13/14th Emléktábla Workshop

Extremal combinatorics / Matroid theory 08.13 - 08.18.2023.

Preliminary Schedule for 14^{th} group:

Sunday

Arrival in the hotel from 14:00.
17:00-18:00 Short introduction/presentation of problems - Part I
18:00 Dinner
19:00 Short introduction/presentation of problems - Part II (if needed)

Monday

7:30-8:30 Breakfast
9:00 Grouping
10:00-17:00 Working in groups
12:30 Lunch
17:00 Presentation of daily progress
18:00 Dinner

Tuesday-Thursday

7:30-8:30 Breakfast
9:00-17:00 Working in groups
12:30 Lunch
17:00 Presentation of daily progress
18:00 Dinner

Friday

7:30-8:30 Breakfast
9:00-11:00 Working in groups
11:00 Final presentation of progress, farewell
12:30 Lunch
13:30-14:00 Check-out

List of Participants

Erika Bérczi-Kovács, ELTE
Kristóf Bérczi, ELTE-MTA
András Frank, ELTE
Panna Gehér, ELTE
Florian Hörsch, CISPA Saarbrücken
András Imolay, ELTE
Tibor Jordán, ELTE
Alpár Jüttner, ELTE
Naonori Kakimura, Keio University
Tamás Király, ELTE
Max Kölbl, Osaka University
Vasilis Livanos, UIUC
Georg Loho, University of Twente
Kevin Long, George Washington University
Péter Madarasi, Egerváry Research Group
Lydia Mirabel Mendoza Cadena, ELTE-MTA
Neil Olver, LSE
Taihei Oki, University of Tokyo
Gyula Pap, ELTE
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Tamás Schwarcz, ELTE
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Kitti Varga, Rényi Institute
Tomohiko Yokoyama, University of Tokyo

Strongly connected re-orientations and polarity

by Ahmad Abdi

Let $D = (V, A)$ be a digraph with m arcs whose underlying undirected graph is 3-edge-connected. A *strongly connected re-orientation* is a subset $J \subseteq A$ such that $D \triangle J$ (the digraph obtained after flipping the orientations of the arcs in J) is strongly connected. Consider the set-system

$$\mathrm{SCR}(D) := \{\chi_J : J \text{ is a strongly connected re-orientation}\} \subseteq \{0,1\}^m.$$

This set-system enjoys several appealing discrete geometric properties. Let $S := \mathrm{SCR}(D)$ for short. For instance,

- S is *antipodally symmetric*, that is, a point belongs to S iff its antipodal point belongs to S: $p \in S$ iff $\mathbf{1} - p \in S$.
- S is *strictly connected*, that is, between every pair of points in S there is a monotone path on the skeleton graph of $\{0,1\}^m$ where all the intermediate nodes also belong to S. This follows from [3] and uses the 3-edge-connectivity of the underlying undirected graph of D.

  Let us describe this property in a different but equivalent way. Denote by G_m the skeleton graph of $\{0,1\}^m$. Then, if $S' \subseteq \{0,1\}^{m'}$ is a *restriction of* S (i.e. it is obtained from S after restricting some coordinates to 0 or 1 and then dropping the coordinates altogether), then the subgraph $G_{m'}[S']$ induced on S' is connected.
- Finally, S is *cube-ideal*, that is, its convex hull is described by hypercube and generalized set covering inequalities. More specifically, it is described by

$$\begin{aligned} x &\geq \mathbf{0} \\ x &\leq \mathbf{1} \\ \sum_{a \in \delta^+(U)} x_a + \sum_{a \in \delta^+(U)} (1 - x_a) &\geq 1 \quad \forall U \subsetneq V, U \neq \emptyset. \end{aligned}$$

  To see this, note first that the integer solutions to this system are precisely the points in S. Secondly, note that the generalized set covering inequalities form a submodular flow system, which in turn is box-TDI and so box-integral. This implies that the system above is integral.

The underpinning theme is to understand which restrictions of S have antipodal points.

Given disjoint $I, J \subseteq A$, consider the set-system obtained from $S \cap \{x : x_a = 0\ \forall a \in I, x_b = 1\ \forall b \in J\}$ after dropping the coordinates in $I \cup J$; we call this the *restriction of* S obtained after 0-restricting I and 1-restricting J.

Problem 1. *Find sufficient conditions on $I, J \subseteq A$ such that the restriction S' of S obtained after 0-restricting I and 1-restricting J, contains antipodal points.*

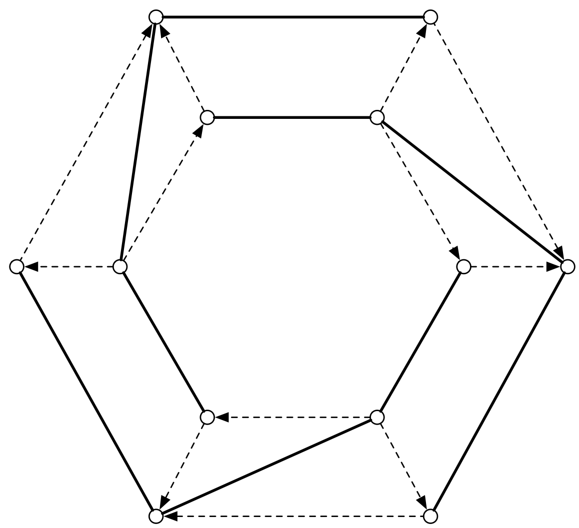
An obvious necessary condition is that the points in S' do not agree on a coordinate, that is, $S' \subseteq \{x : x_i = a\}$ for some $i \in [m']$ and $a \in \{0,1\}$.

Let $S' \subseteq \{0,1\}^{m'}$ be a set-system. We say that S' is *polar* if either $S' \subseteq \{x : x_i = a\}$ for some $i \in [m']$ and $a \in \{0,1\}$, or S' contains antipodal points; otherwise it is *non-polar*.

For instance, the following conjecture has been made in relation to the problem above.

Conjecture 2. *[2] If $A-(I\cup J)$ is a spanning tree of D, then the restriction S' of S obtained after 0-restricting I and 1-restricting J, is polar.*

Not all restrictions of S are polar. For instance, consider the digraph D below, let I be the set of dashed arcs, and let $J := \emptyset$. (The orientation of the solid arcs is irrelevant, you can orient them arbitrarily.) It can be shown that if S' is the restriction of S obtained after 0-restricting I, then the points in S' do not agree on a coordinate (every cut either has an incoming dashed arc or has at least two solid arcs), yet S' does not contain antipodal points (the solid arcs cannot be oriented in such a way such that every cut either has an incoming dashed arc, or has solid arcs crossing in both directions).



S' is *strictly non-polar* if it is non-polar and every restriction is polar. Observe that every non-polar set has a restriction that is strictly non-polar. My hope is that the discrete geometric properties above are helpful in tackling the above problems. More specifically,

Problem 3. *What can be said about strictly non-polar set-systems that are both cube-ideal and strictly connected?*

In [1], cube-ideal strictly non-polar sets were studied, and partly characterized, as they are helpful in generating *ideal minimally non-packing clutters* (for instance, they give rise to 716 such clutters with at most 14 elements). Interestingly, the notion of strict connectivity showed up and was studied there, albeit in a different context. I suspect there is more to be said.

References

[1] Abdi, A., Cornuéjols, G., Guričanová, N., and Lee, D.: Cuboids, a class of clutters. Journal of Combinatorial Theory, Series B, 142:144–209, 2020.

[2] Chudnovsky, M., Edwards, K., Kim, R., Scott, A., and Seymour, P.: Disjoint dijoins. Journal of Combinatorial Theory, Series B, 120:18–35, 2016.

[3] Fukuda, K., Prodon, A., and Sakuma, T.: Notes on acyclic orientations and the shelling lemma. Theor. Comput. Sci., 263:9–16, 2001.

Covering a 2-base matroid by rainbow bases

by Florian Hörsch

For a positive integer k, a *k-base matroid* is a matroid whose ground set can be partitioned into k bases. A *coloring* of a matroid M is a mapping $\phi : E(M) \to \mathbb{Z}_{\geq 0}$. For a positive integer p, we say that ϕ is *p-bounded* if $|\phi^{-1}(n)| \leq p$ for every $n \in \mathbb{Z}_{\geq 0}$. Given a matroid M and a coloring ϕ of M, we say that a subset S of $E(M)$ is *rainbow* with respect to ϕ if $|\phi(S)| = |S|$. The following problem is posed in [1].

Problem 1. *Is there a positive integer μ such that for every 2-base matroid M and every 2-bounded coloring ϕ of M, there is a collection $B_1, \ldots, B_\mu$ of bases of M all of which are rainbow with respect to ϕ and satisfy $\bigcup_{i=1}^{\mu} B_i = E(M)$?*

It is not difficult to see, using matroid intersection, that every element of $E(M)$ is contained in a rainbow basis of M, hence the statement is true if μ is allowed to depend on $|E(M)|$. In [1], it is proven that a number of bases that is logarithmic in $|E(M)|$ suffices and that the statement is true if we replace '2-base matroid' by 'k-base matroid' for some fixed $k \geq 3$.

References

[1] F. Hörsch, T. Kaiser, M. Kriesell, `https://arxiv.org/abs/2206.10322`

Problems

by Naonori Kakimura

Submodular reassignment

Problem 1. *Let $f_1, \dots, f_k : 2^U \to \mathbb{R}$ be k submodular functions. Design an approximation or fixed-parameter algorithm for the following problem:*

$$\min \max_i |X \triangle Y_i| \text{ s.t. } X \subseteq U, Y_i \in \operatorname{argmin} f_i.$$

- The problem is NP-hard by reduction from the closest string problem on binary strings even when each f_i has a unique minimizer.
- The problem of maximizing $\sum_i |X \triangle Y_i|$ instead can be solved in polynomial time by reduction to the maximum flow problem [2].

Set-covering a (common) base

Problem 2. *Design an approximation algorithm or show inapproximability for the following problem: Given a family of subsets $\mathcal{S}$ on the ground set V, find a subfamily $\mathcal{F}$ of $\mathcal{S}$ with minimum size such that $\bigcup_{F \in \mathcal{F}} F$ includes some common base of $\mathcal{M}_1$ and $\mathcal{M}_2$. That is,*

$$\min \ |\mathcal{F}| \ \text{ sub. to } \ \exists B \in \mathcal{B}_1 \cap \mathcal{B}_2, B \subseteq \bigcup_{F \in \mathcal{F}} F.$$

- It is a generalization of the set cover problem and the partial set cover problem that finds a subfamily that covers at least k elements of V.
- The single matroid case (i.e., when $\mathcal{M}_1 = \mathcal{M}_2$) is known as the matroid base cover problem [3]. It also has applications to scheduling problem when $\mathcal{M}$ is a transversal matroid [1]. It is not difficult to see that the problem is a special case of the submodular cover problem $\min |X|$ s.t. $f(X) \geq k$, because $f(X) = r_1(\bigcup_{S \in \mathcal{F}} S)$ is submodular.

References

[1] Erik D. Demaine and Morteza Zadimoghaddam. Scheduling to minimize power consumption using submodular functions. Proceedings of the 22nd Annual ACM Symposium on Parallelism in Algorithms and Architectures (SPAA). 2010.

[2] Kakimura, N., Kamiyama, N., Kobayashi, Y., and Okamoto, Y, Submodular reassignment problem for reallocating agents to tasks with synergy effects. Discrete Optimization, 44, 100631, 2022.

[3] Ildikó Schlotter and Katarína Cechlárová. A connection between sports and matroids: How many teams can we beat? Algorithmica 80 (2018): 258–278.

Opposite vertices of 2-polymatroids

by Tamás Király

The following question was asked by András Frank several years ago, and I am not aware of any significant development since then. A 2-*polymatroid function* on ground set V is an integer-valued monotone increasing submodular set function f with the property that $f(U) \leq 2|U|$ for every $U \subseteq V$. The 2-*polymatroid* defined by f is the integer polyhedron

$$P(f) = \{x \in \mathbb{R}^V : x \geq 0, \sum_{u \in U} x_u \leq f(U) \ \forall U \subseteq V\}.$$

Problem 1. *Is it true that if $P(f)$ is a 2-polymatroid which contains the* $\mathbf{1}$ *(all-ones) vector, then it has a vertex x^* such that $\mathbf{2} - x^*$ is also in $P(f)$?*

The answer is affirmative if f has only even values. In this case, $f/2$ is a matroid rank function, and the matroid union theorem implies that V can be partitioned into a base B and an independent set I. Hence, $x^* = 2\chi^B$ is a vertex of $P(f)$ such that $\mathbf{2} - x^* = 2\chi^I$ is in $P(f)$.

Roots of Ehrhart polynomials of reflexive polytopes

by Max Kölbl

Let P be a d-dimensional lattice polytope (i.e., a polytope which is the convex hull of points in $\mathbb{Z}^d$). Its *lattice point enumerator* is the function

$$E_P(k) = |kP \cap \mathbb{Z}^d|.$$

This function happens to be a polynomial of degree d and is more commonly referred to as the *Ehrhart polynomial* of P.

One popular question in the study of Ehrhart polynomials is to investigate the distribution of their roots. Of particular interest is the case where P is *reflexive*, i.e., where the *dual polytope* [1] of P is also a lattice polytope. In that case, the roots are distributed symmetrically not only across the real line (which follows from the fact that the Ehrhart polynomial has real coefficients), but across the *critical line* $CL := \{z \in \mathbb{C} | \mathfrak{Re}(z) = -1/2\}$ also. Reflexive polytopes whose Ehrhart polynomial roots all lie on the critical line are called *CL-polytopes.* A natural problem that arises is the following.

Problem 1. *Classify all CL-polytopes.*

While there is no hope that this problem will be solved any time soon in full generality, restricting the problem to certain families of polytopes has proven useful. One popular class is that of *symmetric edge polytopes*, which was originaly been defined as a class of graph polytopes but has recently been generalised to regular matroids [1]. Given a regular matroid M with a unimodular representation $U \in \mathbb{Z}^{d\times n}$, its symmetric edge polytope P_M is given by the convex hull of the columns of U and $-U$.

With that in mind, we can ask a more appropriate question.

Problem 2. *Find classes of regular matroids whose symmetric edge polytopes are CL.*

A popular tool for studying CL-ness is the *theory of interlacing polynomials* [2], which was used e.g. in [3].

References

[1] A. D'Alì, M. Juhnke-Kubitzke, M. Koch, On a generalization of symmetric edge polytopes to regular matroids. arXiv preprint arXiv:2307.04933, 2023.

[2] S. Fisk, Polynomials, roots, and interlacing. arXiv preprint math/0612833, 2006.

[3] A. Higashitani, M. Kummer, M. Michałek, Interlacing Ehrhart polynomials of reflexive polytopes. Selecta Mathematica, 23(4):2977–2998, 2017.

[1] The dual polytope of $P \subset \mathbb{R}^d$ is given by $P^\circ := \{x \in (\mathbb{R}^d)^\circ | \forall y \in P\colon \langle x, y\rangle \geq -1\}$ where $(\mathbb{R}^d)^\circ$ is the dual space of $\mathbb{R}^d$.

Selectability in Matroids

by Vasilis Livanos

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid, and $\mathcal{P}(\mathcal{M})$ its corresponding polytope, i.e. the convex hull of the indicator vectors of the bases of $\mathcal{M}$. Let $x \in \mathcal{P}(\mathcal{M})$. Consider the random set $R(x)$ in which each $i \in S$ appears with probability x_i. For $i \in S$, let $R - i$ denote $R(x) \setminus \{i\}$, for brevity.

The problem asks with what probability is an element $i \in S$ not spanned by $R(x) - i$. It is easy to see that no non-trivial lower bound on this probability can be given, for all $i \in S$. Simply consider a uniform matroid of rank 1 with $|S| = 2$ and $x_1 = 1 - \varepsilon, x_2 = \varepsilon$. Therefore, we consider lower bounds that pertain only to some elements in S.

Problem 1. *For every $\mathcal{M}$ and $x \in \mathcal{P}(\mathcal{M})$, there exists an $i \in S$ such that*

$$\Pr[i \notin \operatorname{span}(R(x) - i)] \geq \frac{1}{e}.$$

Along with Kristóf Bérczi and Yutaro Yamaguchi, we have resolved the problem for all matroids of rank at least $\frac{|S|}{2}$, as well as several special cases of matroids (i.e. all uniform matroids). Resolving this problem for all matroids will be a crucial step in the development of optimal online approximation algorithms for matroids [1].

References

[1] Vasilis Livanos. Simple and optimal greedy online contention resolution schemes. In S. Koyejo, S. Mohamed, A. Agarwal, D. Belgrave, K. Cho, and A. Oh, editors, *Advances in Neural Information Processing Systems*, volume 35, pages 9687–9699. Curran Associates, Inc., 2022. URL: `https://proceedings.neurips.cc/paper_files/paper/2022/file/3f45a1768ed452c1706e0a3a272e3bbc-Paper-Conference.pdf`.

QUOTIENTS OF M-CONVEX FUNCTIONS

by Georg Loho

The notion of a *quotient* of two matroids is a fundamental concept in the structure theory of matroids and it forms the building block of flag matroids in the sense of [1].

On one hand, matroids are special M-convex sets, that is, sets of integer points in the base polytope of an integral submodular function, since rank functions of matroids are integral submodular functions. On the other hand, valuated matroids are functions of the form $\phi\colon \binom{[n]}{r} \to \mathbb{R} \cup \{-\infty\}$ such that $\operatorname{argmax}\left\{\phi(z) - c^\top z \mid z \in \binom{[n]}{r}\right\}$ forms the bases of a matroid for each vector $c \in \mathbb{R}^{\binom{[n]}{r}}$. Matroids are those valuated matroids which attain the values zero on the bases and $-\infty$ elsewhere.

The notion of quotient of matroids has recently been generalized to valuated matroids [2] and M-convex sets [3]. M-convex functions in the sense of [4] form a vast generalization of matroids, encompassing valuated matroids and M-convex sets.

Problem 1. *Generalize the notion of quotient to M-convex functions while keeping many cryptomorphic characterizations and desirable properties.*

References

[1] A. V. Borovik, I. M. Gelfand, and N. White. Coxeter matroids, volume 216 of Progr. Math. Boston, MA:Birkhäuser, 2003

[2] M. Brandt, C. Eur, and L. Zhang. Tropical flag varieties. Adv. Math., 384:#107695, 41, 2021

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Simultaneous independent sets in matroids

by Péter Madarasi

Let $S_1, \ldots, S_k$ be finite disjoint sets, and let us be given a matroid M_{ij} on the groundset $S_i \cup S_j$ for all i, j such that $1 \leq i < j \leq k$. Our goal is to find a maximum-size $F \subseteq S_1 \cup \cdots \cup S_k$ such that F is independent in every M_{ij}.

For $k = 2$, the problem asks for a maximum-size independent set in the matroid M_{12} defined on the groundset $S_1 \cup S_2$.

For $k = 3$, both the matching and matroid intersection problems are special cases. It turns out that the case $k = 3$ reduces to the matroid matching problem, and it is polynomial-time solvable provided that the matroids are linear [1, 2]. However, no combinatorial algorithm or nice min-max formula is known, not even in the graphic case.

Problem 1. *Give a combinatorial algorithm and a min-max formula for graphic matroids when $k = 3$.*

The complexity of the non-linear case is open for $k = 3$, and even the linear case is unknown for $k \geq 4$.

Problem 2. *Is the problem solvable in polynomial time for arbitrary matroids when $k = 3$?*

Problem 3. *Is the problem solvable in polynomial time (for graphic matroids) when $k \geq 4$?*

Another natural generalization of the case $k = 3$ to larger k's is when, for every $i \in \{1, \ldots, k\}$, at most two of the matroids are non-free whose groundset includes S_i. In this case, the problem reduces to the matroid matching problem for all k, but no combinatorial algorithm is known.

References

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[2] T. A. Jenkyns, *Matchoids: a Generalization of Matchings and Matroids.* Ph.D. Thesis, University of Waterloo, 1974.

Group-labeled Matroid Problems

by Taihei Oki

Group-labeled combinatorial optimization problems are considered for the shortest path problem in, e.g., [1]. We here consider the group-labeled variants for matroidal problems.

Let Γ be an abelian group, where the operation is denoted in additive notation. A *group-labeled matroid* is a matroid $\mathbf{M}$ whose ground set E is endowed with a group labeling $\psi : E \to \Gamma$. The label of a set $X \subseteq E$ is defined as the sum of labels of the elements in X, i.e., $\psi(X) \coloneqq \sum_{e \in X} \psi(e)$. *Zero* and *non-zero* bases of $\mathbf{M}$ are bases whose labels are the zero element $0_\Gamma \in \Gamma$ and a non-zero element, respectively.

The most basic problems on group-labled matroids are the following:

- Zero Base Problem: given a group-labeled matroid, find its zero base (if it exists).
- Non-Zero Base Problem: given a group-labeled matroid, find its non-zero base (if it exists).

The target group element 0_Γ in the (Non-)Zero Base Problem can be replaced with any other element $g \in \Gamma$ by considering the (Non-)Zero Base Problem for the direct sum of $\mathbf{M}$ and the free matroid over the singleton of a new element having label $-g$.

Non-Zero Base Problem is indeed tractable:

Theorem 1. Non-Zero Base Problem *is polynomial-time solvable.*

Proof (sketch). Take any base B of $\mathbf{M}$. If B is a non-zero base, then we are done. Suppose not. We exhaustively search for $e \in E \setminus B$ and $f \in C(B, e)$[2] such that $B + e - f$ is a zero base. If such e and f are found, output $B + e - f$; otherwise, we can show that $\mathbf{M}$ has no non-zero base in the following way. Let $e \in E \setminus B$. Since $0_\Gamma = \psi(B + e - f) = \psi(B) + \psi(e) - \psi(f) = \psi(e) - \psi(f)$ for all $f \in C(B, e)$, all elements in the fundamental circuit $C(B, e)$ have the same label $\psi(e)$. Thus, all elements in a connected component of $\mathbf{M}$ has the same label. Since any base of $\mathbf{M}$ is the disjoint union of bases of the connected components, all bases have the label $\psi(B)$. $\square$

There are several natural generalizations of Non-zero Base Problem:

- Weighted Non-zero Base Problem: given a group-labeled matroid equipped with a weight function $w : E \to \mathbb{R}$, find a minimum-weight non-zero base.
- Non-zero Common Base Problem: given two group-labeled matroids with the same ground set and the same group labeling, find a non-zero base of both matroids.
- Weighted Non-zero Common Base Problem, Non-Zero Matroid Parity Problem, ...

Problem 1. *Are these problems polynomial-time solvable? If not, what are tractable special cases?*

Zero Base Problem is hard in general:

Theorem 2. Zero Base Problem *is NP-hard.*

[2] The fundamental circuit of e in B, i.e., the unique circuit in $B \cup \{e\}$.

Proof (sketch). Reduction from the subset sum problem to ZERO BASE PROBLEM for the uniform matroid with $\Gamma = \mathbb{Z}$. $\square$

If $\Gamma = \mathbb{Z}_2$, ZERO and NON-ZERO BASE PROBLEMS are the same, thus polynomial-time solvable by Theorem 1.

Problem 2. *Is* ZERO BASE PROBLEM *polynomial-time solvable if* Γ *is a fixed finite abelian groups?*

This is true for regular matroids by algebraic technique (details omitted). More generally, ZERO COMMON BASE PROBLEM and ZERO PARITY BASE PROBLEM for *Pfaffian pairs* [3] (including arborescences) and *Pfaffian parities* [2] are tractable if Γ is a fixed finite abelian group. How about other matroids with $\Gamma = \mathbb{Z}_3$?

References

[1] Y. Iwata and Y. Yamaguchi. Finding a shortest non-zero path in group-labeled graphs. *Combinatorica*, 42:1253–1282, 2022.

[2] K. Matoya and T. Oki. Pfaffian pairs and parities: counting on linear matroid intersection and parity problems. *SIAM J. Discrete Math.*, 36(3):2121–2158, 2022.

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Integrality gap for the common intersection of multiple matroids

by Neil Olver

Let $M_1, M_2, \ldots, M_k$ be k matroids on the same groundset E. Let P_i be the independence polytope of matroid M_i. Consider the problem of, given a weight vector $w : E \to \mathbb{R}_{\geq 0}$, finding a maximum weight set that is independent in all k matroids. We can phrase this as an integer program:

$$\begin{array}{lll} \max & w^T x & \\ \text{s.t.} & x \in P_i & i = 1, 2, \ldots, k \\ & x \in \{0,1\}^E & \end{array}$$

Problem 1. *What is the integrality gap of the LP relaxation obtained by dropping the integrality constraints?*

The conjecture is that it is $k - 1$. For $k = 2$, this is of course true — this simply says that the intersection of two matroid polytopes is integral. For $k = 3$, it also holds [1]; the proof is via an iterative rounding/relaxation approach, also with an invocation of matroid intersection. What about $k > 3$?

A proof of this conjecture would likely yield a $(k-1)$-approximation for this problem. Using local search methods rather than LP-based methods, a $(k - 1 + \epsilon)$-approximation is known, for any $\epsilon > 0$ [2]. This has no implication for the integrality gap, however.

References

[1] A. Linhares, N. Olver, C. Swamy and R. Zenklusen. Approximate multi-matroid intersection via iterative refinement. Mathematical Programming **183**:397–418, 2020.

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Symmetric edge polytopes: degree of the γ-polynomial

by Lilla Tóthmérész

Let $G = (V, E)$ be an undirected graph. Its *symmetric edge polytope* is the following:

$$P_G = conv\{\mathbf{1}_u - \mathbf{1}_v, \mathbf{1}_v - \mathbf{1}_u \mid uv \in E\} \subset \mathbb{R}^V$$

.

This is a nice polytope that is quite actively investigated recently. It is known to be reflexive, hence its Ehrhart h^*-polynomial is symmetric. Thus, one can write the h^*-polynomial in symmetric basis, and obtain the γ-polynomial.

Suppose now that G is bipartite. In this case there is another polynomial associated to G: the interior polynomial. With Tamás Kálmán, we noticed that whenever we computed these two polynomials, their degrees agreed.

Problem 1. *For an undirected graph is it true that the degree of the γ-polynomial of the symmetric edge polytope always agrees with the degree of the interior polynomial?*

γ-positivity of (non-symmetric) edge polytopes

by Lilla Tóthmérész

Let $D = (V, A)$ be a directed graph. Its *edge polytope* is the following:

$$\mathcal{Q}_G = conv\{\mathbf{1}_v - \mathbf{1}_u \mid \overrightarrow{uv} \in A\} \subset \mathbb{R}^V$$

.

A particularly nice (and actively researched) special case is if D is obtained from an undirected graph by substituting each (undirected) edge with two oppositely directed edges. In that case, the polytope is called the *symmetric edge polytope*-

The edge polytope is reflexive if and only if D is strongly connected (in particular, in the symmetric case), hence its Ehrhart h^*-polynomial is symmetric. Thus, one can write the h^*-polynomial in symmetric basis, and obtain the γ-polynomial.

Ohsugi and Tsuchiya conjectures that for a symmetric edge polytope, the coefficients of the γ-polynomial are nonnegative.

For general strongly connected digraphs, γ-positivity does not hold.

Problem 1. *Identify classes of strongly connected digraphs where there is or there isn't γ-positivity.*

Diameter reduction by edge reversals

by Lilla Tóthmérész

Suppose that we have a directed graph $D = (V, A)$. It is known that if we want to make D strongly connected by reversing some of its edges, then we need to reverse at least $\nu(D)$ edges, where

$$\nu(D) = \min |\{F \subset A \mid \text{ F intersects each directed cut }\}|,$$

and this quantity is computable in polynomial time.

We can also think of making a directed graph strongly connected, as decreasing its (directed) diameter from ∞ to at most —V—.

With this philosophy, we can ask:

Problem 1. *What is the minimum number of edges we need to reverse in a directed graph D to obtain a graph with diameter at most k?*