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CONTINUOUS SELECTIONS. I

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1. Introduction

One of the most interesting and important problems in topology is the extension problem: Two topological spaces X and Y are given, together with a closed $A \subset X$, and we would like to know whether every continuous function $g: A \to Y$ can be extended to a continuous function f from X (or at least from some open $U \supset A$) into Y. Sometimes there are additional requirements on f, which frequently (as in the theory of fibre bundles) take the following form: For every x in X, $f(x)$ must be an element of a pre-assigned subset of Y. This new problem, which we call the *selection* problem, is clearly more general than the extension problem, and presents a challenge even when A is the null set or a 1-point set (where the extension problem is trivial). So far, only isolated and special cases of the selection problem have been considered and, with the possible exception of Tong [33] [34], Katětov [19] [20], and C. H. Dowker [8], no attempt has been made to obtain results under minimal hypotheses. This paper and the following ones make such an attempt, and yield the following overall conclusion: Most of the familiar extension theorems, such as Urysohn's characterization of normality [35], Kuratowski's extension theorems for finite dimensional spaces [24; Theorem 1 and Theorem 1′] and the homotopy extension theorem [17; Theorem VI, 5] can be slightly altered (and essentially generalized) to obtain analogous selection theorems. To show how this can be done, and how the resulting theorems can be applied, is the purpose of this sequence of papers.

Let us now introduce some notation. Throughout these papers, X and Y will denote topological spaces, and 2^Y will denote the family of non-empty subsets of Y. Subfamilies of 2^Y will be denoted by script letters such as $\mathcal{S}$ and $\mathcal{F}(Y)$. A function from a subset of X to Y will be denoted by a lower-case Roman letter such as f, g, or h, while a function from a subset of X to 2^Y, which we call a *carrier*, will be denoted by a lower case Greek letter such as ϕ, ψ, or θ. We freely use all conventional notation, such as $\phi: X \to 2^Y$ to denote a function ϕ from X to 2^Y, $\phi \mid A$ to denote the restriction of ϕ to A, Λ to denote the null-set, and R to denote the real line.

After these preliminaries, we now turn to the central concept of these papers. If $\phi: X \to 2^Y$, then a *selection* for ϕ is a continuous $f: X \to Y$ such that

$$f(x) \in \phi(x)$$

[1] Most of the results in this paper were obtained while the author was an A. E. C. Fellow. Some of these results were presented to the American Mathematical Society in December 1952.

for every $x \in X$. The following examples show how some familiar topological problems can be rephrased in terms of selections.

EXAMPLE 1.1. Let $u:Y \to X$ be onto. Define $\phi:X \to 2^Y$ by $\phi(x) = u^{-1}(x)$. Then $f:X \to Y$ is a selection for ϕ if and only if f is continuous and $f(x) \in u^{-1}(x)$ for every x in X.

EXAMPLE 1.2. Let $Y = R \cup \{+\infty\} \cup \{-\infty\}$, and let $g:X \to R \cup \{-\infty\}$ and $h:X \to R \cup \{+\infty\}$ be such that $g(x) \leqq h(x)$ for every x in X. Define $\phi:X \to 2^Y$ by $\phi(x) = \{y \in R \mid g(x) \leqq y \leqq h(x)\}$. Then $f:X \to Y$ is a selection for ϕ if and only if f is continuous and $g(x) \leqq f(x) \leqq h(x)$ for every $x \in X$.

EXAMPLE 1.3. Let $\psi:X \to 2^Y$, let $A \subset X$, and let $g:A \to Y$ be a selection for $\psi \mid A$. Define $\phi:X \to 2^Y$ by $\phi(x) = \{g(x)\}$ if $x \in A$, and $\phi(x) = \psi(x)$ if $x \in X - A$. Then $f:X \to Y$ is a selection for ϕ if and only if f is a selection for ψ which extends g.

The selection problem of the first paragraph can now be rephrased as follows:

(P) *Under what conditions on X, $A \subset X$, Y, and $\phi:X \to 2^Y$, can every selection for $\phi \mid A$ be extended to a selection for ϕ, or at least for $\phi \mid U$ for some open $U \supset A$?*

The first step in answering this question is provided by the following elementary but important necessary condition, which is proved in Section 2.

PROPOSITION 2.2. *If the carrier $\phi:X \to 2^Y$ has the property that, for every $x_0 \in X$, there exists a selection for $\phi \mid U$ (U a neighborhood of x_0) which has a preassigned value $y_0 \in \phi(x_0)$ at x_0, then ϕ is lower semi-continuous.*[2]

The following are the easily verified conditions under which the carriers ϕ in Examples 1.1–1.3 are lower semi-continuous.

EXAMPLE 1.1*. ϕ is lower semi-continuous if and only if u is open.[3]

EXAMPLE 1.2*. ϕ is lower semi-continuous if and only if g is upper semi-continuous[4] and h is lower semi-continuous.[4]

EXAMPLE 1.3*. If ψ is lower semi-continuous, A closed in X, and g continuous, then ϕ is lower semi-continuous.

In view of Proposition 2.2, we shall henceforth restrict our attention to lower semi-continuous carriers. But this is the only "continuity" restriction which will ever be put on a carrier ϕ in our first two papers, and therefore all that need henceforth concern us are the domain and range of ϕ, and the set $A \subset X$. In other words, our problem becomes

(Q) *When do X, closed $A \subset X$, and $\mathcal{S} \subset 2^Y$ satisfy either of the following two conditions:*

(Q_1) *If $\phi:X \to \mathcal{S}$ is lower semi-continuous, then every selection for $\phi \mid A$ can be extended to a selection for ϕ.*

(Q_2) *If $\phi:X \to \mathcal{S}$ is lower semi-continuous, then every selection for $\phi \mid A$ can be extended to a selection for $\phi \mid U$ for some open $U \supset A$.*

Now an obvious *necessary* condition for (Q_1) (resp. (Q_2)) to be satisfied is

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[2] A carrier $\phi:X \to 2^Y$ is *lower semi-continuous* if, whenever $V \subset Y$ is open in Y, $\{x \in X \mid \phi(x) \cap V \neq \Lambda\}$ is open in X. For more details, see Section 2.

[3] This means that $u(V)$ is open in X for every open $V \subset Y$.

[4] As a real-valued function in the ordinary sense.

that every element S of $\mathcal{S}$ be properly behaved; i.e., that every continuous $g:A \to S$ can be extended to a continuous $f:X \to S$ (resp. $f:U \to S$ for some open $U \supset A$). Unfortunately, this condition is *not sufficient*, as is shown by Example 6.1, where every $S \in \mathcal{S}$ is homeomorphic to a closed interval, but (Q_2) (and *a fortiori* (Q_1)) does not hold. To obtain a sufficient condition, we therefore not only need well behaved *elements* of $\mathcal{S}$, but must guarantee that these elements are properly "hooked together". In the present paper, this is accomplished by taking $\mathcal{S}$ to be various collections of *convex* subsets of Banach spaces. This leads us to some new characterizations of such properties of X as paracompactness,[5] normality, etc., thus possibly shedding some new light on the interrelation between these properties.

By taking the elements of $\mathcal{S}$ to be convex sets, we are naturally led to answering problem (Q_1); problem (Q_2) will play no role in the present paper. Fortunately, (Q_1) can be simplified by means of the following easy but powerful result which, unfortunately, seems to have no analogue for problem (Q_2).

Proposition 1.4. *If $\mathcal{S} \subset 2^Y$ contains all one-point subsets of elements of $\mathcal{S}$, then the following two properties of $\mathcal{S}$ are equivalent.*

(a) *Every lower semi-continuous $\phi:X \to \mathcal{S}$ admits a selection.*

(b) (Q_1) *holds for every closed $A \subset X$.*

Proof. That (b) $\to$ (a) is obvious. To show that (a) $\to$ (b), let $\psi:X \to \mathcal{S}$ be lower semi-continuous, let $A \subset X$ be closed, and let g be a selection for $\psi \mid A$; we must extend g to a selection for ψ. Now let $\phi:X \to \mathcal{S}$ be defined as in Example 1.3. Then ϕ is lower semi-continuous by Example 1.3*, hence admits a selection by assumption (a), and this selection has the required properties by Example 1.3.

Proposition 1.4 reduces problem (Q_1), which deals with *extending* a selection, to the simpler problem of merely *finding* one, and all the theorems of this paper will therefore be stated in terms of the latter problem. For the proper understanding of these theorems, the reader should keep in mind the following immediate consequence of Proposition 1.4, where Y is called an *extension space with respect to* X if, for every closed $A \subset X$, every continuous $g:A \to Y$ can be extended to a continuous $f:X \to Y$.

Corollary 1.5. *If $\mathcal{S} \subset 2^Y$ contains every one-point subset of Y and also Y itself, then* (a) *implies* (b) *below:*

(a) *Every lower semi-continuous $\phi:X \to \mathcal{S}$ admits a selection.*

(b) *Y is an extension space with respect to X.*

So much for preliminary considerations. Now for a look at the principal theorems of this paper. All of these theorems will be selection analogues of the following celebrated extension theorem of P. Urysohn [35], and its recent modifications in which R is replaced by a Banach space.

Theorem (Urysohn). *The following properties of a T_1-space X are equivalent:*

(a) *X is normal.*

(b) *The real line R is an extension space with respect to X.*

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[5] This term is defined in the appendix (Section 9).

All our selection theorems, together with the known extension theorems they analogize, will be systematically exhibited in Section 3. Among these selection theorems, the following, while typical, seems especially interesting. (It is also one of the easiest to prove.)

THEOREM 3.2″[6]. *The following properties of a T_1-space X are equivalent*:

(a) *X is paracompact.*[5]

(b) *If Y is a Banach space, then every lower semi-continuous carrier for X to the family of non-empty, closed, convex subsets of Y admits a selection.*

This result seems interesting from several points of view. In the first place, it fits naturally into the general scheme of extension and selection theorems which will be developed in Section 3, and thus highlights the relation of paracompactness to other separation properties. In the second place, it is the first characterization of paracompactness which deals with *continuous functions* rather than *coverings*. Since functions are usually easier to handle than coverings, this feature can be quite helpful in certain proofs; examples of this can be found in [25, Proposition 2.4], and in Section 8 of this paper, where we give a new proof (along the lines of Morita [30]) that every J. H. C. Whitehead *CW*-complex [36] is paracompact. Finally, and most important, the theorem yields the simple fact that every paracompact space, and hence every metric space, has property (b) of Theorem 3.2″.[7] We apply this fact in Section 7, which is mostly devoted to simplifying and strengthening a result of R. Bartle and L. M. Graves [3; Theorem 4]. The following special case of this result, which is particularly interesting and easy to prove, illustrates this kind of application of Theorem 3.2″.

COROLLARY (Bartle-Graves). *If Y and X are Banach spaces, and if u is a continuous linear transformation from Y onto X, then there exists a continuous $f:X \to Y$ such that $f(x) \in u^{-1}(x)$ for every $x \in X$.*

To see that this corollary follows from Theorem 3.2″, notice that we are looking for a selection f for the carrier $\phi:X \to 2^Y$ defined by $\phi(x) = u^{-1}(x)$. Since u is *open* by the open mapping theorem of Banach [2], ϕ is l.s.c. by Example 1.1*. Since X is metric (and hence paracompact), and since $\phi(x) \in \mathfrak{F}(Y)$ for every x in X, the existence of a selection f for ϕ is now guaranteed by Theorem 3.2″.

As the proof will show, Theorem 3.2″ remains true if "Banach space" is replaced by "space of type ($\mathfrak{F}$)" in the sense of [6]. This, however, is the only way known to the author in which part (b) of Theorem 3.2″ can be strengthened and still leave a true theorem. In particular, "Banach space" cannot be replaced by "normed linear space". In the first place, this would make (b) false even if X is the closed unit interval (cf. Example 6.2), and furthermore, since an incomplete normed linear space is not topologically complete [22; Theorem 2.7],

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[6] This theorem, and a slightly weaker form of Theorem 3.1′, were first stated in the author's Abstract (Bull. Amer. Math. Soc., vol. 59 (1953), p. 180). After reading this abstract, T. Kandô [18] obtained two related results (one of which is our Theorem 3.1″ (a) $\leftrightarrow$ (c)), and also gave his own proofs (similar to ours) of the theorems in the abstract.

[7] Using Proposition 2.6, we may even conclude that if X is paracompact, Y a Banach space, and $\phi:X \to 2^Y$ l.s.c., then there exists a continuous $f:X \to Y$ such that $f(x)$ is in the closed, convex hull of $\phi(x)$ for every x in X.

it cannot even be an extension space with respect to every paracompact space [15] [25; Theorem 3.1 (b)]. It follows that, even if X is the closed unit interval, we cannot simply drop the requirement that the sets $\phi(x)$ all be *closed*, for if we could, "Banach space" could clearly be replaced by "normed linear space". However, this requirement that the sets $\phi(x)$ be closed can in some cases at least be weakened; precisely how and when will be seen in the text (Theorem 3.1‴).

A word should be said about the relationship between Theorem 3.2″ above, and the well known extension theorem of Dugundji [10; Theorem 4.1] which asserts that if X is metric, and Y a locally convex topological linear space, then Y is an extension space with respect to X. (This theorem does not fit into our general scheme of extension and selection theorems, since it does not *characterize* anything.) It seems to the author that the significant feature of Dugundji's theorem is not that it is an extension theorem, but that Dugundji obtains a formula which enables him to extend all continuous functions from the same closed subset *simultaneously* in a *linear* fashion. No approximations are used in the proof; the extensions drop fully-grown out of the formula. The situation is quite different in the proof of Theorem 3.2″ and the other selection and extension theorems of Section 3; every function that is produced must be constructed "by hand", using a tailor-made approximation method. It is this feature which explains why we must require Y to be metrizable and complete, while Dugundji does not need these assumptions.

Section 2 is devoted to some general lemmas about lower semicontinuous carriers. The characterization theorems are stated in Section 3; Theorem 3.2″ is proved in Section 4, and the others in Section 5. Section 6 contains three counter-examples, and Sections 7 and 8 are devoted to applications.

2. Lower semi-continuous carriers

According to the definition in Section 1, a carrier $\phi: X \to 2^Y$ is *lower semi-continuous*, or l.s.c. for short, if $\{x \in X \mid \phi(x) \cap V \neq \Lambda\}$ is open in X for every open $V \subset Y$. The following proposition, whose verification is left to the reader, provides two alternative definitions for lower semi-continuity.

PROPOSITION 2.1. *If $\phi: X \to 2^Y$, then the following are equivalent:*

(a) *ϕ is* l.s.c.

(b) *ϕ is continuous with respect to the (non-T_1) topology on 2^Y which is generated by the open collections $\mathcal{V} = \{A \in 2^Y \mid A \cap V \neq \Lambda\}$, with V an open subset of Y.*

(c) *If $x \in X$, $y \in \phi(x)$, and V is a neighborhood of y in Y, then there exists a neighborhood U of x in X such that for every $x' \in U$, there exists a $y' \in \phi(x') \cap V$.*

Perhaps this is the time to point out that there exists an analogous concept of *upper semi-continuity*, for which Proposition 2.1, with the obvious modifications, also holds: A carrier $\phi: X \to 2^Y$ is *upper semi-continuous* if

$$\{x \in X \mid \phi(x) \subset V\}$$

is open in X for every open $V \subset Y$. This concept has many interesting applications, but plays no role in selection theory.

The next proposition was already stated, without proof, in the introduction.

PROPOSITION 2.2. *If $\phi:X \to 2^Y$ is such that, for every $x_0 \in X$ and $y_0 \in \phi(x_0)$, there exists a selection f for $\phi \mid U$ (U some neighborhood of x_0) such that $f(x_0) = y_0$, then ϕ is* l.s.c.

PROOF. Let $V \subset Y$ be open; we must show that $G = \{x \in X \mid \phi(x) \cap V \neq \Lambda\}$ is open in X. For each $x_0 \in G$, pick a $y_0 \in \phi(x_0) \cap V$; then, by assumption, there exists a selection f_0 for $\phi \mid U_0$, for some neighborhood U_0 of x_0, such that $f_0(x_0) = y_0$. Now if $U_0' = U_0 \cap \{x \in X \mid f_0(x) \in V\}$, then U_0' is a neighborhood of x_0 which is contained in G. Hence G is open.

From time to time, we shall need some simple properties of l.s.c. carriers, some of which will be summarized in the following propositions.

PROPOSITION 2.3. *If $\phi:X \to 2^Y$ is* l.s.c., *and if $\psi:X \to 2^Y$ is such that $\overline{\psi(x)} = \overline{\phi(x)}$ for every $x \in X$, then ψ is* l.s.c.

PROOF. This follows from the definitions, and the obvious fact that, if U is open in Y and $B \subset Y$, then $B \cap U \neq \Lambda$ if and only if $\overline{B} \cap U \neq \Lambda$.

PROPOSITION 2.4. *Let $\phi:X \to 2^Y$ be* l.s.c., *let $U \subset Y$ be open, and suppose that $\phi(x) \cap U \neq \Lambda$ for all $x \in X$. Then the carrier $\theta:X \to 2^Y$, defined by $\theta(x) = \phi(x) \cap U$, is* l.s.c.

PROOF. This follows immediately from the definitions.

PROPOSITION 2.5. *Let V be an open entourage for some uniform structure on Y. Suppose that $\phi:X \to 2^Y$ and $\psi:X \to 2^Y$ are* l.s.c. *Let $\theta(x) = \phi(x) \cap V(\psi(x))$, and suppose that $\theta(x)$ is never empty. Then $\theta:X \to 2^Y$ is* l.s.c.

PROOF. Define $\chi:X \to 2^{Y \times Y}$ by $\chi(x) = \psi(x) \times \phi(x)$; then χ is clearly l.s.c. To show that θ is l.s.c., we must prove that $\{x \in X \mid \theta(x) \cap U \neq \Lambda\}$ is open for every open $U \subset Y$. But

$$\{x \in X \mid \theta(x) \cap U \neq \Lambda\} = \{x \in X \mid \chi(x) \cap [V \cap (Y \times U)] \neq \Lambda\},$$

and this latter set is open because χ is l.s.c. This completes the proof.

PROPOSITION 2.6. *If Y is a topological linear space, and if $\phi:X \to 2^Y$ is* l.s.c., *then the carrier $\psi:X \to 2^Y$, defined by $\psi(x)$ = convex hull of $\phi(x)$, is also* l.s.c.

PROOF. This follows immediately from the definitions, using Proposition 2.1 (c).

3. Statement of the extension and selection theorems

In this section, we state all the selection theorems of this paper, as well as the known extension theorems to which they are related. First the extension theorems.

THEOREM 3.1 (Urysohn [35], Dugundji [10], Hanner [14]). *The following properties of a T_1-space X are equivalent:*

(a) *X is normal.*

(b) *The real line R is an extension space with respect to X.*

(c) *Every separable Banach space is an extension space with respect to X.*

THEOREM 3.2 (C. H. Dowker [9]). *The following properties of a T_1-space X are equivalent:*

(a) *X is collectionwise normal.*[5]

(b) *Every Banach space is an extension space with respect to X.*

We are now ready for our selection theorems, all of which should be read with Corollary 1.5 in mind. We shall consider the following families of sets, where Y is a Banach space.

$$\mathcal{K}(Y) = \{S \in 2^Y \mid S \text{ is convex}\},$$

$$\mathfrak{F}(Y) = \{S \in \mathcal{K}(Y) \mid S \text{ is closed}\},$$

$$\mathcal{C}(Y) = \{S \in \mathfrak{F}(Y) \mid S \text{ is compact, or } S = Y\}.$$

In addition to the above, we need (in Theorem 3.1‴) a subfamily $\mathfrak{D}(Y)$ of $\mathcal{K}(Y)$, whose definition is somewhat complicated, and will therefore be postponed until the proof of Theorem 3.1‴. At this point, let us merely state that $\mathfrak{D}(Y)$ contains all elements of $\mathcal{K}(Y)$ which are either finite-dimensional, or closed, or have an interior point. We therefore have the following inclusions, where R is the real line.

$$\mathcal{C}(Y) \subset \mathfrak{F}(Y) \subset \mathfrak{D}(Y) \subset \mathcal{K}(Y) \subset 2^Y$$

$$\mathcal{K}(R) = \mathfrak{D}(R).$$

Our first two selection theorems show how statements (b) and (c) in the above extension theorems can be strengthened to statements about selections.

THEOREM 3.1′.[8] *The following properties of T_1-space are equivalent:*

(a) *X is normal.*

(b) *Every* l.s.c. *carrier $\phi: X \to \mathcal{C}(R)$ admits a selection.*

(c) *If Y is a separable Banach space, then every* l.s.c. *carrier $\phi: X \to \mathcal{C}(Y)$ admits a selection.*

THEOREM 3.2′. *The following properties of a T_1-space are equivalent:*

(a) *X is collectionwise normal.*

(b) *If Y is a Banach space, then every* l.s.c. *carrier $\phi: X \to \mathcal{C}(Y)$ admits a selection.*

Our next two theorems show that $\mathcal{C}(Y)$ can be replaced by $\mathfrak{F}(Y)$ in parts (b) and (c) of the above theorems, provided that one simultaneously strengthens part (a).

THEOREM 3.1″. *The following properties of a T_1-space are equivalent:*

(a) *X is normal and countably paracompact.*[5]

(b) *Every* l.s.c. *carrier $\phi: X \to \mathfrak{F}(R)$ admits a selection.*

(c) *If Y is a separable Banach space, then every* l.s.c. *carrier $\phi: X \to \mathfrak{F}(Y)$ admits a selection.*

THEOREM 3.2″. *The following properties of a T_1-space are equivalent:*

(a) *X is paracompact.*[5]

(b) *If Y is a Banach space, then every* l.s.c. *carrier $\phi: X \to \mathfrak{F}(Y)$ admits a selection.*

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[8] The equivalence of (a) and (b) is a slight generalization of a theorem of H. Tong [33], [34] (see also M. Katětov [19], [20]).

Our last theorem shows how Theorem 3.1″ can be altered to replace $\mathfrak{F}(R)$ by $\mathcal{K}(R)(=\mathfrak{D}(R))$, and $\mathfrak{F}(Y)$ by $\mathfrak{D}(Y)$.

THEOREM 3.1‴. *The following properties of a T_1-space X are equivalent:*

(a) *X is perfectly normal.*[5]

(b) *Every* l.s.c. *carrier $\phi: X \to \mathcal{K}(R)$ admits a selection.*

(c) *If Y is a separable Banach space, then every* l.s.c. *carrier $\phi: X \to \mathcal{K}(Y)$ admits a selection.*

It would be nice if $\mathfrak{D}(Y)$ could be replaced by $\mathcal{K}(Y)$ in (c) above, but Example 6.3 shows that this is impossible, even if X is the unit interval. As for a similar analogue to Theorem 3.2″, the author has been unable to find a really satisfactory one; some partial results do exist, but we shall not bore the reader by stating them.

4. Proof of Theorem 3.2″

(A) We begin by proving that 3.2″ (a) $\to$ (b).

LEMMA 4.1. *If X is paracompact, Y a normed linear space,*[9] *$\psi: X \to \mathcal{K}(Y)$ a* l.s.c. *carrier, and if V is a convex neighborhood of the origin of Y, then there exists a continuous $f: X \to Y$ such that $f(x) \in (\psi(x) + V)$ for every x in X.*

PROOF. For every $y \in Y$, let $U_y = \{x \in X \mid y \in (\psi(x) + V)\}$. Since also

$$U_y = \{x \in X \mid \psi(x) \cap (y - V) \neq \Lambda\},$$

it follows from the definition of lower semi-continuity that every U_y is open in X. Let $\mathfrak{U} = \{U_y\}_{y \in Y}$; then $\mathfrak{U}$ is an open covering of X. Since X is paracompact, $\mathfrak{U}$ has a locally finite refinement, and hence (see, for instance, [26; Proposition 2]) there exists a locally finite partition of unity P on X which is subordinated to $\mathfrak{U}$. This means that P is a collection of continuous functions from X to non-negative reals, such that every $x \in X$ has a neighborhood on which all but finitely many elements of P vanish, $\sum_{p \in P} p(x) = 1$ for every x in X, and every $p \in P$ vanishes outside some $U \in \mathfrak{U}$. Now for each $p \in P$, pick a $y(p)$ in Y such that p vanishes outside $U_{y(p)}$. We can now set $f(x) = \sum_{p \in P} p(x)y(p)$, and it is trivial to check that f satisfies all our requirements. This completes the proof of the lemma.

To prove that 3.2″ (a) $\to$ (b), let $\phi: X \to \mathfrak{F}(Y)$ be l.s.c., and let us find a selection for ϕ. Let $\{V_i\}_{i=1}^{\infty}$ be a base for the neighborhoods of the origin in Y consisting of symmetric, convex sets, such that $V_{i+1} \subset (\frac{1}{2})^i V_i$ for all i. We will construct a sequence $\{f_i\}_{n=1}^{\infty}$ of continuous functions from X into Y such that, for every x in X,

$$\text{(a)} \qquad f_i(x) \in (f_{i-1}(x) + 2V_{i-1}) \qquad (i = 2, 3, \cdots),$$

$$\text{(b)} \qquad f_i(x) \in (\phi(x) + V_i) \qquad (i = 1, 2, \cdots).$$

This will be sufficient, because then (by (a)) $\{f_i\}_{n=1}^{\infty}$ is uniformly Cauchy, and therefore converges uniformly to a continuous $f: X \to Y$, and it then follows from (b) that $f(x) \in \phi(x)$ for every x in X.

We construct $\{f_i\}_{i=1}^{\infty}$ by induction. The existence of an f_1, satisfying (b) for $i = 1$, is guaranteed by Lemma 4.1. Suppose that we have $f_1, \cdots f_k$ satisfying

[9] This lemma and its proof are actually valid for any topological linear space Y.

(a) and (b) for $i = 1, \cdots, k$. We must find a continuous $f_{k+1}: X \to Y$ which satisfies (a) and (b) for $i = k + 1$. Now define $\phi_{k+1}(x) = \phi(x) \cap (f_k(x) + V_k)$; then $\phi_{k+1}(x)$ is never empty, by the induction hypothesis, and ϕ_{k+1} is l.s.c. by Proposition 2.5. By Lemma 4.1 there now exists a continuous $f_{k+1}: X \to E$ such that $f_{k+1}(x) \in (\phi_{k+1}(x) + V_{k+1})$ for every x in X. But then

$$f_{k+1}(x) \in (f_k(x) + V_k + V_{k+1}) \subset (f_k(x) + 2V_k),$$

which is (a), and $f_{k+1}(x) \in (\phi(x) + V_{k+1})$, which is (b).

(B) We shall now prove that 3.2″ (b) $\to$ (a). We assume that X is a T_1-space satisfying 3.3″ (b), and we will show that X is paracompact. To show that, it is sufficient, by [26, Proposition 2], to show that every open covering $\mathfrak{U}$ of X has a partition of unity P subordinated to it; this means that there exists a collection P of continuous functions from X to the non-negative reals such that $\sum_{p \in P} p(x) = 1$ for every x in X, and every p in P vanishes outside some U in $\mathfrak{U}$. So let $\mathfrak{U}$ be an open covering of X. Let $Y = l_1(\mathfrak{U})$,[10] and let

$$C = \{y \in Y \mid y(U) \geqq 0 \text{ for all } U \in \mathfrak{U}; \textstyle\sum y(U) = 1 \text{ for summation over all } U \in \mathfrak{U}\};$$

clearly C is a closed, convex subset of Y. Now, for $x \in X$, let

$$\phi(x) = C \cap \{y \in Y \mid y(U) = 0 \quad \text{for all} \quad U \in \mathfrak{U} \text{ such that } x \notin U\}.$$

Clearly $\phi(x) \in \mathfrak{F}(Y)$ for every $x \in X$. We will now show that ϕ is l.s.c. and we will then apply 3.3″ (b) to prove our result.

Let us first of all show that, for every $y \in C$ and $\varepsilon > 0$, there exists a $y' \in C$ such that $\| y - y' \| < \varepsilon$, and $y'(U) > 0$ for only finitely many $U \in \mathfrak{U}$ (say $U_1, \cdots, U_n$) such that $y(U_i) > 0$ for $i = 1 \cdots n$. To find such a y', we need only pick $U_1, \cdots, U_n \in \mathfrak{U}$ such that $y(U_i) > 0$ for all i and $y(U_1) + \cdots + y(U_n) = \delta > 1 - \varepsilon/2$, and then define $y' \in C$ by $y'(U) = 0$ for $U \notin \{U_1, \cdots, U_n\}$, $y'(U_1) = y(U_1) + (1 - \delta)$, and $y'(U_i) = y(U_i)$ for $i = 2, \cdots n$; clearly $\| y - y' \| \leqq 2(1 - \delta) < \varepsilon$, and therefore y' satisfies all our requirements.

Next we will show that ϕ is l.s.c. By Proposition 2.1, this is equivalent to showing that if $x \in X$, $y \in \phi(x)$, and $\varepsilon > 0$, then there exists a neighborhood U of x in X such that, for every $x' \in U$, there exists a $y' \in \phi(x')$ with $\| y - y' \| < \varepsilon$. Suppose, therefore, that $y \in \phi(x)$ and $\varepsilon > 0$ are given, and let y' and $U_1, \cdots, U_n$ be as in the previous paragraph. Let $U = U_1 \cap \cdots \cap U_n$. Since $y(U_i) > 0$ for $i = 1, \cdots, n$, it follows from the definition of ϕ that $x \in U_i$ for $i = 1, \cdots, n$, and hence U is indeed a neighborhood of x. It also follows from the definition of ϕ that $y' \in \phi(x')$ for every $x' \in U$, and therefore U satisfies all our requirements.

By assumption 3.2″ (b), there now exists a selection f for ϕ. For each $U \in \mathfrak{U}$, define $f_U: X \to R$ by $f_U(x) = [f(x)](U)$. It now follows immediately from the definitions that $\{f_U \mid U \in \mathfrak{U}\}$ is a partition of unity on X, and this partition is subordinated to $\mathfrak{U}$, since f_U vanishes outside U for every $U \in \mathfrak{U}$. This completes the proof of the theorem.

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[10] For any set $l_1 S$, (S) is the Banach space defined by

$$l_1(S) = \{y: S \to R \mid \Sigma_{x \in S} \mid y(x) \mid < \infty\}, \text{ with } \| y \| = \Sigma_{x \in S} \mid y(x) \mid.$$

5. Proof of the remaining theorems of Section 3

Observe first of all that Theorems 3.1 and 3.2 are known extension theorems, and that Theorem 3.2″ was proved in Section 4. To prove the remaining theorems, it is clearly sufficient to prove the implications in the following table:

3.1′ (a) $\to$ (c), 3.2′ (a) $\to$ (b),

3.1″ (b) $\to$ (a) $\to$ (c),

3.1‴ (b) $\to$ (a) $\to$ (c).

In addition, we shall also prove 3.1‴ (a) $\to$ (b), since this is so much easier to prove than the stronger implication 3.1‴ (a) $\to$ (c). The proofs are independent, except that the proof of 3.1′ (a) $\to$ (c) should be read after that of 3.2′ (a) $\to$ (b).

Proof of 3.1″ (b) $\to$ (a). Let us assume that X satisfies 3.1″ (b). Since X is then obviously normal (by Theorem 3.1), it only remains to show that X is countably paracompact. The idea of our proof is taken from the proof of [8; Theorem 4, $\beta \to \alpha$].

By [8; Theorem 2], it suffices to show that, if $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of closed subsets of X with empty intersection, then there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X with empty intersection such that $A_n \subset U_n$ for all n. So let $\{A_n\}_{n=1}^{\infty}$ be given, and define $g: X \to R$ by

$$g(x) = \min \{n \mid x \in X - A_n\}.$$

Then

$$A_n = \{x \in X \mid g(x) > n\}.$$

Now clearly g is upper semi-continuous, and hence the carrier $\phi: X \to \mathfrak{F}(R)$, defined by

$$\phi(x) = \{y \in R \mid g(x) \leqq y\},$$

is l.s.c. by Example 1.2*. By assumption, there now exists a selection f for ϕ; that is, $f: X \to R$ is continuous, and

$$f(x) \geqq g(x) \qquad x \in X.$$

If we now define $U_n \subset X$ by

$$U_n = \{x \in X \mid f(x) > n\},$$

then $\{U_n\}_{n=1}^{\infty}$ satisfies all our requirements.

Proof of 3.1‴ (b) $\to$ (a). Let us assume that X satisfies 3.1‴ (b). Again it is clear that X is normal, and hence it remains to show that every closed $A \subset X$ is a G_δ. Now if we define $\phi: X \to \mathcal{K}(R)$ by $\phi(x) = [0, 1]$ if $x \in A$, and $\phi(x) = (0, 1]$ if $x \in X - A$, then ϕ is l.s.c. by Proposition 2.3. Also, the function $g: A \to R$, defined by $g(x) = 0$, is a selection for $\phi \mid A$. By assumption, g can be extended to a selection f for ϕ. But then $f^{-1}(0) = A$, which implies that A is a G_δ in X.

Proof of 3.1″ (a) $\to$ (c). The proof of 3.2″ (a) $\to$ (b) in Section 4 goes through

almost unchanged. In fact, it is sufficient to prove Lemma 4.1 under the changed assumptions that X is normal and countably paracompact, and that Y has a countable dense subset $\{y_i\}_{i=1}^{\infty}$. In the proof of Lemma 4.1, we now take $\mathfrak{U}$ to be $\{U_{y_i}\}_{i=1}^{\infty}$; since this is a countable covering of the countably paracompact space X, it has a locally finite refinement, and the remainder of the proof goes through as before.

Proof of 3.2′ (a) → (b). The proof is similar to the proof of 3.2″ (a) → (b) in Section 4. In fact, it is sufficient to prove Lemma 4.1 under the changed assumptions that X is collectionwise normal, and that $\phi(x)$ is either totally bounded or is equal to Y.

We begin by considering the sets $y - V$ just as in the proof of Lemma 4.1. Next, however, we observe that $\{y - V\}_{y \in Y}$ is an open covering of the metric space Y, and hence has a locally finite refinement $\mathfrak{W}$. For each $W \in \mathfrak{W}$ let

$$U_W = \{x \in X \mid \phi(x) \cap W \neq \Lambda\},$$

and let $\mathfrak{U} = \{U_W \mid W \in \mathfrak{W}\}$. The proof now proceeds just as the proof Lemma 4.1, provided that we can show that $\mathfrak{U}$ has a locally finite refinement. That is what we shall now do.

Pick a fixed $W_0 \in \mathfrak{W}$, and let $A = X - U_{W_0}$. Then A is closed in X, and if $x \in A$, then $\phi(x)$ is totally bounded. Since $\mathfrak{W}$ is locally finite, this implies that $\{A \cap U_W \mid W \in \mathfrak{W}\}$ is a point-finite covering of the collectionwise normal space A. By [27; Theorem 2], we can therefore find a locally finite refinement $\{R_\alpha\}$ of $\{A \cap U_W \mid W \in W \in \mathfrak{W}\}$. By a result of C. H. Dowker [9], there exists a locally finite open covering $\{S_\alpha\}$ of X such that $S_\alpha \cap A = R_\alpha$ for all α. Now for each α, pick a $W_\alpha \in \mathfrak{W}$ such that $R_\alpha \subset U_{W_\alpha}$, and let $T_\alpha = S_\alpha \cap U_{W_\alpha}$. If we now let $\mathfrak{M}$ be the open covering of X whose elements are all the T_α and the set U_{W_0}, then $\mathfrak{M}$ is clearly a locally finite refinement of $\mathfrak{U}$. This completes the proof.

Proof of 3.1′ (a) → (c). This proof is almost identical with the above proof of 3.2′ (a) → (b). Because of the separability of Y, we can now take all our coverings to be countable. Since the results about *arbitrary* (open!) coverings of collectionwise normal spaces which were used in the above proof ([27] and [9]) remain valid for *countable* coverings of *normal* spaces ([29; Corollary to Theorem 5] and [15; Lemma 7.2] respectively), the proof goes through just as above.

Proof of 3.1‴ (a) → (b). Let $\phi: X \to \mathcal{K}(R)$ be l.s.c., and let us find a selection for ϕ. For every $x \in X$, let $g(x) = \text{g.l.b. } \phi(x)$, $h(x) = \text{l.u.b. } \phi(x)$. It is clearly sufficient to find a continuous $f: X \to R$ such that $g(x) \leqq f(x) \leqq h(x)$ for every $x \in X$, and such that $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$.

Now by Proposition 2.3 and Example 1.2*, g is upper semi-continuous[4] and h is lower semi-continuous.[4] Since every perfectly normal space is countably paracompact [8; Corollary on p. 21], it follows from Theorem 3.1′ (a) → (b) (and Example 1.2*) that there exists a continuous $k: X \to R$ such that $g(x) \leqq k(x) \leqq h(x)$ for every $x \in X$. Now let $A = \{x \in X \mid g(x) = h(x)\}$; then A is certainly closed. Since every subset of X is perfectly normal and hence

countably paracompact, we can find [8; Theorem 4] a continuous $u: (X - A) \to R$ such that $g(x) < u(x) < h(x)$ for every $x \in X - A$. Finally, since X is perfectly normal, we can find a continuous $p: X \to [0, 1]$ such that $p^{-1}(0) = A$. Now define the function $v: X \to R$ by

$$v(x) = 0 \qquad \text{if } x \in A$$

$$v(x) = \frac{p(x)}{1 + |u(x) - k(x)|}(u(x) - k(x)) \qquad \text{if } x \in X - A.$$

It is easy to check that v is continuous. If we now finally define $f: X \to R$ by $f(x) = k(x) + v(x)$, then f satisfies all our requirements. This completes the proof.

Before we can finally prove that 3.1‴ (a) $\to$ (c), we must, of course, define the class $\mathfrak{D}(Y)$. This is done as follows: If K is a closed convex subset of a normed linear space, then a *supporting set* of K is a closed, convex subset S of K, $S \neq K$, such that if an interior point of a segment in K is in S, then the whole segment is in S. The set of all elements of K which are not in any supporting set of K will be denoted by $I(K)$ (suggesting "Inside of K"). We now define the family $\mathfrak{D}(Y)$ for any normed linear space as follows:

$$\mathfrak{D}(Y) = \{B \in \mathfrak{K}(Y) \mid B \supset I(\bar{B})\}.$$

Let us quickly check that, as asserted in Section 3, every element K of $\mathfrak{K}(Y)$ which is either closed, or has an interior point, or is finite dimensional, belongs to $\mathfrak{D}(Y)$. If K is closed, this is obvious. If K has an interior point, and if $y \in \bar{K} - K$, then the Hahn-Banach theorem guarantees the existence of a closed hyperplane $H \subset Y$ which supports $\bar{K}$ at y but does not contain $\bar{K}$; clearly $H \cap \bar{K}$ is a supporting set of $\bar{K}$, and hence $y \notin I(\bar{K})$. Finally, if K is finite dimensional, then K has an interior point with respect to the smallest linear variety V containing K; hence $K \in \mathfrak{D}(V) \subset \mathfrak{D}(Y)$.

To prepare for the proof of 3.1″ (a) $\to$ (c), we need the following two lemmas, the first of which slightly generalizes a result of Klee [21; Theorem 11.1] [23; (2.1)].

LEMMA 5.1. *If K is a non-empty, closed, convex, separable subset of a Banach space Y, then $I(K)$ is not empty. In fact, if $\{y_i\}_{i=1}^{\infty}$ is a dense subset of K, and if*

$$z_i = y_1 + \frac{(y_i - y_1)}{\max(1, \| y_i - y_1 \|)}) \quad \textit{for all } i,$$

$$z = \sum_{i=1}^{\infty} (\tfrac{1}{2})^i z_i,$$

then $z \in I(K)$.

PROOF. Suppose $z \notin I(K)$. Then there exists a supporting set $S \subset K$ such that $z \in S$. Now for every i, z is either an interior point of a segment in K one of whose end points is z_i, or else $z = z_i$; in either case, we must have $z_i \in S$. But, for every i, z_i is either an interior point of the segment $[y_1, y_i]$, or else $z_i = y_i$, so in either case we must have $y_i \in S$. But $\{y_i\}_{i=1}^{\infty}$ is dense in K, and since S is closed, this finally implies $S = K$, which is impossible.

LEMMA 5.2. *If X is perfectly normal, Y a separable Banach space, and if $\phi: X \to$*

$\mathfrak{F}(Y)$ *is* l.s.c., *then there exists a countable collection* F *of selections for* ϕ *such that, for every* $x \in X$, $\{f(x)\}_{f \in F}$ *is dense in* $\phi(x)$.

PROOF. Let $\{y_j\}_{j=1}^{\infty}$ be a countable, dense subset of Y, and let $\{V_k\}_{k=1}^{\infty}$ be a symmetric, convex basis for the neighborhoods of the origin in Y such that $V_{k+1} \subset \frac{1}{2} V_k$. For each j and k, let

$$U_{j,k} = \{x \in X \mid \phi(x) \cap (y_j - V_k) \neq \Lambda\};$$

then $U_{j,k}$ is open in X, and hence

$$U_{j,k} = \bigcup_{i=1}^{\infty} A_{i,j,k}$$

where each $A_{i,j,k}$ is closed in X. Let

$$\phi_{i,j,k}(x) = \begin{cases} \phi(x) & \text{if } x \notin A_{i,j,k}, \\ \overline{\phi(x) \cap (y_j - V_k)} & \text{if } x \in A_{i,j,k}. \end{cases}$$

Then the restriction of $\phi_{i,j,k}$ to $A_{i,j,k}$ is l.s.c. by Proposition 2.4 and 2.2, and since $A_{i,j,k}$ is closed in X, it follows immediately from the definition that $\phi_{i,j,k}$ is l.s.c. Since every perfectly normal space is countably paracompact [8; Theorem 4], it follows from Theorem 3.1″ that there exists a selection $f_{i,j,k}$ for each $\phi_{i,j,k}$. Let F be the collection of all the $f_{i,j,k}$; then F is a countable collection of selections for ϕ, and it only remains to check that $\{f(x)\}_{f \in F}$ is dense in $\phi(x)$ for every $x \in X$.

Let $x \in X$, $y \in \phi(x)$, and let k be a positive integer; we must find an $f \in F$ such that $f(x) \in y + V_k$. Pick some $y_j \in y + V_{k+2}$. Then $x \in U_{j,k+2}$, and hence $x \in A_{i,j,k+2}$ for some i. But then $f_{i,j,k+2}(x) \in y_j + \bar{V}_{k+2} \subset y_j + V_{k+1} \subset y + V_{k+2} + V_{k+1} \subset y + V_k$, which completes the proof.

PROOF OF 3.1‴ (a) $\rightarrow$ (c). Let $\phi: X \rightarrow \mathfrak{D}(Y)$ be l.s.c., and let us find a selection for ϕ. Define $\psi: X \rightarrow \mathfrak{F}(Y)$ by $\psi(x) = \overline{\phi(x)}$; what we must find is a continuous $f: X \rightarrow Y$ such that $f(x) \in I(\psi(x))$ for every $x \in X$. Now by Proposition 2.2, ψ is l.s.c., and hence, by Lemma 5.2, there exists a sequence $\{g_i\}_{i=1}^{\infty}$ of selections for ψ such that $\{g_i(x)\}_{i=1}^{\infty}$ is dense in $\psi(x)$ for every $x \in X$. Now let

$$f_i(x) = g_1(x) + \frac{g_i(x) - g_1(x)}{\max(1, \| g_i(x) - g_1(x) \|)} \qquad \text{for all } i,$$

$$f(x) = \sum_{i=1}^{\infty} (\tfrac{1}{2})^i f_i(x).$$

By Lemma 5.1, $f(x) \in I(\psi(x))$ for every $x \in X$. Since the series defining f converges uniformly in some neighborhood of every $x \in X$, it follows that f is also continuous, and thus has all the required properties.

6. Three counter-examples

This section contains the counter-examples promised in the introduction. Example 6.2 shows that, even if X is the closed unit interval, 3.2″ (b) becomes false if "Banach" is replaced by "normed linear", and Example 6.3 shows that 3.1‴ (a) $\rightarrow$ (c) becomes false if "separable" is omitted.

EXAMPLE 6.1. Let X be the closed unit interval, and Y euclidean 2-space. Then there exists a family $\mathcal{S}$ of closed subsets of Y, all of whose elements are homeomorphic to a closed interval, and an l.s.c. carrier $\phi: X \rightarrow \mathcal{S}$, such that for no neighborhood U of $0 \in X$ does $\phi \mid U$ have a selection.

PROOF. Let Y consist of couples (t, s). Let $Z \subset Y$ consist of the graph of the functions $s = \sin 1/t$ $(t \neq 0)$, together with the closed interval $\{(0, s) \in Y \mid -1 \leqq s \leqq 1\}$. Define $\phi: X \rightarrow 2^Y$ by $\phi(x) = \{(t, s) \in Z \mid \frac{1}{2}x \leqq t \leqq x\}$, and let $\mathcal{S} = \{\phi(x)\}_{x \in X}$. Clearly every element of $\mathcal{S}$ is homeomorphic to the closed unit interval, and ϕ is l.s.c. But for no neighborhood U of 0 in X can there exist a selection for $\phi \mid U$, for such a selection would make Z arcwise connected, which it isn't.

EXAMPLE 6.2. Let X be the closed unit interval. Then there exists a separable, normed linear space Y, and a l.s.c. carrier $\phi: X \rightarrow \mathfrak{F}(Y)$ for which there is no selection.

PROOF. Let Z be the set of rationals in X, and suppose that Z is ordered as a sequence $z_1, z_2, \cdots$. Let $Y = \{y \in l_1(Z) \mid y(x) \neq 0$ for only finitely many $x \in Z\}$.[10] Let $C = \{y \in Y \mid y(x) \geqq 0$ for all $x \in Z\}$. Finally, let

$$\phi(x) = \begin{cases} C & \text{if } x \in X - Z, \\ C \cap \{y \in Y \mid y(z_n) \geqq 1/n\} & \text{if } x = z_n . \end{cases}$$

It is easy to check that ϕ is l.s.c. Let us show that there is no selection for ϕ.

Suppose f were a selection for ϕ. Since f is continuous, each $z_n \in Z$ has a neighborhood U_n in X such that $[f(x)](z_n) > (1/2n)$ whenever $x \in \bar{U}_n$. By induction, pick a sequence $\{n_k\}_{k=1}^{\infty}$ of distinct integers such that $z_{n_{k+1}} \in \bigcap_{i=1}^{k} U_{n_i}$ for all k. Then $\{\bar{U}_{n_k}\}_{k=1}^{\infty}$ is a sequence of closed subsets of X with the finite intersection property, and hence there is an $x_0 \in \bigcap_{k=1}^{\infty} U_{n_k}$. But then $[f(x_0)](z_{n_k}) > 0$ for all k, which is impossible.

EXAMPLE 6.3. There exists a l.s.c. carrier ϕ from the closed unit interval X to the non-empty, open, convex subsets of a Banach space Y for which there exists no selection.

PROOF. Let $Y = l_1(X)$.[10] Let $\phi(x) = \{y \in Y \mid y(x) > 0\}$. We will show that ϕ is l.s.c. and that there exists no selection for ϕ.

To show that ϕ is l.s.c., pick an $x_0 \in X$, and an $\varepsilon > 0$. According to Proposition 2.1, we must find a neighborhood U of x_0 such that, if $x \in U$, then there is a $y \in \phi(x)$ such that $\| y - y_0 \| < \varepsilon$. Since $y_0(x_0) > 0$, and since $y_0(x) < -\varepsilon/2$ for only finitely many $x \in X$, there is a neighborhood U of x_0 such that $y_0(x) \geqq -\varepsilon/2$ whenever $x \in U$. Now if $x \in U$, we can pick $y \in \phi(x)$ such that $y(x) < \varepsilon/2$, and $y(x') = y_0(x')$ for $x' \neq x$. Then $\| y - y_0 \| = | y(x) - y_0(x) | < \varepsilon$.

Suppose now that there existed a selection f for ϕ. Since f is continuous every $x \in X$ has a neighborhood U_x such that $[f(x)](x) > 0$ whenever $x \in U_x$. Since each U_x contains a rational number, some rational number r must be in

uncountably many U_x. But then $[f(r)](x) > 0$ for uncountably many x, which is impossible.

7. Applications of Theorem 3.2″ (a) → (b)

In this section, we shall use Theorem 3.2″ to obtain a refinement of a result of R. Bartle and L. M. Graves [3; Theorem 4] dealing with linear transformations between Banach spaces.[11] In preparation for this, we begin by proving the following refinement of Theorem 3.2″ (a) → (b).

LEMMA 7.1. *Let X be a paracompact space, Y a Banach space, and $\phi: X \to \mathfrak{F}(Y)$ a* l.s.c. *carrier. Let* $m(x) = \text{g.l.b. } \{\| y \| \mid y \in \phi(x)\}$, *and suppose that $p: X \to R$ is lower semi-continuous*[4] *with $p(x) \geqq 0$ for all x, and $p(x) > m(x)$ whenever $m(x) > 0$. Then there exists a selection f for ϕ such that $\| f(x) \| \leqq p(x)$ for all $x \in X$.*

PROOF. Let

$$\psi(x) = \begin{cases} \phi(x) \cap \{y \in E \mid \| y \| < p(x)\} & \text{if } p(x) > 0, \\ \{0\} & \text{if } p(x) = 0. \end{cases}$$

Using Proposition 2.4, it is easily checked that ψ is l.s.c. Hence if we define $\theta: X \to \mathfrak{F}(Y)$ by $\theta(x) = \overline{\psi(x)}$; then θ is also l.s.c. by Proposition 2.3. By Theorem 3.2″, there now exists a selection f for θ, and f satisfies all our requirements.

Before considering the general situation encountered in the theorem of Bartle and Graves, we shall first illustrate our results with an important special case, namely with a refinement of the corollary to Theorem 3.2″ in the introduction.

PROPOSITION 7.2. *Let E and F be real* (resp. *complex*) *Banach spaces, and let u be a continuous linear transformation from E onto F. Then, for any $\lambda > 1$, there exists a continuous $f: F \to E$ such that, for every $x \in F$,*

(a) $f(x) \in u^{-1}(x)$,

(b) $\| f(x) \| \leqq \lambda \, \text{g.l.b.}\{\| y \| \mid y \in u^{-1}(x)\}$,

(c) $f(\alpha x) = \alpha f(x)$ *for all real* (resp. *complex*) *scalars* α.

PROOF. Let $X = \{x \in F \mid \| x \| = 1\}$. Define $\phi: X \to \mathfrak{F}(E)$ by $\phi(x) = u^{-1}(x)$. Since u is open by the open mapping theorem [2], it follows from Example 1.1* that ϕ is l.s.c. Let $m(x) = \text{g.l.b. } \{\| y \| \mid y \in u^{-1}(x)\}$ for every $x \in X$; then m is known to be continuous. We can therefore apply Lemma 7.1, with $p(x) = \lambda m(x)$, to obtain a continuous $g: X \to E$ satisfying (a) and (b) above for every $x \in X$.

Now define $h: F \to E$ by

$$(1) \qquad \begin{aligned} h(x) &= \| x \| \, g(x/\| x \|) && \text{if } x \neq 0, \\ h(x) &= 0 && \text{if } x = 0. \end{aligned}$$

Then h satisfies all our conditions, except that (c) holds only for scalars $\alpha \geqq 0$. To remedy this last deficiency, we proceed as follows: If the field of scalars is the

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[11] The suggestion that this might be possible was made to the author by R. Bartle and L. M. Graves themselves.

reals, we simply set

$$f(x) = \tfrac{1}{2}h(x) - \tfrac{1}{2}h(-x), \tag{2}$$

and f clearly satisfies all our conditions. If, on the other hand, the field of scalars is the complexes, then, following a suggestion by R. Kadison, we set

$$f(x) = \int_C \bar{\gamma} h(\gamma x)\, d\mu(\gamma), \tag{3}$$

where C is the group of complex numbers of absolute value one, μ is the ordinary translation invariant measure on C (normalized such that $\mu(C) = 1$), and where the integrand is, of course, Banach space valued.[12] To check that f is continuous, we must check that if $x_n \to x_0$, then $\bar{\gamma} h(\gamma x_n) \to \bar{\gamma} h(\gamma x_0)$ uniformly; this is true, however, since $\{\gamma x_n \mid n = 0, 1, 2, \cdots, \mid \gamma \mid = 1\}$ is compact, and since h is thus uniformly continuous on this set. Since, by [11; Theorem 1.2.2], $f(x)$ is in the closed convex hull of the set $\{\bar{\gamma} h(\lambda x)\}_{\gamma \in C}$, f satisfies conditions (a) and (b). To see, finally, that f satisfies (c), set $\alpha = r\lambda$, with $\mid \lambda \mid = 1$ and $r \geqq 0$, and notice that

$$\begin{aligned} f(r\lambda x) &= \int_C \bar{\gamma} f(r\lambda\gamma x)\, d\mu(\gamma) \\ &= r \int_C \bar{\gamma} f(\lambda\gamma x)\, d\mu(\gamma) \\ &= r\lambda \int_C \overline{\lambda\gamma} f(\lambda\gamma x)\, d\mu(\gamma) \\ &= r\lambda f(x). \end{aligned}$$

This completes the proof.

REMARK. Part (c) of Proposition 7.2 cannot be strengthened to assert that f is *linear*, for the existence of such a linear f is equivalent to the existence of continuous, linear projection from E onto the null-space of u, and such a projection need not exist [31].

We now prepare to generalize Proposition 7.2 by considering, as in [3], not one but many linear transformations u. Until the end of the section, E and F will denote Banach spaces, and L (resp. L_0) will denote the set of continuous linear transformations from E into (resp. onto) F; we assume that L_0 is not empty. The *norm* and *strong* topologies are defined on L in the usual manner; the norm topology is generated by the norm $\| u \| = \sup_{\|y\| \leqq 1} \| u(y) \|$, and the strong topology is the topology of pointwise convergence on E (= the coarsest topology making all maps of the form $u \to u(y)$, with $y \in E$, continuous). The prefix "norm" (resp. "strong") before a term related to L will mean that L is, on that occasion, assumed to carry the norm (resp. strong) topology.

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[12] Since we are integrating a continuous function on a compact set, all definitions of such integrals coincide.

We now define three important functions. We define $\omega: L_0 \times F \to \mathfrak{F}(E)$ by

(1) $$\omega(u, z) = u^{-1}(z),$$

we define $m: L_0 \times F \to R$ by

(2) $$m(u, z) = \inf\{\| y \| \mid y \in \omega(u, z)\} = \inf\{\| y \| \mid u(y) = z\}.$$

and we define $q: L_0 \to R$ by

(3) $$q(u) = \sup\{m(u, z) \mid \| z \| \leqq 1\}.$$

It follows from [2; p. 38, (1)] (the open mapping theorem) that $q(u) < \infty$ for all $u \in L_0$. A subset A of $L_0 \times F$ will be called *q-bounded* if $\{q(u) \mid (u, z) \in A\}$ is bounded.

In [13; Theorem 1], Graves proves a result which may be rephrased as follows:

(4) if $u_0 \in L_0$, $u \in L$, $\alpha > 1$, and $\| u - u_0 \| < 1/\alpha q(u_0)$,

then $u \in L_0$ and $q(u) \leqq \alpha q(u_0)/\alpha - 1$.

Starting with (4), it is possible to prove the following refinement (which is more precise than is necessary for our purposes) by the use of some tedious but straightforward juggling whose details we omit:

(5) If $u_0 \in L_0$, $u \in L$, $\alpha > 1$,

$\| u - u_0 \| < 1/\alpha q(u_0)$, and $\| z - z_0 \| < \delta$, then $u \in L_0$ and

$$| m(u_0, z_0) - m(u, z) | \leqq \frac{\| z_0 \| + \alpha\delta}{\alpha - 1} q(u_0).$$

We are now ready for the fundamental lemma, which will permit us to apply Theorem 3.2″ to our present situation.

LEMMA 7.3. (a) *ω is norm-*l.s.c., *and m is norm-continuous.*

(b) *If A is a q-bounded subset of $L_0 \times F$, then $\omega \mid A$ is strong-*l.s.c.

PROOF. The second part of (a) follows immediately from (5) above, and the first part of (a) follows immediately from (b) and (4) above. It is therefore sufficient to prove (b).

Suppose, therefore, that $q(u) < M$ for $(u, z) \in A$, and let us show that $\omega \mid A$ is strong-l.s.c. To do this, it is sufficient (by Proposition 2.1) to show that if $(u_0, z_0) \in A$, $y_0 \in \omega(u_0, z_0)$, and $\varepsilon > 0$, then there exists a neighborhood U of (u_0, z_0) in A such that, for every $(u, z) \in U$, there exists a $y \in \omega(u, z)$ with $\| y_0 - y \| < \varepsilon$. Pick a strong neighborhood U_1 of u_0 in L such that

$$\| u(y_0) - u_0(y_0) \| < \varepsilon/2M$$

for every $u \in U_1$, let $U_2 = \{z \in F \mid \| z - z_0 \| < \varepsilon/2M\}$, and let $U = A \cap (U_1 \times U_2)$. Suppose now that $(u, x) \in U$. Remembering that $u_0(y_0) = z_0$, we have $\| u(y_0) - z \| \leqq \| u(y_0) - z_0 \| + \| z_0 - z \| < \varepsilon/M$, and since $q(u) < M$, it follows that there exists a $y \in E$ with $\| y_0 - y \| < \varepsilon$ and $u(y_0 - y) = u(y_0) - z$, whence $u(y) = z$. This completes the proof.

We are now ready for the main theorem of this section, which generalizes Proposition 7.2, and from which our strengthening of the Bartle and Graves theorem will follow as an immediate corollary.

THEOREM 7.4. *There exists, for any* $\lambda > 1$, *a norm-continuous* $k: L_0 \times F \to E$ *such that, for every* $(u, z) \in L_0 \times F$,

(a) $k(u, z) \in u^{-1}(z)$,

(b) $\| k(u, z) \| \leqq \lambda m(u, z)$,

(c) $k(\alpha u, \beta z) = (\beta/\alpha)k(u, z)$ *for all scalars* α *and* β *with* $\alpha \neq 0$.

PROOF. The existence of a continuous $k: L_0 \times F \to E$ satisfying only (a) and (b) follows immediately from Lemmas 7.1 and 7.3. To satisfy (c) as well, we must go through the same kind of skullduggery as was used in the proof of Proposition 7.2.

Let $X = \{(u, z) \in L_0 \times F \mid \| u \| = \| z \| = 1\}$, and let $\phi = \omega \mid X$. Let L_0 carry the norm topology. Now ω is l.s.c. by Lemma 7.3 (a), and hence so is ϕ. Also m is continuous by Lemma 7.3 (a), and hence so is $m \mid X$. We can therefore apply Lemma 7.1 to obtain a continuous $g: X \to E$ satisfying (a) and (b) of our theorem.

Now define $h: L_0 \times F \to E$ by

$$h(u, z) = \| u \|^{-1} \cdot \| z \| \, g(u/\| u \|, z/\| z \|) \qquad \text{if } z \neq 0,$$

$$h(u, z) = 0 \qquad \text{if } z = 0.$$

Then h satisfies all our conditions, except that (c) holds only for scalars $\alpha, \beta > 0$. To remedy this last deficiency, we finally let

$$f(x) = \tfrac{1}{4}[h(u, z) - h(u, -z) - h(-u, z) + h(u, z)] \qquad \text{for real scalars,}$$

$$f(x) = \int_C \int_C \bar{\gamma} \bar{\lambda} h(\gamma u, \lambda z) \, d\mu(\gamma) \, d\mu(\lambda) \qquad \text{for complex scalars,}$$

where C is the group of complex members of absolute value one, and μ is the ordinary translation invariant measure on C (normalized such that $\mu(C) = 1$). That f satisfies all our requirements is now clear in the real case, and is verified just as in the proof of Proposition 7.2 in the complex case.

COROLLARY 7.5. *Let* X *be a topological space,* $h: X \to L_0$ *norm-continuous, and* $g: X \to F$ *continuous. Then, for any* $\lambda > 1$, *there exists a continuous* $f: X \to E$ *such that*

(a) $f(x) \in [h(x)]^{-1}(g(x))$ *for every* $x \in X$,

(b) $\| f(x) \| < \lambda m(h(x), g(x))$ *for every* $x \in X$,

(c) *If* $h(x_1) = \alpha h(x_2)$ *and* $g(x_1) = \beta g(x_2)$ *for* $x_1, x_2 \in X$ *and* α, β *scalars with* $\alpha \neq 0$, *then* $f(x_1) = (\beta/\alpha) f(x_2)$.

PROOF. If $k: L_0 \times F \to E$ is as in Theorem 7.4, then we need only set $f(x) = k(h(x), g(x))$, and it is trivial to verify that f satisfies all our conditions.

The Theorem of Bartle and Graves, as well as Theorem 7.4 and Corollary 7.5 above, dealt only with the case where L_0 carries the norm topology. Our final theorem, which is analogous to Corollary 7.5, deals with the case where L_0 carries

the (coarser!) strong topology. Notice that in this case $L_0 \times F$ need not be paracompact, and that $m: L_0 \times F \to R$ need not be continuous.

PROPOSITION 7.6. *Let X be a paracompact space, $h: X \to L_0$ strong-continuous, $g: X \to F$ continuous, and suppose that* $\sup_{x \in X} q(h(x)) < \infty$. *Then there exists, for any $\lambda > 1$, a continuous $f: X \to E$ such that, for every $x \in X$*

(a) $f(x) \in [h(x)]^{-1}(g(x))$,

(b) $\| f(x) \| \leqq \lambda \sup_{x' \in X} m(h(x'), g(x))$.

PROOF. Define $\theta: X \to \mathfrak{F}(E)$ by $\theta(x) = [h(x)]^{-1}(g(x))$. Then $\theta(x) = \omega(h(x), g(x))$, and hence θ is l.s.c. by Lemma 7.3 (b) and Proposition 2.1. The existence of the required f now follows from Lemma 7.1.

8. An application of the characterization of paracompactness in Theorem 3.2″

In this section, Theorem 3.2″ will be used to show (Theorem 8.2) that, if a topological space X has "sufficiently many" closed, paracompact subsets, then X is paracompact. The paracompactness of every CW-complex [36], which was first proved in full generality by H. Miyazaki [28], is an immediate corollary of this result.[13]

DEFINITION 8.1. Let X be a topological space, and $\mathfrak{B}$ a collection of closed subsets of X. Then $\mathfrak{B}$ *dominates* X if, whenever $A \subset X$ has a closed intersection with every element of some subcollection $\mathfrak{B}_1$ of $\mathfrak{B}$ which covers A, then A is closed.[14]

It follows immediately from Definition 8.1 that, if $\mathfrak{B}$ dominates X, and if $\mathfrak{B}_1 \subset \mathfrak{B}$, then

(a) $\bigcup \mathfrak{B}_1$ is closed;

(b) If Y is any topological space, and if $f: (\bigcup \mathfrak{B}_1) \to Y$ is a function such that $f \mid B$ is continuous for every $B \in \mathfrak{B}_1$, then f is continuous.

THEOREM 8.2. *A topological space X is paracompact if and only if it is dominated by a collection of paracompact subsets.*

PROOF. The "only if" assertion is obvious, since one need only take $\mathfrak{B} = \{X\}$. Let us therefore prove the "if" assertion. By Theorem 3.2″, it is sufficient to show that if Y is a Banach space, and $\phi: X \to \mathfrak{F}(Y)$ is l.s.c., then ϕ admits a selection. So let Y and ϕ be given, and let us find f.

Consider the class $\mathfrak{E}$ of all couples of the form $(\mathfrak{C}, h)$, where $\mathfrak{C} \subset \mathfrak{B}$, and h is a selection for $\phi \mid \bigcup \mathfrak{C}$; we partially order $\mathfrak{E}$ in the obvious manner. By part (b) of the remark following definition 8.1, every simply ordered sub-class of $\mathfrak{E}$ has an obvious upper bound, and hence, by Zorn's Lemma, $\mathfrak{E}$ has a maximal element $(\mathfrak{C}_0, h_0)$. We need only show that $\mathfrak{C}_0 = \mathfrak{B}$, for then we can simply take $f = h_0$.

Suppose $\mathfrak{C}_0 \neq \mathfrak{B}$; then there exists a $B \in \mathfrak{B}$ with $B \notin \mathfrak{C}_0$. Denote $B \cap (\bigcup \mathfrak{C})$ by B'. Now $h \mid B'$ is a selection for $\phi \mid B'$ and hence, by Theorem 3.2″ and Proposi-

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[13] Theorem 8.2 and Corollary 8.3 have been obtained independently by K. Morita [30], whose proof is quite different from ours. These results were obtained by the author at the same time as Theorem 3.2″, in December, 1952, but were not published in the abstract which announced Theorem 3.2″ (see footnote 6).

[14] In K. Morita's [30] terminology, X has the *weak* (= fine!) topology with respect to $\mathfrak{B}$.

tion 1.4, it can be extended to a selection k for $\phi \mid B$. If we now let $\mathcal{C}_1 = \mathcal{C}_0 \cup \{B\}$, and define $h_1 : \bigcup \mathcal{C}_1 \to Y$ by

$$h_1(x) = h_0(x) \qquad x \in \bigcup \mathcal{C}_0,$$

$$h_1(x) = k(x) \qquad x \in B,$$

then $(\mathcal{C}_1, h_1)$ is an element of $\mathfrak{E}$ which is larger than $(\mathcal{C}_0, h_0)$. This contradicts the maximality of $(\mathcal{C}_0, h_0)$, and thus the proof is complete.

Theorem 8.2 remains true if "paracompact" is replaced by "normal", or "perfectly normal", or "normal and countably paracompact". To see this, one need only replace Theorem 3.2″ in the proof by some other appropriate characterization in Section 3. Similarly for dim (X) *sn*, using the functional characterization [1] [7] [16].

Since a CW-complex [36] is dominated by the collection of its finite subcomplexes, each of which is metrizable, one immediately obtains

COROLLARY 8.3. *Every CW-complex is paracompact and perfectly normal.*

9. Appendix on the topological spaces encountered in this paper

A *covering* of a topological space X is, in this paper, a collection of open subsets of X whose union is X. A refinement of a covering $\mathcal{U}$ is a covering $\mathcal{V}$ such that every $V \in \mathcal{V}$ is a subset of some $U \in \mathcal{U}$. A covering $\mathcal{U}$ is *point-finite* if every $x \in X$ is an element of only finitely many $U \in \mathcal{U}$, it is *locally finite* if every $x \in X$ has a neighborhood intersecting only finitely many $U \in \mathcal{U}$. A Hausdorff space X, every covering of which has a locally finite refinement, is called *paracompact* [5; p. 36]; if every countable covering of X has a locally finite refinement (which can then always be chosen to be countable), then X is called *countably paracompact* [8; p. 219]. A T_1-space X is called *collectionwise normal* [4; p. 176] if, for every disjoint, locally finite collection $\{A_\alpha\}$ of closed subsets of X, there exists a disjoint collection $\{U_\alpha\}$ of open subsets of X such that $A_\alpha \subset U_\alpha$ for all α. Finally, a

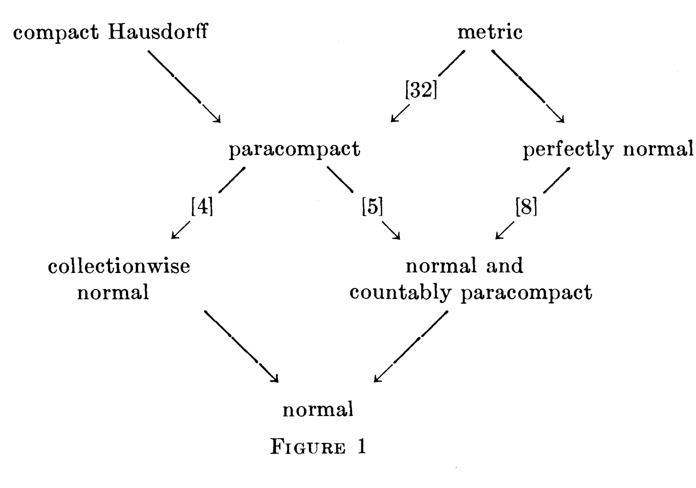

FIGURE 1

normal space is *perfectly normal* if every closed subset is a G_δ. Figure 1 shows the implications between these concepts, together with appropriate references; these are the only implications, except for the unsolved problem [8, 19] of whether every normal or every collectionwise normal space is countably paracompact.

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