17th Emléktábla Workshop

Matroid theory

07.15 - 07.20.2025.

Preliminary Schedule for 17^{th} group

Tuesday

10:00 - 13:00 Arrival to the hotel
12:30 - 13:30 Lunch
15:00 - 18:00 Short introduction/presentation of problems
18:00 Dinner

Wednesday – Saturday

8:00 - 9:00 Breakfast
9:00 - 12:30 Work in groups
12:30 - 13:30 Lunch
13:30 - 17:00 Work in groups
17:00 - 18:00 Presentation of daily progress
18:00 Dinner

Sunday

8:00 - 9:00 Breakfast
9:00 - 12:00 Work in groups
12:00 Final presentations, farewell, check-out

List of Participants

Ahmad Abdi
Yuhang Bai
Kristóf Bérczi
Erika Bérczi-Kovács
Nóra Borsik
Nathan Bowler
Gergely Csáji
Tamás Fleiner
András Frank
Dániel Garamvölgyi
Florian Hörsch
András Imolay
Áron Jánosik
Tibor Jordán
Alpár Jüttner
Csaba Király
Tamás Király
Yusuke Kobayashi
Siyue Liu
Vasilis Livanos
Péter Madarasi
Bence Mátravölgyi
Jannik Matuschke
Mirabel Mendoza
Arturo Merino
Taihei Oki
Neil Olver
Gyula Pap
Kanstantsin Pashkovich
Ildikó Schlotter
Ulrike Schmidt-Kraepelin
Benjamin Schröter
Tamás Schwarcz
András Sebő
Olha Silina
Neta Singer
Dániel Szabó
Eszter Szabó
Shin-ichi Tanigawa
Theophile Thiery
Lilla Tóthmérész
Kitti Varga
Victor Verdugo
Soma Villányi
Yutaro Yamaguchi
Yu Yokoi

Orientation for Maximum Reachability

by Florian Hörsch

Given a digraph or undirected graph D, we use $P(D)$ for the set of ordered pairs of distinct vertices in $V(D)$. Next, we define $\kappa_D : P(D) \to \{0,1\}$ by $\kappa_D(u,v) = 1$ if v is reachable from u in D and $\kappa_D(u,v) = 0$, otherwise. Next, given a digraph D and some $P \subseteq P(D)$, we use $R_P(D)$ for $\sum_{(u,v)\in P} \kappa_D(u,v)$. It is now interesting to find an orientation $\vec{G}$ of a given undirected graph G that maximizes $R_P(\vec{G})$.

It is known that this problem is APX-hard, but there is a sublogartihmic approximation algorithm [1]. The following question has remained open.

Problem 1. *Given an undirected graph G and a set $P \subseteq P(G)$, is there a constant $\alpha > 0$ and a polynomial-time algorithm that computes an orientation $\vec{G}_0$ of G such that $R_P(\vec{G}_0) \geq \alpha R_P(\vec{G})$ holds for every orientation $\vec{G}$ of G?*

Problem 1 can easily be reduced to trees. Moreover, the same question can be asked for orientations of mixed graphs rather than undirected graphs. Here, a sublinear approximation algorithm and a slightly stronger APX-hardness result is known [2] .

It seems to me that the problem should be much harder in mixed graphs than in undirected graphs. In particular, the structure of acyclic mixed graphs is much more complex than the structure of trees. This gives some hope that progress should be possible in some direction. Of course, related questions and restricted classes can also be studied.

References

[1] M. Elberfeld, V. Bafna, I. Gamzu, A. Medvedovsky, D. Segev, D. Silverbush, U. Zwick, and R. Sharan. On the approximability of reachability-preserving network orientations. Internet Mathematics, 7(4):209–232, 2011.

[2] M. Elberfeld, D. Segev, C. R. Davidson, D. Silverbush, and R. Sharan. Approximation algorithms for orienting mixed graphs. Theoretical Computer Science, 483: 96-103, 2013.

Fractional Packing of Perfect Matchings

by Alpár Jüttner

For a given a graph $G = (V, E)$, let $\mathcal{P}$ denote the polyhedron of the perfect matchings.

Problem 1. *For a capacity function $u \in \mathbb{R}^E$, find*

$$\max\{\alpha : x \in \mathcal{P}, \alpha x \leq u\}$$

In other words, we are looking for a maximum fractional packing of perfect matchings. If G is bipartite, then the problem can be reduced to a network flow problem, but no strongly polynomial algorithm is known for the general case. Interestingly, a nice combinatorial algorithm is known for fractional packing of T-joins, see [1].

The ultimate goal would be to have a combinatorial algorithm for the following minimum cost fractional 1-packing problem.

Problem 2. *Let us given a cost function $c \in \mathbb{R}^E$ and a capacity function $u \in \mathbb{R}^E$ on the edges. Find*

$$\min\{cx : x \in \mathcal{P}, x \leq u\}$$

As a motivation, let us consider the following *Budgeted Inverse Optimization Problem*

Problem 3. *Let us given a graph $G = (V, E)$, a weight function $w \in \mathbb{R}^E$, a cost function $c \in \mathbb{R}^E$ and a budget value B. The goal is the increase the weight of the minimum weight perfect matching as much as possible by increasing the components of w. Increasing the weight of edge $e \in E$ by one unit costs $c(e)$, and the total cost of the increment must be at most B. Formally, find*

$$\max\left\{\min\{(w+z)x : x \in \mathcal{P}\} : z \in \mathbb{R}^E, \quad z \geq 0, \quad cz \leq B\right\}.$$

It can be shown that if there exists an algorithm for Problem 2 that runs in time T, then Problem 3 is solvable in $O(T^2)$ time.

References

[1] F. Barahona. Fractional Packing of T-Joins. SIAM Journal on Discrete Mathematics 17(4):661-669, April 2004.

Nowhere-zero Submodular Flows

by Tamás Király

Let $G = (V, E)$ be an undirected graph, $r \in V$, and $k \in \mathbb{Z}_+$. An orientation $\vec{E}$ of E is *r-rooted k-cut-balanced* if $d^{out}_{\vec{E}}(U) \geq \frac{1}{k} d_E(U)$ for every $U \subseteq V$ containing r. At last year's workshop, Karthik Chandrasekaran posed the following problem: *Does every 2-edge-connected graph admit an r-rooted 5-cut-balanced orientation?* As far as I know, no counterexampe is known even for rooted 4-cut-balanced orientations. It can be shown using Seymour's theorem on nowhere-zero flows that an r-rooted 6-cut-balanced orientation always exists (in fact, there is an orientation that is good for any r). See [1] for results on weighted cut-balanced orientation problems.

An interesting feature of rooted cut-balanced orientations is that they are a special case of *nowhere-zero submodular flows*, a natural extension of the notion of nowhere-zero flows that has not yet been systematically studied in the literature. For an integer $k \geq 2$, let K be the set of nonzero integers between $-k + 1$ and $k - 1$. Given a graph $G = (V, E)$ with a reference orientation $\vec{G} = (V, A)$ and a submodular set function $b : 2^V \to \mathbb{Z} \cup \{\infty\}$, a *nowhere-zero submodular k-flow* is vector $x \in K^A$ such that $d^{in}_x(U) - d^{out}_x(U) \leq b(U)$ for every $U \subsetneq V$. We note that the existence of such a vector does not depend on the reference orientation $\vec{G}$. Since deciding the existence of a nowhere-zero 3-flow or a nowhere-zero 4-flow is NP-complete, we cannot expect a general good characterization for the existence of nowhere-zero submodular k-flows. However, if we fix the positive and negative arcs, then the problem becomes a normal submodular flow problem where a simple characterization is known.

Sufficient conditions for nowhere-zero submodular k-flows may turn out to be useful for improved approximations of Woodall's conjecture and for its capacitated versions, since the submodular function offers an additional flexibility compared to cut-balanced orientations.

Problem 1. *Give nontrivial sufficient conditions for the existence of nowhere-zero submodular k-flows.*

References

[1] K. Chandrasekaran, S. Liu, R. Ravi, *Minimum Cost Nowhere-zero Flows and Cut-balanced Orientations*, arXiv preprint arXiv:2504.18767 (2025)

Matchoid Problem

by Yusuke Kobayashi

In the matchoid problem [1], the input consists of a graph $G = (V, E)$ and a matroid M_v on the ground set $\delta(v)$ for each $v \in V$, and the objective is to find an edge subset $F \subseteq E$ of maximum size subject to $F \cap \delta(v)$ is an independent set of M_v for any $v \in V$. When each M_v is given as an independence oracle, this problem is equivalent to the matroid parity problem (or the matroid matching problem), and it requires exponential number of queries. Meanwhile, the following positive results are known.

- If each M_v is a linear matroid and its linear representation is given as input, then the matchoid problem can be solved in polynomial time (see e.g. [3]).
- If each M_v is given as a list of all the independent sets (i.e., its input size is $|\mathcal{I}_v|$), then the matchoid problem can be solved in polynomial time [2].

My question is what we obtain when these two results are combined.

Problem 1. *Can we solve the matchoid problem in polynomial time when each M_v is given as a linear representation or a list of all the independent sets?*

We might be able to use existing algorithms as a black box.

References

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Congruency-constrained Shortest Paths

by Mirabel Mendoza

In the Congruency-Constrained Shortest Path $(\text{ModPath})_{\text{q,m}}$, we are given an undirected graph $G = (V, E)$, $s, t \in V$, a weight function $w : E \to \mathbb{R}$, two positive integers q, m, and the goal is to find a path P from s to t with minimum weight such that it has length $|P| = q \mod m$.

The following property is not difficult to see.

Theorem 1 ([1])**.** *For a fixed m,* $\text{ModPath}_{q,m}$ *reduces to* $\text{ModPath}_{q',m}$ *for any pair $q, q' \in [m]$.*

Last property follows by adding a path from t to a new vertex t' of length $q - q' \mod m$.

When $m = 2$ and $q = 1$, the problem is referred to as the Shortest Odd Path. If the weight w is non-negative, then the problem is solvable in polynomial time [2, 3]. If w is conservative (meaning there are no cycles with negative weight), the problem is NP-hard [4]. However, when the negative edges form a tree, the problem can be solved in polynomial time [5]. There are also two FPT algorithms: one is parameterized by the number of negative edges, and the other is parameterized by the treewidth [5].

For undirected graphs with positive weights, there exists a linear-time algorithm that computes the greatest common divisor of the cycle weights [6]. The paper also gave an algorithm that determines, given two nodes s, t, whether all paths between s and t have length q modulo m. Note that this problem is not equivalent to $\text{ModPath}_{\text{q,m}}$.

Problem 2. *Can we find a polynomial-time algorithm to solve* $(\text{ModPath})_{q,m}$ *for $q = 0$ and $m = 3$ when the weight function is non-negative?*

As Shortest Odd Path is NP-hard when w is a conservative weight function, then $(\text{ModPath})_{\text{q,m}}$ it is also NP-hard when w is conservative. Then, we are interested in the question below.

Problem 3. *Can we find FPT algorithms for* $(\text{ModPath})_{q,m}$ *when $q = 0$ and $m = 3$ for $w \geq 0$?*

Problem 4. *Can we derive FPT algorithms with another parameters for* Shortest Odd Path*?*

References

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Pfaffian Testing and Signing for Matrix Pairs

by Taihei Oki

Let $\mathbb{F}$ be a field of characteristic zero. For an $r \times n$ matrix A and $J \subseteq [n]$, let $A[B]$ denote the submatrix of A obtained by collecting columns in J.

A pair (A_1, A_2) of $r \times n$ matrices over $\mathbb{F}$ is called *Pfaffian* [3] if

$$\det A_1[B] \det A_2[B]$$

is constant for any $B \in \binom{[n]}{r}$ such that both $A_1[B]$ and $A_2[B]$ are nonsingular, i.e., B is a common base of the linear matroids represented by A_1 and A_2. If (A_1, A_2) is a Pfaffian pair, then by the Cauchy–Binet formula,

$$\det A_1 A_2^\top = \sum_{B \in \binom{[n]}{r}} \det A_1[B] \det A_2[B]$$

is proportional to the number of common bases, giving a polynomial-time algorithm for counting common bases. This generalizes Kirchhoff's matrix-tree theorems for counting spanning trees and arborescences [1].

The following have been posed in [3].

Problem 1 (Pfaffian testing)**.** *Given a pair (A_1, A_2) of $r \times n$ matrices, check if it is Pfaffian or not.*

Problem 2 (Pfaffian signing)**.** *Given a pair (A_1, A_2) of $r \times n$ matrices, find an $n \times n$ diagonal matrix S such that every diagonal entry of S is ± 1 and $(A_1 S, A_2)$ is Pfaffian.*

These problems are open even for pairs of totally unimodular matrices. We do not even know whether these are in NP or not. For instances of bipartite matching, i.e., instances where each column of A_k has exactly one entry equal to 1 and all others are 0, these problems are nothing but the so-called Pfaffian orientation and are polynomial-time solvable [2].

References

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Submodular Facility Location on a Line

by Neil Olver

In the submodular facility location problem, we are given a finite metric space (V, d), a partition of V into a set F of *facilities* and a set C of *clients*, and a monotone nonnegative submodular function $f : 2^C \to \mathbb{R}_{\geq 0}$ with $f(\emptyset) = 0$. The goal is to choose an assignment $\pi : C \to F$. The meaning of f is that to open a facility that is assigned a set S of clients, we must pay $f(S)$. Our overall cost is the sum of opening costs and all distances between clients and their assigned facility:

$$\text{cost}(\pi) = \sum_{c \in C} d(c, \pi(c)) + \sum_{v \in F} f(\pi^{-1}(v)).$$

Recently, Abbasi, Adamczyk, Bosch-Calvo, Byrka, Grandoni, Sornat and Tinguely [1] gave an $O(\log \log n)$-approximation, where $n = |V|$, for any metric. Their approach relies on reducing to HSTs, and then slightly extending the approach of Bosman and Olver [2], which (while motivated by a different problem) can be interpreted as giving an $O(\log \log n)$-approximation if (V, d) is a line metric.

Problem 1. *Let's restrict to the case where (V, d) is a line metric. Can we improve the $O(\log \log n)$ approximation to a constant factor? The results mentioned make use of the natural LP relaxation, which uses the Lovász extension of f. Bosman and Olver show that the integrality gap is $O(\log \log n)$, but as far as we know, it could be a constant.*

References

[1] Fateme Abbasi, Marek Adamczyk, Miguel Bosch-Calvo, Jarosław Byrka, Fabrizio Grandoni, Krzysztof Sornat, and Antoine Tinguely. An $O(\log \log n)$-Approximation for Submodular Facility Location. In *Proc. ICALP*, 2024.

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Philosopher Inequalities for Matroids

by Kanstantsin Pashkovich

Problem 1. *Let us consider a scenario where a vendor sells several items and receives offers for items over time. The vendor attempts to maximize their profit subject to structural constraints. The offers correspond to random variables $v_1, v_2, \ldots, v_n$ drawn independently from distributions known to the vendor. At each timestamp t, the vendor may choose to sell their t-th item gaining the value v_t, or the vendor may choose to discard this offer. At the end, the sold items should form an independent set with respect to the underlying matroid. Let us consider the benchmark introduced by the so called* philosopher*. The philosopher does not know the realizations of $v_1, v_2, \ldots, v_n$ but has unlimited computational power. How well can a vendor limited to polynomial time computation compete with the philosopher, where both know only the distributions of $v_1, v_2, \ldots, v_n$ but the latter has unlimited computational power?*

Theorem 2. *[1] showed that for graphic matroids, it is $\mathbb{PSPACE}$-hard for the vendor to approximate the expected gain of the philosopher up to some fixed constant. Moreover, for the graphic matroids there is no arrival order which "substantially increases competitiveness" of the vendor.*

Problem 3. *Is there an arrival order for the graphic matroids corresponding to planar graphs that permits a Polynomial-Time Approximation Scheme (PTAS) for the vendor to approximate the expected gain of the philosopher?*

Problem 4. *Do the graphic matroids corresponding to planar graphs permit a Polynomial-Time Approximation Scheme (PTAS) for the vendor to approximate the expected gain of the philosopher?*

Theorem 5. *[1] provided a PTAS for all laminar matroids with "left-to-right" arrival orders, in which elements from each constraint arrive consecutively. Furthermore, the provided PTAS also holds for arrival orders that are "close" to "left-to-right" orders, i.e., to orders where each element is contained only in constantly many bins on which the arrival order is not "left-to-right."*

Theorem 6. *[2] gave a PTAS for the special cases of bounded-depth laminar matroids and production constrained selection.*

The result in [1], imposes the conditions on the structure of the arrival order but not on the structure of the laminar matroid. The result in [2] for bounded-depth laminar matroids, imposes no conditions on the structure of the arrival order but imposes conditions on the structure of the laminar matroid.

Problem 7. *Do laminar matroids permit a PTAS for the vendor to approximate the expected gain of the philosopher (with no conditions on the arrival order or the matroid structure)?*

References

[1] I. DeHaan, K. Pashkovich, Matroid Bayesian Online Selection, 2024

[2] N. Anari, R. Niazadeh, Rad, A. Saberi, A. Shameli, Ali, Nearly Optimal Pricing Algorithms for Production Constrained and Laminar Bayesian Selection, 2019

Relaxations of matroid polytopes

by Benjamin Schröter

One may associate to a matroid M on the ground set E its *base polytope*

$$\mathcal{P}_M = \left\{ e_I \in \mathbb{R}^E \ : \ I \in \binom{[n]}{r} \right\} = \left\{ x \in [0,1]^E \ : \ \sum_{i \in E} x_i = r, \sum_{i \in F} x_i \leq \mathrm{rk}(F) \text{ for all flats } F \right\}$$

where r is the rank of M and $e_I = \sum_{i \in I} e_i$ is the sum of standard unit vectors indexed by the k element subset I. These 0/1-polytopes are used in a variety of modern results in matroid theory.

We are interested in *relaxations* of a matroid base polytope (in the sense of linear or (mixed) integer optimization) such that the larger polytope is again a matroid base polytope of a matroid on the same ground set. In other words we study matroids on the same ground set for which the identity is a weak map. This idea generalizes hyperplane relaxations, but also the recently introduced notion of relaxation of stressed subsets in [3] which is leads to one of many ways to characterize elementary split matroids.

Problem 1. *Given two (connected) matroids M and N on E . Characterize (in terms of their flats) when there is a set $F \subseteq E$ and a number f such that*

$$\mathcal{P}_M = \mathcal{P}_N \cap \{x \in \mathbb{R}^E \ : \ \sum_{i \in F} x_i \leq f\}.$$

Problem 2. *Describe all connected matroids M for which $\mathcal{P}_N \subseteq \mathcal{P}_M$ implies $N = M$ whenever N is connected.*

A *polyhedral subdivision* of a polytope P is a collection $\mathcal{S}$ of polytopes such that faces of any member in $\mathcal{S}$ are in $\mathcal{S}$, the intersection of any two polytopes in $\mathcal{S}$ is a face of both, and the $\mathcal{S}$ covers P, i.e., $\bigcup_{Q \in \mathcal{S}} Q = P$. We are intersted in *matroid subdivions* of a matroid base polytope P_M, that is a polyhedral subdivision for which all the polytopes in $\mathcal{S}$ are matroid base polytopes.

A variation of the previous problem are the following two problems.

Problem 3. *Find all matroids M such that the only matroid subdivision of P_M consists of its faces.*

We call a subdivision of P *regular* if it is induced by a height function on the vertices. This term leads us to the final variantion of the above problem.

Problem 4. *Describe all (connected) matroids M such that the only regular matroid subdivision of the polytope P_M consists of the faces of P_M.*

There are several related articles, see for example [2] which shows that if there is a weak map from M to N then there is not necessarily a regular subdivision that contains P_N and refines P_M, [4] who relate these problems to realizability space and [1] which shows that binary matroids belong to the matroids in Problem 2.

References

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The Girth of Binary Matroids

by András Sebő

We are interested in the optimal solution of a binary equation, that is, in the problem

$$\min c^\intercal x, \quad x \in \{0,1\}^n : Ax \equiv b \mod 2, \qquad (A \in \{0,1\}^{m\times n}, b \in \{0,1\}^m, c \in \mathbb{Z}^n). \qquad (1)$$

This is exactly the problem of finding a *minimum weight cycle containing* b in the binary matroid represented by the the columns of the matrix (A, b). The minimum weight of a cycle without more requirements is easier in some special cases, but it is also NP-hard in this generality, even if all the weights are equal to 1 [7]. Odd cycles and their hitting sets, or even cycles are also part of this framework, by altering the studied binary matroid. A huge amount of related problems and generalizations are exposed in [3]. These may be grouped to three main complexity behaviors of relevant special cases that have been identified:

- Polynomially solvable special cases: eg. minimum odd cycle in graphs, minimum transversal of odd cycles in planar graphs NP-hard in general, equivalent to MAX CUT. Minimum weight odd cuts and their minimum transversals can be found in polynomial time in any graph.
- Open problems: eg. a very special one is the MAX CUT problem in graphs embeddable to the projective plane, open for all 1 weights as well. This problem has been identified [1] as a special case of Problem 27 of [5], reactivated by [8], and shown to be NP-hard by [4]. We distinguish two kinds of open problems (MAX CUT in the projective plane belongs to 1.1):
  1. Open problems that belong to the class RP [3]:
     - 1.1 either through random sampling of polynomials from determinants of variables
     - 1.2 or through Karger-type minimum cut algorithms
  2. New open problems specializing NP-complete cases
- NP-hard problems

I select three open problems according to their relevance or simplicity:

Problem 1. *Shortest Odd Cycle in ±1-weighted undirected conservative graphs.*

Problem 2. *Shortest odd cycle through a given vertex in ±1-weighted planar graphs if the negative edges form a matching.*

The famous Planar "Back and Forth vertex-disjoint Paths" problem (BFP) [2] can be reduced to this problem as in [4]. BFP has been solved in planar graphs [6]. Guyslain Naves pointed out that I should be more careful in keeping planarity with the reduction and suggested a planarity-keeping gadget. Taking then the planar dual, and slightly generalizing it, the following problem arises for planar graphs. I ask whether this can be solved in general, an interesting problem for its own sake, and an accessible solution would provide such a one for planar BFP [6]:

Problem 3. *Let $G = (V, E)$ an arbitrary graph, suppose $R \subseteq E$, and for every cut C, we have $|C \setminus R| \geq C \cap R|$. Minimize $|E \setminus R| - |E \cap R|$*

a. on T-cuts.

b. on T-cuts containing a given edge of G.

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Directed k-cut

by Daniel P. Szabo

Many versions of k-cut, where we partition the graph into k parts to minimize the sum of the interpartition edges, have been studied. Surprisingly, the following directed generalization has not:

Problem 1. *Given a* directed *graph* $G = (V, E)$ *and a positive integer* k, *find a minimum-cost edge set* $F \subseteq E$ *for which* $\exists t_1, t_2, \ldots, t_k \in V$ *such that there is no* $t_i \to t_j$ *or* $t_j \to t_i$ *path for any* $i \neq j$ *in* $(V, E \setminus F)$

Existing methods seek to minimize over all partitions the sum of the indegrees of the parts, while here we seek a very different partition. One motivation for this problem is the $k = 2$ case [1], where the global bicut problem has a $2-\varepsilon$-approximation (and no hardness result!), which is strictly better than the best possible with fixed terminals, suggesting that this problem is completely different in terms of approximation.

References

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Maximality Problem of Matroids

by Shin-ichi Tanigawa

Maximality Problem. Let E be a finite set and $M = (E, r_M)$ be a matroid on E of rank r_M. We define the class $\mathcal{E}(M)$ of matroids to consist of those whose truncation is equal to M. Formally, $N \in \mathcal{E}(M)$ iff N has the ground set E, the rank r_N of N is at least r_M, and the truncation of N to rank r_M is equal to M.

A general maximality problem asks:

Problem 1. *Characterize a matroid M such that $\mathcal{E}(M)$ has a unique maximal element with respect to weak order.*

Example 1. Let me first give a simple concrete example. Suppose E is the edge set of the complete graph K_n. Let $\mathcal{C}$ be the set of all triangles in K_n. Then $\mathcal{C}$ forms the set of non-spanning circuits in a rank-three paving matroid M on E. (To see this, consider the truncation of the graphic matroid of K_n to rank three.) Now, $\mathcal{E}(M)$ is a class of matroids on E in which any triangle is a circuit and any other 3-sets are independent. One can show that the graphic matroid of K_n is the unique maximal matroid in $\mathcal{E}(M)$.

Case of Paving Matroids. Problem 1 seems somewhat too general, and it is unclear what kind of characterization one might expect. An interesting special case, I believe, is when M is a paving matroid.

Given a finite set E and a collection $\mathcal{C}$ of r-sets in E. We define the *paving closure* $\mathcal{C}^p$ of $\mathcal{C}$ to be a collection of r-sets satisfying

1. $\mathcal{C} \subseteq \mathcal{C}^p$,

2. if $X, Y \in \mathcal{C}^p$ with $|X \cup Y| = r + 1$, then any r-set in $X \cup Y$ is in $\mathcal{C}^p$, and

3. $\mathcal{C}^p$ is minimal among those satisfying 1 and 2.

Indeed, $\mathcal{C}^p$ is uniquely defined as it can be build from $\mathcal{C}$ by greedily adding r-sets to satisfy Condition 2. Now, $\mathcal{C}^p$ forms the collection of non-spanning circuits of a paving matroid, which is denoted by $M_{\mathcal{C}}$.

Problem 2. *Let E be a finite set. Characterize a collection $\mathcal{C}$ of r-sets in E such that $\mathcal{E}(M_{\mathcal{C}})$ has a unique maximal element with respect to weak order.*

Example 2. Let K_n be the complete graph on n vertices and H be a graph. In [1], I and Bill Jackson have worked in the case when $E = E(K_n)$ and $\mathcal{C}$ is the collection $\mathcal{C}_H$ of all isomorphic copies of H in K_n.

If $H = K_3$, then $\mathcal{C}_H$ is the collection of all triangles in K_n, and we recover Example 1.

If $H = K_{1,3}$, $\mathcal{C}_H$ is the collection of all copies of $K_{1,3}$ in K_n. $\mathcal{C}_{K_{1,3}}$ already satisfies Condition 2 above and hence the paving closure does not add any new element. By using the circuit elimination axiom, one can easily check that $\mathcal{E}(M_{\mathcal{C}_{K_{1,3}}}) = \{M_{\mathcal{C}_{K_{1,3}}}\}$.

In [1, 2], we have shown that if

$$H \in \{P_k, K_2 \cup \cdots \cup K_2, C_4, K_3, K_4, K_5, K_4^-, K_5^-, K_6^-, K_{1,3}, K_{1,4}\},$$

$\mathcal{E}(M_{\mathcal{C}_H})$ has the unique maximal element. On the other hand, if

$$H \in \{K_{2,3}, C_t(t \geq 5), K_{1,t}(t \geq 5)\},$$

then $\mathcal{E}(M_{\mathcal{C}_H})$ have more than one maximal element. (The first negative result for C_5 was pointed out by Gyula Pap.) Recently, we have also found that the unique maximality does not hold if $H = K_{t,t}$ $(t \geq 4)$ and the ground set is the edge set of a complete bipartite graph.

Those results are obtained by ad-hoc inspections, and so far there is no theory to understand them. My current impression is that, for most of sufficiently large H, $\mathcal{E}(M_{\mathcal{C}_H})$ has more than one maximal element.

Example 3. The case when $H = K_{1,r+1}$ is particularly interesting. In this case, $\mathcal{E}(M_{\mathcal{C}_{K_{1,r+1}}})$ is a quasi (2nd) symmetric product of the rank-r uniform matroid U_n^r in the sense of Lovász [3] and Mason [4]. (More precisely, a 2nd symmetric product is a matroid on the edge set of a complete graph with a loop at each vertex; here we only consider the case without loops.) The unique maximality problem has been posed by Mason, and Las Vergnas gave a negative answer (in the quasi tensor case). Further negative examples can be found in our recent paper [2].

The class of quasi (2nd) symmetric products of U_n^r is slightly different from $\mathcal{E}(M_{\mathcal{C}_{K_{1,r+1}}})$ since a quasi (2nd) symmetric product of U_n^r may have a circuit of size smaller than r. We do not know how much those two classes are different.

Mason's question for symmetric or non-symmetric tensors has a close connection to Graver's conjecture, which is a long-standing open problem in graph rigidity theory.

To the best of our knowledge, very little is known about the non-uniform case, and it would be interesting to collect both positive and negative examples.

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Eulerian Orientations Minimizing the Number of Arborescences

by Lilla Tóthmérész

It is a classical result that an Eulerian digraph has the same number of arborescences for any root. It seems that the following natural extremal problem was not much investigated yet.

Problem 1. *Given an Eulerian graph, which Eulerian orientation has minimal number of arborescences?*

I am interested in this, because computations suggest the following neat conjecture:

Conjecture 2. *If the Eulerian graph is planar, then the alternating orientation (in- and out-edges alternate around each vertex) has a minimal number of arborescences among Eulerian orientations.*

In the preprint [1], we have solved the problem for some special graph classes (complete, complete bipartite, each edge multiplicity is even) but the conjecture is wide open.

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Sampling Bases in a Matroid

Victor Verdugo

Let $r_i \in (0,1)$ for each $i \in \{1,\ldots,n\}$, and let $k = \sum_{i=1}^n r_i$ be a positive integer. Our goal is to randomly round the value r_i to zero or one, for each $i \in \{1,\ldots,n\}$, to get values $R_1,\ldots,R_n \in \{0,1\}$ satisfying the following:

1. $\mathbb{E}(R_i) = r_i$ for every $i \in \{1,\ldots,n\}$.
2. $\sum_{i=1}^n R_i = \sum_{i=1}^n r_i = k$.

While property (1) is easily attained by independent rounding, we need to correlate the rounding to satisfy (2), i.e., to preserve the summation of the input values. In the past two decades, theoretical computer scientists have studied this question under the name of *dependent randomized rounding* [1], which has proven to be a powerful algorithmic design tool in approximation. However, this question has its origins in statistics under the name of πps *sampling without replacement* [2, 3]; in fact, in 1983, Brewer and Hanif [2] already listed 50 methods to solve this question, or a relaxation of it. In a recent work about dependent rounding and monotone proportional apportionment [4], we studied the method introduced by Sampford in 1967, which works as follows:

(i) Sample a value $i \in \{1,\ldots,n\}$ with probability proportional to r_i; call the sampled value i_1.
(ii) Then, randomly sample $k-1$ values $i_2,\ldots,i_k$, with replacement from $\{1,\ldots,n\}$, choosing each value i with probability proportional to $r_i/(1-r_i)$.
(iii) If the k drawn values are distinct, select them; otherwise, start over.

Sampford's method solves the dependent rounding question (1)-(2). In a matroid language, we are given a feasible point r in the k-uniform matroid polytope (i.e., $r \in [0,1]^n$ with $\sum_{i=1}^n r_i = k$), and we want to sample an extreme point R (i.e., $R \in \{0,1\}^n$ with $\sum_{i=1}^n r_i = k$) such that $\mathbb{E}(R) = r$.

Problem 1. *Generalize Sampford to handle a broader family of matroids.*

This would open the possibility to have a *selection monotone* sampling method for matroids. In the context of k-uniform matroids, this property requires that for any $A \subseteq \{1,\ldots,n\}$ of size k, if r_i increases for each $i \in A$ and r_j remains the same or decreases for each $j \notin A$, then it is more likely that the elements in A are rounded up. Sampford's method is selection monotone [4]. However, this monotonicity property can be easily violated by other rounding methods; e.g., pipage rounding or maximum entropy are not selection monotone [4].

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Fast Construction of Exact Matching(s)

by Yutaro Yamaguchi

The *exact matching problem (EM)* is as follows: given a graph in which each edge is colored by red or blue, find a perfect matching with exactly k red edges. It admits a fairly simple *randomized* polynomial-time algorithm, but any *deterministic* polynomial-time algorithm is not known for more than 40 years since the problem was stated. Recently, Sato and Yamaguchi [3] proposed a fast randomized algorithm for the decision problem as follows.

Theorem 1. *One can find with high probability the set of integers k such that there exists a perfect matching with exactly k red edges in $\mathrm{O}(n^\omega)$ time (field operations) in total, where n denotes the number of vertices in the input graph and $\omega < 2.37134$ denotes the matrix multiplication exponent.*

The key is a reduction of the bottleneck task in the usual randomized approach, computing the pfaffian $\mathrm{pf}\, T$ of the Tutte matrix (including the color information) after random substitution to the edge indeterminates, to computation of the characteristic polynomial $\det(tI - A)$ of a constant matrix A of the same size, which can be done deterministically in $\mathrm{O}(n^\omega)$ time [2].

Let us consider construction of solutions. On the one hand, by a divide-and-conquer strategy with the aid of fast low-rank update of the inverse matrices, one can find with high probability a perfect matching itself in $\mathrm{O}(n^\omega)$ time if any [1]. On the other hand, for the exact matching problem, the above decision algorithm leads to an $\mathrm{O}(n^{\omega+1})$-time construction algorithm for each possible k as follows. In any order, for each vertex v, fix which color should be used to match the vertex v by testing the existence of an exact matching after removing all the red edges incident to v. This raises a natrual question as follows.

Problem 2. *Is there a faster (randomized) algorithm for finding a perfect matching with exactly k red edges? In particular,*

1. $\mathrm{O}(n^{\omega+1})$ *time for all possible k in total,*
2. $\mathrm{O}(n^\omega)$ *time for each possible k, or*
3. $\mathrm{O}(n^\omega)$ *time for all possible k in total.*

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Parity-Constrained 2-Factor

by Yutaro Yamaguchi

In the workshop in 2024, we obtained the following results on the parity-constrained 2-factor problem. Here, ODD/EVEN 2-FACTOR is the decision problem asking the existence of a 2-factor containing at least one odd/even cycle, and ALL-ODD/EVEN 2-FACTOR asks the existence of a 2-factor containing no even/odd cycles.

Theorem 1 ([2])**.** *ODD 2-FACTOR, ALL-ODD 2-FACTOR, and ALL-EVEN 2-FACTOR are* NP*-hard.*

A natrual remaining question is as follows.

Problem 2. *Is EVEN 2-FACTOR in* P *or* RP*, or* NP*-hard?*

There is a possible extension to red-blue graphs (where "odd/even" means the parity of the number of red edges contained in each cycle), which is in fact polynomial-time equivalent. Also, the following problems can be reduced to EVEN 2-FACTOR. EVEN DICYCLE COVER asks the existence of a directed cycle cover containing at least one even dicycle, and EVEN DICYCLE asks the existence of a directed cycle of even length. We know that EVEN DICYCLE reduces to EVEN DICYCLE COVER and EVEN DICYCLE COVER reduces to EVEN 2-FACTOR, but the reverse of each reduction is not known.

It is well-known that EVEN DICYCLE is equivalent to Pólya's permanent problem and in P [3]. Thus, a reverse reduction or another intermediate problem would be welcome. A randomized approach like [1] might be useful.

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Fair Allocation with Subsidy for Matroid-Related Valuations

by Yu Yokoi

Let $N = \{1, 2, ..., n\}$ be the set of agents and E the set of items. Each agent $i \in N$ has a valuation $v_i : 2^E \to \mathbf{R}_+$. An *allocation* is a tuple $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ such that $A_1 \cup A_2 \cup \cdots \cup A_n = E$ and $A_i \cap A_j = \emptyset$ for any distinct $i, j \in N$. The set A_i is the bundle allocated to $i \in N$.

Ideally, we seek an envy-free allocation, i.e., one satisfying $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in N$, but such an allocation may not exist. We therefore consider attaining envy-freeness using *subsidy*. Let $\vec{p} = (p_1, p_2, \ldots, p_n) \in \mathbf{R}^n_+$ denote a subsidy vector. A pair $(\mathcal{A}, \vec{p})$ of an allocation and a subsidy vector is called *envy-free* if

$$v_i(A_i) + p_i \geq v_i(A_j) + p_j \quad (i, j \in N).$$

It was shown in [1] that every instance admits such an envy-free pair. We are interested in the minimum amount of subsidy required to ensure envy-freeness for any instance.

W.l.o.g., we assume that the marginal value of any item is at most 1, i.e., $v_i(X \cup \{e\}) - v_i(X) \leq 1$ for every $i \in N$, $X \subseteq E$, and $e \in E \setminus X$. Clearly, a subsidy of $n-1$ is a lower bound. (If there is only one item, valued at 1 by all agents, $n-1$ agents must each receive a subsidy of 1.)

- For additive valuations, this lower bound $n-1$ is tight [2]. There always exists an envy-free pair $(\mathcal{A}, \vec{p})$ such that $p_i \leq 1$ for every $i \in N$ (and $p_{i'} = 0$ for some $i' \in N$).
- The same holds for matroid rank functions [3] and more general binary marginal functions [4].
- For general monotone functions, the current best upper bound is $n(n-1)/2$ [5].

It may be worth exploring upper and lower bounds for valuations beyond additive and matroid rank functions. For example, one can consider weighted matroid rank functions, i.e., functions of the form $v_i(X) = \max\{w_i(Y) : Y \subseteq X, Y \in \mathcal{I}_i\}$ for some matroid $(E, \mathcal{I}_i)$ and weight $w_i \in \mathbf{R}^E_+$.

Problem 1. *When each v_i is a weighted matroid rank function (with marginal values at most 1), does there exist an envy-free pair $(\mathcal{A}, \vec{p})$ such that $p_i \leq 1$ for every $i \in N$?*

Problem 2. *What about other function classes, such as monotone $M^\natural$-convex or submodular functions? (cf. additive $\subseteq$ weighted matroid rank $\subseteq$ $M^\natural$-concave (gross substitutes) $\subseteq$ submodular)*

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