1st Emléktábla Workshop

Hypergraphs, Set Systems 07. 26-29. 2010.

Preliminary Schedule

Day 1:
9:44 Welcome
9:45 - 10:30 Gyula O.H. Katona
10:45 - 11:30 Gábor Simonyi
11:45 - 12:30 András Gyárfás
Lunch Break
14:00 - 14:45 Zsolt Tuza
15:00 - 15:45 Zoltán Füredi
Traveling together to Gyöngyöstarján.

Other Days:
9:29 Waking up
8:30 - 9:30 Breakfast
9:30 - 12:30 Work in Groups of 3-5
10:45 - 11:00 Coffee Break
12:30 - 14:00 Lunch Break
14:00 - 17:00 Work in Groups of 3-5
15:45 - 16:00 Coffee Break
17:00 - 18:30 Discussion of Results
18:30 - Dinner and other activities

Last Day:
Discussion from 16:00 and then Return to Budapest.

Invited Problems

Graph and Hypergraph Turán problems

by Zoltán Füredi

1 Union-free subfamilies

The following problem is due to L. Moser: Let $A_1, \cdots, A_m$ be any m sets. Take the largest subfamily $A_{i_1}, \cdots, A_{i_r}$ that is union-free, i.e., $A_{i_{j_1}} \cup A_{i_{j_2}} \neq A_{i_{j_3}}$, $1 \leq j_1 \leq r$, $1 \leq j_2 \leq r$, $1 \leq j_3 \leq r$, for every triple of distinct sets $A_{j_1}, A_{j_2}, A_{j_3}$. Put $f(m) = \min r$, where the minimum is taken over all families of m distinct sets. Determine or estimate $f(m)$. Riddel showed that $f(n) > c\sqrt{n}$, and the Erdős and Komlós [5] showed that $\sqrt{n} \leq f(n) \leq 2\sqrt{2}\sqrt{n}$. These bounds were later improved to $\sqrt{2n} - 1 < f(n) < 2\sqrt{n} + 1$ by Kleitman and by Erdős and Shelah [6], resp. They **conjecture** that $f(n) = (2 + o(1))\sqrt{n}$.

One can define $f(\mathcal{F}, \Gamma)$ as the size of the largest subfamily having property Γ,

$$f(\mathcal{F}, \Gamma) := \max\{|\mathcal{F}'| : \mathcal{F}' \subseteq \mathcal{F},\ \ \mathcal{F}' \text{ has property } \Gamma\},$$

for example $\mathrm{ex}_r(n, \mathcal{H})$, the Turán number, is $f(K_r^n, \mathcal{H}\text{-}free)$. Let $f(m, \Gamma) := \min\{f(\mathcal{F}, \Gamma) : |\mathcal{F}| = m\}$.

The case when Γ is the property that

no four distinct sets satisfy $A_1 \cup A_2 = A_3$, $A_1 \cap A_2 = A_4$

is called B_2-*free*. Erdős and Shelah [6] gave an example that $f(m, B_2\text{-}free) \leq (3/2)m^{2/3}$. They also **conjecture** that $f(m, B_2\text{-}free) > c_2 m^{2/3}$, and in fact it seems likely that $f(m, B_2\text{-}free)/m^{2/3}$ tends to a limit.

What about other properties, like when we exclude $A_1 \cup ... \cup A_a = A_{a+1} \cup \cdots \cup A_{a+b}$, or when we exclude a Boolan lattice B_d of dimension d?

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Hanson posed the following problem: Let $g(n)$ be the smallest integer such that the subsets of $[n]$ can be split into $g(n)$ classes, where each of the classes is union-free. Hanson proved that $\Omega(\sqrt{n}) < g(n) \leq n/2 + 2$, and he conjectured that the upper bound is substantially correct. This was proved in [6] showing $g(n) > n/4$. They **conjecture** that $\lim g(n)/n$ exists and it is close to 1/2.

One can define $g(\mathcal{F}, \Gamma)$ as the minimum r such that $\mathcal{F}$ can be decomposed into r subfamilies having property Γ,

$$g(\mathcal{F}, \Gamma) := \min\{r : \mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_r,\ \ \text{each } \mathcal{F}_i \text{ has property } \Gamma\}.$$

Let $g(n, \Gamma) := g(2^{[n]}, \Gamma)$.

Erdős and Shelah [6] showed $g(n, B_2\text{-}free) = \Theta(\sqrt{n})$. What about other properties, like B_d-free, (a, b)-union-free, etc ?

2 Union-free uniform hypergraphs

The classical Turan problem concerning complete subhypergraph, i.e., determining $T(n, l, k)$ seems to be very difficult. An important wider class of problems was proposed by Brown Erdős and T Sós, the so called density questions. (For latest results see, e.g., Sarkozy et al. [10], the citations there, and the papers in Math Reviews citing it). These are still fairly difficult (like Ruzsa Szemeredi's (6,3) theorem), so we further ease the restrictions.

Consider an integer k, $0 < k < n$, and denote by $\mathrm{ex}_k(n)$ the maximal size of a family $\mathcal{F}$ of k-subsets of $[n]$, such that all the $\binom{|\mathcal{F}|}{2}$ unions $F_1 \cup F_2$, $F_1, F_2 \in \mathcal{F}$, are distinct. Let $e(k) = \frac{1}{2}\lceil 4k/3 \rceil$. Solving a problem due to Erdős it was show in [7, 8] that there exist positive constants c_k, c'_k, such that the relation $c_k n^{e(k)} \leq \mathrm{ex}_k(n) \leq c'_k n^{e(k)}$ holds. The proof of the lower bound is based on a construction using the elementary symmetric polynomials.

One can ask more generally the determinations of $\mathrm{ex}_k(n, \Gamma)$ where Γ is a forbidden Boole type relation. For example, $\mathrm{ex}_k(n, F_1 \cup F_2 = F_3 \cup F_4)$ is mentioned in the previous paragraph. What can we say about the relation $F_1 \cup ... \cup F_a = F_{a+1} \cup \cdots \cup F_{a+b}$? Call this (a, b)-union-free.

There are two versions, when repetitions in Γ are allowed and when all the $a + b$ sets must be distinct. The non-uniform case was widely investigated, mainly because of its connections with superimposed codes (see, e.g., Ruszinko [9], the citations there, and the 23 papers in Math Reviews citing it).

3 Covering a complete r-graphs with r-partitions

An r-cut $\mathcal{C}$ of the complete r-uniform hypergraph K_r^n is obtained by partitioning its vertex set $[n]$ into r parts and taking all edges that meet every part in exactly one vertex. In other words it is the edge set of a spanning complete r-partite subhypergraph of K_r^n. An r-cut cover $\mathcal{F} := \{\mathcal{C}_1, \mathcal{C}_2, \dots\}$ is a collection of r-cuts so that each edge of K_r^n is in at least one of the cuts. While in the graph case ($r = 2$) any 2-cut cover on average covers each edge at least $2 - o(1)$ times, when r is odd Cioaba, Kundgen, Timmonsz, and Vysotsky [3] exhibited an r-cut cover in which each edge is covered *exactly once.*

When r is even no such decomposition can exist, but they bounded the minimum of the average number of times an edge is cut in an r-cut cover between $1 + 1/(r + 1)$ and $1 + (1 + o(1))/\log r$.

When r is even, and $n = r + 1$ then K_r^{r+1} has an odd number of edges, but every r-cut of it has size 2, so t the minimum total size of an r-cut system is $r + 2$. It follows from an averaging argument for all n that $\sum |\mathcal{C}_i| \geq \frac{r+2}{r+1}\binom{n}{r}$.

Let $\mathcal{M} := \mathcal{M}(\mathcal{F})$ be the hypergraph of multiple covered r-sets by the cuts from $\mathcal{F}$. We have

$$|\mathcal{M}| \geq T(n, r + 1, r), \tag{1}$$

whenever r **is even**. So we can apply the known bounds for Turan numbers. D. de Caen's lower bound [1] gives that $(T(n, l, k) \geq k^{-1}\binom{l-1}{k-1}^{-1}(n - l + 1)\binom{n}{k-1}$, also see in [11, 12])

$$|\mathcal{M}| \geq \frac{1}{r} \times \frac{n - r}{n - r + 1}\binom{n}{r}. \tag{2}$$

Can we impove this lower bound for $|\mathcal{M}|$, can we improve it for $\sum |\mathcal{C}_i|$? (It seems to be easier than to determine $T(n, r + 1, r)$). Can we determine (asymptotically) the case $r = 4$?

4 Turan's hypergraph problem in geometry

Let P be an n-element set on the plane, no three on a line. Let $h(P)$ denote the number of acute triangles with all three vertices from P. Obviously,

$$h(P) \leq T(n, 4, 3). \tag{3}$$

Turan conjectures (see Sidorenko [11, 12]) that

$$T(n,4,3) \le |\mathcal{H}_n| \qquad (?) \tag{4}$$

where $\mathcal{H}_n$ is the largest n-vertex hypergraph of the form $V(\mathcal{H}_n) = [n] = A_0 \cup A_1 \cup A_2$ $(A_i \cap A_{i+1} = \emptyset)$ and $E(\mathcal{H}_n) := \{a_0a_1a_2 : a_i \in A_i\} \cup \{b'b''c : b', b'' \in A_i,\, c \in A_{i+1}, i = 0,1,2\}$. So (3) and (4) would imply $h(P) \le |\mathcal{H}_n| = \frac{5}{9}\binom{n}{3} + o(n^3)$.

Conway, Croft, Erdős, and Guy [4] gave a planer set P_n example where the acute triangles represents $\mathcal{H}_n$, so we have

$$h(n) := \max h(P) \ge |\mathcal{H}_n|. \tag{5}$$

The best published upper bound on $\lim T(n,4,3)/\binom{n}{3}$ is due to Chung and Lu [2] $\frac{3+\sqrt{17}}{12} = 0.593592\cdots$. Can we prove a better bound for $\lim h(n)/\binom{n}{3}$? Can we prove it is 5/9?

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Large matchings with few colors

by András Gyárfás

5 A generalized Kneser-type problem.

I address here a problem that came from a recent work with Gábor Sárközy and Stanley Selkow [16].

Let K_n^r denote the complete r-uniform hypergraph on n vertices, i.e. all r-sets of an n element ground set. A matching M in a hypergraph is a set of pairwise vertex disjoint edges, the size of M, $|M|$ is the number of edges in M. Several recent Ramsey type results, for example [5], [9], [13],[14], [15], [17] relied on lemmas about sizes of monochromatic matchings. In these applications an additional connectivity property of monochromatic matchings is needed. Nevertheless the starting point is the Ramsey number of matchings stated in the following well-known theorem of Alon, Frankl and Lovász 1986 [2].

Theorem 1. *([2]) Suppose $n = (t-1)(k-1) + kr$ and a coloring of the edges of K_n^r is given with t colors. Then there exists a monochromatic matching M such that $|M| \geq k$.*

Theorem 1 (conjectured by Erdős 1973 [8]) is at the crossroad of combinatorics and topology and were preceeded by several notable special cases. The graph case ($r = 2$) is due to Cockayne and Lorrimer 1975 [6] (no topology in their proof). The 2-color case ($t = 2$) was solved (without topology) 1985 [1], 1985 [12]. The case $k = 2$ is Kneser's conjecture, solved by Lovász 1978 [18] who introduced topological methods, then Bárány 1978 [3], Green 2002 [10], Matousek 2004 [19] gave new proofs. Results generalizing Theorem 1 were obtained by Schrijver 1978 [22], Dolnikov 1988 [7], Sarkaria 1990 [21], Ziegler 2002 [23], see also [20] for further related material.

Theorem 1 is sharp. To describe easily t-colorings of K_n^r we need, consider partition vectors with t positive integer coordinates whose sum is equal to n. We assume that $V(K_n^r) = \{1, 2, \ldots, n\}$. Then $[p_1, p_2, \ldots, p_t]$ represents the coloring obtained by partitioning $V(K_n)$ into parts A_i so that $|A_i| = p_i$ for $i = 1, 2, \ldots, t$ and the color of any edge e is the the minimum j for which e has non-empty intersection with A_j. With this notation, the coloring $[k-1, k-1, \ldots, k-1, kr-1]$ shows that Theorem 1 is sharp.

Recently [16] a possible extension of Theorem 1 emerged, it uses an additional integer parameter s satisfying $1 \leq s \leq t$. A matching with edges colored by at most s distinct colors (out of t colors) is called an *s-colored matching.* Thus 1-colored matchings are the monochromatic matchings. We ask for the smallest n such that in any t-coloring of the edges of K_n^r there is an s-colored matching of size k. For $s = 1$ Theorem 1 provides the answer.

Problem 2. *Find the smallest n such that every t-coloring of K_n^r contains an s-colored matching of size k.*

In fact, Theorem 1 can be iterated to get an upper bound for Problem 2 because one can take largest monochromatic matchings step by step through s steps. However, it would be better to find a tighter, possibly best result - or at least a reasonable conjecture. Based on partition vector

$$[p, p, \ldots, p, pr, pr^2, \ldots, pr^{s-1}, pr^s]$$

where the first $t - s$ coordinates are p-s, the following seems to be a plausible conjecture for certain values of $s \in \{1, 2, \ldots, t\}$ (but not always).

Conjecture 3. *Every t-coloring of K_n^r contains an s-colored matching of size k provided that*

$$n \geq kr + \left\lfloor \frac{(k-1)(t-s)}{1+r+r^2+\dots r^{s-1}} \right\rfloor .$$

Notice that the case $s=1$ is Theorem 1, the case $s=t$ is trivial (and best possible).

5.1 The case $s=2, t=3$.

The smallest case not covered by Theorem 1 is $s=2, t=3, r=2$. Conjecture 3 holds for these parameters and the bound $\lfloor \frac{7k-1}{3} \rfloor$ is best possible for every $k \geq 1$ [16]. The proof is based on a simple idea that works well in certain situations to facilitate induction, for example the proof of Cockayne and Lorrimer [6] for the case $r=2$ of Theorem 1 can be formulated this way. For hypergraphs the same method was applied in [12], and more recently in [17],[15]. I posed the case $t=2, k=r$ on 2007 USAMO and 2007 Schweitzer competition. An early appearance of the argument is the 'bow tie argument': finding a red and a blue triangle with one common vertex (bow tie) drives the inductive argument of the proof of $R(kK_3, kK_3) = 5k$ see [4], [11]. In our case $(s=2, t=3, r=2)$ the 'bow tie' in a 3-coloring of a complete graph K is a subset $X \subset V(K)$ such that $|X| = 7$ and for any choice of two distinct colors $\{i,j\} \subset \{1,2,3\}$ there is a matching $M \subset K[X]$ such that M has three edges, all colored with some color of $\{i,j\}$.

It would be interesting to decide whether Conjecture 3 holds for $s=2, t=3$ and general r, i.e. for hypergraphs, at least for $r=3$.

Conjecture 4. *In every coloring of the edges of K_n^r with three colors there is a 2-colored matching of size at least k provided that $n \geq kr + \lfloor \frac{k-1}{r+1} \rfloor$.*

The smallest test case (it also belongs to the next subsection) is $r=3, k=4$: is it true that in every 3-coloring of the edges of K_{12}^3 there is a 2-colored matching of size four?

5.2 Perfect matchings.

When the second term in Conjecture 3 is zero, i.e.

$$\frac{(k-1)(t-s)}{1+r+r^2+\dots r^{s-1}} < 1$$

then the existence of a perfect s-colored matching is claimed. This happens, in particular, when $s=t-1$ and $k-1 = r+\dots+r^{s-1}$, leading to the following.

Conjecture 5. *Suppose that $n = r+r^2+\dots+r^{t-1}$. In every t-coloring of K_n^r there is a perfect matching colored with at most $t-1$ colors (i.e. missing at least one color).*

Note that Conjecture 5 is sharp in the sense that it is not true if n is increased by r, as the t-coloring of the partition $[1, r, r^2, \dots, r^{t-2}, r^{t-1}+r-1]$ shows.

5.3 The graph case, $r=2$.

- $s=t-1$. Conjecture 3 is probably true ($t=3$ is solved, [16]). In particular, the special case of Conjecture 5 seems challenging: for $n \geq \sum_{i=1}^{t-1} 2^i$ in every t-colored K_n has a perfect matching that misses at least one color. Smallest test case: $t=4$.
- $t=4, s=2$. There is a 2-colored matching if size k in every 4-coloring of K_n if $n \geq \lfloor \frac{8k-2}{3} \rfloor$. Smallest test case: $k=6$.

- $t = 5, s = 2$. There is a 2-colored matching of size k in every 5-coloring of K_n if $n \geq \lfloor \frac{9k-3}{3} \rfloor$. Smallest test case: $k = 6$.

- $t = 6, s = 2$: Here Conjecture 3 fails, the inequality is $n \geq 2k + \lfloor \frac{4(k-1)}{3} \rfloor$. Thus for $k = 9$ we get $n = 28$, i.e. in every coloring of K_{28} with six colors we should have a 2-colored matching of size nine. But the partition vector $[4, 4, 4, 4, 4, 8]$ gives only 2-colored matchings of size eight. The reason for this that for $s = 2$ the partition vector $[p, p, p, p, 2p, 4p]$ on which Conjecture 3 based on is worse than the partition vector $[p, p, p, p, p, 2p]$ of the Cockayne-Lorrimer bound (special case of the bound of Theorem 1).

- What is a plausible conjecture for $s = 2$?

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Miscellany

by Gyula O.H. Katona

Problem 1. Let ξ be a random vector in the d-dimensional Euclidean space, where $d \geq 2$. The modified distribution function of its length is $f(x) = \mathrm{P}(|\xi| \geq x)$. (It is actually 1 minus the distribution function. We use this instead of the real distribution function, because the results have nicer forms.) Let ξ and η be independent, identically distributed copies of ξ. The modified distribution function of the length of their sums is $g(x) = \mathrm{P}(|\xi + \eta| \geq x)$. Determine the "best" functional H such that

$$g(x) \geq H(f(x))$$

holds and the inequality is sharp.

If H can "use" only one specific place of $f(x)$ then the answer is known. For instance

$$g(x) \geq \frac{1}{2} f^2(x)$$

holds when H "uses" the function $f(x)$ at the same place, that is H is a function of $f(x)$. And this is the best lower estimate of this form. The answer is also known when H is a function of $f(cx)$ with a fixed c.

There is a modest lower estimate on $g(x)$ using two places of $f(x)$ namely, $f(\frac{1}{\sqrt{2}}x)$ and $f(\frac{1+\sqrt{3}}{2}x)$. The proofs of these results use elementary (two-dimensional) geometry and Turán type theorems.

Problem 2. Let $\mathcal{C}$ be a one-error-correcting code, that is a family of 0-1 sequences (called *codewords*) of length n with pairwise Hamming distance at least 3. Let p_i denote the number of codewords containing exactly i ones. The vector $(p_0, p_1, \ldots, p_n)$ is called the *profile vector* of the code $\mathcal{C}$. It is a point in the n-dimensional space. Take all the profile vectors of the one-error-correcting codes and consider the convex hull of these points. The ideal goal would be to determine the extreme points of this convex hall. They would be the profile vectors of codes extremal in certain sense. It is hopeless to describe the set of extreme points, since the maximum value in the coordinate i (that is the maximum size of a one-error-correcting code consisting of codewords having i ones) is unknown.

Is the profile vector of the Hamming code (when $n = 2^r - 1$) extreme? Find good hyperplanes (inequalities) giving upper bounds for the convex hull.

Problem 3. (Füredi-Zs. Katona) Let $\mathcal{F} \subset 2^{[n]}$ be a family in which the intersection of any two sets has at least two elements, while the intersection of any three sets has at most 3 elements. Determine the maximum of $|\mathcal{F}|$. Both versions are interesting, when the sets can be repeated and when not. In the first case the following construction might be the best. Take all sets $\{1, 2, i, j\} (3 \leq i < j \leq n)$ twice and the sets $[n] - \{1\}, [n] - \{2\}$.

Problem 4. Let $\mathcal{F} \subset \binom{[n]}{k}$ where $k \geq 3$. Its *second shadow* is defined in the following way.

$$\sigma^2(\mathcal{F}) = \{G : \ |G| = k - 2, G \subset F \text{ for some } F \in \mathcal{F}\}.$$

What is the minimum of $|\sigma^2(\mathcal{F})|$ under the conditions (i) n, k and $|\mathcal{F}|$ are fixed and (ii) the pairwise symmetric differences of distinct members of $\mathcal{F}$ are at least 4.

Is it true that

$$|\mathcal{F}| \geq \frac{\binom{a}{k-1}}{k}$$

for some integer a implies

$$|\sigma^2(\mathcal{F})| \geq \binom{a}{k-2}?$$

This is trivial for $k = 3$.

Problem 5. Let $\mathcal{F} \subset 2^{[n]}$ be a family satisfying the following condition. For any three distinct members A, B, C at most one of their three pairwise intersections can be empty. Determine the maximum of $|\mathcal{F}|$.

We conjecture that the largest family consists of all sets of size at least $\frac{n}{2}$ if n is even. If n is odd then we have to choose all the sets of size at least $\frac{n+1}{2}$ and the $\frac{n-1}{2}$-element sets containing $\{1\}$.

A more general question: find the maximum of $\mathcal{F}$ under the condition that for any r distinct members at most s of the pairwise intersections can be empty. The case $s = \binom{r}{2} - 1$ has been solved by Kleitman, decades ago.

Some Problems in Extremal Combinatorics

by Gábor Simonyi

6 Sandglass Conjecture

In its original form the Sandglass Conjecture states the following (in its simplest, binary case):

If $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are two set systems such that
(1) $\forall A, A' \in \mathcal{A}$ and $\forall B, B' \in \mathcal{B}$ $\quad A \cup B = A' \cup B'$ implies $A = A'$
and
(2) $\forall A, A' \in \mathcal{A}$ and $\forall B, B' \in \mathcal{B}$ $\quad A \cap B = A' \cap B'$ implies $B = B'$,
then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

If true, this bound is sharp: take all supersets of a fixed set as $\mathcal{A}$ and all subsets of the same set as $\mathcal{B}$. The name of the conjecture refers to the "shape" of this configuration.

I first presented this conjecture in Oberwolfach in 1989, it first appeared in print in [1] where a generalization to more general lattices can be found. The conjecture is still open, the best bound for the above is due to Körner and Holzman [5].

By complementing the sets in the first family, one obtains the following simple reformulation, it appears in this form e.g. in Gil Kalai's blog [6]:
If $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are two set systems such that
(1') $\forall A, A' \in \mathcal{A}$ and $\forall B, B' \in \mathcal{B}$ $\quad A \setminus B = A' \setminus B'$ implies $A = A'$
and
(2') $\forall A, A' \in \mathcal{A}$ and $\forall B, B' \in \mathcal{B}$ $\quad B \setminus A = B' \setminus A'$ implies $B = B'$,
then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

The above was generalized by Ron Aharoni, who formulated the following bold statement as a possible stronger conjecture:
Let $\mathcal{A}, \mathcal{B}$ be set systems satisfying conditions (1') and (2'). Then $\Sigma_{A \in \mathcal{A}, B \in \mathcal{B}} 2^{|A \cap B|} \leq 2^n$.
Some other variations can be found in [9].

7 Trifference

The following question was asked by Vera T. Sós during a workshop in Bielefeld in 1991.

What is the minimum number $t = t_n$ for which one can give t edge-colorings of the complete graph K_n so that for every triangle in K_n there is a coloring in which its three edges get three distinct colors?

It is easy to see that $t_n \geq \log_3(n-1)$ (since all pairs among the $n-1$ edges at any given vertex must get different colors in some coloring). In [7] we gave a construction showing that $t_n \leq \lceil \log_2 n \rceil - 1$.

Problem: What is the exact value of t_n? In particular, is the above upper bound tight?

It is shown in [2] that the above upper bound is tight for $n = 9$ which is the first non-trivial case.

8 Shannon capacity of Mycielski graphs

Let G^t denote the following graph exponentiation. $V(G^t) = [V(G)]^t$ and two vertices, $\mathbf{x} = x_1x_2 \dots x_t$ and $\mathbf{y} = y_1y_2 \dots y_t$ are adjacent iff $\exists i: \ \{x_i, y_i\} \in E(G)$.

We define the Shannon capacity of a graph as

$$C(G) := \lim(\sup)_{t\to\infty} \frac{1}{t} \log \omega(G^t).$$

(We note that this definition uses a complementary language compared to the original one, but in certain contexts it is also customary and more convenient.)

Let us denote by M_k the k-chromatic Mycielski graph, that is the graph we obtain after $k-2$ iteration of the Mycielski construction when starting with the one edge graph K_2.

Problem: Is the always existing limit $\lim_{k\to\infty} C(M_k)$ finite or infinite?

The above problem may sound somewhat particular at first glance. Note, however, that it is equivalent to the following question of Erdős.

Let $R(3;k)$ denote the smallest integer n for which every k-edge-coloring of K_n contains a monochromatic triangle. Is $\lim_{k\to\infty} R(3;k)$ finite or infinite?

The main observations showing the latter claimed equivalence can be found in [4] (see also [8]) combined with an observation in [3].

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Coloring problems related to mixed hypergraphs

by Zsolt Tuza

The upper chromatic number with respect to a given type of coloring is the largest possible number of colors.

In the following problems, except the last one, we restrict ourselves to C-coloring, what means that every edge of the hypergraph in question contains two vertices of the same color. (If the hypergraph is a graph, this means that every connected component is monochromatic, and then the upper chromatic number is equal to the number of connected components.) A hypergraph is r-uniform if each of its edges has exactly r vertices.

Conjecture (Bujtás, Tuza) The minimum number of edges in a 3-uniform hypergraph with upper chromatic number 3 is $n^2/6 + o(n^2)$.

Comment: Constructions with approximately $n^2/6$ edges are known, the problem is to prove a matching lower bound.

The independence number of a hypergraph is the maximum number of vertices not containing any edges. It is easy to see that the upper chromatic number cannot exceed the independence number. A hypergraph is C-perfect if in each of its induced subhypergraphs the upper chromatic number is equal to the independence number. (An induced subhypergraph is obtained by specifying a subset of the vertex set and taking all edges entirely contained in this subset.)

Problem (Voloshin) Characterize the 3-uniform C-perfect hypergraphs.

Comment: Six minimally non-C-perfect 3-uniform hypergraphs are known, perhaps this collection is complete.

Conjecture (Bujtás, Tuza) For every fixed r, the number of inclusionwise minimal non-C-perfect r-uniform hypergraphs is finite.

Comment: It is known that the number is finite if we also fix the vertex covering number (or the packing number). Hence the conjecture is equivalent to the assertion that a minimally C-imperfect r-uniform hypergraph cannot have an arbitrarily large number of mutually disjoint edges.

A more general structure class is that of color-bounded hypergraphs. Here a lower and upper bound is specified for each edge, meaning that the number of colors on the vertices of the edge has to be between the two bounds. (Different edges may have different bounds.) An interval hypergraph has a linear order on its vertex set, such that every edge is a set of consecutive vertices.

Problem (Bujtás, Tuza) Does there exist a polynomial-time algorithm that decides whether any given interval hypergraph with lower and upper color bounds on its edges admits at least one coloring? If the hypergraph is colorable, can the upper chromatic number be determined in polynomial time?

Comment: Assuming that the input hypergraph is colorable, the minimum possible number of colors can be determined in linear time.

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Contributed Problems

An Inequality for 2-intersecting Families

by Ameera

Let X be an n-element set, and let 2^X denote the family of all subsets of X. A family $\mathcal{F} \subset 2^X$ is called λ-*intersecting* if we have $|F_1 \cap F_2| = \lambda$ for any distinct $F_1, F_2 \in \mathcal{F}$. For $x \in X$, the *degree* of x, denoted $\deg(x)$, is defined to be the number of sets in $\mathcal{F}$ that contain x. We say $\mathcal{F}$ is *trivial* if there exists $x \in X$ with $\deg(x) = |\mathcal{F}|$, and is *non-trivial* otherwise.

Problem: If $\mathcal{F} \subset 2^X$ is a non-trivial 2-intersecting family of size $m < n$, is it true that

$$\sum_{F \in \mathcal{F}} \binom{|F|}{2} \geq \sum_{x \in X} \binom{\deg(x)}{2} = m(m-1)? \tag{6}$$

Remarks.

Observe that when $\mathcal{F} \subset \binom{X}{k}$ is k-uniform, then (6) is equivalent to proving

$$m \leq \binom{k}{2} + 1. \tag{7}$$

Hall [2] proved that (7) holds for all non-trivial, k-uniform, 2-intersecting families.

We believe that the only non-trivial 2-intersecting family for which (6) does not hold is

$$\hat{\mathcal{F}} := \{\{1,2,4\}, \{1,4,6,7\}, \{1,2,5,7\}, \{1,2,3,6\}, \{2,3,4,7\}, \{1,3,4,5\}, \{2,4,5,6\}\}.$$

Ryser showed that $\hat{\mathcal{F}}$ is the unique non-uniform 2-intersecting family with size $m = n$. Hence, another way to state the problem is:

Restatement: If $\mathcal{F} \subset 2^X$ is a non-trivial 2-intersecting family of size m and $\mathcal{F} \neq \hat{\mathcal{F}}$, then

$$\sum_{F \in \mathcal{F}} \binom{|F|}{2} \geq \sum_{x \in X} \binom{\deg(x)}{2} = m(m-1).$$

If our conjecture is true, it would imply a conjecture of Frankl and Furedi [1] when $\lambda = 2$.

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Can we find the max and min against k lies with $(k+1+\epsilon)n$ comparisons?

by Dani

Suppose we have n elements with all different weights and in one step we can compare any two of them to see which one is bigger. It is well-known that $n-1$ comparisons are needed to find the biggest element and $\lceil \frac{3n}{2} \rceil - 2$ to find the biggest and the smallest. It is also well known that finding the maximum with k lies, i.e. when k comparisons might return a false result, requires $(k+1)n-1$ comparisons in the worst case. Aigner [1] raised the question of what happens if we mix the two models. In [2] we proved that if $k=1$ then $\frac{87}{32}n + \Theta(1)$ is the optimal answer. Hoffmann et al. [3] have shown that $(k+1+C)n$ comparisons are sufficient for a fixed constant C. Can we do better if k is big? Can we guarantee that $C \to 0$ as k grows?

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Do shift-chains have Property B?

by Dömötör

For $A \subset [n]$ denote by a_i the i^{th} smallest element of A.

For two k-element sets, $A, B \subset [n]$, we say that $A \leq B$ if $a_i \leq b_i$ for every i.

A k-uniform hypergraph $\mathcal{H} \subset [n]$ is called a *shift-chain* if for any hyperedges, $A, B \in \mathcal{H}$, we have $A \leq B$ or $B \leq A$. (So a shift-chain has at most $k(n-k)+1$ hyperedges.)

We say that a hypergraph $\mathcal{H}$ has Property B if we can color its vertices with two colors such that no hyperedge is monochromatic.

Is it true that shift-chains have Property B if k is large enough?

Remarks. This question is motivated by decomposition of multiple coverings of the plane by translates of convex shapes, there are many open questions in this area. (For more, see my brand new thesis.)

For $k = 2$ there is a trivial counterexample: (12),(13),(23).

A very magical counterexample was given for $k = 3$ by Rado [1]:
(123),(124),(125),(135),(145),(245),(345),(346),(347),(357),
(367),(467),(567),(568),(569),(579),(589),(689),(789).

If we allow the hypergraph to be the union of two shift-chains (with the same order), then there is a counterexample for any k.

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Is there a rainbow matching in a 6-regular graph if each colorclass is a C_6?

by Padmini

Suppose that we have a $2d$-regular graph whose edges are colored such that the edges of each color form a cycle of length $2d$. (So if the graph has $2n$ vertices, then there are n colors.) Is it true that there always is a perfect matching containing one edge of each color?

Remarks. A positive result would imply a stronger bound for pairing strategies in certain combinatorial games, see [1]. For $d = 2$ there is a simple proof by Zoltán Király who also invented the above formulation of the problem. I do not even know the answer for $d = 3$.

References

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A problem

by Russ

9 The problem

Let $\mathcal{H}$ be a d-uniform hypergraph with the following properties:

1. If E_1 and E_2 are hyperedges then there are vertices $v_1 \in E_1$ and $v_2 \in E_2 \setminus E_1$ such that $(E_1 \setminus v_1) \cup v_2$ is a hyperedge.

2. The "neighborhood hypergraph" with hyperedges

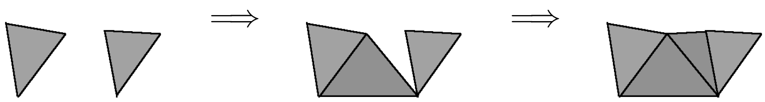
$$\{E \setminus v \,:\, E \text{ an edge containing } v\}$$

satisfies the same condition, recursively defined.

No harm is done in assuming that every vertex of $\mathcal{H}$ is contained in at least one edge. Call a hypergraph satisfying these properties *admissible*.

Notice that admissibility is essentially a very strong connectivity condition. A picture is shown in Figure 1.

Figure 1: The admissibility condition for 2-uniform hypergraphs



Example 6. *If $\mathcal{H}$ is a graph, then the condition implies that every two disjoint edges have an edge connecting them. (I.e., $\mathcal{H}$ has no induced subgraph consisting of two disjoint edges.)*

A *shedding vertex* is a vertex v such that for every edge E with $v \in E$, there is a vertex w_E such that $(E \setminus v) \cup w_E$ is a hyperedge.

Question 7. *For a fixed d, is there a number N_d such that if $\mathcal{H}$ is an admissible hypergraph with more than N_d vertices, then $\mathcal{H}$ has a shedding vertex?*

10 Motivation

A positive answer to Question 7 would solve a conjecture of Wachs.

A simplicial complex Δ is *shellable* if there is an ordering of its facets $F_1, F_2, \ldots, F_m$ such that the intersection of F_i with the complex generated by $F_1, \ldots, F_{i-1}$ is $(\dim F_i - 1)$-dimensional. Shellability is one of the main definitions of algebraic/geometric combinatorics.

A simplicial complex Δ is an *obstruction to shellability* if Δ is not shellable but every proper induced subcomplex is shellable.

Conjecture 8. (Wachs *[2, 3]*) *There are a finite number of obstructions to shellability that are of dimension d.*

The first condition of admissible hypergraphs is somewhat weaker than what is required for shellability and/or obstructions to shellability. It essentially requires only that the induced simplicial complex on the vertex set of two d-faces is shellable.

An obstruction to shellability Δ is a *strong obstruction to shellability* if in addition the simplicial complex generated by $\{F : F \text{ contains } v\}$ is shellable for any fixed v.

Theorem 9. (Hachimori and Kenjiwabara *[1]*) *TFAE:*

1. *The number of obstructions to shellability of dimension $\leq d$ is finite.*

2. *The number of strong obstructions to shellability of dimension $\leq d$ is finite.*

The second condition of admissible hypergraphs is somewhat weaker than what would be required for strong obstructions to shellability.

We consider the simplicial complex consisting of all subsets of hyperedges from an admissible hypergraph $\mathcal{H}$. The connection between the shedding vertex of Question 7 and Conjecture 8 is via the following lemma:

Lemma 10. *If Δ is a simplicial complex with a shedding vertex v such that the induced subcomplex $\Delta \setminus v$ and the complex generated by $\{F : v \in F\}$ are both shellable, then Δ is shellable.*

The *pure d-skeleton* of a simplicial complex is the simplicial complex generated by all faces of dimension d. It is well-known that every pure skeleton of a shellable complex is shellable, hence the pure d-skeleton of an obstruction to shellability must be non-shellable for some d. Moreover, taking pure skeletons clearly commutes with taking induced subcomplexes or "links".

Conjecture 8 is known to hold in dimension ≤ 2 [2], and the obstructions in these dimensions were explicitly enumerated in [1]. Conjecture 8 is also known to hold in the case where the minimal non-faces of Δ form a graph, and the obstructions are in 1-1 correspondence with the cyclic graphs of length $\neq 3, 5$ [4].

11 Why is this a reasonable problem?

Dömötör instructed us "no big conjectures", yet I've proposed this problem. Why is this reasonable? Well, first, it is perhaps not a "big" conjecture (only moderate size). Moreover, it has mainly been of interest to geometric combinatorialists. I am very hopeful that the problem, as rephrased in Section 1, may fall to some of the combinatorial tools (Ramsey Theory, Lovasz Local Lemma, Szeméredi Regularity, ???) in which Emléktábla are well-versed, but geometric people are less expert in.

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Results

First place: Rado - Nagy Zoli
Second place: Padmini - Dömötör
Third place: Cory - Dani
Fourth place: Lindsay - Neal

To Union-free subfamilies problem by Füredi

by Himself, János Barát, Ida Kantor, Younjin Kim, Balázs Patkós

We proved that Erdos and Shelah's conjecture about B_2-free subfamilies is true and established some general lower and upper bounds on $f(m, B_d - free)$. Also, we obtained $f(m, (1, a) - free) \leq (\lceil \sqrt{a+1} \rceil + o(1))\sqrt{m}$. (An easy observation shows that $f(m, (a, b) - free) = a + b - 1$ if both a and b are at least 2.)

To s=2, t=3 case of problem by Gyárfás

by Himself, Ago, Dani, Diana, Dominik, Lindsay, Neal, Tamás Terpai

Theorem 11. *In every 3-edge-colouring of a 3-uniform hypergraph K^3_{12} on 12 vertices, there is a 2-coloured matching of size 4.*

Theorem 12. *In every* 3*-edge-colouring of a* 3*-uniform hypergraph K^3_{16} on* 16 *vertices, there is a* 2*-coloured matching of size* 5*.*

Theorem 13. *In every 3-edge-colouring of a 3-uniform hypergraph K^3_{19} on 19 vertices, there is a 2-coloured matching of size 6.*

For more, see http://kam.mff.cuni.cz/∼ diana/emlektabla/

To Problem 2 about determining profile vectors of codes by Katona

by Cory and Dömötör

We considered the following much simpler problem. Suppose we have linear weights, w_i, and we want to find a 1-error detecting code such that the weighted sum of the codewords is maximal. By a simple induction argument, we have shown that selecting every odd or every even level is optimal.

To Problem 3 about set systems with special intersections by Katona

by Fabrício Benevides, Ameerah Chowdhury, Zoltán Gyenes, Michal Przykucki, Ago-Erik Riet and Manuel Silva

We tried to determine the maximum possible size of $\mathcal{F} \subset 2^{[n]}$ in which the intersection of any two sets has at least two elements, while the intersection of any three sets has at most 3 elements. It was already known, by a simple construction, that $\max |\mathcal{F}|$ is at least $n^2 - 5n + 8$. We proved that $|\mathcal{F}| \leq 1435n^2$ for $n \geq 25$, that is $|\mathcal{F}| = O(n^2)$.

To Sandglass conjecture by Simonyi

by Neal Bushaw, Misha Tyomkyn, Dominik Vu and Russ Woodroofe

We examined the proof of the current best bounds by Holzman and Körner, and have sketched an extension of this to a collection of families of sets where each pair of families is a recovering pair; the bound on the product of the sizes of the families would match that of Holzman and Körner for two families, but the proof requires a rather serious technical lemma which remains unproven. We also have some ideas for extending the proof of Holzman and Körner to truly recovering families rather than cancellative families, but so far this has yielded no improved bounds.

To Trifference by Simonyi

by Zoltán Gyenes, Cory Palmer

We gave a new proof for the upper bound using a simple induction argument.

To Shift-Chains by Dömötör

by Radoslav Fulek, Tamás Hubai, Balázs Keszegh, Zoltán Nagy, Thomas Rothvoß, Máté Vizer

Degree bounded cases

First we see, that any counterexample must have a size that is exponential in k:

Lemma 14. *If $k \geq 2 \log_2 n$ and $n \geq 17$, there is a proper coloring χ.*

Fast moving shift-chains have Property B

An ℓ-shift-chain is a shift-chain in which neighboring hyperedges differ by at least ℓ elements, i.e. $|e_{j+1} \setminus e_j| \geq \ell$ for any j.

Lemma 15. *For $\ell \geq 3 + \log_2 k$, an ℓ-shift chain has Property B.*

Few dangerous vertices

Let us call a vertex, v, *α-dangerous* if $\deg(v) \geq \alpha$ and *normal* otherwise.

Lemma 16. *If the number of c^k-dangerous vertices is at most c^k and $c < \sqrt{2}$, then for big enough k there is a proper coloring.*

Cyclic Shift-Chains

Theorem 17. *For any cyclic shift-chain, H, there is always a non-monochromatic 5-coloring. Moreover, if five colors are needed, H' contains K_5 as a subgraph.*

To Rainbow matching problem by Padmini

by Fabricio Benevides, Michal Przykucki, Tamás Terpai, Mykhaylo Tyomkyn

An example of a 6-regular graph was constructed that splits into edge-disjoint C_6's but does not admit any perfect pairing, let alone a rainbow one. It seems that the example can be extended to $d > 3$ as well, but we do not know what is the case if the graph is required to be bipartite.

To A problem by Russ

by Dömötör, Péter Csorba and Russ

We gave a negative answer to Question 7 for all d. The construction is the following. Take the independence complex of C_n and for each vertex add one complex that contains the vertex. E.g., if $d = 3$ and n is odd, then we can take for each vertex the opposing edge. E.g., for $n = 7$, we take the independent sets of C_7, together with $\{1,4,5\}$,$\{2,5,6\}$,$\{3,6,7\}$,... It is not hard to prove that this will indeed give an admissible hypergraph without shedding vertices. It is even true that any association of each edge of C_n with a non-neighboring vertex gives a similar counterexample, by the same proof. It is possible that some such association gives an infinite family of non-shellable complexes where every $\leq$ 6-vertex subset is shellable. Since shellability of every $\leq$ 7-vertex subset implies shellability, this would be rather interesting.