

On the cardinality of sumsets in torsion-free groups

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To the memory of **Yahya Ould Hamidoune**

Notation

- ▶ G - torsion free group
- ▶ A, B, C are finite subsets of cardinality at least two
- ▶ $|A|$ - cardinality of A
- ▶ AB - product set
- ▶ $A + B$ - sumset if G is Abelian

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- ▶ progression of length k and quotient q - a, aq, \dots, aq^{k-1} ,
where a and q commute

Minimal product-sets in torsion-free groups

Theorem (Kemperman, Freiman-Brailovsky)

$$|AB| \geq |A| + |B| - 1,$$

with equality iff A and B are left and right translates of progressions with the same quotient.

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Goal Conditions to ensure larger additive error

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Goal Conditions to ensure larger additive error

Theorem (Hamidoune, Lladó, Serra)

If A is not contained in the left coset of a cyclic subgroup, and $|B| \geq 4$, then

$$|AB| \geq |A| + |B| + 1.$$

Results if G is Abelian

Theorem (Freiman)

If A is not contained in a progression, then

$$|A + A| \geq 3|A| - 3.$$

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- ▶ Bound $3|A| - 3$ is optimal.
- ▶ If $|A + A| \leq 2|A| + k$ for $0 \leq k \leq |A| - 4$, then A is a subset of a progression of length $|A| + k + 1$.

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Theorem (Ruzsa, Gardner-Gronchi)

If $|A| \leq |B|$, and A is not contained in a progression, then

$$\begin{aligned} |A + B| &\geq |A| + |B| + |A| - 3 \\ |A + B| &\geq |A| + |B| + \sqrt{(|A| - 2)(|B| - 2)} - 1. \end{aligned}$$

Small product sets if G is any torsion-free group

Theorem (B., Pálffy, Serra)

If A is not contained in the left coset of a cyclic subgroup, and $|B| \geq c(k)$ for $k \geq 1$ and $c(k) = 32(k+3)^6$, then

$$|AB| \geq |A| + |B| + k.$$

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Remark

- ▶ $c(k)$ is at least quadratic in k .
- ▶ $c(k) = k^2 + 2k + 3$ works if G is Abelian by Gardner-Gronchi.
- ▶ $c(k) = 4(2k+3)^3$ works if G is a unique product group (an element of AB has a unique representation as a product).

A^2 if G is any torsion-free group

Conjecture (Freiman)

If $|A^2| \leq 3|A| - 4$, then A is contained in a progression of length $2|A| - 3$.

- ▶ More precisely, if $|A^2| \leq 2|A| + k$ for $0 \leq k \leq |A| - 4$, then A is a subset of a progression of length $|A| + k + 1$.

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Theorem (B., Pálffy, Serra)

The conjecture of Freiman holds if

$$k \leq \frac{1}{2} |A|^{1/6} - 3.$$

Proof of the main Theorem

Equivalent form of the Theorem

If G is any torsion-free group, $|B| \geq c(k)$, and

$$|AB| < |A| + |B| + k,$$

then A is contained in the left coset of a cyclic subgroup.

Proof of the main Theorem

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Step 1 A can be assumed to have bounded size by the Isoperimetric method of Serra and Hamidoune.

Step 2 When $|A| = 3$, use Schur's theorem on the central subgroup.

Step 1 - Isoperimetric method

Lemma

If G is any torsion-free group, $|A| \geq n$ for $n \geq 2$, and

$$|AB| < |A| + |B| + k,$$

then there exists C with $|CB| < |C| + |B| + k$ such that

$$n \leq |C| \leq n(2k + 1).$$

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Conjecture (Hamidoune)

$|C| = n$ can be assumed.

- ▶ Holds in unique product (hence in Abelian) groups.

Step 2 - $|A| = 3$

Lemma

If G is any torsion-free group, A with $|A| = 3$ is not contained in the left coset of a cyclic subgroup, and $|B| > 4d^3$ for $d \geq 1$, then

$$|AB| > |B| + d.$$

Step 2 - $|A| = 3$

Lemma

If G is any torsion-free group, A with $|A| = 3$ is not contained in the left coset of a cyclic subgroup, and $|B| > 4d^3$ for $d \geq 1$, then

$$|AB| > |B| + d.$$

- ▶ One can assume that $A = \{1, a, b\}$, where a, b do not commute, and $G = \langle a, b \rangle$.
- ▶ Probably the right condition is that $|B|$ is at least quadratic in d .