A dichotomy for infinitely many $\mathbf{\Sigma}^0_2(\kappa)$ relations on the κ-Baire space

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A dichotomy for infinitely many F_σ relations

$\mathcal{R}$ is a collection of finitary relations on a set X.

$Y \subseteq X$ is $\mathcal{R}$-*independent* if for all $1 \leq k < \omega$ and k-ary $R \in \mathcal{R}$ we have:

$(x_1, \ldots, x_k) \notin R$ for all pairwise distinct $x_1, \ldots, x_k \in Y$.

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Theorem (Kubiś, 2003; Doležal, Kubiś 2015)

Let $\mathcal{R}$ be a countable set of F_σ relations on a Polish space X (i.e., every $R \in \mathcal{R}$ is an F_σ subset of ${}^kX$ for some $1 \le k < \omega$).

If for every $\alpha < \omega_1$, there exists an $\mathcal{R}$-independent $Y \subseteq X$ of Cantor-Bendixson rank $\ge \alpha$ (i.e. $Y^{(\alpha)} \neq \emptyset$),

then there exists a perfect $\mathcal{R}$-independent subset of X.

A dichotomy for infinitely many $\boldsymbol{\Sigma}^0_2(\kappa)$ relations

Assume $\kappa^{<\kappa} = \kappa$ is uncountable. ${}^\kappa\kappa$ denotes the κ-Baire space.

R is a $\boldsymbol{\Sigma}^0_2(\kappa)$ *relation* on a topological space X iff

R is a union of $\leq \kappa$ many closed subsets of ${}^k X$ for some $1 \leq k < \omega$.

Theorem (Sz.)

Assume $\diamondsuit_\kappa$ or κ is inaccessible. Let $\mathcal{R}$ be a collection of $\leq \kappa$ many $\boldsymbol{\Sigma}^0_2(\kappa)$ relations on the κ-Baire space ${}^\kappa\kappa$.

If the κ-version of the statement "there exist $\mathcal{R}$-independent subsets of arbitrarily large Cantor-Bendixson rank" holds,

then there exists a κ-perfect $\mathcal{R}$-independent subset $Y \subseteq {}^\kappa\kappa$,

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then there exists a κ-perfect $\mathcal{R}$-independent subset $Y \subseteq {}^\kappa\kappa$,

i.e., $Y = [T]$ for a κ-perfect tree T (a tree $T \subseteq {}^{<\kappa}\kappa$ of height κ that is $< \kappa$-closed and its set of splitting nodes is cofinal).

A game characterizing κ-perfectness

Definition (Väänänen)

Let $X \subseteq {}^\kappa\kappa$ and let $\omega \leq \delta \leq \kappa$. Then $G_\delta(X)$ is the following game.

I	n_0		n_1		$\dots$		n_α		$\dots$	
II		x_0		x_1		$\dots$		x_α		$\dots$

I plays $n_\alpha < \kappa$ such that $n_\alpha > n_\beta$ for all $\beta < \alpha$, and $n_\alpha = \sup_{\beta<\alpha} n_\beta$ at limits α.

II responds with $x_\alpha \in X$ such that $x_\alpha \restriction n_{\beta+1} = x_\beta \restriction n_{\beta+1}$ but $x_\alpha \neq x_\beta$ for all $\beta < \alpha$.

Player II wins, if she can make all her δ moves.

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When $\delta = \omega$:

- A closed set X contains a perfect subset iff II wins $G_\omega(X)$ (iff I does not win $G_\omega(X)$).

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When $\delta = \kappa$:

- A closed set X contains a κ-perfect subset iff II wins $G_\kappa(X)$.
- X is κ-*scattered* iff Player I wins $G_\kappa(X)$.

Trees as "Cantor-Bendixson ranks" for the κ-Baire space

Definition (Väänänen)

Let $X \subseteq {}^{\kappa}\kappa$, and let T be any tree. $G_T(X)$ is the following game.

I	t_0, n_0		t_1, n_1		$\dots$		t_α, n_α		$\dots$
II		x_0		x_1		$\dots$		x_α		$\dots$

I plays $t_\alpha \in T$ and $n_\alpha < \kappa$ such that $t_\alpha >_T t_\beta$ and $n_\alpha > n_\beta$ for all $\beta < \alpha$, and $n_\alpha = \sup_{\beta<\alpha} n_\beta$ at limits α.

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The first player who can not move loses, and the other player wins.

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- If T consists of just one branch of length δ, then $G_T(X)$ is same game as $G_\delta(X)$.

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For an ordinal α, let

$B_\alpha =$ tree of descending sequences of elements of α.

Claim

The Cantor-Bendixson rank of X is $\geq \alpha$ (i.e. $X^{(\alpha)} \neq \emptyset$)

iff Player I *wins* $G_{B_\alpha}(X)$

iff Player II *does not win* $G_{B_\alpha}(X)$.

Let $X \subseteq {}^\kappa\kappa$, and let T be any tree. $G_T(X)$ is the following game.

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Two ways to generalize Cantor-Bendixson ranks for $X \subseteq {}^\kappa\kappa$ using trees T without κ-branches:

"X is simple iff Player I wins $G_T(X)$"

or

"X is simple iff Player II does not win $G_T(X)$."

Recall: A closed $X \subseteq {}^\kappa\kappa$ contains a κ-perfect subset iff II wins $G_\kappa(X)$.

A dichotomy for infinitely many $\mathbf{\Sigma}_2^0(\kappa)$ relations

Theorem (Sz.)

Assume $\diamondsuit_\kappa$ or κ is inaccessible.

Let $\mathcal{R}$ be a collection of $\leq \kappa$ many $\mathbf{\Sigma}_2^0(\kappa)$ relations on ${}^\kappa\kappa$.

Then either

- *there exists a κ-perfect $\mathcal{R}$-independent subset of ${}^\kappa\kappa$, or*
- *there exists a tree T without κ-branches, $|T| \leq 2^\kappa$,*
  such that
  Player II does not win $G_T(X)$ for any $\mathcal{R}$-independent $X \subseteq {}^\kappa\kappa$.

Step 1

Proposition

Assume $\diamondsuit_\kappa$ *or* κ *is inaccessible.*

Let $\mathcal{R}$ *be a collection of* $\leq \kappa$ *many* $\mathbf{\Sigma}^0_2(\kappa)$ *relations on* ${}^\kappa\kappa$.

If II *wins* $G_\kappa(X)$ *for some* $\mathcal{R}$*-independent* $X \subseteq {}^\kappa\kappa$*, then there exists a* κ*-perfect* $\mathcal{R}$*-independent subset of* ${}^\kappa\kappa$.

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Proof.

We build a κ-perfect tree S by playing κ many simultaneous runs of $G_\kappa(X)$ with II using her winning strategy.

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Proof.

We build a κ-perfect tree S by playing κ many simultaneous runs of $G_\kappa(X)$ with II using her winning strategy.

We can assume that every $R \in \mathcal{R}$ is closed.

Then, by $\diamondsuit_\kappa$ or the inaccessibility of κ, I can play in all the simultaneous runs in such a way that $[S]$ will be $\mathcal{R}$-independent. □

Step 2

Lemma

Let $\kappa^{<\kappa} = \kappa$ *and* $\mathcal{R}$ *an arbitrary set of finitary relations on* ${}^\kappa\kappa$.

If II *does not win* $G_\kappa(X)$ *for any* $\mathcal{R}$*-independent* $X \subseteq {}^\kappa\kappa$*, then*

there exists a tree T *without* κ*-branches,* $|T| \leq 2^\kappa$*,*

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$T_0 =$ the tree of winning strategies τ of II in short games $G_\delta({}^\kappa\kappa)$ (i.e., $\delta < \kappa$) such that the set of all possible τ-moves of II is $\mathcal{R}$-independent

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$T = \sigma T_0$, the tree of ascending chains in T_0. □

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Remark

If I wins $G_\kappa(X)$ for all $\mathcal{R}$-independent $X \subseteq {}^{\kappa}\kappa$, then there exists a tree S without κ-branches, $|S| \leq 2^\kappa$ such that I wins $G_S(X)$ for all $\mathcal{R}$-independent $X \subseteq {}^{\kappa}\kappa$.

The size of the tree T

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Under the above assumptions, we can take

$T_0 =$ *the tree of "$\mathcal{R}$-nice" winning strategies of II in short versions $G^*_\delta({}^\kappa\kappa)$ of the κ-perfect set game,*

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$T = \sigma T_0$, *the tree of ascending chains in T_0.*

Therefore when κ is inaccessible, we can have $|T| \leq \kappa$.

A corollary

Theorem (Väänänen, 1991)

Assume $\mathrm{I}^-(\kappa)$. If $X \subseteq {}^\kappa\kappa$ and $|X| > \kappa$, then Player II wins $G_\kappa(X)$.

$\mathrm{I}^-(\kappa)$*:* there exists a κ^+-complete normal ideal $\mathcal{I}$ on κ^+ and a dense subset $K \subseteq \mathcal{I}^+$ such that every descending sequence of elements of K of length $< \kappa$ has a lower bound in K.

Corollary (Sz., Väänänen,2015)

Assume $\mathrm{I}^-(\kappa)$ and that $\diamondsuit_\kappa$ or κ is inaccessible.

Let A be a κ-analytic subset of ${}^\kappa\kappa$, $\mathcal{R}$ a collection of $\leq \kappa$ many $\mathbf{\Sigma}^0_2(\kappa)$ relations on A.

If there is an $\mathcal{R}$-independent $X \subseteq A$ of size $> \kappa$, then
there exists a κ-perfect $\mathcal{R}$-independent subset of A.

Thank you for your attention!