15/16th Emléktábla Workshop

Extremal combinatorics / Matroid theory

07.14 - 07.18.2024.

Preliminary Schedule for the Extremal Combinatorics group:

Sunday

Arrival at the hotel before 12:30.
12:30-14:00 Lunch
15:00 Short introduction/presentation of problems
18:00 Dinner

Monday-Wednesday

7:30-8:30 Breakfast
9:00-17:00 Working in groups of 3-5
12:20-14:00 Lunch
16:45 Presentations of daily progress
18:00 Dinner

Thursday

7:30-8:30 Breakfast
9:00-17:00 Working in groups of 3-5
12:20-14:00 Lunch
14:15 Summary of the progress
Check-out.

List of Participants

Sam Adriaensen, Vrije Univeriteit Brussel
Péter Ágoston, Rényi Institute
Yandong Bai, Northwestern Polytechnical University, Xi'an
János Barát, University of Pannónia
Anurag Bishnoi, Delft University of Technology
Zoltán Blázsik, Szeged
Calum Buchanan, University of Vermont
Neal Bushaw, Virginia Commonwealth University
Márk Di-Giovanni, Budapest
Panna Tímea Fekete, ELTE
Dániel Gerbner, Rényi Institute
Andrzej Grzesik, Jagiellonian University
Ervin Győri, Rényi Institute
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Gaurav Kucheriya, Charles University
Craig Larson, Virginia University
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Kenneth Moore, University of British Columbia
Dániel Nagy, Rényi Institute
Kartal Nagy, ELTE
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Bo Ning, Nankai University
Cory Palmer, Uni. Montana
Magdalena Prorok, AGH University Kraków
Dömötör Pálvölgyi, ELTE-MTA
Puck Rombach, University of Vermont
Ananthakrishnan Ravi, Delft University of Technology
Nika Salia, King Fahd University
Gabor Somlai, ELTE
Dávid Szabó, ELTE
Casey Tompkins, Rényi Institute
Máté Vizer, Rényi Institute
Shenggui Zhang, Northwestern Polytechnical University, Xi'an
Zeyu Zheng, Carnegie Mellon

The span-closed sets conjecture

by Sam Adriaensen

A family $\mathcal{F}$ of sets is called *union-closed* if for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ is also contained in $\mathcal{F}$. The *union-closed sets conjecture* is a long-standing conjecture in extremal set theory, stating that for every finite union-closed family $\mathcal{F} \neq \{\emptyset\}$, there exists an element x lying in at least half of the members of $\mathcal{F}$. Using Shannon's entropy, it was recently proved that there is an element x lying in at least 38% of the members of $\mathcal{F}$ [1, 2].

Note that there are union-closed families in which no element lies in more than half of its members, e.g. when is the power set of a finite set.

Typically, a problem in combinatorial set theory has a *q-analog*. This means that we translate the problem from a set to an $\mathbb{F}_q$-vector space. The notion of a "subset of size k" is transformed into a "subspace of dimension k", and the notion of "union" is replaced by "(linear) span".

Call a family $\mathcal{F}$ of subspaces of $V := \mathbb{F}_q^n$ *span-closed* if for any two subspaces $A, B \in \mathcal{F}$, their span $\langle A, B \rangle$ is also contained in $\mathcal{F}$. Let $\binom{V}{k}_q$ denote the set of k-dimensional subspaces of V.

Define the quantity $f(\mathcal{F})$ as

$$f(\mathcal{F}) = \max_{x \in \binom{V}{1}_q} \frac{|\{A \in \mathcal{F} \,\|\, x \leq A\}|}{|\mathcal{F}|}.$$

Problem 1 (Span-closed sets conjecture)**.** *Prove (or disprove) that among all span-closed families $\neq \{\mathbf{0}\}$ of $V = \mathbb{F}_q^n$, the quantity $f(\mathcal{F})$ is minimised by taking $\mathcal{F}$ to be the family of all subspaces of V.*

There are however multiple ways to yield sensible q-analog of the union-closed sets conjecture. Let $[n]$ denote the set $\{1, \ldots, n\}$ and let $2^[n]$ denote its power set. The union-closed sets conjecture can be reformulated as follows: For every non-empty finite union-closed family $\mathcal{F} \subseteq 2^[n]$, there exists an $(n-1)$-element subset $U \subset [n]$ containing at most half of the members of $\mathcal{F}$.

Define the quantity $g(\mathcal{F})$ as

$$g(\mathcal{F}) = \min_{U \in \binom{V}{n-1}_q} \frac{|\{A \in \mathcal{F} \,\|\, A \not\leq U\}|}{|\mathcal{F}|}.$$

Problem 2 (Span-closed sets conjecture, second version)**.** *Prove (or disprove) that among all non-empty span-closed families of $V = \mathbb{F}_q^n$, the quantity $g(\mathcal{F})$ is maximised by taking $\mathcal{F}$ to be the family of all subspaces of V.*

Note that by switching to orthogonal complements, the second version can be restated as looking for the 1-dimensional subspace that occurs the least as a subspace of the elements of an *intersection-closed* family in $\mathbb{F}_q^n$.

References

[1] J. Gilmer. A constant lower bound for the union-closed sets conjecture. arXiv:2211.09055, 2022.

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The maximum number of edges in a K_6-minor-free graph of girth 5

by János Barát

We use n for the number of vertices of G, and m for the number of edges. The following is a fundamental question in extremal graph theory:
How many edges can an n-vertex K_t-minor-free graph have and what do the extremal graphs look like?

We would like to add a girth condition.
What is the maximum number of edges in a K_t-minor-free graph with n vertices and girth g?

We obtained some results in this direction. In particular for $t = 4, 5$ and $g = 5$, see [1, 2]. Aigner-Horev and Krakovski [3] proved that any K_6-minor-free graph of girth 6 has at most $3n - 6$ edges. We are still mostly interested in the case $g = 5$.

Problem 1. *What is the maximum number of edges in a K_6-minor-free graph with n vertices and girth 5?*

This is probably a difficult problem. We have a conjecture. If it is true, then it would have consequences to list coloring. There is a recent paper, which could be relevant: [4].

References

[1] János Barát. Extremal K_4-minor-free graphs without short cycles. *Periodica Mathematica Hungarica*, 86: 108–114, 2023.

[2] János Barát. On the number of edges in a K_5-minor-free graph of given girth. *Discrete Mathematics* to appear

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https://doi.org/10.1016/j.jctb.2023.08.005

The trifference problem and its generalizations

by Anurag Bishnoi

A set $C \subseteq \{0,1,2\}^n$ is called a trifferent code (of length n) if for any three distinct x, y, z in C there is an index $i \in [n]$ such that $\{x_i, y_i, z_i\} = \{0,1,2\}$. The trifference problem asks to find the largest size $T(n)$ of a trifferent code of length n. The exact values of $T(n)$ are only known for $n \leq 9$, and the main questions have been to understand the asymptotic behaviour of this function. A folklore upper bound of

$$T(n) \leq 2(3/2)^n$$

can be proved via a simple recursion of $T(n) \leq 3T(n-1)/2$, while the best lower bound, due to Körner and Marton from 1988 is

$$T(n) \geq (9/4)^{n/5}.$$

Recently, the upper bound was improved by Bhandari and Khetan [1] to

$$T(n) \leq cn^{-2/5}(3/2)^n.$$

The main idea is to first prove that for any subset S of $\{0,1,2\}^n$ we have

$$T(n) \leq T_S(n)3^n/|S|,$$

where $T_S(n)$ is the largest trifferent code restricted to S, and then find a special S for which they can prove good upper bounds on $T_S(n)$. For these upper bounds they use the Turán number of bipartite graphs.

Problem 1. *Give further improvements to the upper bound by improving their analysis or finding a better S.*

Another direction is to look at the generalization of the problem where we study the maximum size $T(n,k)$ of a $C \subset \{0,1,2,\}^n$ such that for any three x, y, z in C, there exists at least k indices i for which $\{x_i, y_i, z_i\} = \{0,1,2\}$. It's clear that for $T(n,1) = T(n)$ and that $T(n,n) = 3$. But what about other k's in the middle of these extremes?

Problem 2. *Determine the asymptotic behaviour of $T(n,k)$ as k varies from 2 to $n-1$.*

It would also be interesting to look at the linear version of this problem (see for example [2]), where the code C is restricted to be a vector subspace of $\mathbb{F}_3^n$.

References

[1] S. Bhandari and A. Khetan. Improved Upper Bound for the Size of a Trifferent Code. arXiv:2402.02390, 2024.

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Cliques in a graph and its complement

by Hilal Hama Karim

Given a graph G, let $\mathcal{N}(H,G)$ denote the number of copies (as a subgraph) of H in G. Let $\overline{G}$ denote the complement graph of G, and n denote the number of vertices of G. Starting from the obvious, counting cliques of size two, $\mathcal{N}(K_2,G)+\mathcal{N}(K_2,\overline{G})=\binom{n}{2}$. The next step of counting triangles comes from the work of Goodman [1]:

$$\mathcal{N}(K_3,G)+\mathcal{N}(K_3,\overline{G})=\binom{n}{3}-\frac{1}{2}\sum_{v\in V(G)}\deg(v)(n-1-\deg(v)). \tag{0.1}$$

The main question is that can there be a similar formula for counting larger cliques in a graph and its complement?

Problem 1. *Determine a function f that may depend on n and the degrees or the number of edges in G, or something else, such that for $t\geq 4$*

$$\mathcal{N}(K_t,G)+\mathcal{N}(K_t,\overline{G})=\binom{n}{t}-f.$$

For values of n below the diagonal Ramsey number $R(t,t)$, there can be graphs with the same number of edges but different values for the left hand side, showing that in this case f cannot depend only on the number of edges (besides n). However, one may consider the problem for n large.

Restriction to regular graphs, or further to fixed regularity degrees, is still interesting. This may help in determining exact results in generalised regular Turán problems. For instance, a result in [2] determines the maximum regularity degree, $\mathrm{regex}(n,T)$, among n-vertex regular graphs that do not contain a tree T on t vertices. Together with this, (0.1) helped in finding the exact result for the generalized regular Turán number $\mathrm{rex}(n,K_3,P_t)$, the maximum number of triangles in n-vertex regular graphs that do not contain a path on t vertices [3].

A variant of this problem is to determine the sum of the number of cliques in G and $\overline{G}$ that contain a fixed vertex. For a vertex $u\in V(G)$, let $K_t(u)$, and $\overline{K}_t(u)$ denote the number of t-cliques that contain u in G and $\overline{G}$, respectively. Nair and Vijayakumar [4] resolved the case $t=3$. They showed that if G has m edges, then for every $u\in V(G)$,

$$K_3(u)+\overline{K}_3(u)=\sum_{v\in N(u)}\deg(v)-m+\frac{1}{2}(n-\deg(u)-1)(n-\deg(u)-2),$$

where $N(u)$ is the set of neighbors of u in G.

References

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Linearity of generalized Turán problems

by Dániel Gerbner

Given graphs H and F and a positive integer n, the generalized Turán number $ex(n, H, F)$ is the largest number of copies of H in n-vertex F-free graphs. Alon and Shikhelman [1] determined the graphs F with $ex(n, K_3, F) = O(n)$. Similar characterization for forests in place of K_3 follows from their results. Gerbner and Palmer [3] determined the graphs F with $ex(n, C_k, F) = O(n)$. Gerbner [2] determined the graphs F with $ex(n, K_{2,t}, F) = O(n)$. What about other graphs, in particular larger cliques? We can also replace linearity by some other bound. What are the graphs F with $ex(n, K_3, F) = O(n^2)$? What are the graphs F with $ex(n, K_3, F) = o(n^2)$? In general, for any H we know what graphs F have $ex(n, H, F) = o(n^{|V(H)|})$ by another result of Alon and Shikhelman, and we also know the graphs F with $ex(n, H, F) = o(n)$ (which are the same as graphs with $ex(n, H, F) = O(1)$ by unpublished results of Gerbner and Methuku.

References

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Counting fixed perimeter polygons

Kenneth Moore

One of Erdős's most famous open problems is to bound the number of unit distance pairs that may be spanned by n points in the plane. A variation of this that has been studied is bounding the number of unit-perimeter triples, which is also difficult. We suggest two explorations in this direction.

Problem 1. *What is the maximum number of convex k-gons with perimeter 1 spanned by n points in the plane?*

Take an ellipse focussed at $(0,0)$ and $(\frac{1}{4},0)$, with major axis length $\frac{1}{2}$. Place $\lfloor \frac{n}{2} \rfloor - 1$ points below the x-axis on the ellipse, and $\lceil \frac{n}{2} \rceil - 1$ above. This yields $\Omega(n^2)$ unit perimeter convex quadrilaterals from n points, and similar constructions follow for any even k. The trivial lower bound is therefore $\Omega(n^{\lfloor k/2 \rfloor})$. A better lower bound is known for $k = 3$, the main ideas coming from [2], and later improvements from [1]. These two papers also explain how to achieve the best known upper bound of $O_\varepsilon(n^{9/4+\varepsilon})$ for $k = 3$ using point-curve incidence bounds, and this technique can be applied for larger k as well. It is currently unclear how to use this technique to show anything as strong as $O(n^{k-1})$ however.

Alternatively, sum all distances between pairs of points in the k-tuples. We refer to this sum as the 'total distance' of a k-tuple.

Problem 2. *What is the maximum number of k-tuples with total distance 1 spanned by n points in the plane?*

References

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On graphs without cycles of length 0 modulo k

by Yandong Bai, Binlong Li

For two integers $k > \ell \geqslant 0$ satisfying that $k\mathbb{Z} + \ell$ contains even integers, what is the maximum number of edges in an n-vertex graph containing no cycles of length ℓ mod k. By confirming a conjecture proposed by Burr and Erdős [5] in 1976, Bollobás [2] proved that such a graph can contain at most a linear number of edges. Define $c_{\ell,k}$ to be the smallest constant such that every n-vertex graph with $c_{\ell,k} \cdot n$ edges contains a cycle of length ℓ mod k. For $k > \ell \geqslant 3$, Sudakov and Verstraëte [9] showed that $\frac{ex(k,C_\ell)}{k} \leqslant c_{\ell,k} \leqslant 96 \cdot \frac{ex(k,C_\ell)}{k}$, where $ex(k, C_\ell)$ is the maximum number of edges in a k-vertex graph containing no cycles of length ℓ. It follows that for even $\ell \geqslant 4$ determining $c_{\ell,k}$ is as hard as the famous extremal problem of determining the Turán number of the even cycle C_ℓ. To our best knowledge, precise values of $c_{\ell,k}$ are only known for $k \leqslant 4$.

Theorem 1. *For $4 \geqslant k > \ell \geqslant 0$, the following results hold.*

- $c_{0,2} = \frac{3}{2}$.
- $c_{0,3} = 2$. *Chen and Saito [3].*
- $c_{1,3} = \frac{5}{3}$. *Bai, Li and Pan [1].*
- $c_{2,3} = 3$. *Dean et al. [4]; Saito [8].*
- $c_{0,4} = \frac{19}{12}$. *Győri et al. [7].*
- $c_{2,4} = \frac{5}{2}$. *Gao et al. [6].*

Here we propose the following problem for possible discussion with the workshop participants.

Problem 1. *Whether the following statements hold or not for $k \geqslant 5$:*

(1) $c_{0,k} = k - 1$ *for odd k; the complete bipartite graph $K_{k-1,n-k+1}$ is an extremal graph.*

(2) $c_{0,k} = \frac{k-1}{2}$ *for even k; if $(k-2)|(n-1)$ then any graph whose blocks are K_{k-1} is an extremal graph.*

(3) $c_{1,k} = \frac{k}{2}$ *for odd k; if $(k-1)|(n-1)$ then any graph whose blocks are K_k is an extremal graph.*

(4) $c_{2,k} = k$ *for odd k; the complete bipartite graph $K_{k,n-k}$ is an extremal graph.*

(5) $c_{2,k} = \frac{k+1}{2}$ *for even k; if $k|(n-1)$ then any graph whose blocks are K_{k+1} is an extremal graph.*

References

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Turán problems for edge-ordered paths

Gaurav Kucheriya

Definition 1. *An* edge-ordered graph *is a finite simple graph G together with a linear order on its edge set E. The edge-order is given by an injective labeling $L : E \to \mathbb{R}$. For the purpose of this writeup, we focus on edge-ordered paths, denoted by P_k^L, on $k > 1$ vertices whose edge order is given by the labeling L.*

An isomorphism between edge-ordered graphs must respect the edge-order. We say that the edge-ordered graph G contains *another edge-ordered graph H, if H is isomorphic to a subgraph of G otherwise we say that G* avoids *H.*

For a positive integer n and an edge-ordered graph H, let the Turán number $\mathrm{ex}_<(n, H)$ *be the maximal number of edges in an edge-ordered graph on n vertices that avoids H. Fixing the* forbidden edge-ordered path P_k^L, $\mathrm{ex}_<(n, P_k^L)$ *is a function of n and we call it the* extremal function *of P_k^L.*

A systematic study of the Turán problem for edge-ordered graphs was initiated by Gerbner, Methuku, Nagy, Pálvölgyi, Tardos and Vizer in [3]. We are interested in the following problem:

Problem 1. *Bounds on* $\mathrm{ex}_<(n, P_6^L)$*, where $L \in \{14523, 14532, 15423, 21453\}$.*

The best upper bound on $\mathrm{ex}_<(n, P_6^L)$ for $L \in \{14523, 14532, 15423, 21453\}$ is established in [1].

Theorem 2. $\mathrm{ex}_<(n, P_6^L) = O(n \cdot 2^{O(\sqrt{\log n})})$ *for $L \in \{14523, 14532, 15423, 21453\}$.*

The following theorem is from an upcoming paper [2]. Here, Kucheriya and Tardos proved an upper bound on the extremal function for the edge-ordered path on 6 vertices with the labeling 13254.

Theorem 3. $\mathrm{ex}_<(n, P_6^{13254}) = O(n \log^2 n)$.

Also, [3] established a lower bound of $n \log n$ for many edge-ordered paths which includes, in particular, the path P_5^{1342}. Moreover from [3] we have $\mathrm{ex}_<(n, P_5^{1342} = O(n \log^2 n)$.

Problem 2. *Can we determine* $\mathrm{ex}_<(n, P_5^{1342})$*, in asymptotic?*

References

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Prescribed intersection numbers in planar sets

Zoltán Lóránt Nagy

Let us consider point sets in a finite projective plane Π_q of order q. (Π_q might be desarguesian, up to our taste.)

Definition 1. *Let $f : \mathcal{L} \to \mathbb{N}$ be a function which assigns a non-negative integer to every line $\ell \in \mathcal{L}$ of a projective plane Π_q of order q. Here we do not require that the plane Π_q is desarguesian. An f-avoiding set S in Π_q is a point set where for each line $\ell \in \mathcal{L}$, $|S \cap \ell| \neq f(\ell)$ holds.*

For *f-avoiding set S in* Π_q, we have the following theorem.

Theorem 2 (Héger, Nagy)**.** *There exists an f-avoiding set in every projective plane Π_q for every function f.*

Problem 1. *Suppose that on each line $\ell \in \mathcal{L}$, a set of forbidden values is given by a function $f^* : \mathcal{L} \to 2^{\mathbb{N}}$.*
Determine the maximum value M for which the following holds. For every function $f^* : \mathcal{L} \to 2^{\mathbb{N}}$ which satisfies $|f^*(\ell)| \leq M := M(q)$ for all ℓ, there exists an f^*-avoiding set in Π_q.

$M = 1$ gives back the previous result.

Small subgraphs with given min degree

Zoltán Lóránt Nagy

Erdős, Faudree, Rousseau and Schelp proved the following.

Theorem 1 (Erdős, Faudree, Rousseau and Schelp [3]). *Every graph G on $n \geq k-1$ vertices with at least $(k-1)n - \binom{k}{2} + 1$ edges contains a subgraph with minimum degree at least k.*

Specializing to $k = 3$, this yields that an n-vertex graphs on $2n-2$ edges have 3-cohesive sets. This result has been strengthened in the following two directions.

Theorem 2 (Alon, Friedland and Kalai [1]). *Let p be a prime power and G be a graph having average degree $\overline{d} > 2p-2$ and maximum degree $\Delta(G) \leq 2p-1$. Then G has a p-regular subgraph.*

This celebrated result was obtained by a clever application of a polynomial method, later called Combinatorial Nullstellensatz. Observe that for $k = p$, a dense enough graph G contains not only a subgraph with min degree k but also a subgraph which is k-regular.

Lisa Sauermann recently proved the following strengthening of the theorem of Erdős et al.

Theorem 3 (Sauermann [4]). *For every k there exists an $\varepsilon := \varepsilon_k > 0$ such that for every graph G on n vertices with at least $(k-1)n - \binom{k}{2} + 2$ edges contains a subgraph on at most $(1-\varepsilon)n$ vertices with minimum degree at least k.*

Thus once we have only one more edges, we will find a considerably smaller subgraph with the properties. What if we have much more edges, how small subgraph can we find then with min degree k?

Problem 1. *Determine the best possible $\lambda_{d,k}$ constant, depending on d and k, for which the following holds. Let G be a graph on n vertices with average degree $d > 2k-2$. Then there is a subgraph $H \subset G$ on at most $(\lambda_{r,t} + o(1))n$ vertices with minimum degree $\delta(H) \geq k$.*

In a problem variant, one might impose conditions on the maximum degree as well, just like in the AFK theorem.

References

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Grid drawings of the complete bipartite graph

by Cory Palmer

A three dimensional *grid drawing* of a graph G is a placement of its vertices at integer lattice points so that the edges (when drawn as straight-line segments) are pairwise non-crossing.

The complete graph K_n has a grid drawing in $[n]\times[2n]\times[2n]$ by placing the vertices $i = 1, 2, \ldots, n$ on the moment curve

$$(i, i^2 \bmod p, i^3 \bmod p)$$

for a fixed prime p between n and $2n$. Because no four of these points lie on the same plane, such a straight-line drawing has no crossing edges.

For a grid drawing in $[a]\times[b]\times[c]$ where $a \geq b \geq c$, the *volume* is abc and the *aspect ratio* is a/c, i.e., the ratio of largest and smallest sides. A conjecture of Pach, Thiele and Tóth [1] asks if every graph of maximum degree 3 has a grid drawing of volume $O(n)$. A recent paper of Balogh and White [3] used a nice probabilistic argument to show that volume $O(n \log n)$ is enough. Actually more is proved: max degree 3 can be replaced with D-degenerate.

The goal here is to find efficient grid drawings of the complete bipartite graph $K_{n,n}$. If we draw one class of $K_{n,n}$ as the points $(i, 0, 0)$ and the other class as the points $(0, a, b)$ where a and b are relatively prime, we get

Proposition 1 (Pach, Thiele, Tóth [1])**.** *There is a grid drawing of $K_{n,n}$ into a grid of dimensions $O(n) \times O(\sqrt{n}) \times O(\sqrt{n})$.*

This gives a grid drawing of $K_{n,n}$ of volume $O(n^2)$. By another result of Pach, Thiele, Tóth [1], this is optimal (in the order of magnitude). However, the aspect ratio is $O(\sqrt{n})$, which may be open to improvement.

Problem 1. *Find a grid drawing of $K_{n,n}$ with aspect ratio less than $\sqrt{n}$.*

Minimizing the largest dimension is another reasonable direction.

Problem 2. *Find the smallest $m = m(n)$ such that there is a a grid drawing of $K_{n,n}$ in $[m]\times[m]\times[m]$.*

There are further nice problems by Cohen et al [2], but I don't know their status. This would require some literature review.

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Hyponontiling Wang tiles

by Dömötör Pálvölgyi

Call a finite collection of tiles that can tile the plane if we have to use each tile at least once *tiling*.

Problem 1. *Is there a collection of at least 3 tiles that is not tiling, but such that after removing any one tile from it we get a tiling collection?*

For example, a set of two non-compatible, tiling singleton tiles is such a collection. I intentionally didn't define what I mean by tile because I'm interested in all sorts of constructions - it can be Wang tiles or your favorite geometric connected or disconnected polygonal or (simply) connected region with translations or rotations. It is easy to see that on a line we cannot have such a collection of at least 3 Wang tiles.

Remark. I have also posed this problem on Mathoverflow a couple of month ago.

Nonrepetitive nonhomogenous partition regularity

by Dömötör Pálvölgyi

Problem 1. *Is it true that for every k for every k-coloring of the natural numbers there are naturals $a_1, \dots, a_{2l}$ for some $l \geq 2$ such that $a_1 + a_3 = 2a_2 + 2$, $a_2 + a_4 = 2a_3 + 2$, ..., $a_{2l-2} + a_{2l} = 2a_{2l-1} + 2$ and the color of a_i and a_{l+i} are the same for every i?*

Motivation. Such a sequence is called a repetition. Nonrepetitive colorings have been studied a lot, going back to the Thue sequence. Recently, a variant of the problem was raised for Euclidean spaces at EuroComb'23 by Barsse, Gonçalves, Rosenfeld (`https://journals.muni.cz/eurocomb/article/view/35550`). This is an attempt to solve some questions left open with a trick from this classic paper of P Erdős, R.L Graham, P Montgomery, B.L Rothschild, J Spencer, E.G Straus (`https://doi.org/10.1016/0097-3165(73)90011-3`). That is where the $+2$ comes from, without which the a_i's would form an arithmetic progression, so the statement would be true by van der Waerden, even giving all monochromatic a_i's. We cannot hope to get all monochromatic a_i's when the $+2$ is there, as shown by Erdős et al. There might also be some relation to partition regularity.

Remark. I have also posed this problem on Mathoverflow a couple of month ago.

A favorite problem of mine:
Even Cut Conditions vs Cut Conditions

Nika Salia

Cut Condition (CC): A graph G satisfies the cut condition if, for every partition $V(G) = A \cup B$, the number of edges between A and B, denoted $e(A, B)$, satisfies $e(A, B) \geq \min\{|A|, |B|\}$.

The minimal edge graph satisfying CC is a star.

Even Cut Condition (ECC): A graph G, with an even number of vertices, satisfies the cut condition if, for every partition $V(G) = A \cup B$ where $|A| = |B|$, it holds that $e(A, B) \geq \min\{|A|, |B|\} = \frac{v(G)}{2}$.

Faudree, Gyárfás and Lehel [1] posited that restricting the maximum degree $\Delta(G) < v(G) - 1$ of a graph with CC results in a different minimal edge count:

$$e(G) \geq \frac{3}{2}n + O(1);$$

This conjecture was validated by Jobson, Kézdy, and Lehel [2] through a simple yet elegant proof involving the decomposition of the graph into 2-connected blocks and analyzing the resultant weighted tree. Note that, the optimal construction is not 2-connected.

Theorem 1 ([2], Conjectured by [1]). *Assume a graph G with ECC and no universal vertex, then*

$$e(G) \geq \frac{3}{2}n - 4.$$

Open Question: Is ECC sufficient to achieve a similar minimal edge count? Recent findings by Jobson, Kézdy, and Lehel [2] demonstrated that every 2-connected graph with ECC and without a universal vertex has at least $\frac{3}{2}n - 2$ edges.[1]

This problem, particularly with the condition $\Delta(G) < 100$, remains mistery to me, despite extensive collaboration and efforts.

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[1]Győri noted that this also is a direct corollary of the Győri-Lovász Theorem.

Baby project: Planar Saturation

Nika Salia

Recently Clifton and Salia [1] introduced plane-saturated ratio.

Definition 1. *A plane graph H is a plane-saturated subgraph of a planar graph G if, when any edge (potentially introducing new vertices) is added to H, it either introduces a crossing or causes the resulting graph to no longer be a subgraph of G.*

For a graph G, we define the plane-saturation ratio (G) as the minimum value of $\frac{e(H)}{e(G)}$ over all plane-saturated subgraphs H of G.

For example, the plane-saturation ratio of a tree or a cycle is one. Let G_{k_1,k_2} denote the class of planar graphs where at most k_1 vertices of degree 1 have the same neighborhood and at most k_2 vertices of degree 2 have the same neighborhood.

Theorem 2. *For a positive integer k_1 and a non-negative integer k_2 with $(k_1, k_2) \neq (1, 0), (2, 0)$, every planar graph $G \in \mathcal{G}_{k_1,k_2}$ satisfies*

$$(G) > \frac{1}{9 + k_1 + 6k_2}.$$

Furthermore, for all non-negative integers k_1 and k_2 and every positive ϵ, there exists a graph $G_\varepsilon \in \mathcal{G}_{k_1,k_2}$ such that

$$(G_\epsilon) < \frac{1}{9 + k_1 + 6k_2} + \epsilon.$$

Corollary 3. *Any twin-free planar graph G satisfies $(G) > 1/16$.*

They conjectured,

Conjecture 1. *For non-negative integers k_1, k_2, any graph $G \in \mathcal{G}_{k_1,k_2}$ satisfies*

$$(G) > \frac{1}{9 + k_1 + 6k_2}.$$

This is the best possible bound due to Construction 2 [1]. Of greatest interest is the case $k_1 = k_2 = 0$, where Conjecture 1 would give the following.

Conjecture 2. *For a planar graph G with minimum degree at least 3,*

$$(G) > 1/9.$$

We can also impose further conditions on the plane-saturated subgraph.

Problem 3. *What is the smallest possible value of $\frac{e(H)}{e(G)}$ where G is planar and H is a plane-saturated subgraph of G with no isolated vertices? (i.e. one is not allowed to embed vertices without edges)*

As $e(G) < 3v(G)$ and the minimal degree condition on H gives $e(H) \geq \frac{v(H)}{2} = \frac{v(G)}{2}$, we always have $\frac{e(H)}{e(G)} > \frac{1}{6}$ (We also have a page proof for $\frac{8}{45}$). Our best-known construction in this setting comes from Example 4 which gets arbitrarily close to $\frac{1}{5}$.

Example 4. *Let G_{2n+5} be a twin-free planar graph on $2n + 5$ vertices consisting of a matching of size n, two vertices u_1 and u_2 adjacent to every vertex of the matching, and a triangle $v_1v_2v_3$, with additional edges v_1u_1 and v_2u_2; see the left graph in Figure 1.*

Let H_{2n+5} be the plane graph formed by G_7 and a matching of size $n - 1$ embedded in the plane as in the right graph in Figure 1. The graph H_{2n+5} is a plane-saturated subgraph of G_{2n+5}.

Problem 4. *Which maximal planar graph G_n with n vertices maximizes (G_n)?*

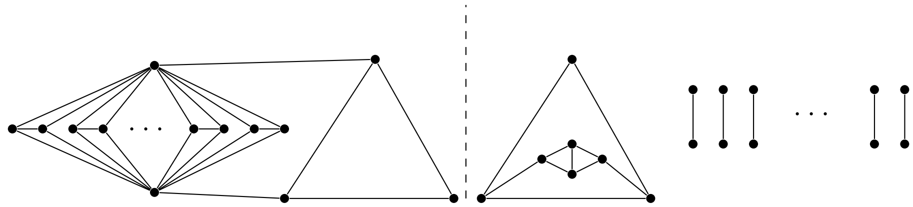

Figure 1: On the left, graph G_{2n+5} and on the right, graph H_{2n+5} from Example 4.

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On properly colored spanning trees in edge-colored complete graphs

Shenggui Zhang

Let T_0 be a fixed tree. We use $\mathcal{T}(n, T_0)$ to denote the collection of n-vertex trees that are subdivisions of T_0. We say that an edge-colored graph is mono-C_3-free if it contains no monochromatic triangles. The following is from a preprint paper of Li, Lu, Su and Zhang [1].

Theorem 1. *Let T_0 be a tree of k edges and let G be a mono-C_3-free edge-colored K_n with $n \geq (k+2)!$. Then there exists a tree $T \in \mathcal{T}(n, T_0)$ such that G contains a PC copy of T.*

In the same paper, they made the following stronger conjecture and confirmed the conjecture when T_0 is a star.

Conjecture 1. *Let T_0 be a fixed tree. Then there is a function $f(T_0)$ such that every mono-C_3-free edge-colored K_n with $n \geq f(T_0)$ contains a PC copy of T for each tree $T \in \mathcal{T}(n, T_0)$.*

Problem 2. *Can we verify the above conjecture for other trees?*

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The Extremal Number of Traces of Graphs

by Casey Tompkins

Given two hypergraphs $\mathcal{H}$ and $\mathcal{F}$, we say that $\mathcal{H}$ contains $\mathcal{F}$ as a trace if there is a set $S \subseteq V(\mathcal{H})$ such that $\{e \cap S : e \in E(\mathcal{H})\}$ contains a sub-hypergraph isomorphic to $\mathcal{F}$.

This notion originates with the classical Sauer–Shelah–Perles theorem about shattering sets. Recently, in the same vein as notions like expansions, suspensions, and Berge copies of graphs, people have been interested in the case when $\mathcal{F}$ is a graph and the underlying hypergraph is uniform.

Let the extremal number for forbidding a graph F as a trace in an r-uniform hypergraph be denoted by $ex_r(n, TF)$. In the case when F is a k-clique and $r < k$, $ex_r(n, TK_k)$ was determined for large n by Mubayi and Zhao [1] and exactly by Pikhurko [2]. The case $k \leq r$ is less understood. There is a conjecture of Mubayi and Zhao (see Gerbner [3] for some progress).

In the case of $r = 3$ and $F = K_{2,t}$, there are asymptotic results (as both t and n go to infinity) by Luo and Spiro [4]. My first problem concerns the case when $F = C_4$. For this case Luo and Spiro proved an upper bound of $\frac{5}{6}n^{3/2} + o(n^{3/2})$ and this was improved by Gerbner [3] to $\frac{3}{4}n^{3/2} + o(n^{3/2})$. The best lower bound is $\frac{1}{2}n^{3/2} + o(n^{3/2})$ with the construction being appending a fixed vertex to each edge of an extremal C_4-free graph (note one can do a tiny bit better by also taking a vertex disjoint set of triangles in the C_4-free graph as hyperedges).

Problem 1. *Determine $ex_3(n, TC_4)$ asymptotically.*

My second problem concerns the case when F is a path (or more generally a tree). Let P_k be the path of length k. For Berge paths the extremal number is well understood for all k and r [6]. When $k > r$, the extremal construction for Berge paths is taking disjoint hypergraph cliques on k vertices. For trace paths in the $r = 3$ case we can take one additional vertex in each clique. Is this optimal?

Problem 2. *Determine $ex_3(n, TP_k)$ asymptotically.*

Note that an argument of Salia shows that we at least have an upper bound of a constant times n (for any tree as well). There is one other paper on trace graphs due to Qian and Ge [5]. Notably, they observed that for trace stars $K_{1,t}$ when $r = 3$, rather than just considering disjoint cliques of maximum size, it is better to take a clique of size one larger and delete a Steiner system from each clique. However, this does not work for avoiding trace paths.

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Problem on the number of Maximal Independent Sets

by Zeyu Zheng

A *maximal independent set* in graph G refers to an independent set I in G such that there does not exist $v \in V(G) \setminus I$ for which $I \cup \{v\}$ is independent. A *k-maximal independent set* refers to a maximal independent set of size k. We denote the number of maximal independent sets in G by $\mathrm{mis}(G)$, and the number of k-maximal independent sets in G by $\mathrm{mis}_k(G)$.

The study of the number of maximal independent sets was initiated independently by Miller, Muller in [3], and Moon, Moser in [4]. They proved the following result

Theorem 1 (Miller-Muller [3], Moon-Moser [4]). *Let G be a graph on n vertices. If $n \geq 2$, then*

$$\mathrm{mis}(G) \leq \begin{cases} 3^{n/3} & n \equiv 0 \mod 3 \\ 4 \cdot 3^{(n-4)/3} & n \equiv 1 \mod 3 \\ 2 \cdot 3^{(n-2)/3} & n \equiv 2 \mod 3 \end{cases},$$

where the bound is achieved if and only if G is a disjoint union of 3-cycles and at most two edges.

Wood has provided a new and short proof of the theorem in [6].

Hujter and Tuza considered $\mathrm{mis}(G)$ when G is a triangle-free graph. They proved

Theorem 2 (Hujter-Tuza [2]). *Let G be a triangle-free graph on n vertices. If $n \geq 4$, then*

$$\mathrm{mis}(G) \leq \begin{cases} 2^{n/2} & n \equiv 0 \mod 2 \\ 5 \cdot 2^{(n-5)/2} & n \equiv 1 \mod 2 \end{cases},$$

where the bound is achieved if and only if G is a disjoint union of a matching and at most one 3-cycle.

Nielsen first studied $\mathrm{mis}_k(G)$ in [5], and he proved

Theorem 3 (Nielsen [5]). *Let G be a graph on n vertices and let $1 \leq k \leq n$. Then*

$$\mathrm{mis}_k(G) \leq \left\lfloor \frac{n}{k} \right\rfloor^{k-(n \mod k)} \left\lceil \frac{n}{k} \right\rceil^{n \mod k},$$

where the bound is achieved if and only if G is a disjoint union of $\lfloor n/k \rfloor$-cliques and $\lceil n/k \rceil$-cliques.

Very recently, He, Nie, Spiro considered $\mathrm{mis}_k(G)$ when G is a triangle-free graph and $k \leq 4$ in [1]. They proved

Theorem 4 (He, Nie, Spiro [1]). *Let G be a triangle-free graph on n vertices. For $n \geq 8$,*

$$\mathrm{mis}_2(G) \leq \left\lfloor \frac{n}{2} \right\rfloor,$$
$$\mathrm{mis}_3(G) = O(n),$$
$$\mathrm{mis}_4(G) = O(n^2),$$

where the first bound is achieved if and only if G is a comatching of order n.

Our problem concerns $\mathrm{mis}_k(G)$ when G is a triangle-free graph and $k = \Theta(n)$.

Problem 1. *Let G be a triangle-free graph on n vertices. Let k be an integer in $[2n/5, n/2]$. Is it true that*

$$\mathrm{mis}_k(G) \leq 2^{5k-2n} \cdot 5^{n-2k},$$

and the bound is achieved if and only if G is a disjoint union of a $(5k-2n)$-matching and $(n-2k)$ 5-cycles?

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