On a Covering Property of Maximal Sperner Families*

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ABSTRACT: In the lattice of subsets of an n-set X, every maximal Sperner family S may be partitioned into disjoint subfamilies $\mathcal{A}$ and $\mathcal{B}$, such that the union of the upset of $\mathcal{A}$ with the downset of $\mathcal{B}$ yields the entire lattice: $\mathcal{U}(\mathcal{A}) \cup \mathcal{D}(\mathcal{B}) = 2^X$. The result generalizes to certain posets.

1. Introduction

Let X be a finite set and 2^X be its power set. If $\mathcal{H}$ is a set system on X (that is $\mathcal{H} \subset 2^X$) then the *downset* $\mathcal{D}(\mathcal{H})$ of the family $\mathcal{H}$ is:

$$\mathcal{D}(\mathcal{H}) := \{E \subset X : \exists H \in \mathcal{H} \{E \subset H\}\}.$$

The *upset* of $\mathcal{H}$ is:

$$\mathcal{U}(\mathcal{H}) = \{E \subset X : \exists H \in \mathcal{H} \{H \subset E\}\}.$$

A set system $\mathcal{S} \subset 2^X$ is called a *Sperner family* if no two elements of $\mathcal{S}$ are comparable under set inclusion. A Sperner family $\mathcal{S}$ is *maximal* if for every Sperner family $\mathcal{S}' \subset 2^X$, $\mathcal{S} \subset \mathcal{S}'$ implies $\mathcal{S} = \mathcal{S}'$. It is easy to see that if $\mathcal{S}$ is a maximal Sperner family then

$$\mathcal{U}(\mathcal{S}) \cup \mathcal{D}(\mathcal{S}) = 2^X. \qquad (1)$$

The purpose of this note is to show that the identity (1) can be achieved more efficiently, namely, by replacing $\mathcal{S}$ in (1) with disjoint subfamilies of $\mathcal{S}$.

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2. Main Result

Let $\mathcal{S}$ be a maximal Sperner family. We now show that $\mathcal{S}$ may be partitioned into two families $\mathcal{S} = \mathcal{A} \cup \mathcal{B}$ such that $\mathcal{U}(\mathcal{A}) \cup \mathcal{D}(\mathcal{B}) = 2^X$. Specifically, we construct disjoint subfamilies $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ which satisfy $\mathcal{D}(\mathcal{S}_1) \cup \mathcal{U}(\mathcal{S}_2) \cup \mathcal{S}_3 = 2^X$. Then, any partition of $\mathcal{S}$ into subfamilies $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \supset \mathcal{S}_1$ and $\mathcal{B} \supset \mathcal{S}_2$ satisfies $\mathcal{U}(\mathcal{A}) \cup \mathcal{D}(\mathcal{B}) = 2^X$.

Theorem 1. *There exist disjoint subfamilies $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ of $\mathcal{S}$ such that*

$$\mathcal{D}(\mathcal{S}_1) \cup \mathcal{U}(\mathcal{S}_2) \cup \mathcal{S}_3 = 2^X \tag{2}$$

holds.

Proof. For every integer i $(0 \leq i \leq |\, X\,|)$ let $<_i$ be an arbitrary ordering of the i-element subsets of X. For example, we may choose the lexicographic ordering for every i. Let the total ordering $<_t$ of the elements of 2^X be defined as follows: If $E, F \subset 2^X$ then

$$E <_t F \Leftrightarrow \begin{cases} |\, E\,| < |\, F\,| & \text{or} \\ |\, E\,| = |\, F\,| \wedge E <_{|\,E|} F. & \end{cases}$$

With every subset $H \in 2^X \setminus \mathcal{S}$ we associate an element $f(H)$ of $\mathcal{S}$ in the following way. If $H \in \mathcal{D}(\mathcal{S}) \setminus \mathcal{S}$ then let $f(H)$ be the greatest element of $\mathcal{S}$ (with respect to the total ordering $<_t$) whose downset contains H. Similarly, if $H \in \mathcal{U}(\mathcal{S})$ then let $f(H)$ be the smallest element of $\mathcal{S}$ (with respect to the total order $<_t$) whose upset contains H. Finally to find the required subfamilies, set:

$$\mathcal{S}_1 = \{f(H) : H \in \mathcal{U}(\mathcal{S}) \setminus \mathcal{S}\}$$
$$\mathcal{S}_2 = \{f(H) : H \in \mathcal{D}(\mathcal{S}) \setminus \mathcal{S}\}$$
$$\mathcal{S}_3 = \mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2).$$

Clearly, $\mathcal{U}(\mathcal{S}_1) \cup \mathcal{D}(\mathcal{S}_2) \cup \mathcal{S}_3 = 2^X$ holds. We merely have to show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint. Suppose there exists some $E \in \mathcal{S}_1 \cap \mathcal{S}_2$. Then there are $H_1, H_2 \subset X$ such that $H_1 \in \mathcal{D}(\mathcal{S})$, $H_2 \in \mathcal{U}(\mathcal{S})$ and $f(H_1) = f(H_2) = E$. Consequently, $H_1 \subset E \subset H_2$. As these inclusions are proper, there exists F, another $|\, E\,|$-element subset of X, satisfying $H_1 \subset F \subset H_2$. Suppose, say, that $E <_t F$. Then $F \notin \mathcal{S}$, otherwise $f(H_1) = E$ is false. Since $\mathcal{S}$ is maximal there is an $F^* \in \mathcal{S}$ such that, say, $F \subset F^*$. But then $f(H_1) = E$ is false, since $|\, F^*\,| > |\, E\,|$ and $H_1 \subset F^*$, a contradiction. On the other hand, if $F^* \subset F$, then $f(H_2) = E$ is false because $|\, F^*\,| < |\, E\,|$ and $F^* \subset H_2$, also a contradiction. We handle the case $F <_t E$ similarly. ∎

A consequence of Theorem 1 is that we may associate at least one monotone Boolean function g with each maximal Sperner family $\mathcal{S}$ with the property that the union of $\mathcal{A}$ the minimal vertices in $g^{-1}(1)$, with $\mathcal{B}$, the maximal vertices in $g^{-1}(0)$, is the Sperner family $\mathcal{S}$.

3. Ranked Posets

Theorem 1 can be generalized to certain partially ordered sets. In this section we discuss such a possible generalization. The definitions not given here can be found in [1].

Let $\mathcal{P} = (P, <_p)$ be a finite poset. We say, that $\mathcal{P}$ is *dense* if every non-empty *open interval* $\{x, y\} = \{z \in P : x <_p z <_p y\})$ contains at least two elements. It is easily shown that every open interval must then contain two incomparable elements. Thus, if P is dense and $z \in \{x, y\}$ then there exists an element $z^* \in \{x, y\}$ incomparable to z. Let us remark, that every Boolean algebra is dense. Furthermore if $\mathcal{P}$ is ranked, the open interval $\{x, y\}$ contains an element z^* such that $\text{rank}(z) = \text{rank}(z^*)$.

For any set $H \subset P$ the definitions of upset $\mathcal{U}(H)$ and downset $\mathcal{D}(H)$ are defined analogously to those given for the the Boolean lattice.

Theorem 2. *Let $\mathcal{P}$ be a finite ranked, dense poset. Then for every maximal Sperner system $S \subset \mathcal{P}$ there exist disjoint subsystems S_1, S_2, and S_3 of S such that*

$$\mathcal{U}(S_1) \cup \mathcal{D}(S_2) \cup S_3 = P \tag{3}$$

holds.

Proof. The proof follows that of Theorem 1. Let $<_t$ be a linear extension of $<_p$ for which every $x, y \in P$, $\text{rank}(x) < \text{rank}(y)$ satisfies $x <_t y$. For every element $x \in P \setminus S$, $x \in \mathcal{D}(S)$ let $f(x)$ be the greatest element $s \in S$ (with respect to the linear extension $<_t$) such that $x <_p s$. If $x \in \mathcal{U}(S)$ then let $f(x)$ be the smallest element $s \in S$ (with respect to the linear extension $<_t$) such that $s <_p x$. Finally, set

$$S_1 = \{f(x) : x \in \mathcal{D}(\mathcal{S})\}$$
$$S_2 = \{f(x) : x \in \mathcal{U}(\mathcal{S})\}$$
$$S_3 = \mathcal{S} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2).$$

By the same argument presented in Theorem 1, one may show, by contradiction, that $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint. ∎

4. Examples and Conjecture

In this section we furnish some posets which satisfy property (3) and some which do not.

Example 1. Every finite geometric lattice satisfies (3). Indeed, a geometric lattice is ranked, and since it is relatively complemented, is dense as well (see [1], Section II.3). Upon application of Theorem 2, the result follows.

We remark that the notion of 'denseness' is not essential. The following lattice is neither dense, nor ranked, yet satisfies (3). The symbol $<$ means 'covers' in the poset:

Example 2. (Faigle, [2]) The lattice $\{a < b < c < d < e; a < f < d; b < g < e\}$ satisfies (3).

On the other hand, there are distributive lattices which do not satisfy (3). For example, the distributive lattice in Figure 2.2 L, given in the book of Aigner [1,p.34], does not satisfy (3).

As Example 2 shows, the 'denseness' of a poset is not necessary, but we believe that it is sufficient. We conjecture:

Conjecture. *Every dense poset satisfies* (3).

It is interesting to remark, that property (3) of a Sperner system bears no relationship to the LYM property or even to the Sperner property of the poset. Indeed, in the disjoint union of some r-chains, which is a LYM poset, no maximal Sperner family satisfies (3). On the other hand, if the ranked posets $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy (3), but the length of the posets are different, then their disjoint union also satisfies (3), but it is not a Sperner poset.

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5. References

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