Studia Scientiarum Mathematicarum Hungarica 22 (1987), 383—393

A 3-PART SPERNER THEOREM

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1. Introduction

Let X be a finite set of n elements and let $\mathcal{F}\subset 2^X$ be a family of different subsets of X such that every pair of members F_1, F_2 of $\mathcal{F}$ $(F_1\neq F_2)$ satisfies $F_1\not\subset F_2$. Sperner [6] proved that in this case

$$|F| \leqq \binom{n}{\left\lfloor \frac{n}{2}\right\rfloor}. \tag{1}$$

If the inequality is realized by equality then either

$$\mathcal{F}=\left\{F\subset X\colon |F|=\left\lfloor\frac{n}{2}\right\rfloor\right\} \quad \text{or} \tag{2}$$

$$\mathcal{F}=\left\{F\subset X\colon |F|=\left\lceil\frac{n}{2}\right\rceil\right\} \quad \text{if} \quad n\equiv 1 \pmod 2.$$

Kleitman [5] and Katona [4] independently proved: If X is divided into two disjoint parts $(X=X_1\cup X_2,\ X_1\cap X_2=\emptyset)$ and the family $\mathcal{F}$ contains no two different members F_1, F_2 such that:

$$F_1\subset F_2 \quad \text{and} \quad F_2\setminus F_1\subset X_i \qquad (i=1 \text{ or } 2) \tag{3}$$

then (1) holds.

By an analogous way, the more-part Sperner problem can be defined. Let X be a finite set of n elements and $X=X_1\cup\ldots\cup X_M$ where $X_i\cap X_j=\emptyset$ $(i\neq j)$. The set $\mathcal{F}\subset 2^X$ of subsets of X is an M-part Sperner family, if no two members of $\mathcal{F}$ satisfy:

$$F_1\subset F_2 \quad \text{and} \quad F_2\setminus F_1\subset X_i \quad \text{for some} \quad i\in\{1,\ldots,M\}. \tag{4}$$

As it is shown in [4], if $M\geqq 3$, then inequality (1) is not true for every M-part Sperner family $\mathcal{F}$. Füredi [2], Griggs, Odlyzko and Shearer [3] found good asymptotic results for the maximum size of M-part Sperner families. But the exact value is not known even for $M=3$.

The aim of this paper is to determine this exact maximum size for the very modest case $M=3$ and $|X_3|=1$. Exactly, we prove

Research partially supported by Hungarian National Foundation for Scientific Research Grant No. 1812.

1980 *Mathematics Subject Classification.* Primary 05A05; Secondary 05C65.
Key words and phrases. Sperner theorem.

THEOREM 1. *Let* $X=X_1\cup X_2\cup X_3$ *be a partition, where* $|X_1|=n_1\leqq|X_2|=n_2$; $|X_3|=1$; $n_1+n_2+1=n$. *Then*

$$\max\{|\mathscr{F}|:\ \mathscr{F}\text{ is a 3-part Sperner family}\}=$$

$$(5)\qquad =\begin{cases}\dbinom{n}{\left\lfloor\frac{n}{2}\right\rfloor} & \textit{if}\quad n_1\not\equiv n_2\pmod 2,\\[2ex] 2\dbinom{n-1}{\frac{n-1}{2}} & \textit{if}\quad n_1\equiv n_2\equiv 1\pmod 2,\\[2ex] 2\dbinom{n-1}{\frac{n-1}{2}}-\left(\dbinom{n_1}{\frac{n_1+2}{2}}-\dbinom{n_1}{\frac{n_1}{2}}\right)\left(\dbinom{n_2}{\frac{n_2+2}{2}}-\dbinom{n_2}{\frac{n_2}{2}}\right) & \textit{if}\quad n_1\equiv n_2\equiv 0\pmod 2.\end{cases}$$

In the proof, the next theorem of Griggs, Odlyzko and Shearer [3] has a fundamental role. (A new proof of this theorem and a generalization of it to the extreme points of the polytop of the M-part Sperner families can be found in [1].)

THEOREM (GOS [3]). *There is an* M*-part Sperner family* $\mathscr{F}$ *such that*

$$|\mathscr{F}|=\max\{|\mathscr{F}'|:\ \mathscr{F}'\text{ is an }M\text{-part Sperner family}\}$$

and $F\in\mathscr{F}$ *implies that all sets* $G\subset X$ *satisfying*

$$|F\cap X_j|=|G\cap X_j|$$

for all j $(1\leqq j\leqq M)$ *belong to* $\mathscr{F}$.

2. Proof of the main theorem

Let $\mathscr{F}\subset 2^X$ be a family of subsets of the set $X=X_1\cup X_2\cup X_3$. The 3-dimensional matrix $P(\mathscr{F})=(p_{i_1,i_2,i_3}(\mathscr{F}))\ i_j=0,\dots,n_j$ is called the *profile-matrix* of $\mathscr{F}$, where

$$(6)\qquad p_{i_1,i_2,i_3}(\mathscr{F}):=\left|\{F\in\mathscr{F}:\ \forall j\,|F\cap X_j|=i_j\}\right|.$$

According to the theorem of Griggs, Odlyzko and Shearer there is a maximum sized 3-part Sperner family $\mathscr{F}$ such that

$$p_{i_1,i_2,i_3}(\mathscr{F})=\begin{cases}\dbinom{n_1}{i_1}\dbinom{n_2}{i_2}\dbinom{n_3}{i_3},\\ \text{or}\quad 0.\end{cases}$$

Let $J=\{(i_1,i_2,i_3):\ p_{i_1,i_2,i_3}\neq 0\}$. Then according to the definition of the 3-part Sperner families the set J is a *partial transversal*, that is, if $(i_1,i_2,i_3),(i_1',i_2',i_3')\in\mathscr{J}$ and they are identical in at least two components; then $(i_1,i_2,i_3)=(i_1',i_2',i_3')$.

To the proof of Theorem 1 we need several lemmas. It is easy to see that the projection of a partial transversal (it is an $(n_1+1)\times(n_2+1)\times 2$ matrix) into its

$(n_1+1)\times(n_2+1)$ "face" has at most 2 elements in each row and each column. This justifies the following definition. $I\subset\{1,\dots,u\}\times\{1,\dots,v\}$ is a *partial 2-transversal* iff no column or row contains more than 2 elements of I. Let the values $a_1\geqq\dots\geqq a_u\geqq 1$, $b_1\geqq\dots\geqq b_v\geqq 1$ be fixed. We will consider the matrix $(a_ib_j)_{1\leqq i\leqq u,1\leqq j\leqq v}$. The partial 2-transversal I will be called *optimal* iff

1) it maximizes

$$(7)\qquad \sum_{i,j\in I}a_ib_j$$

among all partial 2-transversals;

2) It minimizes

$$(8)\qquad \sum_{(i,j)\in I}(i+j)$$

among all partial 2-transversals satisfying 1) and

3) it maximizes

$$(9)\qquad \sum_{(i,j)\in I}i\cdot j$$

among all partial 2-transversals satisfying 1) and 2).

In the proofs of the lemmas the following 3 transformations of partial 2-transversals will be used.

Transformation 1. If I contains at most one element in the i-th row and at most one in the j-th column, add (i,j) to I. This transformation increases (7).

Transformation 2. Move the element $(i,j)\in I$ into (i,k) if $k<j$ and the k-th column of I contains at most one element, or move $(i,j)\in I$ into (l,j) if $l<i$ and the l-th row contains at most one element. This transformation does not decrease (7) but decreases (8).

Transformation 3. Let $i<k$ and $j<l$. Suppose that $(i,l),(k,j)\in I$. The transformation changes I for $I'=(I-\{(i,l),(k,j)\})\cup\{(i,j),(k,l)\}$. It does not decrease (7) because

$$\sum_{(i,j)\in I'}a_ib_j-\sum_{(i,j)\in I}a_ib_j=a_ib_j+a_kb_l-a_ib_l-a_kb_j=(a_i-a_k)(b_j-b_l)\geqq 0.$$

It does not change (8), but it increases (9):

$$\sum_{(i,j)\in I'}i\cdot j-\sum_{(i,j)\in I}i\cdot j=ij+kl-il-kl=(i-k)(j-l)>0.$$

The following lemma is an easy consequence.

LEMMA 2.1. *Transformations 1—3 cannot be applied for an optimal partial 2-transversal.*

In what follows, we will study the structure of the optimal partial 2-transversals.

LEMMA 2.2. *An optimal partial 2-transversal has non-increasing number of elements in the rows (columns).*

PROOF. The number of elements of I in the i-th row (column) is denoted by ϱ_i $(\varkappa_i)$ $(0\leqq\varrho_i\leqq 2,\ 0\leqq\varkappa_i\leqq 2)$. Suppose that $i<j$ and $\varrho_i>\varrho_j$. Consider an element $(i,k)\in I$. Transformation 2 with $(i,k)\to(j,k)$ could be applied contradicting Lemma 2.1. ∎

LEMMA 2.3. *Let $u=v$. An optimal partial 2-transversal satisfies $\varrho_1=\ldots=\varrho_{u-1}=$ $=\varkappa_1=\ldots=\varkappa_{u-1}=2$ and either $\varrho_u=\varkappa_u=2$ or $\varrho_u=\varkappa_u=1$.*

PROOF. Suppose that I is optimal, consequently it satisfies the conditions of Lemma 2.2.

$\zeta_u=0$ implies $|I|\leqq 2u-2$. Hence $\varkappa_u<2$ follows. Transformation 1 could be applied with (u,u). This is a contradiction by Lemma 2.1. $\varrho_u\geqq 1$ is proved. $\varkappa_u\geqq 1$ can be seen in the same way.

Suppose that $\varrho_u=\varkappa_u=1$. Let $\varrho_{u-1}=1$. Either $(u,u)\notin I$ or $(u-1,u)\notin I$ holds, so Transformation 1 could be applied with one of them, contradicting the optimality of I. This proves $\varrho_{u-1}=2$ and $\varkappa_{u-1}=2$ can be proved analogously.

Suppose now that one of ϱ_u and $\varkappa_u$ equals 2. Then $|I|=2u$, thus all ϱ's and $\varkappa$'s are equal to 2. ∎

LEMMA 2.4. *Let $u<v$. An optimal partial 2-transversal satisfies $\varrho_1=\ldots=\varrho_u=$ $=\varkappa_1=\ldots=\varkappa_{u-1}=2$, $\varkappa_{u+2}=\ldots=\varkappa_v=0$ and either $\varkappa_u=\varkappa_{u+1}=1$ or $\varkappa_u=2$, $\varkappa_{u+1}=0$.*

PROOF. Suppose that I is optimal, consequently it satisfies the conditions of Lemma 2.2.

$\varrho_u\leqq 1$ implies $|I|\leqq 2u-1$. Hence $\varkappa_u\leqq 1$ and $\varkappa_{u+1}\leqq 1$ follow. One of (u,u) and $(u,u+1)$ is not in I, thus adding it to I by Transformation 1 it leads to a contradiction. $\varrho_1=\ldots=\varrho_u=2$ is proved.

If $\varkappa_u=2$ then $|I|=\sum_{i=1}^{u}\varrho_i=2u=\sum_{i=1}^{v}\varkappa_i$ implies $\varkappa_1=\ldots=\varkappa_u=2$, $\varkappa_{u+1}=\ldots=\varkappa_v=0$.

Suppose now that $\varkappa_u=1$. It implies $\varkappa_{u+1}=1$. However, $\varkappa_{u-1}=1$ leads to a contradiction. Indeed, $(i,u+1)\in I$ holds for some i. Transformation 2 can be used with $(i,u+1)$ and with either (i,u) or $(i,u-1)$, because both ones cannot belong to I. Hence we have $\varkappa_{u-1}=2$. $\varkappa_{u+2}=\ldots=\varkappa_v=0$ trivially follows. ∎

LEMMA 2.5. *Let $u\leqq v$, $1\leqq i<u$ and $1\leqq j<u-1$. Suppose that I is an optimal partial 2-transversal and its subset $A\subset I$ satisfies the following conditions:*

(10) *A contains at most one element in the ith row, at most one in the jth column and at most one in the $(j+1)$st column,*

and

(11) *$I-A$ has no element of the form (i,k), $k<j$ or (l,j), $l<i$ or $(m,j+1)$, $m<i$.*

Then either $(i,j)\in I$ or $(i,j+1)\in I$ holds. The roles of the rows and columns can be interchanged.

PROOF. We use an indirect way. Suppose that

$$(i,j)\notin I \quad\text{and}\quad (i,j+1)\notin I.$$

Lemmas 2.3, 2.4 and (10) imply the existence of an $(i,k)\in I-A$. $k\neq i,\ i+1$ by the assumption. On the other hand, (11) results in $j+1<k$. The same arguments show the existence of an $(s,j)\in I-A$ and a $(t,j+1)\in I-A$ where $s,t>i$ can be assumed. We distinguish 2 cases:

(i) $s=t$. The j-th column contains two elements of I: (s,j) and $(s,j+1)$. Therefore we have $(s,k)\notin I$. Transformation 3 can be applied for (i,k) and (s,j). This contradiction proves the statement in this case.

(ii) $s\neq t$. (s,k) and (t,k) cannot be both in I. Suppose e.g. that $(s,k)\notin I$. Transformation 3 can be applied for (i,k) and (s,j), again. This case is also settled. ∎

LEMMA 2.6. *Let $u,v\geqq 2$. If I is an optimal partial 2-transversal then $(1,1),(1,2),(2,1)\in I$.*

PROOF. Suppose that $u,v\geqq 4$ and apply Lemma 2.5 with $A=\emptyset$, $i=1$, $j=1$. Either $(1,1)\in I$ or $(1,2)\in I$ can be stated. We distinguish these two cases.

a) $(1,1)\in I$. Apply Lemma 2.5 with $A=\{(1,1)\}$, $i=1$, $j=2$. Two subcases are distinguished: aa) $(1,2)\in I$, ab) $(1,3)\in I$, $(1,2)\notin I$.

aa) $(1,2)\in I$. Lemma 2.5 can be applied, again, with $A=\{(1,1),(1,2)\}$ $i=2$, $j=1$. If we obtain $(2,1)\in I$ we are done. Suppose that $(2,2)\in I$, $(2,1)\notin I$. By Lemmas 2.3 and 2.4 there exists a $(k,1)\in I$, $k>2$. Transformation 3 can be applied with $(2,2)$ and $(k,1)$ because $(k,2)\notin I$. This contradiction proves the lemma in this case.

ab) $(1,3)\in I$. Apply Lemma 2.5 with $A=\{(1,1),(1,3)\}$, $i=2$, $j=1$. Two subcases will be distinguished:

aba) $(2,1)\in I$. We may continue: either $(2,2)\in I$ or $(2,3)\in I$. In the first case the change of $(2,2)$ and $(1,3)$ (Transformation 3) leads to the desired contradiction. In the latter case there is a $(k,2)\in I$ $(k>2)$ by Lemmas 2.3 and 2.4. $(k,2)$ and $(1,3)$ give the contradiction.

abb) $(2,2)\in I$, $(2,1)\notin I$. Transformation 3 with $(2,2)$ $(1,3)$ gives rise to a contradiction unless $(2,3)\in I$. In this latter case there is a $(k,1)\in I$ $(k>2)$. The application of Transformation 3 for $(k,1)$ and $(2,3)$ settles this case.

b) $(1,2)\in I$ but $(1,1)\notin I$. By Lemmas 2.3 and 2.4 there are $(k,1)$, $(l,1)\in I$ $(k,l>1,\ k\neq l)$. One of $(k,2)$ and $(l,2)$, say $(k,2)$, is missing from I, therefore Transformation 3 can be applied with $(k,1)$ and $(1,2)$. A contradiction.

The cases when $v\geqq u=2,3$ can be proved similarly.

LEMMA 2.7. *Let $u,v\geqq 3$. Suppose that I is an optimal partial 2-transversal and $(1,1),(1,2),(2,1)\in I$, $(2,2)\notin I$. Then $(2,3),(3,2),(3,3)\in I$.*

PROOF. Suppose that $u,v\geqq 5$. Use Lemma 2.5 with $A=\{(1,1),(1,2),(2,1)\}$, $i=2$, $j=2$. $(2,2)$ is not in I by the assumption, thus we have $(2,3)\in I$. Apply Lemma 2.5 now with $A=\{(1,1),(1,2),(2,1),(2,3)\}$, $(3,2)$ and $(4,2)$. If $(3,2)\in I$ then Transformation 3 could be used for $(2,3)$ and $(3,2)$ except in the case $(3,3)\in I$. The statement is proved in this case.

If $(4, 2)\in I$ then $(4, 3)\in I$ can be supposed, again by Transformation 3. The third row contains at least two elements, therefore there exists a $(3, k)\in I$ satisfying $k>4$. $(4, k)\notin I$ is obvious, thus Transformation 3 is applicable with $(4, 2)$ and $(3, k)$. This contradiction proves the lemma for $u, v\geqq 5$.

The cases $v\geqq u=3, 4$ can be proved similarly. ∎

LEMMA 2.8. *If $u=1\leqq v$ then the optimal partial 2-transversal I consists of $(1, 1)$ and $(1, 2)$. If $u=2\leqq v$ then $I=\{(1, 1), (1, 2), (2, 1), (2, 2)\}$.*

The proof is trivial.

It is easy to prove by induction, using Lemmas 2.6—2.8, that the optimal partial 2-transversal consists of blocks

$$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$

along the diagonal (i, i) and it might end with an

$$11$$

if exactly one row remains at the end. Not making any additional condition on the a's and b's nothing else can be said about the blocks. However, we want to use these results for binomial coefficients. They are ordered in natural order, therefore we have equal pairs among them. Under this condition the structure of the optimal partial 2-transversal can be described rather well.

First we investigate some further transformations.

Transformation 4.

$$\begin{matrix} 1&1&0&0&0 \\ 1&0&1&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&1 \\ 0&0&0&1&1 \end{matrix} \rightarrow \begin{matrix} 1&1&0&0&0 \\ 1&1&0&0&0 \\ 0&0&1&1&0 \\ 0&0&1&0&1 \\ 0&0&0&1&1. \end{matrix}$$

It is understood that this transformation is made somewhere along the diagonal (i, i) of the matrix $(a_i\cdot b_j)$. Denote by $c_1\geqq\ldots\geqq c_5$ and $d_1\geqq\ldots\geqq d_5$ the values a, resp. b corresponding the rows and columns, resp. The values of the subsums what these submatrices give from $\sum\limits_{(i,j)\in I} a_ib_j$ are

$$c_1d_1+c_1d_2+c_2d_1+c_2d_3+c_3d_2+c_3d_3+c_4d_4+c_4d_5+c_5d_4+c_5d_5$$

and

$$c_1d_1+c_1d_2+c_2d_1+c_2d_2+c_3d_3+c_3d_4+c_4d_3+c_4d_5+c_5d_4+c_5d_5.$$

An easy calculation shows that the second sum is less than or equal to the first sum under the assumption $c_2=c_3$, $d_2=d_3$.

We say that the transformation is *non-increasing*. The *constant* and *non-decreasing* transformations are defined analogously. Easy calculations show the following lemmas.

LEMMA 2.9. *Transformation 4 is non-increasing if $c_2=c_3$, $d_2=d_3$, non-decreasing if $c_3=c_4$, $d_3=d_4$ and constant if $c_2=c_3$, $d_3=d_4$ or $c_3=c_4$, $d_2=d_3$.*

LEMMA 2.10. *Transformation 5 which is defined by*

$$\begin{array}{c|cccccc} c_1 & 1&1&0&0&0&0 \\ c_2 & 1&0&1&0&0&0 \\ c_3 & 0&1&1&0&0&0 \\ c_4 & 0&0&0&1&1&0 \\ c_5 & 0&0&0&1&0&1 \\ c_6 & 0&0&0&0&1&1 \\ & d_1&d_2&d_3&d_4&d_5&d_6 \end{array} \rightarrow \begin{matrix} 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&1&1, \\ & \end{matrix}$$

is non-increasing if $c_2=c_3$, $c_4=c_5$, $d_2=d_3$, $d_4=d_5$, non-decreasing if $c_3=c_4$, $d_3=d_4$ and constant if $c_2=c_3$, $c_4=c_5$, $d_3=d_4$ or $c_3=c_4$, $d_2=d_3$, $d_4=d_5$.

LEMMA 2.11. *Transformation 6 which is defined by*

$$\begin{array}{ccccc} c_1 & 1&1&0&0 \\ c_2 & 1&0&1&0 \\ c_3 & 0&1&1&0 \\ c_4 & 0&0&0&1 \\ & d_1&d_2&d_3&d_4 \end{array} \begin{array}{c} \left(\begin{matrix}0\\0\\0\\1\end{matrix}\right) \\ d_5 \end{array} \rightarrow \begin{matrix} 1&1&0&0 \\ 1&1&0&0 \\ 0&0&1&1 \\ 0&0&1&1 \end{matrix} \left(\begin{matrix}0\\0\\0\\0\end{matrix}\right)$$

is non-decreasing if $c_3=c_4$, $d_2=d_3$, $(d_4=d_5)$ or $c_2=c_3$, $d_3=d_4$ or $c_3=c_4$, $d_3=d_4$.

LEMMA 2.12. *Transformation 7 which is defined by*

$$\begin{array}{ccccc} c_1 & 1&1&0&0 \\ c_2 & 1&0&1&0 \\ c_3 & 0&1&1&0 \\ & d_1&d_2&d_3&d_4 \end{array} \rightarrow \begin{matrix} 1&1&0&0 \\ 1&1&0&0 \\ 0&0&1&1 \\ & \end{matrix}$$

is constant if $c_2=c_3$ and $d_3=d_4$.

LEMMA 2.13. *Transformation 8 which is defined by*

$$\begin{array}{cccc} c_1 & 1&1&0 \\ c_2 & 1&1&0 \\ c_3 & 0&0&1 \end{array} \left(\begin{matrix}0\\0\\1\end{matrix}\right) \rightarrow \begin{matrix} 1&1&0 \\ 1&0&1 \\ 0&1&1 \end{matrix} \left(\begin{matrix}0\\0\\0\end{matrix}\right)$$

is non-decreasing if $c_2=c_3$ and $d_2=d_3$.

Using the above transformations we are now able to describe one of the optimal partial 2-transversals. We know that an optimal partial 2-transversal consists of 2×2 and 3×3 blocks. It is called *superoptimal* iff (i) it minimizes the number of 3×3 blocks and (ii) it minimizes the sum of the coordinates of the starting points of the 3×3 blocks among all optimal ones satisfying (i).

LEMMA 2.14. *Let* $u=n_1+1\geqq n_2+1=v$ *and suppose that* $a_1\geqq\ldots\geqq a_u$ *and* $b_1\geqq\ldots\geqq b_v$ *are the binomial coefficients* $\binom{n_1}{i}$ *and* $\binom{n_2}{i}$, *resp. Then the superoptimal partial 2-transversal consists of* 2×2 *blocks with two exceptions. If* $n_1\equiv n_2\equiv 0$ (mod$_2$) *then the first block is a* 3×3 *one. At the end of the diagonal a block of the form* 1 *or* 1 1 *can occur.*

PROOF. Suppose that I is a superoptimal partial 2-transversal. Three cases will be distinguished in the proof.

1) $n_1\not\equiv n_2$ (mod$_2$). Transformation 5 is constant by Lemma 2.10. It decreases the number of 3×3 blocks. This proves that I cannot contain two neighbouring 3×3 blocks.

However, Transformation 4 is also constant in this case (Lemma 2.9). Applying it backwards, it decreases the sum of the coordinates of starting points of the 3×3 blocks. Therefore I cannot contain a 3×3 block following a 2×2 one. Hence I can have at most one 3×3 block, at the beginning only.

We prove now that even this only one 3×3 block is excluded.

11) $n_1<n_2\equiv 0$ (mod$_2$). As n_1+1 is even, I ends with a block of the form 11. Transformation 4 is constant in this case, we may move the 3×3 block toward the end while I remains optimal. Finally we arrive to the configuration of the left-hand side of Transformation 6. This is non-decreasing, but it decreases the number of 3×3 blocks. I was not superoptimal. This contradiction proves the statement for this case.

12) $n_1<n_2\equiv 1$ (mod$_2$). n_1+1 is odd. We can repeat the argument of case 11), but Transformation 7 should be used in place of Transformation 6. I cannot contain any 3×3 block in this case.

2) $n_1\equiv n_2\equiv 1$ (mod 2). Consider a 3×3 block B_1 following a 2×2 block. Transformation 4 can be applied backwards unless the coordinate of its starting point of B_1 is odd. Thus we may suppose that this is the case. If B_1 is followed by a 3×3 block then Transformation 5 gives rise to a contradiction. Denote by B_2 the first 3×3 block occuring after B_1. It is easy to see that the coordinate of the starting point of B_2 is even. The converse of Transformation 4 leads to a contradiction. Therefore B_1 cannot be followed by a 3×3 block.

The first two blocks cannot be 3×3 ones because of Lemma 2.10. If the first block is a 3×3 one then the first other 3×3 block following it $(=B_1)$ has an even starting coordinate. This contradiction proves that there is at most one 3×3 block in I and its starting coordinate is odd.

Suppose that there is a 3×3 block. By Transformation 4 we can move this block until the end and I remains optimal. n_1+1 is even, thus the end looks like the left-hand side of Transformation 6. Lemma 2.11 gives the contradiction. I cannot contain any 3×3 block in this case.

3) $n_1\equiv n_2\equiv 0$ (mod$_2$). If the first block is a 2×2 then Transformation 4 can be used backwards for the first 3×3 block, with a contradiction. However, all blocks could not be 2×2 because otherwise Transformation 5 would lead to a contradiction, using it backwards. (If $n_1<5$, this argument does not work. For $n_1=2$ and 4 Transformation 8 can be used.) This proves that the first block has to be a 3×3 one. Disregarding the first block, the rest can be treated like case 2). In this case we obtained that the first block is a 3×3 one, all other blocks are 2×2. ∎

PROOF OF THEOREM 1. First we prove that $\max\{|\mathscr{F}|: \mathscr{F}$ is a 3-part Sperner family$\}$ cannot exceed the values given in the theorem. This maximum equals (by Theorem GOS)

$$\max\sum\binom{n_1}{i}\binom{n_2}{j}\binom{1}{k} \tag{12}$$

where we sum over $(i,j,k)\in J$ and the maximum is taken over all partial transversals J in $\{0,\ldots,n_1\}\times\{0,\ldots,n_2\}\times\{0,1\}$. It is easy to see that the set $I=\{(i,j)\colon (i,j,k)\in J\}$ is a partial 2-transversal in $\{0,\ldots,n_1\}\times\{0,\ldots,n_2\}$. Therefore (12) can be upperbounded with

$$\max\sum_{(i,j)\in I}\binom{n_1}{i}\binom{n_2}{j} \tag{13}$$

where the max runs over all partial 2-transversals I. (13) can be determined by Lemma 2.14.

1) One of n_1 and n_2 is odd. Denoting by a_i and b_i the respective binomial coefficients, (13) can be expressed as

$$\sum(a_i+a_{i+1})(b_i+b_{i+1}) = \sum_{i=1,2,\ldots}a_ib_i+\sum_{i=1,3,\ldots}a_ib_{i+1}+\sum_{i=1,3,\ldots}a_{i+1}b_i. \tag{14}$$

If n_1 is odd then $a_1=a_{i+1}$ $(i=1,3,\ldots)$ hence

$$\sum_{i=1,3,\ldots}a_ib_{i+1}=\sum_{i=1,3,\ldots}a_{i+1}b_{i+1}=\sum_{i=2,4,\ldots}a_ib_i$$

and

$$\sum_{i=1,3,\ldots}a_{i+1}b_i=\sum_{i=1,3,\ldots}a_ib_i$$

follow. Substituting these into (14) we obtain $2\sum_{i=1,2,\ldots}a_ib_i$. The case when n_2 is odd can be obtained in the same way.

Let n_1 be odd and n_2 be even. Then

$$\sum_{i=1,2,\ldots}a_ib_i=\binom{n_1}{\frac{n_1-1}{2}}\binom{n_2}{\frac{n_2}{2}}+\binom{n_1}{\frac{n_1+1}{2}}\binom{n_2}{\frac{n_2-2}{2}}+\binom{n_1}{\frac{n_1-3}{2}}\binom{n_2}{\frac{n_2+2}{2}}+$$

$$+\binom{n_1}{\frac{n_1+3}{2}}\binom{n_2}{\frac{n_2-4}{2}}+\ldots=\binom{n_1+n_2}{\frac{n_1+n_2-1}{2}}.$$

Multiplying it by 2 we obtain

$$\binom{n_1+n_2+1}{\frac{n_1+n_2+1}{2}} = \binom{n}{\frac{n}{2}}.$$

The case when n_2 is odd and n_1 is even can be treated analogously.

Finally, if $n_1 \equiv n_2 \equiv 1$ (mod$_2$) then

$$\sum_{i=1,3,\dots} a_i b_i = \binom{n_1}{\frac{n_1-1}{2}}\binom{n_2}{\frac{n_2+1}{2}} + \binom{n_1}{\frac{n_1+1}{2}}\binom{n_2}{\frac{n_2-1}{2}} + \dots = \binom{n_1+n_2}{\frac{n_1+n_2}{2}} = \binom{n-1}{\frac{n-1}{2}}$$

proves the upper bound in this case.

2) $n_1 \equiv n_2 \equiv 0$ (mod 2). Lemma 2.14 gives

$$a_1b_1+a_1b_2+a_2b_1+a_2b_3+a_3b_2+a_3b_3+$$
$$+\sum_{i=4,6,\dots}(a_i+a_{i+1})(b_i+b_{i+1}) =$$
$$= a_1b_1+a_1b_2+a_2b_1+a_2b_3+a_3b_2+a_3b_3+2\sum_{i=4,5,6,\dots} a_ib_i =$$
$$= -a_1b_1+a_1b_2+a_2b_1-2a_2b_2+a_2b_3+a_3b_2-a_3b_3+2\sum_{i=1,2,\dots} a_ib_i =$$
$$= -\binom{n_1}{\frac{n_1}{2}}\binom{n_2}{\frac{n_2}{2}} + \binom{n_1}{\frac{n_1}{2}}\binom{n_2}{\frac{n_2+2}{2}} + \binom{n_1}{\frac{n_1-2}{2}}\binom{n_2}{\frac{n_2}{2}} - 2\binom{n_1}{\frac{n_1-2}{2}}\binom{n_2}{\frac{n_2+2}{2}} +$$
$$+\binom{n_1}{\frac{n_1-2}{2}}\binom{n_2}{\frac{n_2-2}{2}} + \binom{n_1}{\frac{n_1+2}{2}}\binom{n_2}{\frac{n_2+2}{2}} - \binom{n_1}{\frac{n_1+2}{2}}\binom{n_2}{\frac{n_2-2}{2}} + 2\binom{n_1+n_2}{\frac{n_1+n_2}{2}} =$$
$$= 2\binom{n-1}{\frac{n-1}{2}} - \left(\binom{n_1}{\frac{n_1+2}{2}} - \binom{n_1}{\frac{n_1}{2}}\right)\left(\binom{n_2}{\frac{n_2+2}{2}} - \binom{n_2}{\frac{n_2}{2}}\right).$$

We have proved that the right-hand side in the theorem is an upper bound. We need constructions proving the equality.

It is easy to check that the following families are 3-part Sperner families and their size is optimal:

$$\mathscr{F} = \left\{F\colon |F| = \frac{n_1+n_2-1}{2},\ |X_3\cap F| = 0\right\} \cup$$
$$\cup \left\{F\colon |X_2\cap F| - |X_1\cap F| = \frac{n_2-n_1-1}{2},\ |X_3\cap F| = 1\right\}$$

if n_1 is odd, n_2 is even,

$$\mathscr{F} = \left\{F\colon |F| = \frac{n_1+n_2-1}{2},\ |X_3\cap F| = 0\right\} \cup$$
$$\cup \left\{F\colon |X_2\cap F| - |X_1\cap F| = \frac{n_2-n_1+1}{2},\ |X_3\cap F| = 1\right\}$$

if n_1 is even, n_2 is odd,

$$\mathscr{F} = \left\{F\colon |F| = \frac{n_1+n_2}{2},\ |X_3\cap F| = 0\right\} \cup$$
$$\cup \left\{F\colon |X_2\cap F| - |X_1\cap F| = \frac{n_2-n_1}{2},\ |X_3\cap F| = 1\right\}$$

if both n_1 and n_2 are odd and finally

$$\mathscr{F} = \left\{F\colon |F| = \frac{n_1+n_2}{2},\ |X_1\cap F| \neq \frac{n_1}{2}, \frac{n_1}{2}+1,\ |X_3\cap F| = 0\right\} \cup$$
$$\cup \left\{F\colon |X_1\cap F| = \frac{n_1}{2},\ |X_2\cap F| = \frac{n_2}{2}-1,\ |X_3\cap F| = 0\right\} \cup$$
$$\cup \left\{F\colon |X_1\cap F| = \frac{n_1}{2}+1,\ |X_2\cap F| = \frac{n_2}{2},\ |X_3\cap F| = 0\right\} \cup$$
$$\cup \left\{F\colon |X_2\cap F| - |X_1\cap F| = \frac{n_2-n_1}{2},\ |X_3\cap F| = 1\right\}$$

if both n_1 and n_2 are even. ∎

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(Received October 31, 1984)

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