11/12th Emléktábla Workshop

Combinatorics and geometry / Matroid optimization 07.04 - 07.08.2022.

Preliminary Schedule for 11^{th} group:

Monday

Arrival in the hotel from 14:00.
16:30 Short introduction/presentation of problems
18:00 Dinner

Tuesday-Thursday

9:30-17:00 Working in groups of 3-5
12:30 Lunch (except Wednesday)
17:00 Presentations of daily progress
18:00 (or whenever we are done) Dinner

Friday

Check-out at 10:00.

List of Participants

Eyal Ackerman, University of Haifa at Oranim
Péter Ágoston, ELTE
Gergely Ambrus, Rényi Institute
Martin Balko, Charles Univerity, Prague
János Barát, University of Pannónia
Zoltán Blázsik, Rényi Institute
Gábor Damásdi, ELTE
Nóra Frankl, Rényi Institute
Rado Fulek, UCSD
Dániel Gerbner, Rényi Institute
Eric Gottlieb, Rhodes College, Memphis
Anna Halfpap, University of Montana
Attila Jung, ELTE
Balázs Keszegh, Rényi Institute
Nathan Lemons, Los Alamos
Viola Mészáros, Berlin
Padmini Mukkamala, BME
Dániel Nagy, Rényi Institute
Zoltán Lóránt Nagy, ELTE
Márton Naszódi, ELTE
Cory Palmer, University of Montana
Dömötör Pálvölgyi, ELTE-MTA
Rom Pinchasi, Technion
Géza Tóth, Rényi Institute
Russ Woodroofe, University of Primorska
Erika Bérczi-Kovács, ELTE
Kristóf Bérczi, ELTE-MTA
Attila Bernáth, Lufthansa
Gergely Csáji, ELTE
Tamás Fleiner, BUTE
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Dániel Garamvölgyi, ELTE
Zsuzsanna Jankó, Budapest Corvinus University
Attila Joó, Hamburg University
Tibor Jordán, ELTE
Alpár Jüttner, ELTE
Naonori Kakimura, Keio University
Csaba Király, ELTE
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Zoltán Király, ELTE
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Péter Madarasi, ELTE
Jannik Matuschke, KU Leuven
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Gyula Pap, ELTE
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Tamás Schwarcz, ELTE
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Eszter Szabó, ELTE
Lilla Tóthmérész, ELTE
Kitti Varga, Rényi Institute
Yutaro Yamaguchi, Osaka University
Yu Yokoi, NII
Ran Ziv, Tel-Hai College

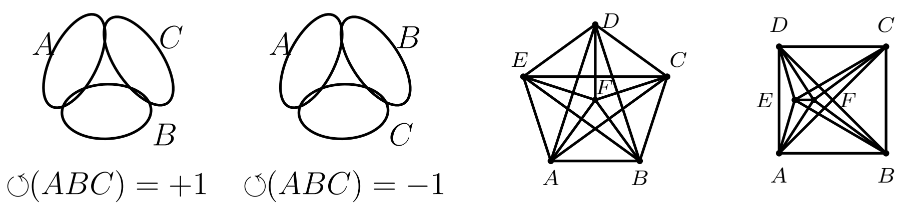
Six-point order types as the orientations of families of convex sets

by Péter Ágoston

Definition 1. *Take a set of elements and assign a number* $\circlearrowleft(ABC) \in \{-1, 1\}$ *to all triples* ABC*, such that* $\circlearrowleft(ABC) = \circlearrowleft(BCA) = \circlearrowleft(CAB) = -\circlearrowleft(ACB) = -\circlearrowleft(CBA) = -\circlearrowleft(BAC)$*. We call such an assignment a total orientation.*

Definition 2. *Call a planar family of pairwise intersecting compact convex sets without any* 3*-intersections a holey family.*

Definition 3. *If* A*,* B *and* C *are members of a holey family, then denote the only finite component of* $\mathbb{R}^2 \setminus (A \cup B \cup C)$ *by* $\curlywedge(ABC)$ *(we call this the hollow of* A*,* B *and* C*).*



(a) Orientations (b) The 6-point order types in question

Definition 4. *Let a C-T3O be a total orientation on a holey family, where* $\circlearrowleft(ABC)$ *is roughly defined as in Figure (a). To be more precise, the orientation depends on what order* A*,* B *and* C *occur on the border of* $\curlywedge(ABC)$*. For more details, see [1].*

Problem 1. *Can the orientation corresponding to the order types seen in Figure (b) be realized as a C-T3O? (All other* 6*-point order types can be realized.) In case this problem proves to be too easy, we can ask similar questions for other types of orientations (for details, see [2]). The question here is the minimum number of points in an order type that does not correspond to such an orientation.*

References

[1] P. Ágoston, G. Damásdi, B. Keszegh, and D. Pálvölgyi. Orientation of convex sets. Preprint, `https://arxiv.org/abs/2206.01721`.

[2] P. Ágoston, G. Damásdi, B. Keszegh, and D. Pálvölgyi. Orientation of good covers. Preprint, `https://arxiv.org/abs/2206.01723`.

Translative coverings via the colourful Bang lemma

by Gergely Ambrus

We say that the convex sets $C_1, \ldots, C_n \subset \mathbb{R}^d$ *permit a translative covering* of a convex body $K \subset \mathbb{R}^d$ if

$$K \subset \bigcup_{i=1}^{n} (C_i + x_i)$$

for some $x_1, \ldots, x_n \in \mathbb{R}^d$. Given K and a fixed convex body C, it is natural to search for a family of homothets of C as "small" as possible which permit a translative covering of K. The following result is proven in [1].

Theorem 1. *Assume that $T \subset \mathbb{R}^d$ is a non-degenerate simplex, and $\lambda_1, \ldots, \lambda_n \geq 0$ are so that the family $-\lambda_1 T, \ldots, -\lambda_n T$ permits a translative covering of T. Then*

$$\sum_{i=1}^{n} \lambda_i \geq d.$$

The proof is based on a generalized, colourful version of Bang's lemma [1]:

Theorem 2. *Assume that all the finite vector sets $U_1, \ldots, U_n \subset \mathbb{R}^d$ contain the origin in their convex hull. Then for any set of vectors $x_1, \ldots, x_n \in \mathbb{R}^d$ we may select $u_i \in U_i$ for each $i \in [n]$ so that setting $u = \sum_i u_i$,*

$$\langle u - x_k, u_k \rangle \geq |u_k|^2$$

holds for every k.

A related conjecture is due to V. Soltan:

Problem 1 (V. Soltan)**.** *Assume that $K \in \mathbb{R}^d$ is a convex body and that $\lambda_1 K, \ldots, \lambda_n K$ permit a translative covering of K with $\lambda_i \in (0, 1)$ for every i. Then*

$$\sum_{i=1}^{n} \lambda_i \geq d.$$

One may try to tackle special cases of the generalization when covering a convex body K with homothetic copies of another convex body L (as in Theorem 1).

The following (probably very hard) conjecture is a generalization of the affine plank problem.

Problem 2. *Assume that the closed, convex sets $C_1, \ldots, C_n \subset \mathbb{R}^d$ permit a translative covering of the convex body $B \subset \mathbb{R}^d$. Then*

$$\sum_{i=1}^{n} r_B(C_i) \geq 1$$

holds, where $r_B(C)$ is the scaling factor of the largest homothet of B contained in C.

The minimal density of a covering of the whole space $\mathbb{R}^d$ with translates of K is called the *translative covering density of K*. The following theorem was proven by Januszewski:

Theorem 3 (Januszewski [2])**.** *The translative covering density of a triangle in the plane is $\frac{3}{2}$.*

His proof is quite technical. It is natural to expect that a simpler proof may be given using the colourful Bang lemma.

Problem 3. *Find a simpler way of determining the translative covering density of the triangle. Can we extend the result to higher dimensional simplices, or other convex discs?*

Plenty of further references are listed in [1].

References

[1] G. Ambrus, *A generalization of Bang's lemma.* `https://arxiv.org/abs/2201.08823`, 2022.

[2] J. Januszewski, *Covering the Plane with Translates of a Triangle*. Discrete & Computational Geometry **43**(1):167–178, 2010.

Saturated 2-planar abstract graphs

by János Barát

Let n denote the number of vertices of a graph. In a drawing of a graph in the plane, vertices are represented by points, edges are represented by curves connecting the points, which correspond to adjacent vertices. A drawing is *simple* if any two edges have at most one point in common, which is either a common endpoint or a crossing.

For any $k \geq 0$, an abstract graph G is *k-planar* if it has a simple drawing in the plane, where each edge contains at most k crossings. A k-planar graph G is *saturated* if adding any edge to G results in a non-k-planar graph.

For saturated 1-planar graphs, the following is known. Brandenburg et al. gave a construction of a family of saturated 1-planar graphs with $\approx 2.647n$ edges [3]. On the other hand, Barát and Tóth proved that any saturated 1-planar abstract graph must have at least $\approx 2.22n$ edges [2].

Problem 1. *How many edges can a saturated 2-planar abstract graph have?*

It is easy to see that such a graph must have at least $n-1$ edges. We believe that this lower bound should be improved.

Auer et al. presented a family of saturated 2-planar graphs with $\approx 2.63n$ edges without proof [1].

References

[1] C. Auer, F. J. Brandenburg, A. Gleissner, K. Hanauer: On Sparse Maximal 2-Planar Graphs, Graph Drawing 2012, Lecture Notes in Computer Science **7704** (2013), 555–556.

[2] J.Barát, G. Tóth. Improvements on the density of maximal 1-planar graphs, J. Graph Theory **88** (2018), 101–109.

[3] F. J. Brandenburg, D. Eppstein, A. Gleissner, M. T. Goodrich, K. Hanauer, J. Reislhuber: On the density of maximal 1-planar graphs, In: International Symposium on Graph Drawing 2012 (pp. 327–338). Springer, Berlin, Heidelberg.

Problems around the Blocking conjecture

by Martin Balko

Let P be a finite set of points in the plane with no three points lying on a common line. A *visibility-blocking set for* P is a set of points Q that is disjoint from P and such that every line segment between two points from P contains at least one point of Q. Let $b(P)$ be the smallest possible size of a visibility-blocking set for P and let $b(n) = \min_P b(P)$, where the minimum is taken over all sets of n points in the plane with no three points lying on a common line; see Figure 1 for an illustration.

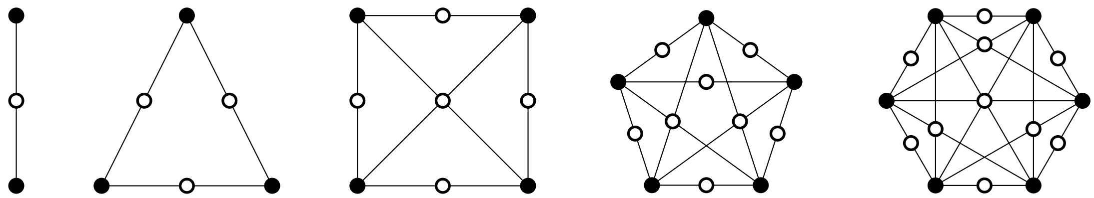


Figure 1: Examples showing the upper bounds $b(2) \leq 1$, $b(3) \leq 3$, $b(4) \leq 5$, $b(5) \leq 8$, and $b(5) \leq 10$. All these bounds are tight [2]. The points from P are black and the points from Q are white.

Problem 1 (The Blocking conjecture [5]). *We have $b(n)/n \to \infty$ as $n \to \infty$.*

In fact, Pinchasi [4] conjectured $b(n) \in \Omega(n \log n)$. There are linear lower bounds on $b(n)$ [1, 2] and it is known that there is a contant c such that $b(n) \leq ne^{c\sqrt{\log n}}$ [2]. If P is a set of n points in convex position, then $b(P) \geq \Omega(n \log n)$ [2].

Problem 2. *Can we improve the bounds on $b(n)$?*

There are numerous open questions around the Blocking conjecture that might be potentially easier to solve. For example, Matoušek suggested to deal with *pseudosegments* instead of straight segments. That is, for a given point set P, we want to construct an arrangement $\mathcal{A}$ of pseudolines and a subset Q of its vertices such that $P \cap Q = \emptyset$, each $p \in P$ is a vertex of $\mathcal{A}$, no three points of P lie on a pseudoline, and any two points of P lie on a common pseudoline $\ell \in \mathcal{A}$ and have a point of Q on the segment of ℓ between them. Then, the lower bound $\Omega(n \log n)$ for point sets in convex position still applies and it is easy to provide here an $O(n \log n)$ upper bound.

Problem 3 ([2]). *Is there a linear upper bound for some P, not in "convex position" in the pseudoline setting?*

Some related blocking-type questions can be found here [3].

References

[1] A. Dumitrescu, J. Pach, and G. Tóth, A note on blocking visibility between points. *Geombinatorics* 19 (2009), no. 2, 67–73.

[2] J. Matoušek, Blocking visibility for points in general position. *Discrete Comput. Geom.* 42 (2009), no. 2, 219–223.

[3] J. Pach, R. Pinchasi, and M. Sharir, On the number of directions determined by a three-dimensional points set. *J. Combin. Theory Ser. A* 108,1(2004) 1–16.

[4] R. Pinchasi, On some unrelated problems about planar arrangements of lines. In *Workshop II: Combinatorial Geometry. Combinatorics: Methods and Applications in Mathematics and Computer Science*. Institute for Pure and Applied Mathematics, UCLA, 2009. `http://11011110.livejournal.com/184816.html`.

[5] A. Pór and D. Wood, On visibility and blockers. *J. Comput. Geom.* 1 (2010), no. 1, 29–40.

Pairwise disjoint perfect matchings

by Zoltán L. Blázsik

Very recently, Mattiolo and Steffen together with Ma and Wolf (in [1, 2]) disproved the conjecture of Thomassen from 2020 for all even values of r, which stated that every r-edge-connected r-regular graph of even order has $r-2$ pairwise disjoint perfect matchings.

For $r \geq 2$, an r-regular graph G is *class 1*, if it has a set of r pairwise disjoint perfect matchings of G. Otherwise it is *class 2*. In [1], one can read the known background of this problem and at the last section they focus on the $r = 5$ case. The following question surprisingly seems to be unsolved.

Problem 1. *Is there any* 5*-edge-connected* 5*-regular class 2 graph?*

For planar graphs, the answer to the above question is "no". Guenin proved that all planar 5-graphs are class 1. Indeed, it is conjectured by Seymour that every planar r-graph is class 1. (so far this conjecture is proved to be true for all $r \leq 8$)

References

[1] Y.Ma, D.Mattiolo, E.Steffen, I.H.Wolf, Pairwise disjoint perfect matchings in r-edge-connected r-regular graphs. arXiv version (2022 June), `https://arxiv.org/pdf/2206.10975.pdf`

[2] D.Mattiolo, E.Steffen, Highly edge-connected regular graphs without large factorizable subgraphs. *J. Graph Theory*, **99**, (2022), 107–116.

Touchings in intersecting pseudocircles

by Gábor Damásdi

An *intersecting arrangement of pseudocircles* is a collection of simple closed curves on the sphere or plane such that any two of the curves either touch in a single point or intersect in exactly two points where they cross.

Problem 1 (Grünbaum). *Every arrangement of n intersecting pseudocircles has at most $2n-2$ touchings.*

Felsner, Roch and Scheucher [2] showed that the conjecture holds for any arrangement, where a triple of pseudocircles is pairwise touching.

References

[1] B. Grünbaum. Arrangements and Spreads, volume 10 of CBMS Regional Conference Series in Mathematics. AMS, 1972.

[2] S. Felsner, S. Roch and M.Scheucher. Arrangements of Pseudocircles: On Digons and Triangles, Eurocg22, 2022

Limit of non-collinear point sets

by Dömötör 2

The following conjecture has been posed recently by Joshua Erde (Graz).

Conjecture 1 (Joshua Erde). *Suppose that $S \subset \mathbb{Z}^2$ is in general position, i.e., no three points of S are on a line.*

$$\liminf \frac{|S \cap \{1,2,\ldots,n\}^2|}{n} = 0.$$

Giving bounds on the growth rate of $\frac{|S \cap \{1,2,\ldots,n\}^2|}{n}$ would also be interesting.

Centrally symmetric crossing numbers

by Nóra Frankl

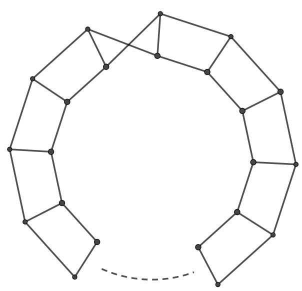
The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a drawing of G in the plane. A drawing is called *rectilinear* if the edges are represented by straight line segments. The *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ is the minimum number of pairwise crossings of edges in a a rectilinear drawing of G. Determining the crossing and rectilinear crossing numbers of K_n and $K_{m,n}$ are difficult open problems. While it is conjectured that

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor,$$

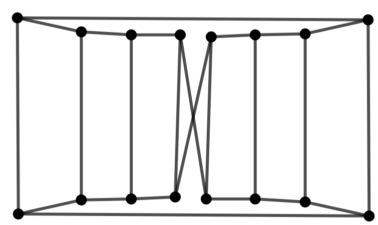
there is no conjectured value for $\overline{\mathrm{cr}}(K_n)$, and the current best known bounds are

$$0.37997\binom{n}{4} \leq \overline{\mathrm{cr}}(K_n) \leq 0.38047\binom{n}{4}.$$

The *rectilinear centrally symmetric crossing number* $\overline{\mathrm{cr}}_{cs}(G)$ is defined as the mininum number of crossings over all centrally symmetric rectilinear drawings of G. It was recently introduced in [1], where they also determined it for K_{2n} exactly: $\overline{\mathrm{cr}}_{cs}(K_{2n}) = 2\binom{n}{4} + \binom{n}{2}^2$. The *centrally symmetric crossing number* $\mathrm{cr}_{cs}(G)$ is defined similarly.



(a) Möbius ladder M_{2n}



(b) Centrally symmetric drawing of M_{16}

The Möbius ladder M_{2n} on $2n$ vertices is the graph shown in Figure (a). $\mathrm{cr}(M_{2n}) = 1$ for any n, and it is easy to draw M_{4n} in a centrally symmetric position with 1 crossing (see Figure (b)), thus $\mathrm{cr}_{cs}(M_{4n}) = 1$. The following problems are from [2].

Problem 1. *Determine* $cr_{cs}(M_{4n+2})$ *and* $\overline{cr_{cs}}(M_{4n+2})$.

Problem 2. *Find good bounds for* $\overline{cr}_{cs}(K_{m,n})$ *and* $cr_{cs}(K_{2n})$.

Note that if G has an odd number of vertices, then one of its vertices has to be in the centre. If in the drawing no edge is allowed to contain a vertex other than its endpoints, certain graphs, such as K_3, cannot be drawn in a centrally symmetric position.

References

[1] B. M. Abrego, J. Dandurand, and S. Fernandez-Merchant. *The crossing number of centrally symmetric complete geometric graphs*. In: Procedia Computer Science 195 (2021). Proceedings of the XI Latin and American Algorithms, Graphs and Optimization Symposium., pp. 275-279

[2] M. Schaefer, *The graph crossing number and its variants: A survey*, Electronic Journal of Combinatorics. Dynamic Survey

Testing Disjointedness of Polygonal Paths

by Radoslav Fulek

For $\varepsilon > 0$ and a polygonal line in the Euclidean plane, that is, a piece-wise linear continuous map $\varphi : [0, 1] \to \mathbb{R}^2$, an *$\varepsilon$-perturbation* of φ is a map $\psi : [0, 1] \to \mathbb{R}^2$ such that $\|\psi - \varphi\| < \varepsilon$, where $\|.\|$ is the supremum norm.

Problem 1. *Does there exist a polynomial time algorithm to decide whether a given pair of polygonal lines can be made disjoint by an arbitrarily small perturbation, or is this problem NP-complete?*

A variant of the problem of finding the smallest ε such that there exists an ε perturbation making a pair of given trajectories disjoint was studied in the field of topology [1, Section 3.2], [4], see also [3, Example 3.3]. Less formally, seeing the trajectories against the background of a road network [2], we seek perturbations of the given trajectories staying on the road.

Problem 1 can be considered as a first step towards a thorough understanding of the problem for general graphs and with metric constraints, which is largely unexplored to the best of our knowledge. It turned out [3, Section 3] that commonly used topological machinery fails to provide an efficient algorithm to decide if a given pair of trajectories can be made disjoint by an arbitrarily small perturbation.

If a polynomial time algorithm in Problem 1 is found, a possible next step is to consider the following problem.

Problem 2. *Does there exist a polynomial time algorithm to decide whether a given pair of closed polygonal lines, that is, polygons, can be made disjoint by an arbitrarily small perturbation, or is this problem NP-complete?*

References

[1] Akhmetiev, PM and Repovš, D and Skopenkov, AB. Obstructions to approximating maps of n-manifolds into R^{2n} by embeddings. Topology and its Applications, 123, 1, 3-14, 2002.

[2] Güting, Ralf Hartmut and De Almeida, Victor Teixeira and Ding, Zhiming. Modeling and querying moving objects in networks. The VLDB Journal, 15, 2, 165–190, 2006.

[3] Skopenkov, Mikhail. On approximability by embeddings of cycles in the plane. Topology and its Applications, 134, 1,–22, 2003

[4] Spież, S and Toruńczyk. H. Moving compacta in R^m apart. Topology and its Application, 41, 3, 193–204, 1991.

Helly number of an integer set

by Attila Jung

For an $S \subset \mathbb{R}^2$, let $H(S)$ be the smallest number such that the following is true for every finite family $\mathcal{C}$ of convex sets from $\mathbb{R}^2$. If the intersection of any $H(S)$ members of $\mathcal{C}$ contains a point from S, then the intersection of all the members from $\mathcal{C}$ contains a point from S. If there is no such number, let $H(S) = \infty$. Dillon posed the following question in [1].

Problem 1. *What is $H(\{2^n : n \in \mathbb{N}\}^2)$?*

As a special case of Doignon's Theorem [2], we know that $H(\mathbb{Z}^2) = 4$. In [1], it is proved, that $H(\{p(n) : n \in \mathbb{N}\}^2) = \infty$ if p is a polinomial of degree at least 2.

References

[1] T. Dillon. Discrete Helly-type Theorems with boxes. Adv. Appl. Math., 129 (2021), https://doi.org/ 10 .1016/j.aam.2021.102217, article 102217.

[2] J.-P. Doignon. Convexity in cristallographical lattices. J. Geom., 3 (1973), pp.71-85.

Proper coloring directed hypergraphs

by Balázs Keszegh

A directed hypergraph is a hypergraph in which the vertex set of each hyperedge is partitioned into two parts, the head-vertices and the tail-vertices of the hyperedge. The following is conjectured by B. Keszegh and D. Pálvölgyi:

Conjecture 1. *Suppose that in a directed hypergraph $\mathcal{H}$ in every hyperedge the number of head-vertices is less than the number of tail-vertices, and for every pair of hyperedges $H_1, H_2 \in \mathcal{H}$, if $H_1 \cap H_2 = \{v\}$, then v is a head-vertex in at least one of the hyperedges. Then $\mathcal{H}$ admits a proper 2-coloring.*

If v is required to be a head-vertex in both hyperedges, then the conjecture is true. Also, it is true for graphs, 3-uniform hypergraphs and linear hypergraphs [1]. What about hypergraphs that have hyperedges of size 2 and 3? What about 4-uniform hypergraphs? Also, in general it might be the case that the conjecture is not true.

References

[1] B. Keszegh, Coloring directed hypergraphs, `https://arxiv.org/abs/2205.11271`

Unavoidable intersections of given size

by Zoltán Lóránt Nagy

Let $G(n, e)$ denote a graph on n vertices and e edges. Erdős, Füredi, Rothschild and T. Sós initiated the investigation of the following problem [1]. Fix a positive integer m and a pair of integers (n, e), such that $0 \leq e \leq \binom{n}{2}$. For which f does it hold that any n-vertex subgraph with e edges contain an induced subgraph on m vertices having f edges? Equivalently, we are seeking pairs (m, f) such that m-vertex subgraphs with f edges are unavoidable in graphs of form $G(n, e)$.

We propose a geometric (or q-analogue, if you wish) variant of this problem.

Notation 1. $\mathrm{AG(n,q)}$ *denotes the affine geometry of dimension n over the q-element field.*

Definition 2. *Let $S(m) \subseteq \mathrm{AG(n,2)}$ denote a point set of cardinality m in the affine geometry $\mathrm{AG(n,2)}$, or an m-set in brief. We say that a k-dimensional t-set is* ***unavoidable*** *in m-sets if for every $S(m)$ $\exists$ an affine subgeometry $H \subseteq \mathrm{AG(n,2)}$ s.t. $H \sim \mathrm{AG(k,2)}$ and $|H \cap S(m)| = t$.*
The property that k-dimensional t-set is unavoidable is denoted by $[n, m] \to [k, t]$.

Problem 1. *For fixed $[k, t]$, determine the density of the unavoidability for k-dim t-sets as follows:*

$$\rho_n(k,t) := \frac{|\{m : [n,m] \to [k,t]\}|}{2^n + 1} \quad \textit{or its limit} \quad \rho(k,t) := \lim_{n\to\infty} \rho_n(k,t).$$

Note that if we investigate the case corresponding to $\mathrm{AG(n,3)}$, then avoiding $[1, 3]$ (i.e., a full line) corresponds to the famous *cap-set problem*.

Some preliminary results:

Proposition 3. .

- $\rho_n(k,t) = \rho_n(k, 2^n - t)$.
- $\rho_n(1,0) = \rho_n(1,1) = \frac{2^n-1}{2^n+1}$.
- $\rho(k,0) = 1$ *moreover* $[n,m] \to [k,0]$ *for every* $m < 2^n - 2 \cdot 2^{(1-2^{-k+1})n}$.

References

[1] Erdős, Füredi, Rothschild and T. Sós (1999). Induced subgraphs of given sizes. Discrete mathematics, 200(1-3), 61-77.

Counting maximal independent sets

by Cory Palmer

Recall that a vertex set in a (hyper)graph is *independent* if it contains no edge. An independent set is *maximal* if it is not a proper subset of a larger independent set. Let $\text{mis}(G)$ denote the number of maximal independent sets (MIS) in a graph G. Miller and Muller and independently Moon and Moser showed that for all n-vertex graphs G

$$\text{mis}(G) \leq 3^{n/3}$$

which is sharp as given by the vertex-disjoint union of triangles. When triangles are forbidden from G, Hujter and Tuza [2] showed

$$\text{mis}(G) \leq 2^{n/2}$$

which is achievable by a matching. If we allow at most t vertex-disjoint triangles, then Palmer and Patkós showed that the best is (roughly) to take t vertex-disjoint triangles and a matching on the remaining vertices.

There are several natural generalizations of these problems to hypergraphs, especially 3-graphs.

Problem 1. *Determine the maximum number of MIS in an n-vertex 3-graph.*

Taking vertex-disjoint copies of K_5^3 beats copies of K_4^3, so perhaps that has the maximum number of MIS. Copies of K_5^3 (given by Tomescu) gives a lower bound of about 1.5849^n and Lonca and Truszczyński [3] gave an upper bound of about 1.6702^n. We can also forbid 3-graphs á la Hujter–Tuza:

Problem 2. *Determine the maximum number of MIS in an n-vertex K_4^3-free 3-graph.*

There are also several generalizations in the graph setting. Nielsen [4] showed that the maximum number of MIS of size k in an n-vertex graph is asymptotic to $(n/k)^k$. He, Nie and Spiro [1] examined the question when G is taken to be K_t-free. Among others they constructed an n-vertex triangle-free graph with $\Omega(n^{k/2})$ MIS of size $k \geq 4$ and asked for a matching upper-bound:

Problem 3. *Show that maximum number of MIS of size $k \geq 4$ in a triangle-free graph is $O(n^{k/2})$.*

One more (less natural) question that seems to be very closely related to Problem 2.

Problem 4. *Determine the maximum number of maximal triangle-free sets in an n-vertex K_4-free graph.*

References

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Variations on the thrackle conjecture

by Dömötör Pálvölgyi

The following variants of the famous thrackle conjecture are from my very recent joint papers with Ágoston, Damásdi and Keszegh [1, 2].

Conjecture 1 ([1])**.** *Suppose that we are given n points in the plane, $\mathcal{P}$, and m subsets $S_1, \ldots, S_m \subset P$, with convex hulls $C_i = conv(S_i)$, such that the following hold:*

- $1 < |S_i| < n$ *for any* i;
- $C_i \cap C_j \neq \emptyset$ *for any* $i \neq j$;
- $C_i \cap C_j \cap C_k \subset \mathcal{P}$ *for any* $i \neq j \neq k \neq i$.

We conjecture that $m \leq n$.

Note that if $|S_i| = 2$ for every i, then we get back a linear thrackle (in this case we can get rid of the condition that $C_i \cap C_j \cap C_k \subset \mathcal{P}$), in which case it is known that $m \leq n$. This also shows that the conjecture would be sharp for any n. An interesting, and probably easier, special case is when $\mathcal{P}$ is in convex position.

We can strengthen the original thrackle conjecture in a seemingly different direction, but in fact the below variant is also related to Conjecture 1 (see [2] for the connection).

Conjecture 2 ([2])**.** *Suppose that we are given in the plane n points, $\mathcal{P}$, and m topological trees that pairwise intersect exactly once such that each leaf of each tree is from $\mathcal{P}$.*
We conjecture that $m \leq n$.

Note that if each tree consists of a single edge, then we get back the original thrackle conjecture. Instead of trees, our conjecture might even hold for forests. We could also weaken our conjecture by requiring that the branching points of the trees also need to be from $\mathcal{P}$. Another interesting, and probably easier, special case is when we also require that each edge of each tree needs to be a segment between two points of $\mathcal{P}$.

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Packing and Hitting Rectangles

by András Sebő

The *packing number*, ν, of a given family of sets is the maximum number of pairwise disjoint sets in the family, while their *hitting number* is the minimum number, τ, of points meeting all of them (with a non-empty intersection). Obviously, $\tau \geq \nu$. A more than half century old conjecture of Wegner (1965, [2]) asks whether $\tau \leq 2\nu - 1$ for rectangles in the plane. Replacing "2" by any larger constant the validity of the conjecture is also not known.

We studied the simplest special cases of this conjecture with Marco Caoduro [1]: by taking small values of some parameters, or by studying squares. I state here two of the most frustrating questions that we could not answer with Marco. The first concerns axis-parallel rectangles where none of the points is covered more than twice, the second concerns squares.

Problem 1. *Given a set of axis-parallel rectangles such that every point of the plane is contained in at most two of them, is it true that there are always* $\lfloor \frac{n}{2} \rfloor$ *disjoint ones among them?*

Trying to prove Wegner's conjecture for the case when each point is covered at most twice, an easy induction shows its equivalence with this statement for sets of axis-parallel rectangles with a factor-critical intersection-graph.

The packing and hitting problems are NP-hard also for axis-parallel unit squares. Wegner's conjecture is easy for them, but for sets of axis-parallel squares of arbitrary size it is not known. There are no better examples for axis-parallel squares than those giving the 3/2 ratio.

For not necessarily axis-parallel unit squares τ can be as large as 3 and is always at most 4. But can it be 4? (For unit disks in the plane the exact bound of 3 is known.) The target of the following problem is to understand the difference between maximum clique of the interesection graph and the maximum number of sets containing a given point. An example would improve the lower bound for arbitrary large ν by taking disjoint copies of the example.

Problem 2. *We ask the following questions both for (not necessarily axis-parallel) squares, and unit squares. How large can be the minimum size of a hitting set of pairwise intersecting squares? If no point is contained in more than two squares what is the maximum number of pairwise intersecting squares?*

References

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Simplifying drawings

by Géza Tóth

A drawing of a graph is simple, if any two edges have at most one point in common, which is either a common endpoint or a crossing.

Michael Hoffmann, Liu Chih-Hung, Meghana M. Reddy, and Csaba D. Tóth proved, that if a graph G is drawn in the plane such that there are at most k crossings on each edge, then it has a *simple* drawing with at most $6k^{3/2}3^k$ crossings on each edge.

This bound is almost surely very far from optimal, the best lower bound is linear in k. It would be very interesting to improve this bound.

Schaefer and Štefankovič proved that if a graph G is drawn in the plane such that each edge is crossed by at most m other edges (but in arbitrarily many points), then it has another (not necessarily simple) drawing such that there are at most 2^m crossings on each edge.

Combining them we obtain the following statement.

If a graph G is drawn in the plane such that each edge is crossed by at most m other edges, then it has a *simple* drawing with at most $6 \cdot 2^{3m/2}3^{2^m}$ crossings on each edge.

This doubly exponential bound is completely ridiculous, surely very far from optimal, so it would be very interesting to improve it.

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TV coloring

by Tomáš Valla

This is a graph coloring problem, that comes from a realworld application, which is a rare case in our field. One day, a guy from a television company came to our department, trying to find a solution for the following commercials scheduling problem in their TV program. There are some fixed shows during the day and empty slots to be filled with k types of commercials. However, the number of occurrences of each commercial type is given and moreover, two commercials of the same type cannot occur too close.

Formally: We have an undirected graph $G = (V, E)$, positive integer k, a partial vertex coloring $\gamma : V \to [k]$, and nonnegative integers $n_1, \ldots, n_k$. The task is to find a total coloring $c : V \to [k]$ such that

- c extends γ,
- for each two vertices $u, v \in V$ at distance at most 2, $c(u) \neq c(v)$, and
- for each $i \in [k]$ the number of occurences of color i is n_i, that is, $|\{v \in V;\ c(v) = i\}| = n_i$.

This problem was very briefly discussed with V. Blažej, D. Knop, J. Malík and O. Suchý more than two years ago and then abandoned due to covid lockdowns and replaced by other urgent problems to be solved.

The original TV scheduling problem corresponds to the case where the graph G is a path. There is a simple dynamic programming algorithm solving this problem efficiently if k is small. We had some crude argument showing that a certain simple greedy process can always color a path with precolored endpoints. We also had some idea how to generalise it to trees and bounded treewidth graphs.

The research task could thus be as follows:

Problem 1. *Provide an algorithm for bounded treewidth graphs.*

Problem 2. *Solve the problem where G is disjoint union of paths. Can some simple greedy algorithm be devised for paths with precolored endpoints?*

Problem 3.

How about generalisation of the distance condition, that is, two vertices at distance at most d must have different colors? And of course, try to study the problem for other graph classes.