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Some Generalizations of Property B and the Splitting Property*

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Abstract. The set system $\mathcal{H} \subseteq 2^X$ satisfies *Property B* if there exists a partition $X_1 \cup X_2 = X$ such that any element of $\mathcal{H}$ intersects both classes. Here, we study the following problem: We are given k set systems on the underlying set X, and we are seeking a k-partition of X such that any element of the ith set system intersects X_i for every i.

Keywords: Property B, splitting property

1. Introduction

A set system $\mathcal{H} \subseteq 2^X$ is said to have *Property B* if there exists a partition $X_1 \cup X_2 = X$ such that every element of $\mathcal{H}$ intersects both classes. E.W. Miller introduced this notion in 1937 [10], and the letter B was given in honor of Felix Bernstein due to his seminal result.

Theorem 1.1. (Bernstein's Lemma) *If $\mathcal{H}$ is a countable system of infinite sets, then $\mathcal{H}$ has Property B.*

In (infinite) combinatorial set theory, several analogous results were proved (e.g., [3, 7]).

In 1961, Paul Erdős and A. Hajnal revived the study of Property B on finite and infinite set systems [3]; in general, they were interested to find conditions when some set systems do not have Property B. (For a thorough survey, see [4, Chapter 4].) On the other hand, in 1979, L. Lovász proved [9, Problem 13.33] the following "positive" theorem:

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Theorem 1.2. *Let $\mathcal{H}$ be a hypergraph on the finite set X such that there are no edges E, F of $\mathcal{H}$ with $|E \cap F| = 1$. Then there is a 2-partition of the underlying set X such that every edge intersects both partition classes.*

In [5] this problem and result were generalized for two families. The purpose of this paper is to prove further generalizations of Lovász's theorem for more families.

2. Splitting Property

In this section we give the above mentioned generalization of Property B for two set systems, and show its connection to the notion of splitting in partially ordered sets (posets). In [5] it was pointed out that an earlier result of Ahlswede et. al. (see [1]) can be formalized as an analogous result to Theorem 1.2 for two simultaneous set systems.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two set systems on the finite set X. Assume that, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$,*

$$|A \cap B| \neq 1. \tag{2.1}$$

Then there is a 2-partition $X_a \cup X_b = X$ such that every element of $\mathcal{A}$ intersects X_a and every element of $\mathcal{B}$ intersects X_b.

We call such a partition a *split* of X (with respect to the set systems $\mathcal{A}$ and $\mathcal{B}$). In order to be self-contained we give a short proof of Theorem 2.1 here.

Proof. Let the underlying set be $X = \{x_1, x_2, \dots, x_n\}$. We assume the elements of X are totally ordered and $x_i < x_j$ whenever $i < j$. Define the class X_a as $X_a = \{\min(A) : A \in \mathcal{A}\}$. We claim that the partition X_a and $X_b = X \setminus X_a$ is a split. For this end it is enough to show that, for every $B \in \mathcal{B}$, its maximal element $\max(B)$ belongs to X_b. On the contrary, assume there is a set $B \in \mathcal{B}$ such that $\max(B) \notin X_b$. Then there is an $A \in \mathcal{A}$ such that $\min(A) = \max(B)$ which clearly contradicts condition (2.1). ∎

Lovász's theorem is an immediate consequence of this result: Let both $\mathcal{A}$ and $\mathcal{B}$ be equal to $\mathcal{H}$. We remark that this result can be generalized for infinite set systems on infinite underlying sets (using some additional conditions) (see [5]).

In the rest of this section we show the connection of the previous result to the notion of splitting in posets. In the partially ordered set $\mathcal{P} = (P, <)$, a subset A is called an *antichain* if and only if its elements are pairwise incomparable. Let G be a subset of P. Then its *downset* is $\mathcal{D}(G) = \{x \in P : \exists g \in G \text{ s.t. } x \leq g\}$, and its *upset* is $\mathcal{U}(G) = \{x \in P : \exists g \in G \text{ s.t. } g \leq x\}$. A subset G is called a *generator system* if $\mathcal{D}(G) \cup \mathcal{U}(G) = P$. It is easy to see that every maximal (for inclusion) antichain is a generator system. The generator system G satisfies the *splitting property* if there is a partition $G_1 \cup G_2 = G$ such that $\mathcal{D}(G_1) \cup \mathcal{U}(G_2) = P$. Finally, the antichain $A \subseteq P$ is *dense* in $\mathcal{P}$ if and only if, for all $x \in \mathcal{D}(A) \setminus A$ and $y \in \mathcal{U}(A) \setminus A$,

$$|\mathcal{U}(x) \cap \mathcal{D}(y) \cap A| \neq 1. \tag{2.2}$$

Ahlswede et. al. proved the following in [1]:

Theorem 2.2. *In every finite partially ordered set, every dense maximal antichain satisfies the splitting property.*

Proof. The proof is straightforward from Theorem 2.1. Let A be a maximal, dense antichain in $\mathcal{P}$. Then for every $x \in \mathcal{D}(A) \setminus A$, let the set $A(x) = \mathcal{U}(x) \cap A$, and for every $y \in \mathcal{U}(A) \setminus A$, let the set $B(y) = \mathcal{D}(y) \cap A$. Then due to Theorem 2.1, the underlying set A has the splitting property with respect to the set systems $\{A(x) : x \in \mathcal{D}(A) \setminus A\}$ and $\{B(y) : y \in \mathcal{U}(A) \setminus A\}$. This gives us the partition $A = A_a \cup A_b$ where A_a intersects every element of $\{A(x)\}$ and A_b intersects every element of $\{\mathrm{B(x)}\}$. Now it is easy to see that by choosing $A_1 = A_a$ and $A_2 = A_b$, we obtain an appropriate split of the antichain A in the poset. ∎

It is worth mentioning that Theorems 2.1 and 2.2 are essentially equivalent. In [5] Theorem 2.1 was proved from Theorem 2.2. On the other hand, in [1] some other splitting results for posets are proved, which do not have analogs for set systems yet.

Finally there is an analog of Theorem 2.2 where the role of the maximal antichains is played by certain generator systems. Let $C(x)$ denote the *cone* of x in the poset $\mathcal{P}$, that is, $C(x) = \mathcal{D}(x) \cup \mathcal{U}(x)$. The generator system G has the *cone-splitting* property with respect to the pair (P_1, P_2) (where $P_1 \cup P_2 \subseteq P \setminus G$) if and only if there exists a partition $G_1 \cup G_2 = G$ such that every $x \in P_1$ belongs to $C(G_1)$ and every $y \in P_2$ belongs to $C(G_2)$. (We remark that P_1 and P_2 are not necessarily disjoint.) The generator system G is *cone-dense* with respect to the pair (P_1, P_2) if and only if, for all elements $x \in P_1$ and $y \in P_2$, we have $|C(x) \cap C(y) \cap G| \neq 1$. Now one can easily prove (like in the proof of Theorem 2.2) the following result:

Theorem 2.3. *Let the generator system G be cone-dense with respect to the pair (P_1, P_2) in the partially ordered set $\mathcal{P}$. Then G has the cone-splitting property with respect to that pair.*

The proof is left to the diligent reader.

3. The k-splitting Property

In this section we generalize the notion of splitting for several simultaneous set systems.

Definition 3.1. *Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ be set systems on the finite underlying set X. Then X has a k-split with respect to these families if and only if there is a k-partition $X_1 \cup X_2 \cup \dots \cup X_k = X$ such that, for every $i = 1, \dots, k$ and for all $A \in \mathcal{A}_i$, we have $X_i \cap A \neq \emptyset$. In the case of $i = 2$ we get back the original notion of splitting.*

The following result is a direct generalization of Theorem 2.1.

Theorem 3.2. *Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ be set systems on the finite set X, with $k \geq 2$. Assume that for every $i = 1, \dots, k$ and for all set systems $\mathcal{A}_{j_1}, \dots, \mathcal{A}_{j_i}$ (where $1 \leq j_1 < \dots < j_i \leq k$), we have for all $A_l \in \mathcal{A}_{j_l}$, $l = 1, \dots, i$*

$$|A_1 \cap A_2 \cap \dots \cap A_i| \notin \{1, \dots, i-1\}. \tag{3.1}$$

Then X has a k-split with respect to these families.

Proof. Let the underlying set be $X = \{x_1, x_2, \dots, x_n\}$ and let X be ordered by the subscripts. We apply induction on k; our base case is $k = 2$, that is, Theorem 2.1. In the inductive step we assume the statement has already been proved for less than k set systems and we are proceeding to prove it for k set systems.

For that end, let $X_1 = \{\min(A_1) : A_1 \in \mathcal{A}_1\}$ and let $\overline{X} = X \setminus X_1$. Furthermore, for every $j = 2, \dots, k$, let $\overline{\mathcal{A}}_j = \{\overline{A}_j : A_j \in \mathcal{A}_j\}$ where $\overline{A}_j = A_j \cap \overline{X} = A_j \setminus X_1$. We remark that from Theorem 2.1 we know that no $\overline{A}_j$ is empty, since the families $\mathcal{A}_1$ and $\mathcal{A}_j$ satisfy condition (3.1) and so does condition (2.1).

Claim: The set systems $\overline{\mathcal{A}}_2, \dots, \overline{\mathcal{A}}_k$ on the underlying set $\overline{X}$ satisfy condition (3.1).

On the contrary, assume there are $\overline{A}_{i_2}, \dots, \overline{A}_{i_l}$ which violate condition (3.1). Then we have

$$1 \le \left|\overline{A}_{i_2} \cap \cdots \cap \overline{A}_{i_l}\right| \le l - 2.$$

Denote the previous intersection by $\overline{H}$ and let $A_{i_2}, \dots, A_{i_l}$ be the original sets with intersection H. Let $H \setminus \overline{H} = \{y_1, \dots, y_t\}$ with maximum element y_t. Since $y_t \in X_1$, therefore, there exists an $A_1 \in \mathcal{A}_1$ with $\min(A_1) = y_t$. Let us check the intersection of A_1 with H. This contains the element y_t but none of $y_1, \dots, y_{t-1}$, because y_t was minimal in A_1. Therefore,

$$1 \le \left|A_1 \cap A_{i_2} \cap \cdots \cap A_{i_l}\right| \le l - 1,$$

contradicting condition (3.1). This proves the claim and the correctness of our inductive step as well. ∎

We believe that a stronger form of this theorem is also true.

Conjecture 3.3. *Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ be set systems on the finite set X. Assume that for $i = 1, \dots, k$ and for every set systems $\mathcal{A}_{j_1}, \dots, \mathcal{A}_{j_i}$ (where $1 \le j_1 < \cdots < j_i \le k$), we have for all $A_l \in \mathcal{A}_{j_l}$, $l = 1, \dots, i$*

$$|A_1 \cap A_2 \cap \cdots \cap A_i| \ne i - 1. \tag{3.2}$$

Then X has a k-split with respect to these set systems.

We can prove the following weaker result:

Theorem 3.4. *Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ be set systems on the finite set X. Assume (3.2) holds for these families. Furthermore assume that for every $j = 1, \dots, k$ and for every $(k-1)$-element subset Y of X,*

$$\text{there exists } A_j(Y) \in \mathcal{A}_j \text{ such that } Y \subseteq A_j(Y). \tag{3.3}$$

Then X has the k-splitting property with respect to these set systems.

Proof. Again, let the underlying set be $X = \{x_1, \dots, x_n\}$ and let X be ordered by the subscripts. We apply double induction on k and on the total number of subsets in the set systems. Our base case is $k = 2$, where the total number of the subsets is arbitrary. This case is exactly Theorem 2.1 (since in the case of $k = 2$ conditions (2.1) and (3.2) coincide). In the general case we assume the statement has already been proved for fewer than k set systems and for k families with fewer subsets in total.

Let $A^1 \in \mathcal{A}_1$ be a subset such that

$$\max\{\min(A) : A \in \mathcal{A}_1\} = \min(A^1).$$

Let

$$\overline{\mathcal{A}}_1 = \mathcal{A}_1 \setminus \{A \in \mathcal{A}_1 : \min(A^1) \in A\}.$$

(We remark that the set system $\overline{\mathcal{A}}_1$ may be empty.) Furthermore, let $\overline{X} = X \setminus \{\min(A^1)\}$, and for all $A \in \mathcal{A}_i$, let $\overline{A} = A \cap \overline{X} = A \setminus \{\min(A^1)\}$ (for every $i = 2, \ldots, k$). Finally, for $i > 1$, let $\overline{\mathcal{A}}_i = \{\overline{A} : A \in \mathcal{A}_i\}$.

Claim: The set systems $\overline{\mathcal{A}}_1, \ldots, \overline{\mathcal{A}}_k$ on the underlying set $\overline{X}$ satisfy the conditions of Theorem 3.4.

First of all we claim that no element of any system $\overline{\mathcal{A}}_i$ is the empty set. Assume the opposite: Let an $\overline{A} \in \overline{\mathcal{A}}_i$ be empty. Then, by the construction, $i > 1$. Furthermore, we have removed exactly one element of X (namely, $\min(A^1)$), therefore, (since originally A was not empty) $A \cap A^1 = \{\min(A^1)\}$ would occur, contradicting condition (3.2). On the other hand, the set system $\overline{\mathcal{A}}_i$ ($i \neq 1$) may become a multiset. In that case the simplest solution is to keep exactly one copy of every multiple subset in $\overline{\mathcal{A}}_i$. At the end this copy will intersect X_i, and the same holds for every additional copy as well.

By assumption, condition (3.3) trivially holds for every $(k-1)$-element (and therefore, every $0 < l < k$ element) subset of $\overline{X}$.

To prove condition (3.2), assume the contrary: We have sets $\overline{A}_1 \in \overline{\mathcal{A}}_{j_1}, \ldots, \overline{A}_i \in \overline{\mathcal{A}}_{j_i}$, where $1 \leq j_1 < \cdots < j_i \leq k$ such that

$$\left|\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_i\right| = i - 1. \tag{3.4}$$

Let all A_l's denote those sets from which $\overline{A}_l$ was derived. Now, we distinguish two cases with respect to j_1. If $j_1 = 1$, then $\min(A^1) \notin \overline{A}_1$. Therefore, $A_1 = \overline{A}_1$, and we have

$$\overline{A}_1 \cap \cdots \cap \overline{A}_i = A_1 \cap \cdots \cap A_i.$$

The cardinality of the right-hand side is $i-1$, contradicting condition (3.2).

The other case is $j_1 > 1$. Then either $|A_1 \cap \cdots \cap A_i| = i - 1$ or for all l the relation $\min(A^1) \in A_l$ holds. In the first subcase we have a contradiction with Condition (3.2). In the second subcase we have at most $k-1$ subsets, thus, $i - 1 \leq k - 2$. Now let us denote this intersection by K. Then $A_1 \cap \cdots \cap A_i = K \cup \{\min(A^1)\} := K'$. Choose an integer $t \in \{1, \ldots, |X|\} \setminus \{j_1, \ldots, j_i\}$ and take the subset $A_t(K') \in \mathcal{A}_t$. (We have such an element due to condition (3.3).) Now

$$\left|A_t(K') \cap A_1 \cap \cdots \cap A_i\right| = |K'| = i$$

contradicts condition (3.2). ∎

Theorem 3.4 is not the only way in which one can vary the conditions of Theorem 3.2. Levon Khachatrian proved the following result [8]:

Theorem 3.5. *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_k$ be set systems on the finite set X. For $l = 1, \ldots, k-1$, assume that for arbitrary $A_i \in \mathcal{A}_i$ where $i = l, \ldots, k$, we have*

$$|A_l \cap \cdots \cap A_k| \notin \{1, \ldots, k-l\}. \tag{3.5}$$

Furthermore, assume that for every $j = 2, \ldots, k$, we have

$$\bigcup_{A \in \mathcal{A}_j} A = X. \tag{3.6}$$

Then X has a k-split with respect to these set systems.

It is also interesting to remark that P. Frankl and G.O.H. Katona [6] proved the following nice theorem which clearly has a close connection to our Conjecture:

Theorem 3.6.[6] *Let $\mathcal{H}$ be a set system on X. Assume any k elements of $\mathcal{H}$ has a non-$(k-1)$-element intersection for every $1 \le k \le |\mathcal{H}|$. Then $|\mathcal{H}| \le |X|$.*

While this result supports our conjecture in a very special case, the real connection still should be discovered.

4. k-splitting in Posets

It is easy to see that Theorems 3.2 and 3.4 imply some direct generalizations of Theorem 2.3. We give a possible generalization applying Theorem 3.2 here.

Let $\mathcal{P} = (P, <)$ be a finite partially ordered set. Let G be a generator system in $\mathcal{P}$. Let $P_1, \ldots, P_k$ be subsets of P which satisfy $P_1 \cup \cdots \cup P_k \subseteq P \setminus G$. The generator system G has the *k-cone-splitting* property with respect to those subsets if and only if there exists a partition $G_1 \cup G_2 \cup \cdots \cup G_k = G$ such that, for every $i = 1, \ldots, k$, we have $P_i \subseteq C(G_i)$. We say that G is *k-cone-dense* with respect to these subsets if, for every $i = 1, \ldots, k$ and $1 \le j_1 < \cdots < j_i \le k$ taking arbitrary elements $x_l \in P_l$, we have

$$|C(x_1) \cap \cdots \cap C(x_i) \cap G| \notin \{1, \ldots, i-1\}.$$

Theorem 4.1. *Let the generator system G be k-cone-split with respect to the subsets $P_1, \ldots, P_k$. Then G has a k-cone-split with respect to those subsets.*

The proof is based on Theorem 3.2 and is very similar to the proof of Theorem 2.3. It is left to the diligent reader. We remark that if G is a maximal antichain and the subsets belong entirely (independently from each other) to $\mathcal{D}(A)$ or $\mathcal{U}(A)$, then in the definitions and in the theorem, it is sufficient to consider operations $\mathcal{U}$ and $\mathcal{D}$ respectively, but we do not discuss it here in detail.

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