$\mathrm{SL}(m,\mathbb{C})$-equivariant and translation covariant continuous tensor valuations[*]

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Abstract

The space of continuous, $\mathrm{SL}(m,\mathbb{C})$-equivariant, $m \geq 2$, and translation covariant valuations taking values in the space of real symmetric tensors on $\mathbb{C}^m \cong \mathbb{R}^{2m}$ of rank $r \geq 0$ is completely described. The classification involves the moment tensor valuation for $r \geq 1$ and is analogous to the known classification of the corresponding tensor valuations that are $\mathrm{SL}(2m,\mathbb{R})$-equivariant, although the method of proof cannot be adapted.

1 Introduction

Let $n \geq 2$, let $\mathbb{V}$ be a vector space of real dimension n, and let $\mathcal{A}$ be an abelian semigroup. Denote by $\mathcal{K}(\mathbb{V})$ the space of convex bodies in $\mathbb{V}$ (i.e., compact and convex sets in $\mathbb{V}$) equipped with the Hausdorff metric. An operator $Z : \mathcal{K}(\mathbb{V}) \to \mathcal{A}$ is called a *valuation* if

$$Z(K \cup L) + Z(K \cap L) = Z(K) + Z(L)$$

whenever $K, L \in \mathcal{K}(\mathbb{V})$ satisfy that $K \cup L \in \mathcal{K}(\mathbb{V})$. Here ' $+$ ' denotes the operation of the semigroup $\mathcal{A}$.

One of the principal aims in the theory of valuations is to obtain characterization results for known operators as the only valuations satisfying certain simple

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geometric and topological properties. Nowadays valuations taking values in different semigroups have been largely studied. The first classification theorem goes back to 1952, when Hadwiger proved that, for $\mathbb{V} = \mathbb{R}^n$, the linear combinations of intrinsic volumes are the only continuous real-valued valuations being invariant under rigid motions of $\mathbb{R}^n$ (see [31]).

Hadwiger's result can be generalized in different directions. For instance, we can change the group acting on $\mathcal{K}(\mathbb{V})$ and classify the continuous real-valued valuations invariant under the action of some group (acting transitively on the unit sphere). This direction of study gave rise to the development of the theory of continuous and translation invariant real-valued valuations and has important consequences in integral geometry. We refer the reader to [7, 8, 14, 17, 18, 20, 25, 51] and references therein for some results in this direction.

Another important and more recent generalization of Hadwiger's theorem consists on changing the target space. For instance, valuations taking values in the space of convex bodies, in concave (or other spaces of) functions, etc. have been considered (see, e.g., [13, 21, 22, 36, 38, 39, 40]). In these cases, the action of a group G acting both on $\mathcal{K}(\mathbb{V})$ and $\mathcal{A}$ is also considered and usually a characterization result for different groups G and actions is studied. The related problem of tensor valuations on lattice polytopes is discussed in the pioneering paper of Ludwig and Silverstein [43].

In this paper, we will focus on the study of tensor-valued valuations. We prove a Hadwiger-type theorem for the continuous, $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation covariant valuations taking values in the space of real symmetric tensors of any given rank. Next we fix the notation to be used.

For $n \geq 2$, $r \in \mathbb{N}$ and an n-dimensional real vector space $\mathbb{V}$, we write $\mathbb{T}^r(\mathbb{V})$ to denote the $\binom{n+r-1}{r}$-dimensional space of symmetric r-tensors of $\mathbb{V}$ over $\mathbb{R}$. In particular, $\mathbb{T}^0(\mathbb{V}) = \mathbb{R}$ and $\mathbb{T}^1(\mathbb{V}) = \mathbb{V}$. We write S_r to denote the group of all permutations of $\{1, \dots, r\}$. For $r \geq 2$, the *symmetric tensor product* of $x_1, \dots, x_r \in \mathbb{V}$ is defined by

$$x_1 \odot \dots \odot x_r = \frac{1}{r!} \sum_{\sigma \in S_r} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(r)}.$$

We set $x^r = x \odot \dots \odot x = x \otimes \dots \otimes x$ for $x \in \mathbb{V}$. In addition, the group $\mathrm{GL}(\mathbb{V}, \mathbb{R})$ acts naturally on $\mathbb{T}^r(\mathbb{V})$ as follows: For $\varphi \in \mathrm{GL}(\mathbb{V}, \mathbb{R})$ the natural action on $\mathbb{T}^r(\mathbb{V})$ is given by

$$\varphi \cdot (x_1 \odot \dots \odot x_r) = \varphi x_1 \odot \dots \odot \varphi x_r$$

for $x_1, \dots, x_r \in \mathbb{V}$. We note that in this paper, tensor product is always over the reals even if the vector space has a complex structure, say possibly $\mathbb{V} = \mathbb{C}^m$ where $n = 2m$, and $\mathbb{T}^r(\mathbb{C}^m)$ still means symmetric r-tensors over the reals.

Given an action of a closed subgroup $G \subset \mathrm{GL}(\mathbb{V}, \mathbb{R})$ on $\mathbb{V}$, we say that a valuation $Z : \mathcal{K}(\mathbb{V}) \to \mathbb{T}^r(\mathbb{V})$ is *G-equivariant* if $Z(\varphi(K)) = \varphi Z(K)$ holds for any $\varphi \in G$ and $K \in \mathcal{K}(\mathbb{V})$. If $r = 0$, then G-equivariance is equivalent with G invariance.

In the following, for $\mathbb{V} = \mathbb{R}^n$, we set $\mathcal{K}(\mathbb{V}) = \mathcal{K}^n$. We say that a tensor valuation $Z : \mathcal{K}^n \to \mathbb{T}^r(\mathbb{V})$ is *translation covariant* if for every $K \in \mathcal{K}^n$, we have

$$Z(K + y) = \sum_{j=0}^{r} Z^{r-j}(K) \odot \frac{y^j}{j!} \tag{1}$$

as a function of $y \in \mathbb{R}^n$ where each Z^{r-j} is a tensor valuation of rank $r - j$ with $Z = Z^r$. We observe that if $r = 0$, then translation covariance is equivalent with translation invariance. If $r > 0$ and Z is G-equivariant for a closed subgroup $G \subset \mathrm{GL}(n, \mathbb{R})$, then so is each Z^{r-j}.

The reason for the normalization in (1) introduced by McMullen [46] is that for $j = 0, \dots, r - 1$, we have

$$Z^{r-j}(K + y) = \sum_{m=0}^{r-j} Z^{r-j-m}(K) \odot \frac{y^m}{m!}, \tag{2}$$

and hence $Z^{r-j}(K)$ is also a translation covariant valuation.

For $r \geq 0$, a basic example of translation covariant tensor-valued valuation is the *moment tensor valuation*

$$M^r(K) = \frac{1}{r!} \int_K x^r \, dx,$$

which is $\mathrm{SL}(n, \mathbb{R})$-equivariant. For $K \in \mathcal{K}^n$, we write $V(K)$ to denote the volume of K, and hence for $y \in \mathbb{R}^n$, we have

$$M^r(K + y) = \sum_{j=0}^{r} M^{r-j}(K) \odot \frac{y^j}{j!} \text{ where } M^0(K) = V(K). \tag{3}$$

Haberl and Parapatits [30] characterized the moment tensor valuation as continuous, $\mathrm{SL}(n, \mathbb{R})$-equivariant, and translation covariant tensor valuation. More precisely, they characterized all measurable $\mathrm{SL}(n, \mathbb{R})$-equivariant tensor valuations on polytopes containing the origin. As a special case of the main result of [30], we have the following.

Theorem 1.1 (Haberl, Parapatits) *Let $n \geq 2$ and $r \geq 0$. An operator $Z : \mathcal{K}^n \to \mathbb{T}^r(\mathbb{R}^n)$ is an $\mathrm{SL}(n, \mathbb{R})$-equivariant and translation covariant continuous valuation if and only if $Z = c \cdot M^r$ for a $c \in \mathbb{R}$, if $r \geq 1$, and $Z = c_1 + c_2 V$ for $c_1, c_2 \in \mathbb{R}$, if $r = 0$.*

The main result in [30] culminates a series of papers devoted to the study of tensor valuations that are affine-equivariant. The weakening of the continuity hypothesis to the measurability was an important aim after the results for upper semi-continuous valuations. We refer the reader to [4, 19, 28, 29, 32, 35, 37, 41, 42, 52] for results in this direction and on tensor valuations.

In this paper, we consider $\mathbb{V} = \mathbb{C}^m \cong \mathbb{R}^{2m}$ and $\mathrm{SL}(m, \mathbb{C})$ acting on $\mathbb{V}$. We prove that the moment tensor valuation is again essentially the only $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation covariant tensor-valued valuation. More precisely, we prove the following result.

Theorem 1.2 *Let $m \geq 2$ and $r \geq 0$. An operator $Z : \mathcal{K}(\mathbb{C}^m) \to \mathbb{T}^r(\mathbb{C}^m)$ is an $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation covariant continuous valuation if and only if $Z = c\, M^r$ for a $c \in \mathbb{R}$, if $r \geq 1$, and $Z = c_1 + c_2 V$ for $c_1, c_2 \in \mathbb{R}$, if $r = 0$.*

We first notice that the case $r = 0$ is not new since using $\mathrm{SU}(m) \subset \mathrm{SL}(m, \mathbb{C})$, it can be obtained as a direct consequence of characterization of the $\mathrm{SU}(m)$-invariant and translation invariant real-valued valuations by Alesker [10] if $m = 2$ and by Bernig [14]) if $m \geq 3$.

Theorem 1.3 (Alesker, Bernig) *Let $m \geq 2$. An operator $Z : \mathcal{K}(\mathbb{C}^m) \to \mathbb{R}$ is an $\mathrm{SL}(m, \mathbb{C})$ and translation invariant continuous valuation if and only if $Z = c_1 + c_2 V$ for $c_1, c_2 \in \mathbb{R}$.*

Based on the evenness of the valuation proved only in Section 7, we provide a direct argument leading to Theorem 1.3 (the $r = 0$ case) in Section 5 with the aim to enlighten the general case $r \geq 1$ in Theorem 1.2.

With the tools used in this paper, to weaken the continuity hypothesis in Theorem 1.2 to measurability or even upper-semicontinuity is, in the opinion of the authors, out of reach. Indeed, results from the theory of continuous and translation invariant valuations together with the fact that, in some contexts, continuity implies smoothness are heavily used, for instance, to differentiate some functions appearing on the proof of Theorem 1.2. We also note that the method of the proof of Theorem 1.1 by Haberl and Parapatits [30], which led to results under only measurability assumptions, does not seem to be adaptable to Theorem 1.2. One of the main ideas in [30] is the use of double pyramids, which can be seen as a generalization of simplices. As the group $\mathrm{SL}(n, \mathbb{R})$ acts transitively on the space of simplices, the study of the image of a fixed simplex suffices to determine the image of every simplex. Since the group $\mathrm{SL}(m, \mathbb{C})$ acts no longer transitively on the space of simplices in $\mathbb{R}^{2m}$ a similar argument does not seem to work for Theorem 1.2.

It was the paper by Abardia and Bernig [1] that first considered valuations intertwining $\mathrm{SL}(m,\mathbb{C})$ by providing a generalization of the seminal characterization result for the projection body operator obtained by Ludwig [38].

The paper is organized as follows: In Section 2, we present the main steps for the proof of Theorem 1.2, and reduce it to showing the non-existence of non-trivial even or odd, continuous, $\mathrm{SL}(m,\mathbb{C})$-equivariant and translation invariant tensor-valued valuations, see Proposition 2.3. Starting from Section 3, the sole task of the paper is to prove Proposition 2.3. Section 3 reviews the fundamental properties of translation invariant continuous valuations, and Section 4 discusses real subspaces of $\mathbb{C}^m$. Theorem 1.3 (the case $r=0$ of Theorem 1.2) is proved in Section 5. For even valuations, the proof of Proposition 2.3 is treated in Section 6. In the case of odd valuations, Proposition 2.3 is verified in Section 7. In both cases, the section is divided into subsections according to the degree j of homogeneity of the valuation. Putting together the result obtained for the different homogeneity degrees, the result in the odd and even cases follows, and Proposition 2.3 is, in this way, proved (cf. page 38).

2 Proof of Theorem 1.2

In this section, we present the main ideas of the proof of Theorem 1.2, and how to reduce it to Proposition 2.3.

We start with the following fact for tensor-valued valuations, which was shown by McMullen [46] if $s=r$ and by Alesker [5] for $s<r$.

Theorem 2.1 (McMullen, Alesker) *If $n\geq 2$, $r\geq 1$, $0\leq s\leq r$ and the valuations $Z:\mathcal{K}^n\to\mathbb{T}^r(\mathbb{R}^n)$ and $Z^{r-j}:\mathcal{K}^n\to\mathbb{T}^{r-j}(\mathbb{R}^n)$, $j=0,\dots,s$ satisfy*

$$Z(K+y)=\sum_{j=0}^{s} Z^{r-j}(K)\odot\frac{y^j}{j!}$$

for $K\in\mathcal{K}^n$ and $y\in\mathbb{R}^n$, then Z^{r-s} is translation invariant.

The first new result that we need for the proof of Theorem 1.2 is the following.

Proposition 2.2 *If $m\geq 2$, $r\geq 1$ and $Z:\mathcal{K}(\mathbb{C}^m)\to\mathbb{T}^r(\mathbb{C}^m)$ is an $\mathrm{SL}(m,\mathbb{C})$-equivariant translation covariant continuous valuation such that $Z^0\equiv c$ for a constant $c\in\mathbb{R}$; namely, if*

$$Z(K+y)=c\cdot y^r+\sum_{j=0}^{r-1} Z^{r-j}(K)\odot\frac{y^j}{j!}$$

for every $y\in\mathbb{C}^m$ and $K\in\mathcal{K}(\mathbb{C}^m)$, then $c=0$.

Proof: Let $v_1, \dots, v_m$ be a complex basis of $\mathbb{C}^m$, and let $\mathbb{V} = \mathrm{lin}_{\mathbb{R}}\{v_1, \dots, v_m\}$. We observe that $\mathrm{SL}(\mathbb{V}, \mathbb{R}) \subset \mathrm{SL}(m, \mathbb{C})$ is a closed subgroup, and the action of $\varphi \in \mathrm{SL}(\mathbb{V}, \mathbb{R})$ on $\mathbb{C}^m$ is defined by $\varphi(iv) = i\varphi(v)$ for $v \in \mathbb{V}$. For $\varrho = 1, \dots, r$, we consider the basis of $\mathbb{T}^\varrho(\mathbb{C}^m)$ induced by the real basis $v_1, iv_1, \dots, v_m, iv_m$ of $\mathbb{C}^m$. The induced action of $\mathrm{SL}(\mathbb{V}, \mathbb{R})$ on $\mathbb{T}^\varrho(\mathbb{C}^m)$ leaves $\mathbb{T}^\varrho(\mathbb{V})$ invariant, and $\mathbb{T}^\varrho(\mathbb{V})$ has an $\mathrm{SL}(\mathbb{V}, \mathbb{R})$-invariant direct complement subspace spanned by the elements of the basis of $\mathbb{T}^\varrho(\mathbb{C}^m)$ containing at least one of $iv_1, \dots, iv_m$, which subspace in turn is the kernel of a linear projection $\psi : \mathbb{T}^\varrho(\mathbb{C}^m) \to \mathbb{T}^\varrho(\mathbb{V})$ commuting with the action of $\mathrm{SL}(\mathbb{V}, \mathbb{R})$.

For $K \in \mathcal{K}(\mathbb{V})$ and $j = 0, \dots, r-1$, we set $\widetilde{Z}^{r-j}(K) = \psi Z^{r-j}(K)$ and $\widetilde{Z}(K) = \psi Z(K)$. In particular, $\widetilde{Z} : \mathcal{K}(\mathbb{V}) \to \mathbb{T}^r(\mathbb{V})$ is an $\mathrm{SL}(m, \mathbb{R})$-equivariant, translation covariant and continuous valuation such that if $K \in \mathcal{K}(\mathbb{V})$, then

$$\widetilde{Z}(K + y) = c \cdot y^r + \sum_{j=0}^{r-1} \widetilde{Z}^{r-j}(K) \odot \frac{y^j}{j!}$$

for $y \in \mathbb{V}$.

On the other hand, Theorem 1.1 and (3) yield that $\widetilde{Z}^0 = c_0 V_m$ for a constant $c_0 \in \mathbb{R}$ where V_m is the m-dimensional volume on $\mathbb{V}$. Therefore $c = c_0 V_m(K)$ for all $K \in \mathcal{K}(\mathbb{V})$, proving that $c = 0$. □

The following statement is the main novel ingredient of the proof of Theorem 1.2, and the rest of this paper will be devoted to its proof.

Proposition 2.3 *Let $m \geq 2$. If $r \geq 1$ and $Z : \mathcal{K}^{2m} \to \mathbb{T}^r(\mathbb{R}^{2m})$ is an odd or even $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous valuation, then Z is constant zero. In addition, if $Z : \mathcal{K}^{2m} \to \mathbb{R}$ is an odd $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous valuation, then again Z is constant zero.*

Proof of Theorem 1.2 based on Propositions 2.2 and 2.3: Let $Z : \mathcal{K}^{2m} \to \mathbb{T}^r(\mathbb{R}^{2m})$, $r \geq 1$ be an $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation covariant continuous valuation, and hence

$$Z(K + y) = \sum_{j=0}^{r} Z^{r-j}(K) \odot \frac{y^j}{j!}, \quad y \in \mathbb{R}^{2m},$$

where each $Z^{r-j}(K)$ is an $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation covariant tensor valuation of rank $r - j$, $j = 0, \dots, r$. According to Theorem 1.3, $Z^0 = c_1 + c_2 V$ for $c_1, c_2 \in \mathbb{R}$. It follows from (3) that $\widetilde{Z} := Z - c_2 M^r$ is an $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation covariant tensor valuation of rank r, and

$$\widetilde{Z}(K + y) = c_1 \cdot \frac{y^r}{r!} + \sum_{j=0}^{r-1} \widetilde{Z}^{r-j}(K) \odot \frac{y^j}{j!}. \tag{4}$$

We suppose that $\widetilde{Z}$ is not constant zero, and seek a contradiction. First Proposition 2.2 yields $c_1 = 0$. Therefore there exists a maximal $j \in \{0, \ldots, r-1\}$ such that $\widetilde{Z}^{r-j}$ is not constant zero. For $\varrho = r - j \geq 1$, we deduce from Theorem 2.1 that the $\mathbb{T}^{\varrho}(\mathbb{R}^{2m})$-valued $\mathrm{SL}(m, \mathbb{C})$-equivariant continuous valuation $\widetilde{Z}^{\varrho}$ is actually translation invariant.

Now we consider the $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous $\mathbb{T}^{\varrho}(\mathbb{R}^{2m})$-valued valuations

$$\begin{aligned} Z^{+}(K) &= \frac{1}{2}(\widetilde{Z}^{\varrho}(K) + \widetilde{Z}^{\varrho}(-K)), \\ Z^{-}(K) &= \frac{1}{2}(\widetilde{Z}^{\varrho}(K) - \widetilde{Z}^{\varrho}(-K)). \end{aligned}$$

It follows that Z^+ is even and Z^- is odd, and both are translation invariant, continuous and $\mathrm{SL}(m, \mathbb{C})$-equivariant where the last property is a consequence of the fact that $-\mathrm{id}_{\mathbb{R}^{2m}}$ commutes with all elements of $\mathrm{SL}(m, \mathbb{C})$. Therefore Proposition 2.3 yields that Z^+ and Z^-, and in turn $\widetilde{Z}^{\varrho} = Z^+ + Z^-$ is constant zero. This is absurd, thus $\widetilde{Z}$ is constant zero, completing the proof of Theorem 1.2. □

Therefore all we are left to prove is Proposition 2.3.

3 Translation invariant continuous valuations

Let $\mathbb{V}$ be a finite dimensional real vector space. In this section, we survey known properties for continuous and translation invariant valuations $Z : \mathcal{K}^n \to \mathbb{V}$ for $n \geq 2$. Our discussion is mostly based on Alesker [12], and provide arguments using well-known ideas only when the statement we need has not been explicitly stated or proved before. We recall that $\mathcal{K}^n$ denotes the space of compact convex bodies in $\mathbb{R}^n$ and fix a real scalar product on $\mathbb{R}^n$. For general results in the theory of convex bodies and valuations, we refer, e.g., to the books [26, 27, 48].

We write Val to denote the Fréchet space of continuous and translation invariant valuations $Z : \mathcal{K}^n \to \mathbb{R}$ (see Alesker [6] for a description of the Fréchet structure). Hence the Fréchet space of continuous and translation invariant valuations $Z : \mathcal{K}^n \to \mathbb{V}$ is $\mathrm{Val} \otimes \mathbb{V}$ (remember that tensor products are always over $\mathbb{R}$ in this paper). We say that a valuation $Z : \mathcal{K}^n \to \mathbb{V}$ is *homogeneous of degree* j or simply j-homogeneous if $Z(\lambda K) = \lambda^j Z(K)$ for every $\lambda \geq 0$ and $K \in \mathcal{K}^n$. We denote by $\mathrm{Val}_j \subset \mathrm{Val}$ the subset of j-homogeneous real-valued valuations. Moreover, $Z : \mathcal{K}^n \to \mathbb{V}$ can be written uniquely in the form $Z = Z^+ + Z^-$ where Z^+ is even and Z^- is odd; namely, $Z^+(-K) = Z^+(K)$ and $Z^-(-K) = -Z^-(K)$. Val^+ (resp. Val^-) denote the subspace of even (resp. odd) valuations in Val. A typical example of an even valuation with degree of homogeneity j is the jth intrinsic volume V_j, which coincides with the j-dimensional Lebesgue measure on compact convex sets of dimension at most j.

We define an action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathrm{Val}\otimes\mathbb{V}$ as follows.

Definition 3.1 *Let* $\mathrm{GL}(n,\mathbb{R})$ *act on the finite dimensional vector space* $\mathbb{V}$ *and denote this action by* $\varphi\cdot v$*,* $\varphi\in\mathrm{GL}(n,\mathbb{R})$*,* $v\in\mathbb{V}$*. Then, the* action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathrm{Val}\otimes\mathbb{V}$ *is given by*

$$(\varphi Z)(K)=\varphi\cdot Z(\varphi^{-1}K), \tag{5}$$

where $Z\in\mathrm{Val}\otimes\mathbb{V}$*,* $\varphi\in\mathrm{GL}(n,\mathbb{R})$ *and* $K\in\mathcal{K}^n$*.*

Definition 3.2 *Let* $\mathrm{GL}(n,\mathbb{R})$ *act on the finite dimensional vector space* $\mathbb{V}$ *and let* $G\subset\mathrm{GL}(n,\mathbb{R})$ *be a closed subgroup. We say that a valuation* $Z:\mathcal{K}^n\to\mathbb{V}$ *is* G-equivariant *if it is invariant under the action (5) over* G*.*

We denote by $(\mathrm{Val}\otimes\mathbb{V})^G$ *the Fréchet subspace of* G*-equivariant valuations.*

In this paper, $\mathbb{V}$ is always a finite dimensional real vector space. As stated in the introduction, our main focus is the case

$$\mathbb{V}=\mathbb{T}(\mathbb{R}^{2m})=\mathbb{T}(\mathbb{C}^m)=\oplus_{r\geq 0}\mathbb{T}^r(\mathbb{C}^m)$$

and $G=\mathrm{SL}(m,\mathbb{C})$ where $\mathbb{T}^r(\mathbb{C}^m)$ is the real $\binom{2m+r-1}{r}$-dimensional space of rth symmetric tensor power of $\mathbb{C}^m$ over $\mathbb{R}$.

Another essential notion in the theory of valuations and in this paper is that of smoothness.

Definition 3.3 *We say that a valuation* $Z:\mathcal{K}^n\to\mathbb{V}$ *is* smooth *if the action (5) defines a smooth map* $\mathrm{GL}(n,\mathbb{R})\to\mathrm{Val}\otimes\mathbb{V}$*. Equivalently,* Z *is smooth if* $\varphi\mapsto Z\circ\varphi^{-1}$ *is a smooth map* $\mathrm{GL}(n,\mathbb{R})\to\mathrm{Val}\otimes\mathbb{V}$*.*

In this paper, we use the terms smooth and C^∞ interchangeably. We write Val^∞ to denote the Fréchet subspace of smooth elements of Val, which is a dense subspace according to Alesker's Irreducibility Theorem (see [7]).

Theorem 3.4 (Alesker's irreducibility theorem) *The natural representations of the group* $\mathrm{GL}(n,\mathbb{R})$ *in* Val_j^+ *and in* Val_j^- *are irreducible, i.e., there is no proper closed* $\mathrm{GL}(n,\mathbb{R})$ *invariant subspace.*

The Fréchet subspace of smooth elements of $\mathrm{Val}\otimes\mathbb{V}$ is denoted by $(\mathrm{Val}\otimes\mathbb{V})^\infty$. It follows from classical results in representation theory (see, e.g., [50, p. 32]) that

$$(\mathrm{Val}\otimes\mathbb{V})^\infty=\mathrm{Val}^\infty\otimes\mathbb{V}. \tag{6}$$

If $Z:\mathcal{K}^n\to\mathbb{R}$ is invariant under a closed subgroup $G\subset O(n)$ acting transitively on S^{n-1}, then

(i) Z is smooth according to Alesker [9];

(ii) Z is even according to Bernig [15].

Normal cycles provide a natural way to represent smooth valuations. If $K \in \mathcal{K}^n$, the *normal cycle* of K is defined as the set $\mathrm{nc}(K) \subset S\mathbb{R}^n := \mathbb{R}^n \times S^{n-1}$ given by

$$\mathrm{nc}(K) = \{(x, v) \in S\mathbb{R}^n \,:\, x \in K, \langle v, x - y\rangle \geq 0\, \forall y \in K\}.$$

We say that an $(n-1)$-form $\omega \in \Omega^{n-1}(S\mathbb{R}^n) \otimes \mathbb{V}$ is translation invariant if it depends only on its components on S^{n-1}.

Corollary 3.5 $(\mathrm{Val}\otimes\mathbb{V})^\infty$ *is a dense subspace of* $\mathrm{Val}\otimes\mathbb{V}$*. Moreover, the elements of* $(\mathrm{Val} \otimes \mathbb{V})^\infty$ *are given by integration over the normal cycle of a translation invariant form, i.e., if* $Z \in (\mathrm{Val} \otimes \mathbb{V})^\infty$*, then there exists a translation invariant* $\omega \in \Omega^{n-1}(S\mathbb{R}^n) \otimes \mathbb{V}$ *such that*

$$Z(K) = \int_{nc(K)} \omega \quad \text{for } K \in \mathcal{K}^n.$$

Proof: The first statement simply follows from (6) and Alesker's Irreducibility Theorem [9].

The second statement is proved by Alesker [11] if $\mathbb{V} = \mathbb{R}$, and hence it follows again by (6). $\square$

For a closed subgroup $G \subset \mathrm{GL}(n, \mathbb{R})$, we recall that $(\mathrm{Val} \otimes \mathbb{V})^G$ denotes the subspace of G-equivariant valuations in $\mathrm{Val}\otimes\mathbb{V}$. Similarly to the real-valued case, as observed by Alesker and Bernig (private communication), any $Z \in (\mathrm{Val}\otimes\mathbb{V})^G$, with G a closed subgroup $G \subset O(n)$ acting transitively on S^{n-1}, is smooth. For the convenience of the reader, we give a proof following the arguments of Corollary 3.3 in Fu [24] and Theorem 4.1 in Bernig [16].

Proposition 3.6 (Alesker, Bernig) *Let a closed subgroup* $G \subset O(n)$ *act transitively on* S^{n-1}*, and let* $\mathrm{GL}(n, \mathbb{R})$ *act on a finite dimensional real vector space* $\mathbb{V}$*. Then,* $\dim(\mathrm{Val}\otimes\mathbb{V})^G < \infty$ *and* $(\mathrm{Val}\otimes\mathbb{V})^G \subset (\mathrm{Val}\otimes\mathbb{V})^\infty$*, that is, if a continuous translation invariant valuation* $Z : \mathcal{K}^n \to \mathbb{V}$ *is G-equivariant, then Z is smooth.*

Proof: Let $\mathbb{W}$ be the space of all valuations $Z \in \mathrm{Val} \otimes \mathbb{V}$ such that there exists a translation invariant and G invariant $(n-1)$-form $\omega \in \Omega^{n-1}(S\mathbb{R}^n) \otimes \mathbb{V}$ satisfying $Z(K) = \int_{\mathrm{nc}(K)} \omega$ for $K \in \mathcal{K}^n$. In particular, $\mathbb{W} \subset (\mathrm{Val} \otimes \mathbb{V})^\infty$.

First we claim that $\mathbb{W}$ is dense in the closed subspace $(\mathrm{Val} \otimes \mathbb{V})^G$. For this, let $Z \in (\mathrm{Val} \otimes \mathbb{V})^G$ and hence, $Z \in \mathrm{Val} \otimes \mathbb{V}$. It follows from Corollary 3.5 that there exists a convergent sequence $\{Z_{(k)}\}_{k\in\mathbb{N}} \subset (\mathrm{Val}\otimes\mathbb{V})^\infty$ that converges to Z. For every $k\in\mathbb{N}$, define $\widetilde{Z}_{(k)} \in (\mathrm{Val}\otimes\mathbb{V})^{\infty,G}$ by

$$\widetilde{Z}_{(k)}(K) = \int_G \varphi Z_{(k)}(K)d\mu(\varphi)$$

where μ is the probability Haar measure on G. Since Z is G-equivariant, it follows that the sequence $\{\widetilde{Z}_{(k)}\}$ also converges to Z. In addition, each $\widetilde{Z}_{(k)}$ is a smooth and G-equivariant $\mathbb{V}$-valued valuation. We deduce from the second statement of Corollary 3.5 that each $\widetilde{Z}_{(k)}$ is given by integrating an $(n-1)$-form $\omega_{(k)} \in \Omega^{n-1}(S\mathbb{R}^n)\otimes\mathbb{V}$ on the corresponding normal cycle. As $\widetilde{Z}_{(k)}$ is G invariant, we can assume that $\omega_{(k)}$ is also G invariant, proving that $\mathbb{W}$ is dense in $(\mathrm{Val}\otimes\mathbb{V})^G$.

Next we prove that $\mathbb{W}$ is finite dimensional. For the argument, we fix a base point $e\in S^{n-1}$. Since the group G acts transitively on S^{n-1}, the form $\omega_{(k)}$ is determined by the knowledge of it in a single point (note that $\omega_{(k)}$ is translation invariant). In particular, it is enough to know $\omega_{(k)}|_{(o,e)} \in \Lambda^{n-1}(T_{(o,e)}S\mathbb{R}^n)\otimes\mathbb{V}$ where $T_{(o,e)}S\mathbb{R}^n$ stands for the tangent space at (o,e) and $\Lambda^{n-1}(T_{(o,e)}S\mathbb{R}^n)\otimes\mathbb{V}$ is a finite dimensional vector space. Therefore $\mathbb{W}$ is finite dimensional, as well.

As $(\mathrm{Val}\otimes\mathbb{V})^G$ is the closure of the finite dimensional $\mathbb{W}$, we have $(\mathrm{Val}\otimes\mathbb{V})^G = \mathbb{W}$, verifying Proposition 3.6. □

In the following, we study the decomposition of Val and $\mathrm{Val}\otimes\mathbb{V}$ in terms of the degree of the homogeneity of the valuations and describe some of these spaces.

McMullen [44] proved the following useful polynomial behavior of certain valuations:

Theorem 3.7 (McMullen decomposition) *Let $Z:\mathcal{K}^n\to\mathbb{V}$ be a continuous and translation invariant valuation, $K\in\mathcal{K}^n$, and $\lambda\geq 0$. Then,*

$$Z(\lambda K) = \sum_{j=0}^n \lambda^j Z_j(K) \tag{7}$$

where Z_j is a translation invariant continuous valuation homogeneous of degree j, $j=0,\dots,n$ ($Z_j(\lambda K)=\lambda^j Z_j(K)$ for $K\in\mathcal{K}^n$ and $\lambda\geq 0$). In particular,

$$\mathrm{Val} = \oplus_{j=0}^n \mathrm{Val}_j,$$

where Val_j denotes the Fréchet space of continuous and translation invariant valuations homogeneous of degree j, $j=0,\dots,n$.

For $G\subset \mathrm{GL}(n,\mathbb{R})$ a closed subgroup, if Z is G-equivariant, then the same holds for each Z_j.

Let us consider the coefficients occurring in (7) for a continuous and translation invariant valuation $Z : \mathcal{K}^n \to \mathbb{V}$. We have that Z_0 is constant, and, as proved by Hadwiger [31], Z_n is a constant multiple of the volume of K, that is, there exists $c \in \mathbb{V}$ such that

$$Z_n(K) = c \cdot V(K) \text{ for } K \in \mathcal{K}^n. \tag{8}$$

The valuation Z_{n-1} can also be described. A direct extension of McMullen's representation result, proved in [45], gives us the following representation.

Theorem 3.8 (McMullen) *Let $Z_{n-1} : \mathcal{K}^n \to \mathbb{V}$ be a continuous and translation invariant valuation homogeneous of degree $n-1$. Then, there exists a continuous 1-homogeneous function $f : \mathbb{R}^n \to \mathbb{V}$ ($f(\lambda x) = \lambda f(x)$ for $x \in \mathbb{R}^n$ and $\lambda \geq 0$) such that*

$$Z_{n-1}(K) = \int_{S^{n-1}} f \, dS_K \textit{ for } K \in \mathcal{K}^n, \tag{9}$$

where S_K denotes the surface area measure of K (see Schneider [48]). Moreover, f is unique up to a linear function. In other words, for continuous 1-homogeneous functions $f, \tilde{f} : \mathbb{R}^n \to \mathbb{V}$, we have

$$\int_{S^{n-1}} f \, dS_K = \int_{S^{n-1}} \tilde{f} \, dS_K \textit{ for all } K \in \mathcal{K}^n \tag{10}$$

if and only if $f - \tilde{f}$ is a linear function on $\mathbb{R}^n$.

In addition, f is odd if Z_{n-1} is odd.

We recall that if h_C is the support function of a $C \in \mathcal{K}^n$, then

$$\int_{S^{n-1}} h_C \, dS_K = nV(K, \ldots, K, C).$$

Here $V(K, \ldots, K, C)$ denotes the mixed volume with $(n-1)$-times the convex body K and once the convex body C (see [48, Section 5] for more information on mixed volumes). We note that if φ is a volume preserving linear transformation, then

$$\begin{aligned} \int_{S^{n-1}} h_C \, dS_{\varphi K} &= nV(\varphi K, \ldots, \varphi K, \varphi(\varphi^{-1} C)) = nV(K, \ldots, K, \varphi^{-1} C) \\ &= \int_{S^{n-1}} h_{\varphi^{-1} C} \, dS_K = \int_{S^{n-1}} h_C \circ \varphi^{-t} \, dS_K \end{aligned}$$

where φ^{-t} stands for the inverse of the transpose of φ and $\mathbb{R}^n$ is identified with its dual using the inner product. Since any continuous 1-homogeneous function

$f : \mathbb{R}^n \to \mathbb{V}$ can be approximated by differences of support functions (see [48, Lemma 1.7.8]), we deduce that if $\varphi \in \mathrm{GL}(n, \mathbb{R})$ with $\det \varphi = \pm 1$ and $K \in \mathcal{K}^n$, then

$$\int_{S^{n-1}} f \, dS_{\varphi K} = \int_{S^{n-1}} f \circ \varphi^{-t} \, dS_K. \tag{11}$$

The following Proposition 3.9 is also observed by Alesker and Bernig (private communication). Below we provide an argument due to Alesker. Here and later in the paper o denotes the origin of $\mathbb{R}^n$.

Proposition 3.9 (Alesker, Bernig) *Using the notation of Theorem 3.8, Z_{n-1} is smooth if and only if f is smooth on $\mathbb{R}^n \backslash \{o\}$.*

Proof: If f is smooth, then readily the same holds for Z.

We may assume that $\mathbb{V} = \mathbb{R}$. Let $C(S^{n-1})$ be the Banach space of continuous functions on S^{n-1} with the L_∞ norm, and let Val_{n-1} be the Fréchet space of $(n-1)$-homogeneous continuous translation invariant valuations on $\mathcal{K}^n$. We write C_0 to denote the closed subspace of $C(S^{n-1})$ orthogonal to the n-dimensional subspace L_0 of $C(S^{n-1})$ linear maps in terms of the L_2 scalar product of functions induced by the integral of their product; namely, $g \in C_0$ holds for $g \in C(S^{n-1})$ if and only if

$$\int_{S^{n-1}} g(u) \cdot u \, d\mathcal{H}^{n-1}(u) = o \tag{12}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Since $\mathcal{H}^{n-1}$ is invariant under $\mathrm{SO}(n)$, we observe that

$$g \circ \varphi \in C_0 \text{ for any } g \in C_0 \text{ and } \varphi \in \mathrm{SO}(n). \tag{13}$$

Let us consider the positive definite matrix

$$M = \int_{S^{n-1}} u \otimes u \, d\mathcal{H}^{n-1}(u) = \int_{S^{n-1}} u \cdot u^t \, d\mathcal{H}^{n-1}(u).$$

It follows that for any $\psi \in C(S^{n-1})$, there exists a unique $l \in L_0$ such that $\psi - l \in C_0$, namely, $l(u) = \langle c_\psi, u \rangle$ for

$$c_\psi = M^{-1} \int_{S^{n-1}} \psi(u) \cdot u \, d\mathcal{H}^{n-1}(u).$$

Therefore (10) yields that the continuous linear map $\Omega : C_0 \to \mathrm{Val}_{n-1}$ is bijective where

$$\Omega(g)(K) = \int_{S^{n-1}} g \, dS_K \quad \text{for } g \in C_0.$$

It follows from the open mapping theorem that the linear map Ω^{-1} is also continuous, therefore it is smooth.

Now for the smooth valuation Z_{n-1}, we consider $f = \Omega^{-1}(Z_{n-1}) \in C_0$ that satisfies (9). The map $F : \mathrm{SO}(n) \to \mathrm{Val}_{n-1}$ defined by

$$F(\varphi)(K) = Z_{n-1}(\varphi^{-1}K) = \int_{S^{n-1}} f\, dS_{\varphi^{-1}K} = \int_{S^{n-1}} f\circ\varphi^{-1}\, dS_K \text{ for } \varphi \in \mathrm{SO}(n)$$

is smooth (compare (11)), and hence $\Omega^{-1} \circ F$ satisfying

$$\Omega^{-1} \circ F(\varphi) = f \circ \varphi^{-1} \text{ for } \varphi \in \mathrm{SO}(n) \text{ is smooth, as well.} \tag{14}$$

Finally, since f is 1-homogeneous, it is enough to prove that the restriction of f to S^{n-1} is smooth. However, for orthogonal $u, v \in S^{n-1}$, the directional derivative of f in the direction of v at u can be calculated using rotations around the $(n-2)$-dimensional linear subspace orthogonal to $\mathrm{lin}\{u, v\}$, showing that f is C^∞ as well. □

We end this section with two useful results about the determination of j-homogenous valuations by knowing its value on some convex bodies.

Theorem 3.10 (Schneider–Schuster [49]) *Let $j \in \{1, \ldots, n-1\}$ and let $Z_j : \mathcal{K}^n \to \mathbb{V}$ be a continuous and translation invariant valuation homogeneous of degree j. Then,*

$$Z_j(K) = o \textit{ for all } K \in \mathcal{K}^n \textit{ if } Z_j(K) = o \textit{ for all } K \in \mathcal{K}^n \textit{ with } \dim K = j+1. \tag{15}$$

For even valuations we have more information. Again let $Z_j : \mathcal{K}^n \to \mathbb{V}$ be a continuous, j-homogeneous and translation invariant valuation for $j \in \{1, \ldots, n-1\}$. For any linear subspace L of dimension j, Hadwiger's theorem (8) provides the existence of $\mathrm{Kl}_{Z_j}(L) \in \mathbb{V}$ such that

$$Z_j(K) = \mathrm{Kl}_{Z_j}(L) V_j(K) \text{ for } K \subset L.$$

The Grassmannian manifold $\mathrm{Gr}_j(\mathbb{R}^n)$ of linear subspaces of dimension j of $\mathbb{R}^n$ is a smooth real algebraic subvariety of the real projective space over $\Lambda^j(\mathbb{R}^n)$. In this sense, the Klain function $\mathrm{Kl}_{Z_j} : \mathrm{Gr}_j(\mathbb{R}^n) \to \mathbb{V}$ is continuous, and it is smooth if Z_j is smooth. We recall that for a compact topological space X, $C(X)$ denotes the normed space of continuous functions on X with the maximum norm.

Theorem 3.11 (Klain's injectivity theorem [33]) *The map $\mathrm{Kl} : \mathrm{Val}_j^+ \to C(\mathrm{Gr}_j(\mathbb{R}^n))$ is injective.*

In particular, if $Z_j : \mathrm{K}^n \to \mathbb{V}$ is an even, continuous, translation invariant and j-homogenous valuation and there exists $c \in \mathbb{V}$ such that

$$\mathrm{Kl}_{Z_j}(L) = c \textit{ for any linear } j\textit{-dimensional subspace } L\textit{, then } Z_j = c \cdot V_j. \tag{16}$$

4 Real vector subspaces of $\mathbb{C}^m$

In this section, we introduce the notation for linear subspaces in $\mathbb{C}^m$ and some properties of their bases.

We identify the complex vector space $\mathbb{C}^m$, of real dimension $2m$, with $\mathbb{R}^{2m}$ by using the bijection $\mathbb{C}^m \to \mathbb{R}^{2m}$ given by

$$(z_1, \ldots, z_m) = (x_1 + iy_1, \ldots, x_m + iy_m) \mapsto (x_1, \ldots, x_m, y_1, \ldots, y_m).$$

If $L \subset \mathbb{C}^m \cong \mathbb{R}^{2m}$ is a real vector subspace, then $\mathbb{C}L$ denotes the minimal complex linear subspace of $\mathbb{C}^m$ containing L. Hence $\dim_{\mathbb{C}}\mathbb{C}L$ is the maximal number of vectors in L independent over $\mathbb{C}$. We say that a j-dimensional real subspace $L \subset \mathbb{C}^m \cong \mathbb{R}^{2m}$ is of *maximal complex rank* if $\dim_{\mathbb{C}}\mathbb{C}L = \min\{j, m\}$.

Next we describe a natural basis of a real subspace L of $\mathbb{C}^m \cong \mathbb{R}^{2m}$. We observe that $\mathbb{C}^m$, $m \geq 2$, has a natural Hermitian inner product, whose real part is a scalar product on the underlying $\mathbb{R}^{2m}$.

Lemma 4.1 *Let L be a real vector subspace of $\mathbb{R}^{2m} = \mathbb{C}^m$ for $m \geq 2$ with $\dim_{\mathbb{R}}L = j \geq 1$, and let d be the maximal number of vectors in L independent over $\mathbb{C}$. Then, there exist $v_1, \ldots, v_d \in L$ independent over $\mathbb{C}$ such that $v_1, \ldots, v_d$ is a real orthonormal basis of L, if $j = d$, and $v_1, \ldots, v_d, iv_1, \ldots, iv_{j-d}$ is a real orthonormal basis of L if $j > d$.*

Proof: Let $U = L \cap iL$ be a complex subspace of $\mathbb{R}^{2m} = \mathbb{C}^m$ with $k = \dim_{\mathbb{C}}U$, and let W be the real orthogonal complement of U inside L with $t = \dim_{\mathbb{R}}W$, and hence $j = 2k + t$. If $k \geq 1$, then we choose a Hermitian basis $u_1, \ldots, u_k$ of U, and if $t \geq 1$, then we choose a real orthonormal basis $w_1, \ldots, w_t$ of W. We claim that if $t \geq 1$, then

$$\alpha_1 w_1 + \ldots + \alpha_t w_t \in U \text{ for } \alpha_1, \ldots, \alpha_t \in \mathbb{C} \text{ yields } \alpha_1 = \ldots = \alpha_t = 0. \tag{17}$$

We write $\beta_l = \mathrm{Re}\,\alpha_l$ and $\gamma_l = \mathrm{Im}\,\alpha_l$ for $l = 1, \ldots, t$, and set $w = \gamma_1 w_1 + \ldots + \gamma_t w_t \in W$. It follows from the condition in (17) that $iw \in L$, and hence $iw \in L \cap iL = U$. However, U is a complex subspace, thus $w \in U \cap W$. We conclude that $w = 0$, and hence $\gamma_1 = \ldots = \gamma_t = 0$. Therefore the condition in (17) implies $\beta_1 = \ldots = \beta_t = 0$, proving (17).

If $U = L$, and hence $d = k$ and $j = 2k$, then we choose $v_l = u_l$ for $l = 1, \ldots, d$. If $k = 0$, or equivalently, $U = \{o\}$, then $j = d = t$ by (17), and we choose $v_l = w_l$ for $l = 1, \ldots, d$.

Finally, if both $k \geq 1$ and $t \geq 1$, then $\mathbb{C}L = U + \mathbb{C}W$ imples that $d \leq k + t$, and hence $j = 2k + t$ and $d = k + t$ by (17). Therefore we choose $v_l = u_l$ for $l = 1, \ldots, k$ and $v_{k+l} = w_l$ for $l = 1, \ldots, t$. □

Now we show that for our purposes, we may assume that $d = \min\{j, m\}$ in Lemma 4.1.

Lemma 4.2 *If $m \geq 2$ and $j = 1, \ldots, 2m-1$, then the subset of all j-dimensional real subspaces of maximal complex rank constitutes a dense subset of $\mathrm{Gr}_j(\mathbb{R}^n)$.*

Proof: If $j = 1$, then the statement readily holds, thus we assume $j > 1$. Let $k = \min\{j, m\}$. We call a j-dimensional real subspace L of $\mathbb{R}^{2m}$ of *lower complex rank* if $\dim_{\mathbb{C}} \mathbb{C}L < k$.

We recall that the Grassmannian manifold $\mathrm{Gr}_j(\mathbb{R}^{2m})$ of linear subspaces of dimension j of $\mathbb{R}^{2m}$ is a connected smooth real algebraic subvariety of the real projective space over $\Lambda^j(\mathbb{R}^{2m})$, and in particular, locally it can be parametrized by the real wedge product of j independent vectors over $\mathbb{R}$. Now if an $L \in \mathrm{Gr}_j(\mathbb{R}^{2m})$ is represented by $v_1 \wedge \ldots \wedge v_j \in \Lambda^j(\mathbb{R}^{2m})$ for vectors $v_1, \ldots, v_j \in L$ independent over $\mathbb{R}$, then L is of lower complex rank if and only if for any $1 \leq i_1 < \ldots < i_k \leq j$, the complex wedge product

$$v_{i_1} \wedge_{\mathbb{C}} \ldots \wedge_{\mathbb{C}} v_{i_k} = 0 \in \Lambda^k(\mathbb{C}^m).$$

Therefore real j-dimensional subspaces of lower complex rank form a real projective algebraic subvariety X of $\mathrm{Gr}_j(\mathbb{R}^{2m})$. Since there exists some real j-dimensional subspace L of maximal complex rank, and $\mathrm{Gr}_j(\mathbb{R}^{2m})$ is smooth and connected, the real dimension of X is smaller than that of $\mathrm{Gr}_j(\mathbb{R}^{2m})$. We conclude that j-dimensional subspaces of maximal complex rank form a dense subset of $\mathrm{Gr}_j(\mathbb{R}^{2m})$. $\square$

According to Lemma 4.2, the next corollary follows from Klain's Injectivity Theorem 3.11 if the valuation Z_j is even, and from McMullen's Theorem 3.8 and Schneider's and Schuster's Theorem 3.10 if the valuation Z_j is odd.

Corollary 4.3 *For $m \geq 2$, $j = 1, \ldots, 2m-1$ and finite dimensional real vector space $\mathbb{V}$, let $Z_j : \mathcal{K}^{2m} \to \mathbb{V}$ be a continuous translation invariant valuation homogeneous of degree j.*

- **(i)** *If Z_j is even and $\mathrm{Kl}_{Z_j}(L) = 0$ for every real subspace $L \in \mathrm{Gr}_j(\mathbb{R}^{2m})$ of maximal complex rank, then Z_j is constant zero.*
- **(ii)** *If Z_j is odd and for every real subspace $L \in \mathrm{Gr}_{j+1}(\mathbb{R}^{2m})$ of maximal complex rank, the continuous function f on L associated to the restriction of Z_j to L by* (9) *is linear, then Z_j is constant zero.*

5 Real valued $\mathrm{SL}(m,\mathbb{C})$ and translation invariant continuous valuations

In this section we give a direct proof of Theorem 1.3, basing on ideas in Abardia, Bernig [1], Abardia [2, 3]. The main motivation to treat this particular case is that some of the main ideas to prove the general case (see Sections 6 and 7) are already contained in this section. We note that the crucial statement about real valued odd valuations will be only treated in Section 7 together with odd tensor valued valuations in order to shorten the paper.

Let $m \geq 2$ and let $Z : \mathcal{K}^{2m} \to \mathbb{R}$ be an $\mathrm{SL}(m,\mathbb{C})$ and translation invariant continuous valuation. First we observe that Z can be written as the sum of an even Z^+ and an odd Z^- translation and $\mathrm{SL}(m,\mathbb{C})$ invariant continuous valuation (see the proof of Theorem 1.2 in Section 2), thus $Z = Z^+$ is even as $Z^- \equiv 0$ by the case of Proposition 2.3 when $r = 0$ and the valuation is odd.

We next reduce the proof of Theorem 1.3 by using McMullen's decomposition and Klain's injectivity theorem as follows. From the McMullen's decomposition, it follows that $Z = \sum_{j=0}^{2m} Z_j$ where each Z_j is an even $\mathrm{SL}(m,\mathbb{C})$ and translation invariant continuous valuation homogeneous of degree j, $j = 0,\ldots,2m$. As we have described, $Z_0 = c_1\chi$ for a constant $c_1 \in \mathbb{R}$, and $Z_{2m} = c_2 V$ for a constant $c_2 \in \mathbb{R}$.

Therefore we have to verify that $Z_j \equiv 0$ for $j = 1,\ldots,2m-1$. As Z_j is even, continuous and translation invariant, Corollary 4.3(i) applies, and Theorem 1.3 follows if for each $j = 1,\ldots,2m-1$,

$$\mathrm{Kl}_{Z_j}(L) = 0 \quad \text{for all } L \in \mathrm{Gr}_j(\mathbb{R}^{2m}) \text{ of maximal complex rank.} \tag{18}$$

Lemma 5.1 *If $m \geq 2$, $j < m$, and L is a j-dimensional real vector subspace of $\mathbb{R}^{2m} = \mathbb{C}^m$ of maximal complex rank, then $\mathrm{Kl}_{Z_j}(L) = 0$.*

Proof: By definition, there exist $v_1,\ldots,v_j \in L$ independent over $\mathbb{C}$ such that $v_1,\ldots,v_j$ is a real basis of L. We extend $v_1,\ldots,v_j$ to a complex basis $v_1,\ldots,v_m$ of $\mathbb{C}^m$. Since $j < m$, there exists a $\varphi \in \mathrm{SL}(m,\mathbb{C})$ such that $\varphi v_l = 2v_l$ for $l = 1,\ldots,j$. For the j-dimensional simplex K with vertices $o, v_1,\ldots,v_j$, we have $V_j(K) > 0$ and

$$\mathrm{Kl}_{Z_j}(L)V_j(K) = Z_j(K) = Z_j(\varphi K) = \mathrm{Kl}_{Z_j}(L)V_j(\varphi K) = 2^j\mathrm{Kl}_{Z_j}(L)V_j(K),$$

and hence $\mathrm{Kl}_{Z_j}(L) = 0$. □

Proof of Theorem 1.3: According to Lemma 5.1, we need to prove (18) where Z_j is a j-homogeneous valuation with

$$j \in \{m,\ldots,2m-1\}.$$

Hence Lemma 4.1 yields that there exists a complex basis $v_1, \ldots, v_m$ of $\mathbb{C}^m = \mathbb{R}^{2m}$ such that writing $v_{l+m} = iv_l$ for $l = 1, \ldots, m$, the vectors $v_1, \ldots, v_j \in L$ form a real basis of L.

Let $K \subset L$ be a j-dimensional crosspolytope with vertices $\pm v_1, \ldots, \pm v_j$. We claim that if $\psi \in \mathrm{GL}(m, \mathbb{C})$ with $\det_{\mathbb{C}} \psi \in \mathbb{R}\backslash\{0\}$, then

$$Z_j(\psi K) = |\det_{\mathbb{C}} \psi|^{\frac{j}{m}}\, Z_j(K). \tag{19}$$

To prove (19), first we assume that $\det_{\mathbb{C}} \psi > 0$. In this case, we set $D = \det_{\mathbb{C}} \psi$, and hence $\varphi = D^{\frac{-1}{m}} \psi \in \mathrm{SL}(m, \mathbb{C})$ satisfies

$$Z_j(\psi K) = Z_j(\varphi D^{\frac{1}{m}} K) = Z_j(D^{\frac{1}{m}} K) = D^{\frac{j}{m}} Z_j(K),$$

proving (19) if $\det_{\mathbb{C}} \psi > 0$.

If $\det_{\mathbb{C}} \psi < 0$ in (19), then we consider $\tilde{\psi} \in \mathrm{GL}(m, \mathbb{C})$ defined by

$$\tilde{\psi}(v_m) = -\psi(v_m) \text{ and } \tilde{\psi}(v_l) = \psi(v_l) \text{ for } l = 1, \ldots, m-1.$$

It follows that $\det_{\mathbb{C}} \tilde{\psi} = |\det_{\mathbb{C}} \psi|$. Since the complex linear map $v_m \mapsto -v_m$ and $v_l \mapsto v_l$ for $l = 1, \ldots, m-1$ leaves K invariant, we have $\tilde{\psi} K = \psi K$. Thus we deduce

$$Z_j(\psi K) = Z_j(\tilde{\psi} K) = \left(\det_{\mathbb{C}} \tilde{\psi}\right)^{\frac{j}{m}} Z_j(K) = |\det_{\mathbb{C}} \psi|^{\frac{j}{m}}\, Z_j(K),$$

completing the proof of (19).

To finish the proof of Theorem 1.3, we distinguish two cases depending on whether $j > m$ or $j = m$.

Case $m < j < 2m$**:**

For every $\lambda > 0$, we define $\psi \in \mathrm{GL}(m, \mathbb{C})$ by $\psi v_m = \lambda v_m$ and $\psi v_l = v_l$ for $l = 1, \ldots, m-1$. In particular, (19) yields that

$$Z_j(\psi K) = \lambda^{\frac{j}{m}} Z_j(K) = \lambda^{\frac{j}{m}} \mathrm{Kl}_{Z_j}(L) V_j(K).$$

On the other hand, we observe that $\psi(iv_l) = iv_l$, $l = 1, \ldots, j-m$, and hence ψ maps L into L. The real determinant of the restriction of ψ to L is λ. Thus

$$Z_j(\psi K) = \mathrm{Kl}_{Z_j}(L) V_j(\psi K) = \lambda \mathrm{Kl}_{Z_j}(L) V_j(K).$$

We deduce that, for every $\lambda > 0$,

$$(\lambda^{\frac{j}{m}} - \lambda) \mathrm{Kl}_{Z_j}(L) V_j(K) = 0.$$

Hence, using $j > m$, we obtain $\mathrm{Kl}_{Z_j}(L) = 0$.

Case $j = m$:

For $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, let K_t be the m-dimensional crosspolytope with vertices $\pm[(\sin t)v_1 + (\cos t)iv_2], \pm v_2, \ldots, \pm v_m$. We consider the complex linear map ψ_t defined by $\psi_t(v_1) = (\sin t)v_1 + (\cos t)iv_2$ and $\psi_t(v_l) = v_l$ for $l = 2, \ldots, m$. Thus

$$\det_{\mathbb{C}}\psi_t = \sin t.$$

Since $\det_{\mathbb{R}}\psi_0 = 0$, we introduce the associated $\varphi_t \in \mathrm{GL}(2m, \mathbb{R})$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, defined by $\varphi_t(v_1) = (\sin t)v_1 + (\cos t)iv_2$, $\varphi_t(iv_2) = (-\cos t)v_1 + (\sin t)iv_2$, $\varphi_t(iv_1) = iv_1$, $\varphi_t(v_l) = v_l$ for $l \geq 2$ and $\varphi_t(iv_l) = iv_l$ for $l > 2$, which satisfies that $K_t = \psi_t K = \varphi_t K$.

We claim that

$$Z_m(\varphi_t K) = Z_m(\psi_t K) = |\sin t| \cdot \mathrm{Kl}_{Z_m}(L) \cdot V_m(K). \tag{20}$$

Formula (20) follows from (19) if $\sin t \neq 0$, and hence by the continuity of Z_m if $\sin t = 0$.

Now Z_m is smooth because it is invariant under $\mathrm{SU}(m)$ (see Proposition 3.6), and $\varphi_t \in \mathrm{GL}(2m, \mathbb{R})$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, is a C^∞ family of $2m \times 2m$ matrices, thus $Z_m(\varphi_t K)$ is a C^∞ function of t. Since $Z_m(\varphi_t K)$ is differentiable at $t = 0$, but the right-hand-side of (20) is differentiable only if it vanishes, we conclude $\mathrm{Kl}_{Z_m}(L) = 0$ by (20).

In turn, we deduce (18) for $j = m, \ldots, 2m - 1$. Since Lemma 5.1 verifies (18) for $j = 1, \ldots, m - 1$, the proof Theorem 1.3 is now complete. $\square$

6 Z is even

Let $r \geq 1$ and $m \geq 2$. For the whole section, fix an even, $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous valuation $Z : \mathcal{K}^{2m} \to \mathbb{T}^r(\mathbb{R}^{2m})$. By the McMullen decomposition (7), we have $Z = \sum_{j=0}^{2m} Z_j$ where Z_j is a j-homogeneous even $\mathrm{SL}(m, \mathbb{C})$ and translation invariant continuous valuation for $j = 0, \ldots, 2m$. We note that in this section, we do not use the inner product on $\mathbb{R}^{2m}$ at all.

Proposition 2.3 for even valuations will directly follow after proving that the even valuation Z_j is constant zero for each $0 \leq j \leq 2m$, which we prove in the following.

Recall that $\mathrm{Gr}_j(\mathbb{R}^n)$ denotes the family of all real linear j-dimensional subspaces L of $\mathbb{R}^{2m}$, $j = 0, \ldots, 2m$. For $j = 0, \ldots, 2m$ and $L \in \mathrm{Gr}_j(\mathbb{R}^n)$, we consider the Klain constant $\mathrm{Kl}_{Z_j}(L) \in \mathbb{T}^r(\mathbb{R}^{2m})$ such that

$$Z_j(K) = \mathrm{Kl}_{Z_j}(L) V_j(K) \text{ for every } K \in \mathcal{K}^{2m} \text{ with } K \subset L.$$

We recall that $V_j(K)$ is the j-dimensional volume of a compact convex $K \subset L$.

Since Z_j is even and continuous, Klain's injectivity theorem (16) applies, and Proposition 2.3 for even valuations follows if

$$\mathrm{Kl}_{Z_j}(L) = 0 \text{ for all } j = 0, \dots, 2m \text{ and } L \in \mathrm{Gr}_j(\mathbb{R}^{2m}). \tag{21}$$

More precisely, by Corollary 4.3, we can reduce the problem to study only real j-planes of maximal complex rank in (21). Hence, to prove Theorem 2.3 for even valuations, all we have to show is that if $j \in \{0, \dots, 2m\}$ and $Z_j : \mathcal{K}^{2m} \to \mathbb{T}^r(\mathbb{R}^{2m})$ is a j-homogeneous even $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous valuation, then

$$\mathrm{Kl}_{Z_j}(L) = 0 \text{ for all } L \in \mathrm{Gr}_j(\mathbb{R}^{2m}) \text{ of maximal complex rank.} \tag{22}$$

Hence, from now on, we always assume that

$$\text{the } L \in \mathrm{Gr}_j(\mathbb{R}^n) \text{ in (21) is of maximal complex rank if } j = 1, \dots, 2m.$$

According to Lemma 4.1, there exists a complex basis $v_1, \dots, v_m$ for $\mathbb{R}^{2m} = \mathbb{C}^m$ such that setting $v_{m+l} = iv_l$ for $l = 1, \dots, m$, we have that $v_1, \dots, v_m, v_{m+1}, \dots, v_{2m}$ form an $\mathbb{R}$-basis of $\mathbb{R}^{2m}$, and

$$v_1, \dots, v_j \text{ form a real basis of } L \in \mathrm{Gr}_j(\mathbb{R}^n).$$

We write I to denote the family of all $\theta : \{1, \dots, 2m\} \to \mathbb{N}$ such that

$$\sum_{l=1,\dots,2m} \theta(l) = r. \tag{23}$$

It follows that $\mathrm{Kl}_{Z_j}(L)$ can be written in the form

$$\mathrm{Kl}_{Z_j}(L) = \sum_{\theta \in I} c_\theta \odot_{l=1}^{2m} v_l^{\theta(l)} \tag{24}$$

where each $c_\theta := c_{\theta,Z,j,L} \in \mathbb{R}$ depends on θ, Z, j, L.

For $j = 0, \dots, 2m$ and $L \in \mathrm{Gr}_j(\mathbb{R}^n)$, let $\psi \in \mathrm{SL}(m, \mathbb{C})$ satisfy $\psi(L) = L$. Writing ψ_L to denote the ($\mathbb{R}$-linear) restriction of ψ to L, the core of our argument is the claim that

$$|\det_{\mathbb{R}} \psi_L| \cdot \mathrm{Kl}_{Z_j}(L) = \psi \cdot \mathrm{Kl}_{Z_j}(L) \tag{25}$$

where we also prove that $|\det_{\mathbb{R}}\psi_L| = 1$ if $j = m, 2m$, and set $|\det_{\mathbb{R}}\psi_L| = 1$ if $j = 0$. Indeed, choose any j-dimensional compact convex set $K \subset L$, and hence

$$\begin{aligned}\left(\psi \cdot \mathrm{Kl}_{Z_j}(L)\right) V_j(K) &= \psi \cdot Z_j(K) = Z_j(\psi K) = \mathrm{Kl}_{Z_j}(L) V_j(\psi K) \\ &= |\det_{\mathbb{R}}\psi_L| \cdot \mathrm{Kl}_{Z_j}(L)\, V_j(K),\end{aligned}$$

proving (25). Now if $j = 2m$, then $\det_{\mathbb{R}}\psi_L = \det_{\mathbb{R}}\psi = |\det_{\mathbb{C}}\psi|^2 = 1$. Finally, if $j = m$, then $\psi L = L$ yields that each entry of the matrix of $\psi \in \mathrm{SL}(m, \mathbb{C})$ with respect to the complex basis $v_1, \ldots, v_m$ of $\mathbb{C}^m$ is a real number, therefore $\det_{\mathbb{R}}\psi_L = \det_{\mathbb{C}}\psi = 1$.

We observe that if the map ψ in (25) is the diagonal transformation with $\psi(v_l) = \lambda_l v_l$ for $l = 1, \ldots, m$ where each $\lambda_l > 0$ and $\lambda_1 \cdot \ldots \cdot \lambda_m = 1$, then $\psi(v_{m+l}) = \lambda_l v_{m+l}$ for $l = 1, \ldots, m$ and $\psi(L) = L$. In this case, (25) is equivalent with the statement that for each $\theta \in I$, we have

$$\begin{aligned} c_\theta &= \left(\prod_{l=1}^m \lambda_l^{\theta(l)+\theta(m+l)}\right) \cdot c_\theta \quad \text{if } j = 0, m, 2m; \\ \left(\prod_{l=1}^j \lambda_l\right) \cdot c_\theta &= \left(\prod_{l=1}^m \lambda_l^{\theta(l)+\theta(m+l)}\right) \cdot c_\theta \quad \text{if } j = 1, \ldots, m-1; \\ \left(\prod_{l=1}^{j-m} \lambda_l\right) \cdot c_\theta &= \left(\prod_{l=1}^m \lambda_l^{\theta(l)+\theta(m+l)}\right) \cdot c_\theta \quad \text{if } j = m+1, \ldots, 2m-1. \end{aligned} \tag{26}$$

We also note that $\lambda_1 \cdot \ldots \cdot \lambda_m = 1$ yields

$$\prod_{l=1}^{m} \lambda_l^{\theta(l)+\theta(m+l)} = \prod_{l=1}^{m-1} \lambda_l^{\theta(l)+\theta(m+l)-\theta(m)-\theta(2m)}. \tag{27}$$

Combining (26) and (27), we deduce the following statements.

Corollary 6.1 *If $j = 1, \ldots, m-1$ and $c_\theta \neq 0$ in (24), then*

$$\begin{aligned}\theta(l) + \theta(m+l) &= \theta(m) + \theta(2m) + 1 \ \text{ for } l = 1, \ldots, j, \\ \theta(l) + \theta(m+l) &= \theta(m) + \theta(2m) \ \text{ for } l = j+1, \ldots, m.\end{aligned}$$

In particular, $r = m(\theta(m) + \theta(2m)) + j$.

Corollary 6.2 *If $j = m+1, \ldots, 2m-1$, $k = j - m$, and $c_\theta \neq 0$ in (24), then*

$$\begin{aligned}\theta(l) + \theta(m+l) &= \theta(m) + \theta(2m) + 1 \ \text{ for } l = 1, \ldots, k, \\ \theta(l) + \theta(m+l) &= \theta(m) + \theta(2m) \ \text{ for } l = k+1, \ldots, m.\end{aligned}$$

In particular, $r = m(\theta(m) + \theta(2m)) + k$.

In the following subsections, we prove that $Z_j \equiv 0$ for every $j = 0, \ldots, 2m$ by distinguishing the different behaviors of Z_j depending on j.

6.1 Case $m+1 \leq j \leq 2m-1$

Lemma 6.3 *Z_j is constant zero for $j = m+1, \ldots, 2m-1$.*

Proof: Let $k = j - m$, $k \in \{1, \ldots, m-1\}$. As in (24), we write

$$\mathrm{Kl}_{Z_j}(L) = \sum_{\theta \in I} c_\theta \odot_{l=1}^{2m} v_l^{\theta(l)}.$$

It follows from Corollary 6.2 that if $c_\theta \neq 0$ for $\theta \in I$, then

$$\theta(l) + \theta(m+l) = \begin{cases} \theta(m) + \theta(2m) + 1 & \text{for } l = 1, \ldots, k, \\ \theta(m) + \theta(2m) & \text{for } l = k+1, \ldots, m \end{cases},$$

and hence $\theta(1)+\theta(m+1)+\theta(m)+\theta(2m)$ is odd. We consider the $\psi \in \mathrm{SL}(m, \mathbb{C})$ defined by $\psi(v_1) = -v_1$, $\psi(v_m) = -v_m$ and $\psi(v_l) = v_l$ if $1 < l < m$, which therefore satisfies

$$\psi \cdot \mathrm{Kl}_{Z_j}(L) = (-1)^{\theta(1)+\theta(m+1)+\theta(m)+\theta(2m)} \mathrm{Kl}_{Z_j}(L) = -\mathrm{Kl}_{Z_j}(L).$$

This together with (25) implies $\mathrm{Kl}_{Z_j}(L) = 0$, and in turn we conlcude Lemma 6.3 by Corollary 4.3 (i) for $j = m+1, \ldots, 2m-1$. □

6.2 Case $1 \leq j \leq m-1$

Lemma 6.4 *Z_j is constant zero for $j = 1, \ldots, m-1$.*

Proof: As in the proof of Lemma 6.3, we define $\psi \in \mathrm{SL}(m, \mathbb{R})$ given by $\psi(v_1) = -v_1$, $\psi(v_m) = -v_m$ and $\psi(v_l) = v_l$ for $2 \leq l \leq m-1$. Applying (25) to this ψ and using the relations for θ given in Corollary 6.1, we obtain that $\mathrm{Kl}_{Z_j}(L) = -\mathrm{Kl}_{Z_j}(L)$, therefore $\mathrm{Kl}_{Z_j}(L) = 0$. Using Corollary 4.3 (i), the statement of the lemma follows. □

6.3 Case $j = m$

In order to show that Z_0, Z_m and Z_{2m} are constant zero, we shall make use of the First Fundamental Theorem of classical invariant theory on $\mathrm{SL}(m, \mathbb{R})$-invariants of several vectors. We describe it in the following.

For $n \geq 2$, let $\mathbb{V}$ be an n-dimensional $\mathbb{R}$ vector space, and let $\mathbb{T}(\mathbb{V})$ be the direct sum of all $\mathbb{T}^r(\mathbb{V})$, $r \geq 0$. Hence $\mathbb{T}(\mathbb{V})$ is an $\mathbb{R}$-algebra where the "product" is the symmetric tensor product. We observe that $\mathbb{T}(\mathbb{V})$ can be naturally identified with the $\mathbb{R}$-algebra of polynomial functions on $\mathbb{V}^*$ where $\mathbb{T}^r(\mathbb{V})$ corresponds to the homogeneous polynomials of degree r, and the identification respects the $\mathrm{GL}(\mathbb{V},\mathbb{R})$-action.

For $m \geq 2$, we consider the diagonal action of $\mathrm{SL}(m,\mathbb{R})$ on the direct sum $\mathbb{V} = \mathbb{R}^m \oplus \mathbb{R}^m$. As the $\mathbb{R}$-algebras of symmetric tensors and polynomials can be identified, we have the following consequence of the First Fundamental Theorem on vector invariants of $\mathrm{SL}(m,\mathbb{R})$ (see, e.g., Dolgachev [23, Chapter 2], Kraft, Procesi [34, Section 8.4] or Procesi [47, Chapter 11.1.2] for the general statement).

Theorem 6.5 (First Fundamental Theorem) *Let $m \geq 2$, $r \geq 1$ and $\mathbb{V} = \mathbb{R}^m \oplus \mathbb{R}^m$, and let $\Theta \in \mathbb{T}^r(\mathbb{V})$ be invariant under the natural action of $\mathrm{SL}(m,\mathbb{R})$.*

(a) If $m \geq 3$ or r is odd, then $\Theta = 0$;

(b) if $m = 2$ and r is even, then

$$\Theta = c(v_1 \odot w_2 - v_2 \odot w_1)^{r/2}$$

where $c \in \mathbb{R}$ and v_1, v_2 form a basis of the first copy of $\mathbb{R}^2$, and w_1, w_2 is the corresponding basis of the second copy of $\mathbb{R}^2$.

Lemma 6.6 *Z_m is constant zero.*

Proof: According to Corollary 4.3, it is sufficient to prove that if $v_1, \ldots, v_m$ is a complex basis of $\mathbb{C}^m$, and $L = \mathrm{lin}_{\mathbb{R}}\{v_1, \ldots, v_m\}$, then $\mathrm{Kl}_{Z_m}(L) = 0$. We observe that $\mathbb{C}^m = L \oplus_{\mathbb{R}} iL$ where $iv_1, \ldots, iv_m$ is the corresponding real basis of iL.

It follows from (25) that

$$\mathrm{Kl}_{Z_m}(L) = \psi \cdot \mathrm{Kl}_{Z_m}(L)$$

for any $\psi \in \mathrm{SL}(m,\mathbb{R})$. We deduce from the First Fundamental Theorem 6.5 that $\mathrm{Kl}_{Z_m}(L) = 0$ if $m \geq 3$ or r is odd.

Therefore, we assume in the following that $m = 2$ and r is even. According to the First Fundamental Theorem 6.5, there exists $c \in \mathbb{R}$ such that writing $w_1 = iv_1$ and $w_2 = iv_2$, we have

$$\mathrm{Kl}_{Z_2}(L) = c(v_1 \odot w_2 - v_2 \odot w_1)^{r/2}. \tag{28}$$

We suppose that $c \neq 0$ in (28), and seek a contradiction. Let $K \subset L$ be the 2-simplex with vertices o, v_1, v_2. For $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we define K_t to be the 2-dimensional simplex with vertices $o, v_2, (\sin t)v_1 + (\cos t)w_2$.

Claim 1: *If* $c \neq 0$*, then* $r = 2$*,* $Z(K_0) \neq 0$*, and for any* $t \in [0, \frac{\pi}{2})$*, we have*

$$Z_2(K_t) = (\sin t)\mathrm{Kl}_{Z_2}(L)V_2(K) + (\cos t)Z_2(K_0). \tag{29}$$

For $t \in [0, \frac{\pi}{2})$, we consider the complex linear map φ_t defined by $\varphi_t(v_1) = (\sin t)v_1 + (\cos t)(iv_2) = (\sin t)v_1 + (\cos t)w_2$ and $\varphi_t(v_2) = v_2$, thus

$$K_t = \varphi_t K \text{ and } \det_{\mathbb{C}}\varphi_t = \sin t.$$

If $t \in (0, \frac{\pi}{2})$, then $\psi_t = (\sin t)^{\frac{-1}{2}}\varphi_t \in \mathrm{SL}(2,\mathbb{C})$ satisfies

$$\begin{aligned} \psi_t(w_1) &= \psi_t(iv_1) = (\sin t)^{\frac{-1}{2}}\Big((\sin t)w_1 - (\cos t)v_2\Big) \\ \psi_t(w_2) &= \psi_t(iv_2) = (\sin t)^{\frac{-1}{2}}\, w_2. \end{aligned}$$

Since Z_2 is 2-homogeneous, we deduce that if $t \in (0, \frac{\pi}{2})$, then

$$\begin{aligned} Z_2(\varphi_t K) &= Z_2((\sin t)^{\frac{1}{2}}\psi_t K) = (\sin t)Z_2(\psi_t K) \\ &= (\sin t)\psi_t \cdot Z_2(K) = (\sin t)V_2(K)\psi_t \cdot \mathrm{Kl}_{Z_2}(L). \end{aligned}$$

For $t \in (0, \frac{\pi}{2})$, we have

$$\begin{aligned} \psi_t \cdot \mathrm{Kl}_{Z_2}(L) &= c(\sin t)^{\frac{-r}{2}}\left((\sin t\, v_1 + \cos t\, w_2) \odot w_2 - v_2 \odot (\sin t\, w_1 - \cos t\, v_2)\right)^{\frac{r}{2}} \\ &= c(\sin t)^{\frac{-r}{2}}\left((\sin t(v_1 \odot w_2 - v_2 \odot w_1) + \cos t(w_2 \odot w_2 + v_2 \odot v_2)\right)^{\frac{r}{2}}, \end{aligned}$$

implying the formula

$$\begin{aligned} Z_2(\varphi_t K) &= cV_2(K)(\sin t)^{\frac{2-r}{2}} \\ &\quad \left((\sin t(v_1 \odot w_2 - v_2 \odot w_1) + \cos t(w_2 \odot w_2 + v_2 \odot v_2)\right)^{\frac{r}{2}} \end{aligned} \tag{30}$$

Since $Z_2(K_t) = Z_2(\varphi_t K)$ is a continuous function of $t \in [0, \frac{\pi}{2})$, it follows that

$$\lim_{t \to 0^+} Z_2(\varphi_t K) = Z_2(\varphi_0 K) = Z_2(K_0). \tag{31}$$

Combining $r \geq 2$, (30), $cV_2(K) \neq 0$ and

$$\lim_{t \to 0}\left(\sin t(v_1 \odot w_2 - v_2 \odot w_1) + \cos t(w_2 \odot w_2 + v_2 \odot v_2)\right)^{\frac{r}{2}} = (w_2 \odot w_2 + v_2 \odot v_2)^{\frac{r}{2}} \neq 0,$$

we conclude that the limit in (31) exists only if $r = 2$. Therefore $r = 2$, and deduce from (30) and (31) that

$$Z_2(K_0) = Z_2(\varphi_0 K) = cV_2(K)(w_2 \odot w_2 + v_2 \odot v_2) \neq 0. \tag{32}$$

We conclude (29) if $t \in (0, \frac{\pi}{2})$ from (28), (30) and (32), and in turn if $t = 0$ by continuity.

Claim 2: *If $c \neq 0$, then for any $t \in (-\frac{\pi}{2}, 0)$, we have*

$$Z_2(\varphi_t K) = |\sin t| \mathrm{Kl}_{Z_2}(L) V_2(K) - (\cos t) Z_2(K_0). \tag{33}$$

In this case, we have $\sin t < 0$. The argument is similar as above only we modify the definition of φ_t in order to have positive determinant and make use of the fact that we already know that $r = 2$. For $t \in (-\frac{\pi}{2}, 0)$, now the complex linear map φ_t is defined by $\varphi_t(v_1) = v_2$, $\varphi_t(v_2) = (\sin t)v_1 + (\cos t)iv_2$. It follows that again $\varphi_t K = K_t = \varphi_t K$ and

$$\det{}_{\mathbb{C}} \varphi_t = |\sin t|.$$

Now $\psi_t = |\sin t|^{\frac{-1}{2}} \varphi_t \in \mathrm{SL}(2, \mathbb{C})$ satisfies

$$\begin{aligned} \psi_t(w_1) &= \psi_t(iv_1) = |\sin t|^{\frac{-1}{2}} w_2 \\ \psi_t(w_2) &= \psi_t(iv_2) = |\sin t|^{\frac{-1}{2}} \Big((\sin t) w_1 - (\cos t) v_2 \Big). \end{aligned}$$

Since Z_2 is 2-homogeneous, we deduce

$$Z_2(\varphi_t K) = |\sin t| Z_2(\psi_t K) = |\sin t| V_2(K) \psi_t \cdot \mathrm{Kl}_{Z_2}(L).$$

As we already know that $r = 2$ by Claim 1, in this case we have

$$\begin{aligned} \psi_t \cdot \mathrm{Kl}_{Z_2}(L) &= c|\sin t|^{-1} \left(v_2 \odot (\sin t\, w_1 - \cos t\, v_2) - (\sin t\, v_1 + \cos t\, w_2) \odot w_2 \right) \\ &= c|\sin t|^{-1} \left(-\sin t (v_1 \odot w_2 - v_2 \odot w_1) - \cos t (w_2 \odot w_2 + v_2 \odot v_2) \right), \end{aligned}$$

implying the formula

$$Z_2(\varphi_t K) = c V_2(K) \left(|\sin t| (v_1 \odot w_2 - v_2 \odot w_1) - \cos t (w_2 \odot w_2 + v_2 \odot v_2) \right).$$

In turn, we conclude (33) and Claim 2 if $t \in (-\frac{\pi}{2}, 0)$ by (32).

It follows from Claim 1, the continuity of Z_2 and Claim 2 that

$$Z(K_0) = \lim_{t \to 0^+} Z_2(\varphi_t K) = \lim_{t \to 0^-} Z_2(\varphi_t K) = -Z(K_0),$$

and hence $Z(K_0) = 0$. This contradicts Claim 1, therefore proves $\mathrm{Kl}_{Z_2}(L) = 0$ in (28) for the case $m = 2$ and r is even, concluding the proof of Lemma 6.6. □

6.4 Case $j \in \{0, 2m\}$

Lemma 6.7 *Z_0 and Z_{2m} are constant zero.*

Proof: Let $j \in \{0, 2m\}$. According to (25), there exists a $\Theta \in \mathbb{T}^r(\mathbb{R}^{2m})$ such that $Z_j(K) = \Theta V_j(K)$ for any $K \in \mathcal{K}(\mathbb{R}^{2m})$ and

$$\Theta = \psi \cdot \Theta \tag{34}$$

for any $\psi \in \mathrm{SL}(m, \mathbb{C})$. In particular, we have that $\Theta \in \mathbb{T}^r(\mathbb{R}^m \oplus \mathbb{R}^m)$ is invariant under the natural action of $\mathrm{SL}(m, \mathbb{R})$. We deduce from the First Fundamental Theorem 6.5 that $\Theta = 0$ if $m \geq 3$ or r is odd.

Therefore, we assume in the following that $m = 2$ and r is even. In this case, we choose a complex basis v_1, v_2 of $\mathbb{C}^2$, and define $w_l = iv_l$ for $l = 1, 2$. It follows from the First Fundamental Theorem 6.5 that

$$\Theta = c(v_1 \odot w_2 - v_2 \odot w_1)^{r/2} \tag{35}$$

for a $c \in \mathbb{R}$. Since, by (34), Θ is not only invariant under $\mathrm{SL}(2, \mathbb{R})$ but also under $\mathrm{SL}(2, \mathbb{C})$, we consider $\psi \in \mathrm{SL}(2, \mathbb{C})$ given by $\psi(v_1) = v_1$ and $\psi(v_2) = iv_1 + v_2 = w_1 + v_2$, and hence $\psi(w_1) = w_1$ and $\psi(w_2) = -v_1 + w_2$. A computation shows that

$$\psi \cdot \Theta = c(v_1 \odot w_2 - v_2 \odot w_1 - v_1 \odot v_1 - w_1 \odot w_1)^{r/2}. \tag{36}$$

If $c \neq 0$, then any term in Θ (see (35)) contains equal number of indices 1 and 2, while $\psi \cdot \Theta$ contains the term $(v_1 \odot v_1)^{r/2}$ with non-zero coefficient (compare (36)), contradicting the invariance of Θ (see (34)). Thus $c = 0$, concluding the proof of Lemma 6.7 $\square$

7 Z is odd

Let $m \geq 2$, $r \geq 0$, and let $Z : \mathcal{K}^{2m} \to \mathbb{T}^r(\mathbb{R}^{2m})$ be an odd $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous valuation, which we fix through the section. Similarly to Section 6, McMullen's decomposition theorem yields that $Z = \sum_{j=0}^{2m} Z_j$ where each Z_j is an odd $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous j-homogeneous valuation. We prove in the following that $Z_j \equiv 0$ for every $0 \leq j \leq 2m$.

Let $j \in \{0, 2m\}$. According to Hadwiger's theorem (8), there exists a constant $c_j \in \mathbb{T}^r(\mathbb{R}^{2m})$ such that $Z_j(K) = c_j V_j(K)$ for any compact convex set K in $\mathbb{R}^{2m}$. Since Z_j is odd and V_j is even, we have

$$Z_j \equiv 0 \ \text{ if } j = 0, 2m. \tag{37}$$

Therefore we may assume that $j \in \{1, \dots, 2m-1\}$. By Corollary 4.3 (ii), it is sufficient to prove the following.

Lemma 7.1 *If $j = 1, \dots, 2m-1$, $m \geq 2$, $Z_j : \mathcal{K}^{2m} \to \mathbb{T}^r(\mathbb{R}^{2m})$ is an odd $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous j-homogeneous valuation, $L \in \mathrm{Gr}_{j+1}(\mathbb{R}^{2m})$ is of maximal complex rank, and the continuous 1-homogeneous function $f : L \to \mathbb{T}^r(\mathbb{R}^{2m})$ satisfies (cf. (9))*

$$Z_j(K) = \int_{S^{2m-1} \cap L} f \, dS_{K,L} \text{ for } K \in \mathcal{K}(L), \tag{38}$$

where $S_{K,L}$ denotes the surface area measure of K with respect to L, then

$$f(x+y) = f(x) + f(y) \text{ for any } x, y \in L. \tag{39}$$

In order to prove that the function f in Lemma 7.1 is linear, we distinguish three different cases depending on whether $j \leq m-1$, $m \leq j \leq 2m-2$ or $j = 2m-1$, but the idea described next is followed in all cases.

It follows from Proposition 3.6 and $\mathrm{SU}(m) \subset \mathrm{SL}(m, \mathbb{C})$ that Z_j is smooth, and hence applying Proposition 3.9 to the restriction of Z_j to $\mathcal{K}(L)$ yields that

$$\text{the function } f \text{ in (38) is } C^\infty \text{ on } L \backslash \{o\}. \tag{40}$$

As $L \in \mathrm{Gr}_{j+1}(\mathbb{R}^{2m})$ has maximal complex rank, Lemma 4.1 yields the existence of an orthonormal complex basis $v_1, \dots, v_m$ for $\mathbb{R}^{2m} = \mathbb{C}^m$ such that setting $v_{m+l} = iv_l$ for $l = 1, \dots, m$, we have that $v_1, \dots, v_m, v_{m+1}, \dots, v_{2m}$ form an $\mathbb{R}$-basis of $\mathbb{R}^{2m}$, and

$$v_1, \dots, v_j \text{ form an orthonormal real basis of } L \in \mathrm{Gr}_j(\mathbb{R}^n). \tag{41}$$

We note that the following ideas apply to any complex basis $v_1, \dots, v_m$ for $\mathbb{R}^{2m} = \mathbb{C}^m$ satisfying (41) where $v_{m+l} = iv_l$ for $l = 1, \dots, m$.

Similarly to the case of even valuations, we write I to denote the family of all $\theta : \{1, \dots, 2m\} \to \mathbb{N}$ such that

$$\sum_{l=1,\dots,2m} \theta(l) = r. \tag{42}$$

It follows that if $r \geq 1$, then for any $x \in L$, $f(x)$ can be written in the form

$$f(x) = \sum_{\theta \in I} f_\theta(x) \odot_{l=1}^{2m} v_l^{\theta(l)}$$

where $f_\theta(x) \in \mathbb{T}^r(\mathbb{R}^{2m})$ for $x \in L$ and $\theta \in I$, and

$$\text{each } f_\theta,\ \theta \in I, \text{ is } C^\infty \text{ on } L \backslash \{o\} \tag{43}$$

according to (40). If $r = 0$, then the only element θ of I is the constant zero map, and we set $f_\theta = f$.

In order to prove (39), we use the $\mathrm{SL}(m, \mathbb{C})$-equivariance of Z_j as follows. Let $\varphi \in \mathrm{SL}(m, \mathbb{C})$ satisfy that $\varphi(L) = L$, and let $\Delta = |\det_{\mathbb{R}}(\varphi|_L)|$. It follows that $\varphi|_L = \Delta^{\frac{1}{j+1}} \tilde{\varphi}$ where $\tilde{\varphi} \in \mathrm{GL}(L, \mathbb{R})$ satisfies $\det_{\mathbb{R}} \tilde{\varphi} = \pm 1$. Since $\varphi \cdot Z_j(K) = Z_j(\varphi K)$ for any $K \in \mathcal{K}(L)$, we deduce from (11) and (38) that

$$\begin{aligned}
\int_{S^{2m-1}\cap L} \varphi \cdot f \, dS_{K,L} &= \int_{S^{2m-1}\cap L} f \, dS_{\varphi K,L} = \int_{S^{2m-1}\cap L} f \, dS_{\Delta^{\frac{1}{j+1}}\tilde{\varphi}K,L} \\
&= \int_{S^{2m-1}\cap L} \Delta^{\frac{j}{j+1}} f \, dS_{\tilde{\varphi}K,L} \\
&= \int_{S^{2m-1}\cap L} \Delta^{\frac{j}{j+1}} f \circ \tilde{\varphi}^{-t} \, dS_{K,L}
\end{aligned}$$

where $\varphi \cdot f = f$ if $r = 0$. We conclude that

$$\Phi := \varphi \cdot f - |\det_{\mathbb{R}}(\varphi|_L)|^{\frac{j}{j+1}} f \circ \tilde{\varphi}^{-t} \tag{44}$$

is linear by (10) where $\varphi \cdot f = f$ holds if $r = 0$. In particular, setting $k = j+1-m$ provided $j \geq m$, if $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m \in \mathbb{R}$, then

$$\begin{aligned}
\Phi\left(\sum_{q=1}^{j+1} \alpha_q v_q\right) - \sum_{q=1}^{j+1} \Phi(\alpha_q v_q) &= 0 \quad \text{if } j \leq m-1, \\
\Phi\left(\sum_{q=1}^{m} \alpha_q v_q + \sum_{q=1}^{k} \beta_q i v_q\right) - \sum_{q=1}^{m} \Phi(\alpha_q v_q) - \sum_{q=1}^{k} \Phi(\beta_q i v_q) &= 0 \quad \text{if } j \geq m.
\end{aligned} \tag{45}$$

The fact that (45) holds for some suitable family of possible $\varphi \in \mathrm{SL}(m, \mathbb{C})$ will lead to (39).

We note that the argument is somewhat simpler if $r = 0$. Since in this case, the only $\theta \in I$ is constant zero, we have $\delta_{p,l} = 1$ in Lemma 7.3, $\delta = 0$ in Lemmas 7.4, 7.6, 7.7, and there is no need for Lemma 7.5.

7.1 Case $m \leq j \leq 2m-2$

The whole section is devoted to prove the following statement.

Lemma 7.2 *If $j = m, \ldots, 2m-2$, then Z_j is constant zero.*

We prove Lemma 7.2 by a series of lemmas where we use the notation above set up around Lemma 7.1. In particular, we fix a complex basis $v_1, \ldots, v_m$ of $\mathbb{C}^m$

such that $v_1, \ldots, v_k, v_{k+1}, \ldots, v_m, iv_1, \ldots, iv_k$ is a real orthonormal basis of its $\mathbb{R}$-linear span L, where L is a $(j+1)$-dimensional real subspace with $j+1 = k+m$. For $q = 1, \ldots, m$, we set $v_{q+m} = iv_q$.

For $\lambda > 0$, $p \in \{1, \ldots, k\}$ and $l \in \{k+1, \ldots, m\}$, we frequently consider the map $\varphi_{p,l} \in \mathrm{SL}(m, \mathbb{C})$ defined by

$$\begin{aligned}
\varphi_{p,l}(v_p) &= \lambda v_p, \\
\varphi_{p,l}(v_l) &= \lambda^{-1} v_l, \\
\varphi_{p,l}(v_q) &= v_q \text{ if } q = 1, \ldots, m \text{ and } q \neq p, l
\end{aligned}$$

(we do not signal the dependence of $\varphi_{p,l}$ on λ). In this case, $\varphi_{p,l}|_L$ is an $\mathbb{R}$-linear map of the $(k+m)$-dimensional subspace L into L whose determinant is λ. It follows that $\varphi_{p,l}|_L = \lambda^{\frac{1}{m+k}} \tilde{\varphi}_{p,l}$ where $\tilde{\varphi}_{p,l} \in \mathrm{SL}(L, \mathbb{R})$ satisfies

$$\begin{aligned}
\tilde{\varphi}_{p,l}(v_p) &= \lambda^{\frac{m+k-1}{m+k}} v_p \text{ and } \tilde{\varphi}_{p,l}(v_{p+m}) = \lambda^{\frac{m+k-1}{m+k}} v_{p+m}, \\
\tilde{\varphi}_{p,l}(v_l) &= \lambda^{-\frac{m+k+1}{m+k}} v_l, \\
\tilde{\varphi}_{p,l}(v_q) &= \lambda^{-\frac{1}{m+k}} v_q \text{ if } q = 1, \ldots, m+k \text{ and } q \neq p, l, p+m,
\end{aligned}$$

therefore $j = m+k-1$ implies

$$\begin{aligned}
\tilde{\varphi}_{p,l}^{-t}(v_p) &= \lambda^{\frac{-j}{m+k}} v_p \text{ and } \tilde{\varphi}_{p,l}^{-t}(v_{p+m}) = \lambda^{\frac{-j}{m+k}} v_{p+m}, \\
\tilde{\varphi}_{p,l}^{-t}(v_l) &= \lambda^{-\frac{j+2}{m+k}} v_l, \\
\tilde{\varphi}_{p,l}^{-t}(v_q) &= \lambda^{\frac{1}{m+k}} v_q \text{ if } q = 1, \ldots, m+k \text{ and } q \neq p, l, p+m.
\end{aligned}$$

Lemma 7.3 *If $w_q \in \mathbb{C}v_q \cap L$ for $q = 1, \ldots, m$, then for any $\theta \in I$, we have*

$$f_\theta(w_1 + \ldots + w_k) = f_\theta(w_1) + \ldots + f_\theta(w_k), \tag{46}$$

$$f_\theta(w_{k+1} + \ldots + w_m) = f_\theta(w_{k+1}) + \ldots + f_\theta(w_m). \tag{47}$$

Remark We observe that $w_q = \alpha_q v_q + \beta_q i v_q$ for $\alpha_q, \beta_q \in \mathbb{R}$ if $q = 1, \ldots, k$, and $w_q = \alpha_q v_q$ for $\alpha_q \in \mathbb{R}$ if $q = k+1, \ldots, m$.

Proof: To verify (46) and (47), it is sufficient to prove by induction for $p = 1, \ldots, k$ and $l = k+1, \ldots, m$ that if $w_q \in \mathbb{C}v_q \cap (L \backslash \{o\})$ for $q = 1, \ldots, m$, then

$$f_\theta(w_1 + \ldots + w_p) = f_\theta(w_1) + \ldots + f_\theta(w_p), \tag{48}$$

$$f_\theta(w_{k+1} + \ldots + w_l) = f_\theta(w_{k+1}) + \ldots + f_\theta(w_l) \tag{49}$$

by the continuity of f_θ.

For (48), the case $p = 1$ of the induction argument trivially holds, therefore we assume that $p > 1$ and that (48) holds for $p-1$. We deduce from (45) that for every $\theta \in I$, we have

$$\begin{aligned} \lambda^{\theta(p)+\theta(p+m)-\theta(l)-\theta(l+m)} \left(f_\theta(w_1+\ldots+w_p) - f_\theta(w_1) - \ldots - f_\theta(w_p)\right) - \quad (50) \\ -\lambda^{\frac{j}{m+k}} \left[f_\theta\left(\lambda^{\frac{-j}{m+k}} w_p + \sum_{q=1}^{p-1} \lambda^{\frac{1}{m+k}} w_q\right) - f_\theta\left(\lambda^{\frac{-j}{m+k}} w_p\right) - \sum_{q=1}^{p-1} f_\theta\left(\lambda^{\frac{1}{m+k}} w_q\right)\right] = 0. \end{aligned}$$

After multiplying (50) by $\lambda^{\theta(l)+\theta(l+m)-\theta(p)-\theta(p+m)}$, using the constant

$$\delta_{p,l} = \theta(l) + \theta(l+m) - \theta(p) - \theta(p+m) + 1,$$

and using that $f(\alpha v) = \alpha f(v)$ for any $\alpha \in \mathbb{R}$ and $v \in L$, we deduce that (45) is equivalent with

$$\begin{aligned} f_\theta(w_1+\ldots+w_p) - f_\theta(w_1) - \ldots - f_\theta(w_p) - \quad (51) \\ -\lambda^{\delta_{p,l}} \left[f_\theta\left(\lambda^{-1} w_p + \sum_{q=1}^{p-1} w_q\right) - f_\theta(\lambda^{-1} w_p)\right] + \lambda^{\delta_{p,l}} \sum_{q=1}^{p-1} f_\theta(w_q) = 0. \end{aligned}$$

If $\delta_{p,l} < 0$, then letting λ tend to infinity in (51), we deduce (48).

If $\delta_{p,l} > 0$, then f_θ is differentiable at w_p as $w_p \neq 0$ (compare (40)). In particular, if λ is small, then

$$\begin{aligned} f_\theta\left(\lambda^{-1} w_p + \sum_{q=1}^{p-1} w_q\right) - f_\theta(\lambda^{-1} w_p) &= \lambda^{-1}\left(f_\theta\left(w_p + \lambda \sum_{q=1}^{p-1} w_q\right) - f_\theta(w_p)\right) \\ &= \lambda^{-1} O\left(\lambda \sum_{q=1}^{p-1} w_q\right) = O(1), \end{aligned}$$

therefore letting λ tend to zero in (51) implies (48).

Finally, if $\delta_{p,l} = 0$, then (51) reads

$$f_\theta(w_1+\ldots+w_p) = (1-\lambda^{-1}) f_\theta(w_p) + f_\theta\left(\lambda^{-1} w_p + \sum_{q=1}^{p-1} w_q\right).$$

Here letting λ tend to infinity and applying the induction hypothesis, we complete the proof of (48), and in turn (46).

For (49), the case $l = k + 1$ of induction argument trivially holds, therefore we assume that $l > k + 1$ and (49) holds for $l - 1$. We deduce from (45) that for every $\theta \in I$, we have

$$\lambda^{\theta(p)+\theta(p+m)-\theta(l)-\theta(l+m)} \left(f_\theta(w_{k+1} + \ldots + w_m) - f_\theta(w_{k+1}) - \ldots - f_\theta(w_m)\right) \quad (52)$$
$$-\lambda^{\frac{j}{m+k}} \left[f_\theta \left(\lambda^{\frac{j+2}{m+k}} w_l + \lambda^{\frac{1}{m+k}} \sum_{q=k+1}^{l-1} w_q \right) - f_\theta \left(\lambda^{\frac{j+2}{m+k}} w_l \right) - \lambda^{\frac{1}{m+k}} \sum_{q=k+1}^{l-1} f_\theta(w_q) \right] = 0.$$

After multiplying (52) by $\lambda^{\theta(l)+\theta(l+m)-\theta(p)-\theta(p+m)}$, and using that $f(\alpha v) = \alpha f(v)$ for any $\alpha \in \mathbb{R}$ and $v \in L$, we deduce that (45) is equivalent with

$$f_\theta(w_{k+1} + \ldots + w_l) - f_\theta(w_{k+1}) - \ldots - f_\theta(w_l) - \quad (53)$$
$$-\lambda^{\delta_{p,l}} \left[f_\theta \left(\lambda w_l + \sum_{q=k+1}^{l-1} w_q \right) - f_\theta(\lambda w_l) \right] + \lambda^{\delta_{p,l}} \sum_{q=k+1}^{l-1} f_\theta(w_q) = 0.$$

If $\delta_{p,l} > 0$, then letting λ tend to zero in (53), we deduce (49).

If $\delta_{p,l} < 0$, then f_θ is differentiable at w_l as $w_l \neq 0$. In particular, if λ is large, then

$$f_\theta \left(\lambda w_l + \sum_{q=k+1}^{l-1} w_q \right) - f_\theta(\lambda w_l) = \lambda \left(f_\theta \left(w_l + \frac{\sum_{q=k+1}^{l-1} w_q}{\lambda} \right) - f_\theta(w_l) \right)$$
$$= \lambda \, O \left(\frac{\sum_{q=k+1}^{l-1} w_q}{\lambda} \right) = O(1),$$

therefore letting λ tend to infinity in (53) implies (49).

Finally, if $\delta_{p,l} = 0$, then (53) reads

$$f_\theta(w_{k+1} + \ldots + w_l) = (1 - \lambda) f_\theta(w_l) + f_\theta \left(\lambda w_l + \sum_{q=k+1}^{l-1} w_q \right).$$

Here letting λ tend to zero and applying the induction hypothesis complete the proof of (49), and in turn (47). □

Lemma 7.4 *If $v \in \mathrm{lin}_{\mathbb{R}}\{v_1, iv_1, \ldots, v_k, iv_k\}$ and $w \in \mathrm{lin}_{\mathbb{R}}\{v_{k+1}, \ldots, v_m\}$, then $f_\theta(v + w) = f_\theta(v) + f_\theta(w)$.*

Proof: We may assume that $v, w \neq 0$. As $L \cap iL = \mathrm{lin}_{\mathbb{R}}\{v_1, iv_1, \ldots, v_k, iv_k\}$ is a complex subspace of $\mathbb{C}^m$ of complex dimension k, we may choose a complex basis $\tilde{v}_1, \ldots, \tilde{v}_k$ of $L \cap iL$ such that $v = \alpha_1 \tilde{v}_1$ for $\alpha_1 \in \mathbb{R}\backslash\{0\}$ and $\{\tilde{v}_1, i\tilde{v}_1, \ldots, \tilde{v}_k, i\tilde{v}_k\}$ is a real orthonormal basis. Similarly, we may choose a real orthonormal basis $\tilde{v}_{k+1}, \ldots, \tilde{v}_m$ of $\mathrm{lin}_{\mathbb{R}}\{v_{k+1}, \ldots, v_m\}$ such that $w = \alpha_m \tilde{v}_m$ for $\alpha_m \in \mathbb{R}\backslash\{0\}$. In particular, $\tilde{v}_1, \ldots, \tilde{v}_m$ is a complex basis of $\mathbb{C}^m$ such that $\tilde{v}_1, \ldots, \tilde{v}_m, i\tilde{v}_1, \ldots, i\tilde{v}_k$ form a real orthonormal basis of L.

For $\lambda > 0$, we consider $\varphi \in \mathrm{SL}(m, \mathbb{C})$ defined by $\varphi(\tilde{v}_1) = \lambda \tilde{v}_1$, $\varphi(\tilde{v}_m) = \lambda^{-1}\tilde{v}_m$ and $\varphi(\tilde{v}_q) = \tilde{v}_q$ if $q \neq 1, m$, and hence the $\mathbb{R}$-linear map $\varphi_{1,m}|_L$ is of determinant λ. Again, we do not signal the dependence of φ on λ. We have $\varphi|_L = \lambda^{\frac{1}{m+k}} \tilde{\varphi}$ where $\tilde{\varphi} \in \mathrm{SL}(L, \mathbb{R})$ satisfies

$$\begin{aligned}
\tilde{\varphi}^{-t}(\tilde{v}_1) &= \lambda^{\frac{-j}{m+k}} \tilde{v}_1 \text{ and } \tilde{\varphi}^{-t}(\tilde{v}_{m+1}) = \lambda^{\frac{-j}{m+k}} \tilde{v}_{m+1}, \\
\tilde{\varphi}^{-t}(\tilde{v}_m) &= \lambda^{\frac{j+2}{m+k}} \tilde{v}_m \text{ and } \tilde{\varphi}^{-t}(\tilde{v}_{2m}) = \lambda^{-\frac{j}{m+k}} \tilde{v}_{2m}, \\
\tilde{\varphi}^{-t}(\tilde{v}_q) &= \lambda^{\frac{1}{m+k}} \tilde{v}_q \text{ if } q \neq 1, m, m+1, 2m,
\end{aligned}$$

and hence $\tilde{\varphi}^{-t}(v) = \lambda^{\frac{-j}{m+k}} v$ and $\tilde{\varphi}^{-t}(w) = \lambda^{\frac{j+2}{m+k}} w$.

Based on the basis $\tilde{v}_1, i\tilde{v}_1, \ldots, \tilde{v}_m, i\tilde{v}_m$ of $\mathbb{R}^{2m}$, f can be written in the form

$$f(x) = \sum_{\theta \in I} \tilde{f}_\theta(x) \odot_{q=1}^{2m} \tilde{v}_q^{\theta(q)}$$

for $x \in L$ and $\tilde{f}_\theta(x) \in \mathbb{R}$. It is sufficient to prove that $\tilde{f}_\theta(v + w) = \tilde{f}_\theta(v) + \tilde{f}_\theta(w)$ for $\theta \in I$.

We deduce from the analogous of (45) that for every $\theta \in I$, we have

$$\begin{aligned}
\lambda^{\theta(1)+\theta(1+m)-\theta(m)-\theta(2m)} \left(\tilde{f}_\theta(v+w) - \tilde{f}_\theta(v) - \tilde{f}_\theta(w)\right) - \qquad (54) \\
-\lambda^{\frac{j}{m+k}} \left[\tilde{f}_\theta\left(\lambda^{\frac{-j}{m+k}} v + \lambda^{\frac{j+2}{m+k}} w\right) - \tilde{f}_\theta\left(\lambda^{\frac{-j}{m+k}} v\right) - \tilde{f}_\theta\left(\lambda^{\frac{j+2}{m+k}} w\right)\right] &= 0.
\end{aligned}$$

After multiplying (54) by $\lambda^{\theta(m)+\theta(2m)-\theta(1)-\theta(1+m)}$, using the constant

$$\delta = \theta(m) + \theta(2m) - \theta(1) - \theta(1+m),$$

and using that $\tilde{f}(\alpha v) = \alpha \tilde{f}(v)$ for any $\alpha \in \mathbb{R}$ and $v \in L$, we deduce that (45) is equivalent with

$$\begin{aligned}
\tilde{f}_\theta(v+w) - \tilde{f}_\theta(v) - \tilde{f}_\theta(w) - \qquad (55) \\
-\lambda^\delta \left[\tilde{f}_\theta(v + \lambda^2 w) - \lambda^2 \tilde{f}_\theta(w)\right] + \lambda^\delta \tilde{f}_\theta(v) &= 0.
\end{aligned}$$

If $\delta > 0$, then letting λ tend to zero in (55) yields $\tilde{f}_\theta(v+w) = \tilde{f}_\theta(v) + \tilde{f}_\theta(w)$.

If $\delta < 0$, then $\tilde{f}_\theta$ is differentiable at w as $w \neq 0$, and hence

$$\begin{aligned} \tilde{f}_\theta(v+\lambda^2 w) - \lambda^2 \tilde{f}_\theta(w) &= \lambda^2 \left(\tilde{f}_\theta \left(\frac{v}{\lambda^2} + w \right) - \tilde{f}_\theta(w) \right) \\ &= \lambda^2 \, O\left(\frac{v}{\lambda^2} \right) = O(1) \end{aligned}$$

holds for large λ. Therefore letting λ tend to infinity in (55) implies $\tilde{f}_\theta(v+w) = \tilde{f}_\theta(v) + \tilde{f}_\theta(w)$.

Finally, if $\delta = 0$, then (55) reads

$$\tilde{f}_\theta(v+w) = (1-\lambda^2)\tilde{f}_\theta(w) + \tilde{f}_\theta(v+\lambda^2 w).$$

Letting λ tend to zero completes the proof of $\tilde{f}_\theta(v+w) = \tilde{f}_\theta(v) + \tilde{f}_\theta(w)$. $\square$

Having Lemmas 7.3 and 7.4 at hand, to prove that f_θ is linear, all we have to verify is that if $p = 1, \ldots, k$, $\alpha_p, \beta_p \in \mathbb{R}\backslash\{0\}$ and $\theta \in I$, then

$$f_\theta(\alpha_p v_p + \beta_p i v_p) = f_\theta(\alpha_p v_p) + f_\theta(\beta_p i v_p). \tag{56}$$

Lemma 7.5 *If $\theta(p) + \theta(p+m) \neq \theta(m) + \theta(2m)$ for some $p \in \{1, \ldots, k\}$ and $\theta \in I$, then* (56) *holds for all $\alpha_p, \beta_p \in \mathbb{R}$.*

Proof: We may assume that $p = 1$, and hence according to Lemma 7.4 and the continuity of f, Lemma 7.5 is equivalent with the statement that if $\theta(1) + \theta(1+m) \neq \theta(m) + \theta(2m)$ for some $\theta \in I$, then

$$f_\theta(\alpha_1 v_1 + \beta_1 i v_1 + v_m) = f_\theta(\alpha_1 v_1) + f_\theta(\beta_1 i v_1) + f_\theta(v_m) \tag{57}$$

for $\alpha_1, \beta_1 \in \mathbb{R}\backslash\{0\}$.

It follows from (45) that

$$\begin{aligned} \lambda^{\theta(1)+\theta(1+m)-\theta(m)-\theta(2m)} f_\theta(\alpha_1 v_1 + \beta_1 i v_1 + v_m) - \\ \lambda^{\theta(1)+\theta(1+m)-\theta(m)-\theta(2m)} \left(f_\theta(\alpha_1 v_1) + f_\theta(\beta_1 i v_1) + f_\theta(v_m) \right) - \\ -\lambda^{\frac{j}{m+k}} f_\theta \left(\lambda^{\frac{-j}{m+k}} (\alpha_1 v_1 + \beta_1 i v_1) + \lambda^{\frac{j+2}{m+k}} v_m \right) - \\ +\lambda^{\frac{j}{m+k}} \left[f_\theta \left(\lambda^{\frac{-j}{m+k}} \alpha_1 v_1 \right) + f_\theta \left(\lambda^{\frac{-j}{m+k}} \beta_1 i v_1 \right) + f_\theta \left(\lambda^{\frac{j+2}{m+k}} v_m \right) \right] &= 0. \end{aligned} \tag{58}$$

After multiplying (58) by $\lambda^{\theta(m)+\theta(2m)-\theta(1)-\theta(1+m)}$, using the constant

$$\delta = \theta(m) + \theta(2m) - \theta(1) - \theta(1+m) \neq 0, \tag{59}$$

and using that $f(\alpha v) = \alpha f(v)$ for any $\alpha \in \mathbb{R}$ and $v \in L$, we deduce that (45) is equivalent with

$$\begin{aligned} f_\theta(\alpha_1 v_1 + \beta_1 i v_1 + v_m) - f_\theta(\alpha_1 v_1) - f_\theta(\beta_1 i v_1) - f_\theta(v_m) - \\ -\lambda^\delta \left[f_\theta\left(\alpha_1 v_1 + \beta_1 i v_1 + \lambda^2 v_m\right) - f_\theta(\lambda^2 v_m)\right] + \\ +\lambda^\delta (f_\theta(\alpha_1 v_1) + f_\theta(\beta_1 i v_1)) &= 0. \end{aligned} \tag{60}$$

If $\delta > 0$, then letting λ tend to zero in (60) implies

$$f_\theta(\alpha_1 v_1 + \beta_1 i v_1 + v_m) - f_\theta(\alpha_1 v_1) - f_\theta(\beta_1 i v_1) - f_\theta(v_m) = 0, \tag{61}$$

verifying (57).

According to (59), we may assume that $\delta < 0$. In this case f_θ is differentiable at v_m by (40), thus if $\lambda > 0$ is small, then

$$\begin{aligned} f_\theta\left(\alpha_1 v_1 + \beta_1 i v_1 + \lambda^2 v_m\right) - f_\theta(\lambda^2 v_m) &= \lambda^2 \left(f_\theta\left(v_m + \frac{\alpha_1 v_1 + \beta_1 i v_1}{\lambda^2}\right) - f_\theta(v_m)\right) \\ &= \lambda^2 \, O\left(\frac{\alpha_1 v_1 + \beta_1 i v_1}{\lambda^2}\right) = O(1), \end{aligned}$$

therefore letting λ tend to infinity in (60) leads to (61), completing the proof of (57). □

Proof of Lemma 7.2: According to Lemmas 7.3, 7.4 and 7.5, to prove that the function f in Lemma 7.1 is linear, all we have to verify is that if $p \in \{1, \dots, k\}$, $\alpha_p, \beta_p \in \mathbb{R}\backslash\{0\}$ and $\theta \in I$ satisfy $\theta(p) + \theta(p+m) = \theta(m) + \theta(2m)$, then

$$f_\theta(\alpha_p v_p + \beta_p i v_p) = f_\theta(\alpha_p v_p) + f_\theta(\beta_p i v_p). \tag{62}$$

In order to prove the linearity for f_θ, we define $\varphi \in \mathrm{SL}(m, \mathbb{C})$ by $\varphi(v_p) = -v_p$, $\varphi(v_m) = -v_m$ and $\varphi(v_q) = v_q$ if $q \neq p, m$. In this case, $\varphi|_L$ is an $\mathbb{R}$-linear map of the $(m+k)$-dimensional subspace L into L whose determinant is -1. Moreover, $(\varphi|_L)^{-t} = \varphi|_L$. It follows from (45) that

$$\begin{aligned} (-1)^{\theta(p)+\theta(p+m)+\theta(m)+\theta(2m)} \left[f_\theta(\alpha_p v_p + \beta_p i v_p) - f_\theta(\alpha_p v_p) - f_\theta(\beta_p i v_p)\right] - \\ - \left[f_\theta(-\alpha_p v_p - \beta_p i v_p) - f_\theta(-\alpha_p v_p) - f_\theta(-\beta_p i v_p)\right] &= 0 \end{aligned}$$

Here $(-1)^{\theta(p)+\theta(p+m)+\theta(m)+\theta(2m)} = 1$ as $\theta(p)+\theta(p+m) = \theta(m)+\theta(2m)$. Hence the fact that f is odd yields

$$2\left(f_\theta(\alpha_p v_p + \beta_p i v_p) - f_\theta(\alpha_p v_p) - f_\theta(\beta_p i v_p)\right) = 0.$$

We conclude (62), and in turn Proposition 7.2. □

7.2 Case $j = 2m - 1$

The case $j = 2m - 1$ is handled similarly as the case $j = m, \dots, 2m - 2$.

Lemma 7.6 *Z_{2m-1} is constant zero.*

Proof: According to Lemma 7.1, it is sufficient to prove that the $f : \mathbb{R}^{2m} \to \mathbb{T}^r(\mathbb{R}^{2m})$ in Lemma 7.1 satisfies $f(x + y) = f(x) + f(y)$ for any $x, y \in \mathbb{C}^m$. We claim that there exists a complex hermitian basis $v_1, \dots, v_m$ of $\mathbb{C}^m$ such that

$$x, y \in \mathrm{lin}_{\mathbb{R}}\{v_1, iv_1, v_2\}. \tag{63}$$

Indeed, for (63), we may assume that x and y are complex independent. In particular, there exists a complex hermitian basis $v_1, \dots, v_m$ of $\mathbb{C}^m$ such that $x = \alpha\, v_1$ for $\alpha > 0$, and the hermitian projection of y onto the complex $(m-1)$-dimensional subspace of $\mathbb{C}^m$ complex orthogonal to v_1 is γv_2 for $\gamma > 0$, proving (63).

It follows from (63) that it is sufficient to prove that if $v_1, \dots, v_m$ is a complex hermitian basis of $\mathbb{C}^m$, $\theta \in I$ and $\alpha_1, \beta_1, \alpha_2 \in \mathbb{R}\backslash\{0\}$, then

$$f_\theta(\alpha_1 v_1 + \beta_1 iv_1 + \alpha_2 v_2) = \alpha_1 f_\theta(v_1) + \beta_1 f_\theta(iv_1) + \alpha_2 f_\theta(v_2), \tag{64}$$

where we use the notation leading up to (45).

For $\lambda > 0$, we define $\varphi_\lambda \in \mathrm{SL}(m, \mathbb{C})$ by $\varphi_\lambda(v_1) = \lambda v_1$, $\varphi_\lambda(v_2) = \lambda^{-1} v_2$ and $\varphi_\lambda(v_l) = v_l$ for $l > 2$. It follows that

$$\begin{array}{rlcrcl}
\varphi_\lambda^{-t}(v_1) = & \lambda^{-1} v_1 & \text{and} & \varphi_\lambda^{-t}(iv_1) & = & \lambda^{-1} iv_1, \\
\varphi_\lambda^{-t}(v_2) = & \lambda v_2 & \text{and} & \varphi_\lambda^{-t}(iv_2) & = & \lambda v_2, \\
\varphi_\lambda^{-t}(v_l) = & v_l & \text{and} & \varphi_\lambda^{-t}(iv_l) & = & iv_l \ \text{ for } l > 2.
\end{array}$$

We observe that for $\varphi = \varphi_\lambda$, we have $\varphi|_L = \tilde{\varphi} = \varphi_\lambda$ in (45). Writing

$$\delta = \theta(1) + \theta(m + 1) - \theta(2) - \theta(m + 2)$$

for $\theta \in I$, (45) yields that

$$\begin{aligned}
\lambda^\delta \left[f_\theta(\alpha_1 v_1 + \beta_1 iv_1 + \alpha_2 v_2) - \alpha_1 f_\theta(v_1) - \beta_1 f_\theta(iv_1) - \alpha_2 f_\theta(v_2) \right] - & \\
- f_\theta(\lambda^{-1}\alpha_1 v_1 + \lambda^{-1}\beta_1 iv_1 + \lambda\alpha_2 v_2) + & \\
+ f_\theta(\lambda^{-1}\alpha_1 v_1) + f_\theta(\lambda^{-1}\beta_1 iv_1) + f_\theta(\lambda\alpha_2 v_2) & = 0.
\end{aligned}$$

After dividing by λ^δ, we deduce that if $\alpha_1, \beta_1, \alpha_2 \in \mathbb{R}\backslash\{0\}$, then

$$f_\theta(\alpha_1 v_1 + \beta_1 iv_1 + \alpha_2 v_2) - \alpha_1 f_\theta(v_1) - \beta_1 f_\theta(iv_1) - \alpha_2 f_\theta(v_2) - \tag{65}$$

$$-f_\theta(\lambda^{-\delta-1}\alpha_1 v_1 + \lambda^{-\delta-1}\beta_1 i v_1 + \lambda^{1-\delta}\alpha_2 v_2) + \tag{66}$$
$$+f_\theta(\lambda^{-\delta-1}\alpha_1 v_1) + f_\theta(\lambda^{-\delta-1}\beta_1 i v_1) + f_\theta(\lambda^{1-\delta}\alpha_2 v_2) \;=\; 0 \tag{67}$$

for all $\lambda > 0$.

Case 1: *f_θ is linear if $\delta \geq 0$.*

Since f_θ is differentiable at $\alpha_2 v_2$ by $\alpha_2 > 0$ by (43), there exists some $\Omega(\lambda) \in \mathbb{R}$ such that

$$\begin{aligned} f_\theta(\lambda^{-2}\alpha_1 v_1 + \lambda^{-2}\beta_1 i v_1 + \alpha_2 v_2) &= f_\theta(\alpha_2 v_2) + \Omega(\lambda) \\ \lim_{\lambda\to\infty} \lambda^{1-\delta}\Omega(\lambda) &= 0. \end{aligned}$$

Letting λ tend to infinity in (66) and (67), we conclude from the 1-homogeneity of f_θ that

$$\begin{aligned} &-f_\theta(\lambda^{-\delta-1}\alpha_1 v_1 + \lambda^{-\delta-1}\beta_1 i v_1 + \lambda^{1-\delta}\alpha_2 v_2) + \\ &+f_\theta(\lambda^{-\delta-1}\alpha_1 v_1) + f_\theta(\lambda^{-\delta-1}\beta_1 i v_1) + f_\theta(\lambda^{1-\delta}\alpha_2 v_2) \quad = \\ &\qquad -\lambda^{1-\delta}\Omega(\lambda) + f_\theta(\lambda^{-\delta-1}\alpha_1 v_1) \;+\; f_\theta(\lambda^{-\delta-1}\beta_1 i v_1) \end{aligned}$$

tends to zero, yielding (64).

Case 2: *f_θ is linear if $\delta < -1$.*

Let λ tend to zero. Since both (66) and (67) tend to zero, we conclude (64).

Case 3: *f_θ is linear if $\delta = -1$.*

First we observe that if $\delta = -1$ and $\alpha_1, \beta_1, \alpha_2 \in \mathbb{R}$, then

$$f_\theta(\alpha_1 v_1 + \beta_1 i v_1 + \alpha_2 v_2) = f_\theta(\alpha_1 v_1 + \beta_1 i v_1) + \alpha_2 f_\theta(v_2) \tag{68}$$

Indeed, letting λ tend to zero, (65), (66) and (67) yield (68).

Hence, it remains to prove that for $\alpha_1, \beta_1 \in \mathbb{R}$, we have

$$f_\theta(\alpha_1 v_1 + \beta_1 i v_1) = \alpha_1 f_\theta(v_1) + \beta_1 f_\theta(i v_1).$$

In this case, we set $\mu = \alpha_1$, and consider $\varphi \in \mathrm{SL}(m, \mathbb{C})$ by $\varphi(v_1) = v_1 + \mu v_2$ and $\varphi(v_l) = v_l$ for $l \geq 2$. We observe that $\varphi|_L = \tilde{\varphi} = \varphi$ in (45). Since $\varphi(i v_1) = i v_1 + \mu i v_2$ and $\varphi(i v_l) = i v_l$ for $l \geq 2$, we have

$$\begin{aligned} \varphi^{-t}(v_1) &= v_1 && \text{and} & \varphi^{-t}(i v_1) &= i v_1 \\ \varphi^{-t}(v_2) &= -\mu v_1 + v_2 && \text{and} & \varphi^{-t}(i v_2) &= -\mu i v_1 + i v_2 \\ \varphi^{-t}(v_l) &= v_l && \text{and} & \varphi^{-t}(i v_l) &= i v_l \text{ for } l > 2. \end{aligned}$$

For $p, q \in \mathbb{N}$, $0 \leq p \leq \theta(2)$ and $0 \leq q \leq \theta(m+2)$, let $\theta_{pq} \in I$ be such that

$$\begin{aligned}
\theta_{pq}(1) &= \theta(1) + p \text{ and } \theta_{pq}(m+1) = \theta(m+1) + q, \\
\theta_{pq}(2) &= \theta(2) - p \text{ and } \theta_{pq}(m+2) = \theta(m+2) - q, \\
\theta_{pq}(l) &= \theta(l) \text{ and } \theta_{pq}(m+l) = \theta(m+l) \text{ for } l = 3, \dots, m \text{ provided } m \geq 3.
\end{aligned}$$

In particular, $\theta = \theta_{00}$. We observe that if $0 \leq p \leq \theta(2)$ and $0 \leq q \leq \theta(m+2)$, then

$$\delta_{pq} = \theta_{pq}(1) + \theta_{pq}(m+1) - \theta_{pq}(2) - \theta_{pq}(m+2) = \delta + 2p + 2q = -1 + 2p + 2q,$$

and the coefficient of $\odot_{l=1}^{2m} v_l^{\theta(l)}$ in $\varphi \cdot f$ is

$$\sum_{p=0}^{\theta(2)} \sum_{q=0}^{\theta(m+2)} \binom{\theta(1)+p}{p} \binom{\theta(m+1)+q}{q} \mu^{p+q} f_{\theta_{pq}}.$$

We deduce from Case 1 that $f_{\theta_{pq}}$ is $\mathbb{R}$-linear on $\mathbb{R}^{2m}$ unless $p = q = 0$. Therefore it follows from (45) and the properties of φ^{-t} (like $\varphi^{-t}(v_2) = -\mu v_1 + v_2$) applied to the linear combination $\alpha_1 v_1 + \beta_1 i v_1 + v_2$ that

$$\begin{aligned}
f_\theta(\alpha_1 v_1 + \beta_1 i v_1 + v_2) - \alpha_1 f_\theta(v_1) - \beta_1 f_\theta(i v_1) - f_\theta(v_2) - \\
-f_\theta((\alpha_1 - \mu) v_1 + \beta_1 i v_1 + v_2) + f_\theta(\alpha_1 v_1) + f_\theta(\beta_1 i v_1) + f_\theta(-\mu v_1 + v_2) &= 0.
\end{aligned}$$

Since $f_\theta(\alpha_1 v_1 + \beta_1 i v_1 + v_2) = f_\theta(\alpha_1 v_1 + \beta_1 i v_1) + f_\theta(v_2)$ by (68), and

$$-f_\theta((\alpha_1 - \mu) v_1 + \beta_1 i v_1 + v_2) + f_\theta(\alpha_1 v_1) + f_\theta(\beta_1 i v_1) + f_\theta(-\mu v_1 + v_2) = 0$$

by $\mu = \alpha_1$ and (68), we conclude Case 3 and the proof of Lemma 7.6. □

7.3 Case $1 \leq j \leq m-1$

The goal of this section is to prove

Lemma 7.7 *If $j = 1, \dots, m-1$, then Z_j is constant zero.*

Proof: According to Lemma 7.1, it is sufficient to show for any $L \in \mathrm{Gr}_{j+1}(\mathbb{R}^{2m})$ of maximal complex rank $j+1$, if $x, y \in L$, and $f : L \to \mathbb{T}^r(\mathbb{R}^{2m})$ is the function of Lemma 7.1, then $f(x+y) = f(x) + f(y)$. There exists some real orthonormal basis $v_1, \dots, v_{j+1}$ of L such that $x, y \in \mathrm{lin}_{\mathbb{R}}\{v_1, v_2\}$, and $v_1, \dots, v_{j+1}$ can be extended to a complex basis $v_1, \dots, v_m$ of $\mathbb{C}^m = \mathbb{R}^{2m}$. Therefore it is sufficient to prove that for any $\alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2). \tag{69}$$

As f is continuos, we may assume that $\alpha_1, \alpha_2 \in \mathbb{R}\backslash\{0\}$ in (69). We also note that f is C^∞ on $L\backslash\{o\}$ by (40).

To prove (69), we use the notation leading up to (45). For $\lambda > 0$, we define $\varphi \in \mathrm{SL}(m, \mathbb{C})$ by $\varphi(v_1) = \lambda v_1$, $\varphi(v_2) = \lambda^{-1} v_2$ and $\varphi(v_l) = v_l$ for $l > 2$ where we do not signal the dependence on λ. It follows that

$$\begin{aligned} \varphi^{-t}(v_1) &= \lambda^{-1} v_1 \\ \varphi^{-t}(v_2) &= \lambda v_2, \text{ moreover } \varphi^{-t}(v_l) = v_l \text{ for } l > 2. \end{aligned}$$

For $\theta \in I$, writing

$$\delta = \theta(1) + \theta(m+1) - \theta(2) - \theta(m+2),$$

(45) yields

$$\begin{aligned} \lambda^\delta f_\theta(\alpha_1 v_1 + \alpha_2 v_2) - \lambda^\delta \alpha_1 f_\theta(v_1) - \lambda^\delta \alpha_2 f_\theta(v_2) - & \\ - f_\theta(\lambda^{-1}\alpha_1 v_1 + \lambda \alpha_2 v_2) + \alpha_1 f_\theta(\lambda^{-1} v_1) + \alpha_2 f_\theta(\lambda v_2) &= 0. \end{aligned}$$

After dividing by λ^δ, we deduce that if $\alpha_1, \alpha_2 \in \mathbb{R}\backslash 0$, then

$$\begin{aligned} f_\theta(\alpha_1 v_1 + \alpha_2 v_2) - \alpha_1 f_\theta(v_1) - \alpha_2 f_\theta(v_2) - & \\ - f_\theta(\lambda^{-\delta-1}\alpha_1 v_1 + \lambda^{1-\delta}\alpha_2 v_2) + f_\theta(\lambda^{-\delta-1}\alpha_1 v_1) + f_\theta(\lambda^{1-\delta}\alpha_2 v_2) &= 0. \end{aligned} \tag{70}$$

Case 1: *f_θ is linear if $\delta \geq 0$.*

Since f_θ is differentiable at $\alpha_2 v_2$ by $\alpha_2 > 0$, there exists some $\Omega(\lambda) \in \mathbb{R}$ such that

$$\begin{aligned} f_\theta(\lambda^{-2}\alpha_1 v_1 + \alpha_2 v_2) &= f_\theta(\alpha_2 v_2) + \Omega(\lambda), \\ \lim_{\lambda \to \infty} \lambda^{1-\delta}\Omega(\lambda) &= 0. \end{aligned}$$

Letting λ tend to infinity in (70), we conclude from the 1-homogeneity of f_θ that

$$\begin{aligned} \lambda^{1-\delta}\left(-f_\theta(\lambda^{-2}\alpha_1 v_1 + \alpha_2 v_2) + f_\theta(\alpha_2 v_2)\right) + \lambda^{-1-\delta} f_\theta(\alpha_1 v_1) &= \\ -\lambda^{1-\delta}\Omega(\lambda) + \lambda^{-1-\delta} f_\theta(\alpha_1 v_1) & \end{aligned}$$

tends to zero, yielding (69).

Case 2: *f_θ is linear if $\delta \leq -1$.*

If λ tends to zero, then (70) tends to zero, and hence we conclude (69). $\square$

We end this paper by summarizing up the results obtained for even and odd valuations, to give a proof of Proposition 2.3.

Proof of Proposition 2.3: Let $Z : \mathcal{K}^{2m} \to \mathbb{T}^r(\mathbb{R}^{2m})$ be an $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous valuation. McMullen's decomposition Theorem 3.7 yields that Z can be written as $Z = \sum_{j=0}^{2m} Z_j$ where Z_j is an $\mathrm{SL}(m, \mathbb{C})$-equivariant and translation invariant continuous j-homogeneous valuation for $j = 0, \dots, 2m$. If in addition, Z is even (resp. odd), then Z_j, $j = 0, \dots, 2m$, is also even (resp. odd).

By Lemmas 6.3, 6.4, 6.6 and 6.7, we have that if Z is even, then each Z_j in McMullen's decomposition is constant zero, and we conclude Proposition 2.3 for even valuations.

If Z is odd, then combining (37) and Lemmas 7.2, 7.6 and 7.7 shows that each Z_j is constant zero. Hence, Proposition 2.3 is also proved for odd valuations. $\square$

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