors must be different, writing $|G| = |E(G)| + |V(G)|$, we obtain the trivial lower bound $\lceil \log(|G| + 1) \rceil \leq ssc(G)$.

Conjecture 8. *There exists an absolute constant C such that the inequality $ssc(G) \leq \lceil \log(|G| + 1) \rceil + C$ holds for all finite graph G.*

Theorem 9. *For the complete graph the following holds:*

$$ssc(K_n) \leq 1 + 2\lceil \log(n+1) \rceil.$$

Theorem 10. *Let G be a graph with degeneracy number d. The following inequality holds:*

$$ssc(G) \leq \lceil \log(|G|+1) \rceil + \lceil \log(d+1) \rceil.$$

Corollary 11. *For any tree T on n vertices the following holds:*

$$\lceil \log(2n) \rceil \leq ssc(T) \leq \lceil \log(2n) \rceil + 1.$$

Corollary 12. *With high probability the following holds:*

$$ssc(G(n, 1/n)) \leq \log n + \log \log n + 1.$$

To Boxes and the art of ϵ-net maintenance

Corrections

Problem 4 omits the graph description and Janos clarified it should say "Put an edge $\{R, S\}$ in G_2 if the two rectangles R & S do not cross in four points, and R & S have a point in common not contained by any other rectangle." For Problem 5, Y should indeed be S.

Minor results

Dömötör, Viktor, Máté, János, H. Tamás, Marek, Miloš, Géza, Vajk to Crossing numbers: If $e \geq 4n$, then we have two independent edges crossing a third, from which it is possible to slightly improve the bound for the pair-crossing number.

Csíkvári, Hubai, Ervin et al to Strong set coloring: Using computer checked trees on 16 vertices, for P_{16} there is no even stronger set coloring (meaning sets of vertices contain first element while sets of edges do not).

János, Dani, Gergő, Ági to Edge-connectivity and claw-decomposition: Instead of $K_{1,3}$, the conjecture holds for tree on five vertices with diam 3 and ?

Dani and Dömötör to Universal set of lines for trees: If lines are all vertical, then embedding deg 3 root binary tree with 10 vertices would give $K_{3,3}$ (if top four vertices are mapped to the center four lines, siblings to right and left). Turned out that Rado already knew this...