18th Emléktábla Workshop

Extremal combinatorics with geometric connections 07.21 - 07.25.2025.

Preliminary Schedule for the Extremal Combinatorics group:

Monday

Arrival at the hotel before 12:30.
12:30-14:00 Lunch
15:00 Short introduction/presentation of problems

Tuesday-Thursday

7:30-8:30 Breakfast
9:00-17:00 Working in groups of 3-5
12:20-14:00 Lunch
16:45 Presentations of daily progress
18:00 Dinner

Friday

7:30-8:30 Breakfast
9:00-12:29 Working in groups of 3-5
12:30-14:00 Lunch
14:15 Summary of the progress

List of Participants

Azem Berivan Adibelli, Umea University
Péter Ágoston, Rényi Institute
Martin Balko, Charles University, Prague
János Barát, University of Pannónia, Veszprém
Zoltán Blázsik, University of Szeged
Simona Boyadzhiyska, Rényi Institute
Recep Altar Ciceksiz, Umea University
Nóra Frankl, Open University, Milton Keynes
Andrea Freschi, Rényi Institute
Dániel Gerbner, Rényi Institute
Andrzej Grzesik, Jagiellonian University
Hilal Hama Karim, BME
Nina Kamčev, University of Zagreb
Gyula Károlyi, Rényi Institute
Gergely Kiss, Rényi Institute
Balázs Keszegh, Rényi Institute
Nathan Lemons, Los Alamos National Lab
Benedek Kovács, ELTE
Csenge Lili Ködmön, ELTE
Zoltán Lóránt Nagy, ELTE
Dömötör Pálvölgyi, ELTE-MTA
Balázs Patkós, Rényi Institute
Laurentiu Ploscaru, Rényi Institute
Magdalena Prorok, AGH University Kraków
Nika Salia, King Fahd University
Dániel Simon, Rényi Institute
Gábor Somlai, ELTE
Balázs Szabó, ELTE
Dávid Szabó, ELTE
Levente Szemerédi, ELTE
Géza Tóth, Rényi Institute
Tadej Petar Tukara, University of Zagreb
Máté Vizer, BME
Ji Zeng, Rényi Institute
Zeyu Zheng, Carnegie Mellon, Pittsburgh
Kristóf Zólomy, ELTE

Ranges of two-distance graphs

by Péter Ágoston

Suppose that a graph has edges coloured by red and blue and its vertices can be represented by distinct points of the plane such that a pair connected by a red edge always corresponds to a pair of points with distance 1 and a pair connected by a blue edge always corresponds to a pair of points with distance d for some fixed d. We call such a graph (along with the colouring) a $(1,d)$-graph and if there exists a colouring and a d for which a graph is a $(1,d)$-graph, we call it a two-distance graph. Also, for a (red-blue) edge-bicoloured graph, let its *range* be the set of numbers d for which it is a $(1,d)$-graph (see an illustration of the notion in Figure 1).

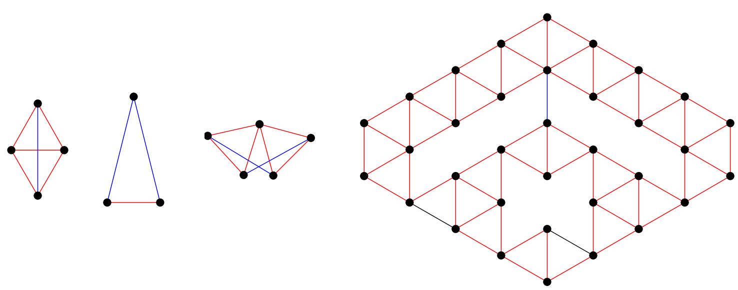

Figure 1: Examples of edge-bicoloured graphs with ranges $\left\{\sqrt{3}\right\}$, $\left[\frac{1}{2},+\infty\right)$, $(0,2]\setminus\{1\}$ and $\{1,2,3,4\}$ (from left to right)

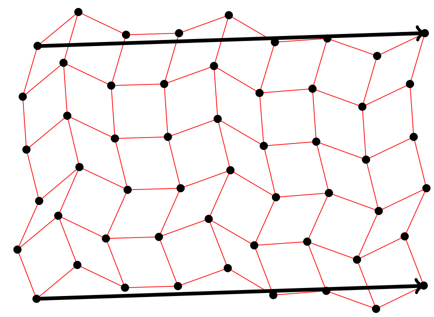

Figure 2: Two vectors connected by a grid, which is a crucial element of the constructions in Theorem 1

Theorem 1. *As a simple consequence of the Tarski–Seidenberg theorem, the range of an edge-bicoloured graph is always a semialgebraic set (in $\mathbb{R}$, that is a set whose only boundary points are finitely many algebraic numbers). I proved [1] that any semialgebraic set in $\mathbb{R}$ is in fact the range of some edge-bicoloured graph with the condition that the set has a positive lower bound and a finite upper bound.*

Problem 1. *Can we construct edge-bicoloured graphs for all semialgebraic sets in $\mathbb{R}$ without the boundedness limitations?*

Problem 2. *What if the points representing different vertices do not have to be distinct?*

Note that the constructions used in the proof of Theorem 1 do not work for any of the variants above: one of the reasons is that these constructions heavily rely on the fact that in any graph, one can assure that a pair of vertices has the same vector as another pair of vertices by connecting these pairs by a grid (see Figure 2).

Problem 3. *What are the possible ranges if the colouring of the edges is not given in advance?*

Problem 4. *Can we generalize the result in some other way (more than two distances, more than two dimensions)?*

References

[1] P. Ágoston: On the range of two-distance graphs, preprint, `https://agostonpeter.web.elte.hu/1dgrafok.pdf`

Ordered Ramsey numbers of paths

by Martin Balko

An *ordered graph* $G^<$ is a pair $(G,<)$ where G is a graph and $<$ is a total ordering of its vertices. The *ordered Ramsey number* $R_<(G^<)$ is the minimum $N \in \mathbb{N}$ such that every 2-coloring of the edges of the ordered complete graph $K_N^<$ on N vertices contains a monochromatic copy of $G^<$. That is, we want to find G as a monochromatic subgraph with a fixed order of vertices; see Figure 1.

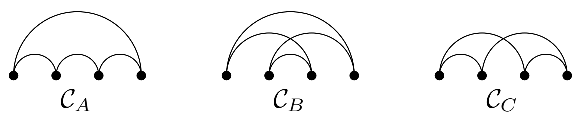

Figure 1: Pairwise non-isomorphic ordered cycles on four vertices. The ordered Ramsey numbers are $R_<(\mathcal{C}_A) = 14$, $R_<(\mathcal{C}_B) = 10$, and $R_<(\mathcal{C}_C) = 11$. The Ramsey number of (unordered) C_4 is 6.

Let $R(G)$ be the Ramsey number of G. It is easy to see that $R(G) \leq R_<(G^<)$ for each vertex-ordering $R_<$ of G. We also have $R_<(G^<) \leq R_<(K_n^<) = R(K_n)$ and thus ordered Ramsey numbers are always finite and at most exponential in the number of vertices.

In the 1980s, Chvátal, Rödl, Szemerédi, and Trotter [4] showed that the Ramsey number $R(G)$ of every n-vertex graph G with constant maximum degree is linear in n. In sharp contrast to this result, Balko, Cibulka, Král, and Kynčl [2] and, independently, Conlon, Fox, Lee, and Sudakov [3] showed that there are ordered matchings $M_n^<$ on n vertices with $R^<(M_n^<)$ superpolynomial in n. In particular, these results give $R_<(P_n^<) \geq n^{\Omega(\log n/\log\log n)}$ for some ordering $P_n^<$ of a path P_n. It is that every ordered path $P_n^<$ on n vertices satisfies $R_<(P_n^<) \leq n^{O(\log n)}$.

The growth rate of the ordered Ramsey numbers of $G^<$ decreases significantly if we bound the *interval chromatic number* of $G^<$, which is the minimum number of intervals the vertex set of $G^<$ can be partitioned into so that there is no edge of $G^<$ within one of the intervals. It is known that the ordered Ramsey number $R_<(P^<)$ of any ordered path $P^<$ on n vertices with interval chromatic number 2 is at most $O(n^3)$ while the best known lower bound is $\Omega((n/\log n)^2)$ and was actually proved for ordered matchings. Geneson, Holmes, Liu, Neeidinger, Pehova, and Wass [5] asked whether this can be improved.

Problem 1 ([5])**.** *Is it true that $R_<(P^<) \leq O(n^2)$ for every ordering $P^<$ of the path on n vertices with interval chromatic number 2?*

The quadratic estimate is trivial for ordered matchings. This problem can be considered with various modifications (lower bound, larger interval chromatic number, other sparse ordered graphs, and so on). There are many open problems in ordered Ramsey theory that we can also consider; see a recent survey [1].

References

[1] M. Balko. A Survey on Ordered Ramsey Numbers, `https://arxiv.org/abs/2502.02155`, 2025.

[2] M. Balko, J. Cibulka, K. Král, J. Kynčl. Ramsey numbers of ordered graphs, *Electron. Notes Discrete Math.*, **49**, 419–424, 2015.

[3] D. Conlon, J. Fox, C. Lee, B. Sudakov. Ordered Ramsey numbers, *J. Combin. Theory Ser. B* **122**, 353–383, 2017.

[4] V. Chvátal, V. Rödl, E. Szemerédi, W.T. Trotter Jr. The Ramsey number of a graph with bounded maximum degree, *J. Combin. Theory Ser. B* **34**(3), 239–243, 1983.

[5] J. Geneson, A. Holmes, X. Liu, D. Neeidinger, Y. Pehova, and I. Wass. Ramsey numbers of ordered graphs under graph operations, `https://arxiv.org/pdf/1902.00259`, 2019.

Ramsey non-nested matching

by János Barát

We consider the complete graph on the ordered vertex set of the first n positive integers, denoted as $[n]$. We refer to an edge xy assuming also $x < y$. Two independent edges xy and uv are *nested* if $x < u < v < y$ or $u < x < y < v$. A set F of edges are non-nested if any two independent edges of F are non-nested. We also 2-color the edges by red and blue. A subgraph S is monochromatic if all edges of S are red or all edges are blue.

Problem 1. *Does every* 2*-colored ordered complete graph on* $[3n-1]$ *contains a monochromatic non-nested matching of size* n*?*

It is easy to see that on $4n-2$ vertices, one can always find a monochromatic non-nested matching of size n. This question grew out of our research on the twisted drawing of the complete graph [2]. Plane (non-crossing) subgraphs of the twisted drawing correspond to non-nested subgraphs. We proved the following for spanning trees:

Theorem 1. *In every* 2*-coloring of the ordered complete graph on* n *vertices, there exists a monochromatic non-nested spanning tree.*

This is a subcase of a more general problem. Simple drawings are drawings of graphs in which the edges are Jordan arcs and each pair of edges shares at most one point (a proper crossing or a common endpoint).

Problem 2. *Is there a monochromatic plane spanning tree in every* 2*-colored simple drawing of the complete graph?*

This holds for geometric graphs [3] and cylindrical drawings [1].

References

[1] Oswin Aichholzer, Michael Hoffmann, Johannes Obenaus, Rosna Paul, Daniel Perz, Nadja Seiferth, Birgit Vogtenhuber, Alexandra Weinberger. Plane Spanning Trees in Edge-Colored Simple Drawings of K_n. GD 2020. LNCS Volume 12590. 482–489, (2020). `https:\doi.org/10.1007/978-3-030-68766-3_37`

[2] J. Barát, A. Gyárfás, G. Tóth. Monochromatic spanning trees and matchings in ordered complete graphs. *J. Graph Theory* 105 (4), 523–541, (2024).

[3] G. Károlyi, J. Pach, G. Tóth. Ramsey-Type Results for Geometric Graphs, I. *Discrete Comput. Geom.* 18, 247–255. (1997).

Small diameter covers

by János Barát

We 2-color the edges of a graph G by red and blue. A subgraph S is monochromatic if all edges of S are red or all edges are blue. Recently, Gyárfás and Sárközy proved the following

Theorem 1. *For any 2-coloring of the edges of a graph G with $\alpha(G) = 2$, the vertices of G can be covered by two monochromatic subgraphs of diameter at most 4.*

Problem 1. *Is there a 2-coloring of the edges of a graph G with $\alpha(G) = 2$ such that we cannot cover $V(G)$ by two monochromatic subgraphs G_1 and G_2 each of diameter at most 3?*

For general α, they proved

Theorem 2. *For any 2-coloring of the edges of a graph G, the vertices of G can be covered by $\lfloor 3\alpha/2 \rfloor$ monochromatic components, each with diameter at most 4.*

Problem 2. *Can we improve this upper bound on the number of monochromatic components?*

References

[1] A. Gyárfás, G.N. Sárközy. Bounded diameter variations of Ryser's conjecture. https://arxiv.org/pdf/2505.02564

Circuits on planar points

by János Barát, Zoli Nagy

Two years ago at Emléktábla, the group of János Barát, Andrzej Grzesik, Attila Jung, Zoli Nagy, Dömötör Pálvölgyi worked on a cycle cover problem of Nika Salia. It could be reformulated to multicolor Hamiltonain cycles as well, as follows.

Problem 1. *Let $\mathcal{C}$ be a set of colors. We assign a subset of $\mathcal{C}$ to every edge of a complete graph K_n, $n \geq 3$, such that*

- *for each color $c \in \mathcal{C}$, the edges colored c induce a clique,*
- *for any set of vertices $X \subseteq V(K_n)$ of size at least two, there exist at least $|X|$ colors assigned to at least one edge of the clique induced by X.*

The obtained colored multigraph contains a rainbow Hamiltonian cycle.

For the results and background we refer to [1]. Some interesting geometric questioned popped up, which can be considered as applications of the original graph-theoretic problem and its variant.

Problem 2. *Given n points and n halfplanes in the plane such that for every $k \geq 2$ for any k points there are at least k halfplanes that contain at least two of them, can we order the points and halfplanes $p_1, H_1, p_2, H_2, \ldots, p_n, H_n$ such that $p_i, p_{i+1} \in H_i$ for $i \in \{1, 2, \ldots, n-1\}$ and $p_n, p_1 \in H_n$?*

Another question is the following.

Problem 3. *Consider n points and n (directed) lines in the plane such that for any k points there are at least $k-1$ lines, which intersect the convex hull of the k points, and ask if we can order the points and lines $p_1, l_1, p_2, l_2, \ldots, p_n, l_n$ such that p_i and p_{i+1} are separated by l_i? Additionally, can we even demand that p_i should be on the left, and p_{i+1} on the right side of the directed line l_i?*

References

[1] Barát, J., Grzesik, A., Jung, A., Nagy, Z. L., Pálvölgyi, D. (2024). The double Hall property and cycle covers in bipartite graphs. Discrete Mathematics, 347(9), 114079.

Bisections of graphs

by Simona Boyadzhiyska

This problem is taken from a recent paper of Ma and Wu [5].

All graphs are finite, simple, and undirected. For a given graph G and a subset $A \subseteq V(G)$, we write $G[A]$ for the subgraph induced by A. We will be interested in problems of the following type: given a graph G satisfying some property, is there a partition $A \sqcup B$ of the vertex set $V(G)$ such that the graphs $G[A]$ and $G[B]$ and the bipartite graph between A and B satisfy certain desirable properties?

A classic result from graph theory shows that, for every graph G with minimum degree at least $2k-1$, there exists a partition $A \sqcup B$ of $V(G)$ such that every vertex has degree at least k to the *opposite* part. Thomassen [7] considered a similar problem where we are interested in degrees within each vertex's *own* part. More precisely, he showed the existence of a function $f(k)$ such that, for every graph G with minimum degree at least $f(k)$, there is a partition $A \sqcup B$ such that every vertex has at least k neighbors in its own part; this function $f(k)$ was determined to be equal to $2k+1$ by Stiebitz [6]. It is natural to also wonder whether it is possible to require that each vertex have many neighbors *both* in its own part and in the opposite part. This was proven to be false by Kühn and Osthus [4], but a number of relaxations are known to be true. These results, in addition to being reasonably natural, also have some applications, for example to problems in structural graph theory. More about these types of problems and their history can be found for instance in [4, 5] and the references therein.

Here we are interested in partitions where the sizes of the parts are as equal as possible, that is, partitions $A \sqcup B$ such that $|\,|A| - |B|\,| \leq 1$. We call such a partition a *bisection*. Inspired by the aforementioned results, one might hope to find a bisection where each vertex has degree at least (roughly) $\deg(v)/2$ to the opposite side. Refuting a stronger conjecture of Bollobás and Scott [1], Ji, Ma, Yan, and Yu [3] conjectured that every graph G has a bisection in which every vertex v has degree at least $\deg(v)/2 - c$ to the opposite side (here $c > 0$ is an absolute constant). The best known general results in this direction are due to Ma and Wu [5] and show that we can always achieve $\deg(v)/4 - o(1)$ in each of the two settings.

Theorem 1. *Every graph G has a bisection such that every vertex v has at least $\frac{\deg(v)}{4} - o(\deg(v))$ neighbors in the opposite part and a bisection such that every vertex v has at least $\frac{\deg(v)}{4} - o(\deg(v))$ neighbors in its own part.*

Naturally, it would be extremely interesting to improve this result and replace the constant 1/4 by 1/2, even in this somewhat weaker asymptotic setting. More specifically, Ma and Wu formulated the following problem.

Problem 1 ([5, Question 19])**.** *Is there a function $f(k) = 2k + o(k)$ such that every graph with minimum degree at least $f(k)$ has a bisection where each vertex has at least k neighbors in its own part, as well as a bisection where each vertex has at least k neighbors in the opposite part?*

A related problem that is worth considering is concerned with connectivity instead of minimum degrees. Recall that a graph G on at least $k+1$ vertices is k-connected if, for every subset $S \subseteq V(G)$ of size at most $k-1$, removing S leaves a connected graph, that is, the graph $G[V(G) - S]$ is connected.

Thomassen [7] showed that there exists a function $f(s,t)$ such that any $f(s,t)$-connected graph G has a partition $A \sqcup B$ such that $G[A]$ is s-connected and $G[B]$ is t-connected. The bounds on this function were later improved for example by Hajnal [2]. It would be interesting to determine whether a similar statement holds for bisections.

Problem 2 ([5, Question 22])**.** *Is there a function $h(t)$ such that every $h(t)$-connected graph G has a bisection $A \sqcup B$ such that both $G[A]$ and $G[B]$ are t-connected? If so, what bounds can we obtain?*

Naturally, the above question can also be considered in the asymmetric setting, as in the original result of Thomassen.

References

[1] B. Bollobás and A. Scott, Problems and results on judicious partitions. Random Structures & Algorithms. 21(3-4):414–430, 2002.

[2] P. Hajnal, Partition of graphs with condition on the connectivity and minimum degree. Combinatorica, 3:95–99, 1983.

[3] Y. Ji, J. Ma, J. Yan, and X. Yu, On problems about judicious bipartitions of graphs. Journal of Combinatorial Theory, Series B, 139:230–250, 2019.

[4] D. Kühn and D. Osthus, Partitions of graphs with high minimum degree or connectivity. Journal of Combinatorial Theory, Series B, 88(1):29-43, 2003.

[5] J. Ma and H. Wu, Bisections of graphs under degree constraints. `arXiv:2504.15096`, 2025.

[6] M. Stiebitz, Decomposing graphs under degree constraints. Journal of Graph Theory, 23(3):321–324, 1996.

[7] C. Thomassen, Graph decomposition with constraints on the connectivity and minimum degree. Journal of Graph Theory, 7(2): 165–167, 1983.

Monochromatic and rainbow monotone paths

by Andrea Freschi

For every $n \in \mathbb{Z}^+$, we write $[n]$ to denote the set of the first n positive integers. We say a path $x_1x_2 \dots x_s$ with vertices in $[n]$ is monotone if $x_1 < x_2 < \dots < x_s$. Given an edge-colouring of the complete graph with vertex set $[n]$, an ℓ-flash is a monochromatic monotone path with ℓ edges. Similarly, a k-rainbow is a rainbow monotone path with k edges. We let $f(\ell, k)$ be the smallest $n \in \mathbb{Z}^+$ such that every edge-colouring of the complete graph with vertex set $[n]$ yields either an ℓ-flash or a k-rainbow.

Lefmann, Rödl and Thomas proposed the following conjecture.

Conjecture 1 (Lefmann, Rödl and Thomas [3]). *For every $k, \ell \in \mathbb{Z}^+$ we have $f(\ell, k) = \ell^{k-1} + 1$.*

The following construction shows the conjecture is sharp. Label the vertices of $[\ell^{k-1}]$ with strings $\{1, 2, \dots, \ell\}^{k-1}$ in lexicographic order. For every $x, y \in [n]$ with $x < y$, colour the edge xy with colour $i \in [k-1]$ so that the ith bit of the string associated to x is strictly smaller than the ith bit of the string associated to y. It is easy to check that such edge-colouring does not yield an ℓ-flash nor a k-rainbow.

Lefmann, Rödl and Thomas proved that the conjecture is true provided ℓ is substantially larger than k. This was recently improved by Girão et al.

Theorem 1 (Girão, Illingworth, Michel, Savery and Scott [1]). *For all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that if $\ell \geq C_\varepsilon k^3 (\log k)^{1+\varepsilon}$, then $f(\ell, k) = \ell^{k-1} + 1$.*

Conversely, the conjeture remains open when ℓ is small compared to k. Lefmann, Rödl and Thomas verified the conjecture for $\ell \leq 2$ and $k \leq 4$. The case $\ell = 3$ is still open and of interest.

Problem 2. *Show that $f(3, k) = 3^{k-1} + 1$.*

Girão et al. [1] also introduced a natural generalisation of Conjecture 1 to tournaments. Recall that a tournament is a complete graph whose edges are assigned an orientation. A walk $x_1x_2 \dots x_s$ within a tournament is oriented if the edge x_ix_{i+1} is oriented from x_i to x_{i+1}, for every i. Given an edge-colouring of a tournament, an ℓ-flash is a monochromatic oriented walk with ℓ edges.[1] Similarly, a k-rainbow is a rainbow oriented walk with k edges. We let $t(\ell, k)$ be the smallest $n \in \mathbb{Z}^+$ such that every edge-colouring of an n-vertex tournament yields either an ℓ-flash or a k-rainbow.

Conjecture 3 (Girão, Illingworth, Michel, Savery and Scott [1]). *For every $k, \ell \in \mathbb{Z}^+$ we have $t(\ell, k) = \ell^{k-1} + 1$.*

Note that the special case of Conjecture 3 where the tournament is transitive corresponds precisely to Conjecture 1. Conjecture 3 is wide open for any $\ell \geq 2$ and $k \geq 3$.

Problem 4. *Show that $t(2, k) = 2^{k-1} + 1$.*

In [1] the authors state that resolving Problem 4 would immediately imply an analogue of Theorem 1 for $t(\ell, k)$, using their proof methods.

It is worth to remark that researchers have also proved upper bounds for $f(\ell, k)$ and $t(\ell, k)$ that hold for every $k, \ell \in \mathbb{Z}^+$, see [2] and [1].

[1] By ℓ edges, we mean that the walk has the form $x_1x_2 \dots x_{\ell+1}$ where an edge may appear multiple times along the walk.

References

[1] A. Girão, F. Illingworth, L. Michel, M. Savery and A. Scott. Flashes and rainbows in tournaments. *Combinatorica* **44** (2024), 675–690.

[2] T. Jiang and D. Mubayi, New upper bounds for a canonical Ramsey problem. *Combinatorica* **20** (2000), 141–146.

[3] H. Lefmann, V. Rödl and R. Thomas, Monochromatic Vs multicolored paths. *Graphs and Combinatorics* **8** (1992), 323–332.

The Anti-Ramsey number of perfect matching in k-graphs

by Nina Kamčev, Tadej Petar Tukara

The anti-Ramsey number $ar(n,k,M_s)$ of an s-matching M_s is the smallest integer c such that any (surjective) c-colouring of the complete n-vertex k-graph $K_n^{(k)}$ contains a *rainbow* s-matching. We propose the following conjecture of Guo, Lu and Peng.

Problem 1. *For some $\epsilon = \epsilon(k)$, there is n such that for $n \leq ks(1+\epsilon)$,*

$$ar(n,k,M_s) = \begin{cases} \binom{k(s-1)-1}{k} + 2, & n \in (ks, ks(1+\epsilon)), \\ \binom{k(s-1)-1}{k} + \binom{k}{k/2-1} + 2, & n = ks, k \text{ even}, \\ \binom{k(s-1)-1}{k} + \frac{1}{2}\binom{k+1}{(k+1)/2} + 2, & n = ks, k \text{ odd}. \end{cases}$$

The conjecture is proved for $k = 3$ by Guo, Lu and Peng[1], and we suggest proving it for arbitrary k and $n = ks$ (perfect matchings). We believe that the case $n = ks$ is the most interesting because of the surprising lower-bound construction (see below) which distinguishes it from all other cases. Problem 2 is an elementary problem which we think implies Problem 1. Before stating it, let us discuss the problem more closely.

The lower bound $ar(n,k,M_s) > \binom{k(s-1)-1}{k} + 1$ comes from the following construction proposed by Özkahya and Young. If G is an M_{s-1}-free k-graph (e.g. a clique on $k(s-1)-1$ vertices), then one can colour G injectively, and $K_n^{(k)} \setminus G$ with a new colour, say '*' to obtain a lower bound of

$$ar(n,k,M_s) > ex(n,k,M_{s-1}) + 1. \tag{0.1}$$

For $n = ks$, Guo, Lu and Peng found a better construction. We summarise this construction for $k = 3$, which is also tight: let $W = \{n, n-1, n-2, n-3\}$. Colour the edges in $[n] \setminus W$ injectively (note that these edges also form the maximal M_{s-1}-free hypergraph). Then introduce three now colours, $c_S = c_{W \setminus S}$ for $S \subset W$ with $|S| = 2$, and colour each e with $e \cap W = S$ by c_S. The remaining edges all get colour *. This construction uses three more colours than (0.1) due to Özkahya and Young.

The following is an elementary problem which *should* imply Problem 1, but this implication should certainly be checked.

Problem 2. *Let W be a set of size $k+1$, and let $P(W)$ denote the family of all subsets of W, excluding $\emptyset$ and W. Let $f : P(W) \to \mathbb{N}$ be a colouring such that for any partition $S_1 \sqcup S_2 \sqcup \cdots \sqcup S_t$ of W,*

$$|\{f(S_1), f(S_2), \ldots, f(S_t)\}| \leq t - 1.$$

Then the image of f contains at most

$$\begin{cases} \binom{k}{k/2-1} + 1, & k \text{ even}, \\ \frac{1}{2}\binom{k+1}{(k+1)/2} + 1, & k \text{ odd} \end{cases}$$

colours.

Remark $n > ks(1+\epsilon)$. There is a close connection between $ex(n,k,M_s)$, the extremal number of M_s (i.e. the Erdős matching conjecture), and the corresponding anti-Ramsey problem, as indicated by (0.1). However, for larger n there is another competing construction for $ex(n,k,M_s)$, so the problem gets perhaps more complex.

References

[1] Guo, M., Lu, H., Peng, X. (2023). Anti-Ramsey number of matchings in 3-uniform hypergraphs. SIAM Journal on Discrete Mathematics, 37(3), 1970-1987.https://arxiv.org/pdf/2305.15631

An Ore-type theorem for Berge cycles in hypergraphs

by Nina Kamčev

Ore's theorem is a strengthening of Dirac's theorem. It states that if a graph G has the property that $d(u) + d(v) \geq |V(G)|$ for any two non-adjacent vertices u and v, then G is Hamiltonian.

There are numerous hypergraph extensions of Dirac's theorem, as cycles and degrees can be defined in multiple ways. Ore's theorem is less explored, and Li, Lu and Luo recently proposed the following problem on Berge cycles [1].

A *Berge cycle* of length t is a collection of t distinct edges $e_1, e_2, \dots, e_t \in E(\mathcal{H})$ and t distinct vertices $v_1, v_2, \dots, v_t \in V(\mathcal{H})$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for every $i \in [t]$, where $v_{t+1} = v_1$. A *Hamiltonian Berge cycle* of a hypergraph $\mathcal{H}$ is a Berge cycle with $t = n$.

The 2-shadow (or shadow) of a hypergraph $\mathcal{H}$, denoted by $\partial\mathcal{H}$, is the simple 2-uniform graph with $V(\partial\mathcal{H}) = V(\mathcal{H})$ and $uv \in E(\partial\mathcal{H})$ if and only if $\{u, v\} \subset e$ for some $e \in \mathcal{H}$.

Problem 1. *Let $r \geq 3$ be a fixed integer. There exists $d_0 = d_0(r)$ and $n_0 = n_0(r)$ such that for every $n \geq n_0$ the following holds: if $\mathcal{H}$ is an n-vertex r-uniform hypergraph such that every pair of nonadjacent vertices $u, v \in V(\mathcal{H})$ satisfies*

$$d_{\partial\mathcal{H}}(u) + d_{\partial\mathcal{H}}(v) \geq n + d_0,$$

then $\mathcal{H}$ contains a Hamiltonian Berge cycle.

Li, Lu and Luo proved the conjecture for $r = 3$, but they also conjectured that in this case, the statement should hold with $d_0 = 1$. They remark that their methods do not seem to address the case $r \geq 4$.

If true, the statement would be sharp, by considering a complete $n-1$-vertex hypergraph with an additional vertex of degree 1.

References

[1] Li, Y., Lu, L., Luo, R. (2025). An Ore-type theorem for [3]-graphs. arXiv preprint arXiv:2505.12035.

Covering the permutohedron by hyperplanes

by Gyula Károlyi

Consider those points in the n-dimensional space whose coordinates form a permutation of the first n positive integers. The elements of this set P_n are the vertices of a convex $(n-1)$-dimensional polytope called the permutohedron (spelled also as permutahedron) Π_{n-1}. This polytope has many fascinating properties and can be used to illustrate various concepts in geometry, combinatorics and group theory. An almost cover of a finite set in the affine space is a collection of hyperplanes that together cover all points of the set except one. An application of the polynomial method yields the following analogue of the Alon-Füredi theorem [1]; see [2].

Theorem 1. *Every almost cover of the vertices of Π_{n-1} consists of at least $\binom{n}{2}$ hyperplanes. This bound is sharp.*

In comparison, all vertices of Π_{n-1} are contained in just one hyperplane H_0 of equation $f_0(x) = \sum_{i=1}^{n} x_i - \binom{n+1}{2} = 0$, and even when Π_{n-1} is embedded into $\mathbb{R}^{n-1}$, its vertices can be covered by n hyperplanes.

Problem 1. *How many affine hyperplanes different from H_0 are needed to cover all points of P_n?*

It is clear that the hyperplanes of equation $x_1 = i$, $1 \le i \le n$ cover P_n. Each of them contains exactly $(n-1)!$ points of P_n, and their intersections with P_n are pairwise disjoint. The same is true for the hyperplanes of equation $x_i = 1$. If $n \ge 4$ is even, then one can do even better: The hyperplanes of equation $x_1 + x_j = n + 1$, $2 \le j \le n$ cover P_n. Each of them contains exactly $n(n-2)!$ points of P_n, and their intersections with P_n are pairwise disjoint. We believe that these examples are extremal in the following sense.

Conjecture 2. *Suppose that the vertex set of Π_{n-1} is contained in the union of the hyperplanes $H_1, \ldots, H_m$ different from H_0. If n is odd, then $m \ge n$. If $n \ge 4$ is even, then $m \ge n-1$.*

Consider a hyperplane H not parallel to H_0, it intersects H_0 in a 1-codimensional affine subspace. To cover P_n by n or less such hyperplanes one needs to find such an H which intersects P_n in at least $(n-1)!$ points. H has an equation of the form $f(x) = 0$, where f is a linear polynomial, in this case we write $H = H(f)$. Then $H(g) \cap H_0 = H(f) \cap H_0$ if and only if there exist $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ such that $g = \alpha f + \beta f_0$. Apart from such equivalence, it seems that $H(f)$ intersects P_n in more than $(n-1)!$ points if and only if n is even and $f = x_i + x_j - (n+1)$ for some $i \neq j$. $|H(f) \cap P_n| = (n-1)!$ occurs in each dimension for $f = x_i - x_j - 1$, and also for $f = x_i + x_j - n$, $f = x_i + x_j - (n+1)$, $f = x_i + x_j - (n+2)$ when n is odd. From these examples one can construct various economical hyperplane covers of P_n. For example, the hyperplanes of equation $x_n = 1$, $x_n - x_i = 1$, $1 \le i \le n-1$ cover P_n for every n, whereas the hyperplanes of equation $x_1 = (n+1)/2$, $x_1 + x_j = n + 1$, $2 \le j \le n$ cover P_n when n is odd.

Conjecture 3. *If n is odd, then every hyperplane different from H_0 contains at most $(n-1)!$ points of P_n.*

Were this true, it would easily imply Conjecture 2.

References

[1] N. Alon, Z. Füredi. Covering the cube by affine hyperplanes. European J. Comb., 14:79–83, 1993.

[2] G. Hegedüs, Gy. Károlyi. Covering the permutohedron by affine hyperplanes. Acta Math. Hungar., 174:453–461, 2024.

Zarankiewicz problem for polygon visibility graphs

by Balázs Keszegh

Given a simple polygon P and two points $p, q \in P$ we say that p and q are *mutually visible* with respect to P (or that they 'see' each other), if the straight-line segment $\overline{pq}$ is disjoint from the exterior of P. The *visibility graph* of P consists of vertices that correspond to the vertices of P and edges that correspond to pairs of mutually visible vertices.

Problem 1. *Does every n-vertex $K_{t,t}$-free polygon visibility graph have $O_t(n)$ edges?*

Induced subgraphs of polygon visibility graphs are equivalent to *curve visibility graphs* which are visibility graphs of points on a Jordan curve. Two vertices in such a graph are adjacent if the straight-line segment between their corresponding points is disjoint from the exterior of the curve. Du and McCarty [3] mention the following open problem which is a generalization of the above problem:

Problem 2 ([3]). *Does every n-vertex $K_{t,t}$-free curve visibility graph have $O_t(n)$ edges?*

With Ackerman [1] we proved various almost-linear upper bounds for both cases and linear upper bounds in some special cases. On the other hand we showed that the answer is false for the even more general class of curve pseudo-visibility graphs. See also the survey of Davies et al. [2] for other related problems.

References

[1] E. Ackerman, B. Keszegh, The Zarankiewicz Problem for Polygon Visibility Graphs, https://arxiv.org/abs/2503.09115

[2] J. Davies, T. Krawczyk, R. McCarty, B. Walczak, Coloring polygon visibility graphs and their generalizations, Journal of Combinatorial Theory, Series B 161 (2023), 268–300, https://doi.org/10.1016/j.jctb.2023.02.010

[3] X. Du and R. McCarty, A survey of degree-boundedness, European Journal of Combinatorics (2024), https://doi.org/10.1016/j.ejc.2024.104092

Balanced tricolor hexagon free colorings

by Zoltán Lóránt Nagy

Definition 1. *A balanced tricolor hexagon is a C_6 edge-colored with 3 colors, in which opposite edges are colored with the same color.*

Problem 1. *Is it true that if $n \in \mathbb{N}$ is large enough that there exists a 1-factorisation (a.k.a. proper $(2n-1)$-coloring of the edge set) of K_{2n} with the following property: it does not contain balanced tricolor hexagons.*

A variant of this just asks for infinitely many values of n, or a dense subset $A \subset \mathbb{N}$ where it holds for all $n \in A$.

Context:

An analogue of this problem asks for the existence of a 1-factorisation of K_{2n} that does not contain balanced bicolor quadrilaterals. (Perfect 1-factorisations work fine, for instance, where a 1-factorisation is perfect if every pair of 1-factors of the factorization induces a Hamiltonian cycle; see e.g. [4].)

Motivation: to prove a Conjecture of Füredi and Ruszinkó [1] on the linear Turán number of grids $ex^{(3)}_{lin}(n, G_{3\times 3})$ grids; if the answer for Problem is affirmative, I can deduce the statement of the conjecture.

Closely related: Erdős-Brown-Sós girth problem, and a conjecture of Linial, which can be seen as 1-factorization problem in a bipartite setting. Linial defines a *cycle* in a Latin square L to be a set of rows A, a set of columns B, and a set of symbols C, with $|A| + |B| + |C| > 3$, such that the $A \times B$ subarray of L contains at least $|A| + |B| + |C| - 2$ symbols in the set C. He defines the girth of a Latin square L to be the minimum of $|A| + |B| + |C|$ over all such cycles in L. These definitions are motivated by the Brown–Erdős–Sós problem in extremal hypergraph theory, and in particular by an old conjecture of Erdős on the existence of high-girth Steiner triple systems, which we recently proved in [3].

Conjecture 2 (Linial, proven in [2] by Kwan, Sah, Sawhney, & Simkin)**.** *If $N > N_{(g)}$, then there exists an $N \times N$ Latin square without "cycles" shorter than g.*

References

[1] Füredi, Z., Ruszinkó, M. (2013). Uniform hypergraphs containing no grids. Advances in Mathematics, 240, 302-324.

[2] Kwan, M., Sah, A., Sawhney, M., Simkin, M. (2023). Substructures in Latin squares. Israel Journal of Mathematics, 256(2), 363-416.

[3] Kwan, M., Sah, A., Sawhney, M., Simkin, M. (2024). High-girth Steiner triple systems. Annals of Mathematics, 200(3), 1059-1156.

[4] Rosa, A. (2019). Perfect 1-factorizations. Mathematica Slovaca, 69(3), 479-496.

Chromatic number of Davenport-Schinzel graph

by Dömötör

Problem 1. *Given a finite collection of planar segments, define a graph whose vertices are the segments and uv is an edge if segments u and v intersect such that there is no segment directly under their intersection point, i.e., on the vertical downward halfline starting at $u \cap v$. Is the chromatic number of this graph bounded?*

It is known that such a graph, whose edges correspond to the complexity of the lower envelope, can have a slightly superlinear number of edges; see `https://en.wikipedia.org/wiki/Davenport-Schinzel_sequence`. The connection to Davenport-Schinzel sequences gives the following almost equivalent reformulation.

Problem 2. *Given a sequence of symbols, with no $A..B..A..B$ pattern, define a graph whose vertices are the letters and AB is an edge if the letters A and B are somewhere adjacent. Is the chromatic number of this graph bounded?*

Small Teaching Sets

by Dömötör

This is a long-standing open problem:

Problem 1. *If the VC-dimension of a family $\mathcal{F}$ is d, then is there a set $F \in \mathcal{F}$ and a set of $O(d)$ elements, $X = \{x_1, \ldots, x_{O(d)}\}$ such that in the restriction of $\mathcal{F}$ to X no other set is mapped to F's image?*

The best bound that is known is that there is an X of size $O(d^2)$. For a summary of results, and why the greedy algorithm doesn't work, see Lower Bounds for Greedy Teaching Set Constructions by Spencer Compton, Chirag Pabbaraju, Nikita Zhivotovskiy `https://arxiv.org/abs/2505.03223`.

Poset saturation

by Dömötör

For a finite poset $\mathcal{P}$, a family $\mathcal{F}$ of subsets of $[n]$ is called $\mathcal{P}$-saturated if it does not contain an induced copy of $\mathcal{P}$, yet adding any other set to $\mathcal{F}$ creates such a copy. The minimum size of a $\mathcal{P}$-saturated family is denoted by $\mathrm{sat}^*(n, \mathcal{P})$. The star is there because $\mathrm{sat}(n, P)$ stands for non-induced saturation, but that is less interesting. Somewhat selfishly, I recommend Balázs Keszegh, Nathan Lemons, Ryan R. Martin, Dömötör Pálvölgyi, Balázs Patkós: Induced and non-induced poset saturation problems https://arxiv.org/abs/2003.04282 as an introduction.

My favorite open problems about this are the following.

Problem 1. *Is it decidable for a poset P whether sat$^*(n, P)$ is bounded or not?*

We know almost nothing about this, except that certain glueing operations don't change boundedness; see Gluing Posets and the Dichotomy of Poset Saturation Numbers by Maria-Romina Ivan, Sean Jaffe https://arxiv.org/abs/2503.12223.

However, the following is still open.

Problem 2. *Is sat$^*(n, P_2 * P_1)$ bounded if and only if sat$^*(n, P_1)$ and sat$^*(n, P_2)$ are both bounded?*

Here $P_2 * P_1$ means that the poset P_1 is over the poset P_2 (or maybe the other way around, right now I don't remember).

Problem 3. *For which k is it true that if we take the poset of k independent chains on two elements, kC_2, then sat$^*(n, kC_2)$ is bounded?*

It was shown in Poset saturation of unions of chains by Shengjin Ji, Balázs Patkós, Erfei Yue https://arxiv.org/abs/2505.23128 that this is bounded for all $k = \binom{2t}{t} + 1$, while unboundedness is only known for $k = 2$; the best lower bound is $\frac{3n+1}{2}$, proved in Induced Saturation of the Poset $2C_2$ by Ryan R Martin, Nick Veldt https://arxiv.org/abs/2408.14648.

Problem 4. *Is sat$^*(n, P) = O(1)$ or $\Theta(n)$ for every poset P?*

Here from below the best bound is $\Omega(\sqrt{n})$ (see The induced saturation problem for posets by Andrea Freschi, Simón Piga, Maryam Sharifzadeh, Andrew Treglown https://arxiv.org/abs/2207.03974) while from above only $O(n^c)$ is known, where $c = c(P)$ (see A Polynomial Upper Bound for Poset Saturation by Paul Bastide, Carla Groenland, Maria-Romina Ivan, Tom Johnston https://arxiv.org/abs/2310.04634). Of particular interest is the so-called diamond poset, for which very recently a lower bound of $\frac{n+1}{5}$ was proved; see The Saturation Number for the Diamond is Linear by Maria-Romina Ivan, Sean Jaffe https://arxiv.org/abs/2507.05122.

Planar Lattices and Equilateral Odd-gons

by Máté Vizer

Theorem 1 (Ball, 1973)**.** *The square lattice $\mathbb{Z}^2$ does not contain an equilateral n-gon if n is odd. The lattice $\mathbb{Z}^2$ contains a convex equilateral n-gon if n is even.*

Definition 2. *A planar lattice L is called an* integral lattice *if for every $x, y \in L$, the inner product $x \cdot y$ is an integer.*

Definition 3. *For a planar lattice L, let $D(L)$ denote the area of a fundamental parallelogram of the lattice L. The square-free part of $D(L)^2$ is denoted by $\nu(L)$.*

Theorem 4 (Maehara)**.** *A planar lattice L contains a convex equilateral n-gon for some $n \neq 4$ if and only if L is similar to an integral lattice.*

Theorem 5 (Maehara)**.** *Every planar integral lattice L contains a convex equilateral n-gon for every even $n \geq 4$. A planar integral lattice L contains an equilateral n-gon for some odd $n \geq 3$ if and only if $\nu(L) \equiv 3 \pmod 4$.*

Theorem 6 (Maehara)**.** *For a planar lattice L, the following three are equivalent:*

1. *L contains an equilateral triangle.*
2. *L contains a convex equilateral n-gon for every $n \geq 3$.*
3. *L is similar to an integral lattice L' with $\nu(L') = 3$.*

Theorem 7 (Iino, Sakiyama)**.** *Let $n \geq 3$ be an odd number. If a planar integral lattice L contains an equilateral n-gon, not necessarily convex, then $n \geq p$ for every prime factor p of $\nu(L)$.*

Problem 1. *Let n be an odd integer with $n \geq 3$. Find the condition on a planar lattice L so that L contains an equilateral n-gon.*

For small values of n, this problem has been solved with the help of computer-aided searches:

Theorem 8. *Let n be an odd integer with $3 \leq n < 29$. For a planar lattice L, the following three conditions are equivalent:*

1. *L contains an equilateral n-gon.*
2. *L contains a convex equilateral k-gon for every integer $k \geq n$.*
3. *L is similar to an integral lattice L' such that $\nu(L') \equiv 3 \pmod 4$ and the largest prime factor p of $\nu(L')$ satisfies $p \leq n$.*

Problem 2. *Find a non-computer-aided proof of the above theorem.*

However, for larger values of n, a complete answer remains unknown.

Conjecture 3. *The above theorem holds for all odd integers $n \geq 3$.*

Additional problem for planar lattices:

Problem 4. *Is the existence of a convex and a non-convex equilateral n-gon in a planar integral lattice always equivalent? That is, does the existence of one imply the existence of the other for all n and all planar integral lattices? What about non-integral lattices?*

References

[1] D. G. Ball, *The constructibility of regular and equilateral polygons on a square pinboard*, The Mathematical Gazette, 57:119–122, 1973.

[2] A. Iino, M. Sakiyama, *Planar lattices and equilateral odd-gons*, **arXiv:** `https://arxiv.org/abs/2503.01911`

[3] H. Maehara, *Equilateral polygons and planar integral lattices*, Education and Development, 6:121–132, 2012.

[4] H. Maehara, *Planar lattices and equilateral polygons*, European Journal of Combinatorics, 80:277–286, 2019.

On the order of intersecting hypergraphs

by Máté Vizer

Definition 1. *The* order *of a hypergraph $\mathcal{H}$ is the number of non-isolated vertices in $\mathcal{H}$, denoted by $Ord(\mathcal{H})$. A hypergraph is called* intersecting *if every pair of distinct hyperedges has a non-empty intersection. A hypergraph $\mathcal{H}$ is called λ-intersecting if every pair of distinct hyperedges shares exactly λ vertices.*

Theorem 2 (Cambie, Kim, Lee, Liu, Tran)**.** *Let $k, \lambda \in \mathbb{N}$ with $\lambda < k$ and $\mathcal{H}$ be a λ-intersecting k-uniform hypergrah that is not a sunflower. Then*

$$Ord(\mathcal{H}) \leq \frac{4}{27}(k-\lambda)^3 + O\big(\lambda(k-\lambda)^2 + (k-\lambda)^{5/2}\big).$$

Moreover, this bound is asymptotically tight when $\lambda = o(k)$.

There is a similar notion of sunflower for intersecting hypergraphs, namely *trivial intersecting hypergraphs*, in which all edges share a common vertex. Note that being a non-trivial intersecting k-graph is a stricter condition than being a non-sunflower.

Problem 1. *For fixed λ and sufficiently large k, is it true that a non-trivial λ-intersecting k-uniform hypergraph has order bounded by $(1+o(1))\frac{4k^3}{27\lambda}$?*

References

[1] S. Cambie, J. Kim, H. Lee, H. Liu, T. Tran, *On the order of intersecting hypergraphs*, arXiv: `https://arxiv.org/abs/2504.05162`

Erdős–Szekeres with colored pairs

by Ji Zeng

Let $f(n)$ be the smallest number N such that within any N points in the plane in general position with their pairs colored red or blue, there exists n points in convex position with their pairs all having the same color.

Problem 1. *Give proper estimate of $f(n)$.*

Let $R(P_n, P_n, P_4, P_4, K_4)$ be the **3-uniform vertex-ordered Ramsey number** for an n-path in the first color or the second color, a 4-path in the third color or the fourth color, and a 4-clique in the fifth color. I can encode the geometric and chromatic information to argue that $f(n) < R(P_n, P_n, P_4, P_4, K_4)$. I can also argue that $R(P_n, P_n, P_4, P_4, K_4) < 2^{O(n^2 \log n)}$.

Problem 2. *Is is true that $R(P_n, P_n, K_4) < 2^{O(n)}$?*

Let me remark that the Erdős–Szekeres cup-cap theorem is essentially $R(P_n, P_n) = \binom{2n-4}{n-2} = 2^{O(n)}$. Of course we can also pivot the direction and work on variations of this problem.