

Stability of some versions of the Prékopa-Leindler inequality

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Abstract

Two consequences of the stability version of the one dimensional Prékopa-Leindler inequality are presented. One is the stability version of the Blaschke-Santaló inequality, and the other is a stability version of the Prékopa-Leindler inequality for even functions in higher dimensions, where a recent stability version of the Brunn-Minkowski inequality is also used in an essential way.

1 The problem

Our main theme is some consequences of the Prékopa-Leindler inequality in one dimension. The inequality itself, due to A. Prékopa [28] and L. Leindler [23], was generalized in A. Prékopa [29] and [30], C. Borell [9], and in H.J. Brascamp, E.H. Lieb [11]. Various applications are provided and surveyed in K.M. Ball [1], F. Barthe [4], and R.J. Gardner [16]. The following multiplicative version from [1], is often more useful and is more convenient for geometric applications.

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THEOREM 1.1 (Prékopa-Leindler) *If m, f, g are non-negative integrable functions on \mathbb{R} satisfying $m(\frac{r+s}{2}) \geq \sqrt{f(r)g(s)}$ for $r, s \in \mathbb{R}$, then*

$$\int_{\mathbb{R}} m \geq \sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g}.$$

S. Dubuc [13] characterized the equality case if the integrals of f, g, m above are positive, and K.M. Ball, K.J. Böröczky [3] even provided the following stability version.

THEOREM 1.2 *There exists a positive absolute constant c with the following property: If m, f, g are non-negative integrable functions with positive integrals on \mathbb{R} such that m is log-concave, $m(\frac{r+s}{2}) \geq \sqrt{f(r)g(s)}$ for $r, s \in \mathbb{R}$, and*

$$\int_{\mathbb{R}} m \leq (1 + \varepsilon) \sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g},$$

for $\varepsilon > 0$, then there exist $a > 0$, $b \in \mathbb{R}$ such that

$$\begin{aligned} \int_{\mathbb{R}} |f(t) - a m(t + b)| dt &\leq c \cdot \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}} \cdot \int_{\mathbb{R}} m(t) dt \\ \int_{\mathbb{R}} |g(t) - a^{-1} m(t - b)| dt &\leq c \cdot \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}} \cdot \int_{\mathbb{R}} m(t) dt. \end{aligned}$$

Remark If f and g are log-concave probability distributions then $a = 1$ can be assumed, and if in addition f and g have the same expectation, then even $b = 0$ can be assumed.

As it was observed by C. Borell [9], and later independently by K.M. Ball [1], assigning to any function $H : [0, \infty] \rightarrow [0, \infty]$ the function $h : \mathbb{R} \rightarrow [0, \infty]$ defined by $h(x) = H(e^x)e^x$, we have the version Theorem 1.3 of the Prékopa-Leindler inequality. We note that if H is log-concave and decreasing, then h is log-concave.

THEOREM 1.3 *If $M, F, G : [0, \infty] \rightarrow [0, \infty]$ integrable functions satisfy $M(\sqrt{rs}) \geq \sqrt{F(r)G(s)}$ for $r, s \geq 0$, then*

$$\int_0^\infty M \geq \sqrt{\int_0^\infty F \cdot \int_0^\infty G}.$$

Therefore we deduce the following statement by Theorem 1.2:

COROLLARY 1.4 *There exists a positive absolute constant c with the following property: If $M, F, G : [0, \infty] \rightarrow [0, \infty]$ are integrable functions with positive integrals such that M is log-concave and decreasing, $M(\sqrt{rs}) \geq \sqrt{F(r)G(s)}$ for $r, s \in [0, \infty]$, and*

$$\int_0^\infty M \leq (1 + \varepsilon) \sqrt{\int_0^\infty F \cdot \int_0^\infty G},$$

for $\varepsilon > 0$, then there exist $a, b > 0$, such that

$$\begin{aligned} \int_0^\infty |F(t) - a M(bt)| dt &\leq c \cdot \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}} \cdot \int_0^\infty M(t) dt \\ \int_0^\infty |G(t) - a^{-1} M(b^{-1}t)| dt &\leq c \cdot \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}} \cdot \int_0^\infty M(t) dt. \end{aligned}$$

Remark If in addition F and G are decreasing log-concave probability distributions then $a = b$ can be assumed. The condition that M is log-concave and decreasing can be replaced by the one that $M(e^t)$ is log-concave.

2 A stability version of the Blaschke-Santaló inequality

Based on the approach in the PhD thesis K.M. Ball [1], in this section we show how to provide a stability version of the Blaschke-Santaló inequality using the stability version Corollary 1.4 of the Prékopa-Leindler inequality in one dimension.

We write o to denote the origin of \mathbb{R}^n , $\langle \cdot, \cdot \rangle$ to denote the standard scalar product. We write $|\cdot|$ to denote the Lebesgue measure in \mathbb{R}^n , where the Lebesgue measure of the empty set is 0. Let B^n be the unit Euclidean ball with volume $\kappa_n = |B^n|$. A convex body K in \mathbb{R}^n is a compact convex set with non-empty interior. If $z \in \text{int}K$, then the polar of K with respect to z is the convex body

$$K^z = \{x \in \mathbb{R}^n : \langle x - z, y - z \rangle \leq 1 \text{ for any } y \in K\}.$$

It is easy to see that $(K^z)^z = K$, and the volume product $|K| \cdot |K^z|$ is invariant under affine maps fixing z . According to L.A. Santaló [32] (see also M. Meyer

and A. Pajor [25]), there exists a unique $z \in \text{int}K$ minimizing $|K^z|$, which is called the Santaló point of K . In this case z is the centroid of K^z . The celebrated Blaschke-Santaló inequality states that if z is the Santaló point (or centroid) of K , then

$$|K| \cdot |K^z| \leq \kappa_n^2, \quad (1)$$

with equality if and only if K is an ellipsoid. The inequality was proved by W. Blaschke [6] for $n \leq 3$, and by L.A. Santaló [32] for all n . The case of equality was characterized by J. Saint-Raymond [31] among o -symmetric convex bodies, and by C.M. Petty [27] among all convex bodies (see also M. Meyer and A. Pajor [25], D. Hug [20], and M. Meyer and S. Reisner [26] for simpler proofs).

A natural tool is the Banach-Mazur distance $\delta_{\text{BM}}(K, M)$ of the convex bodies K and M , which is defined by

$$\delta_{\text{BM}}(K, M) = \ln \min\{\lambda \geq 1 : K - x \subset \Phi(M - y) \subset \lambda(K - x) \text{ for } \Phi \in \text{GL}(n), x, y \in \mathbb{R}^n\}.$$

Here, unlike in K.M. Ball, K.J. Böröczky [3], we introduce a logarithm into the definition of the Banach-Mazur distance to simplify various formulae. In particular, if K and M are o -symmetric, then $x = y = o$ can be assumed, and in this case $\delta_{\text{BM}}(K, M) = \delta_{\text{BM}}(K^o, M^o)$. It follows from a theorem of F. John [21] that $\delta_{\text{BM}}(K, B^n) \leq \ln n$ for any convex body K in \mathbb{R}^n (see also K.M. Ball [2]).

K.J. Böröczky [10] proved a stability version of the Blaschke-Santaló inequality. One of the main tools in that paper is to reduce the problem to o -symmetric convex bodies with axial rotational symmetry; namely, combining Theorem 1.4 and Lemma 2.1 in [10] yields the following.

LEMMA 2.1 *For any $n \geq 2$ there exists $\tilde{\gamma} > 0$ depending only on n , such that if K is a convex body in \mathbb{R}^n with Santaló point z , then one finds an o -symmetric convex body C with axial rotational symmetry, and satisfying $\delta_{\text{BM}}(C, B^n) \geq \tilde{\gamma} \delta_{\text{BM}}(K, B^n)^2$ and $|C| \cdot |C^o| \geq |K| \cdot |K^z|$.*

Remark: If K is o -symmetric, then even $\delta_{\text{BM}}(C, B^n) \geq \tilde{\gamma} \delta_{\text{BM}}(K, B^n)$.

Now we are ready to prove our main result in this section:

THEOREM 2.2 *If K is a convex body in \mathbb{R}^n , $n \geq 3$, with Santaló point z , and*

$$(1 + \varepsilon)|K| \cdot |K^z| > \kappa_n^2 \text{ for } \varepsilon > 0,$$

then for some $\gamma > 0$ depending only on n , we have

$$\delta_{\text{BM}}(K, B^n) < \gamma \varepsilon^{\frac{1}{3(n+1)}} |\log \varepsilon|^{\frac{4}{3(n+1)}}.$$

Remark: If K is o -symmetric, then the exponent $\frac{1}{3(n+1)}$ occurring in Theorem 2.2 can be replaced by $\frac{2}{3(n+1)}$.

Taking K to be the convex body resulting from B^n by cutting off two opposite caps of volume ε shows that the exponent $1/(3(n+1))$ cannot be replaced by anything larger than $2/(n+1)$ even for o -symmetric convex bodies with axial rotational symmetry. Therefore the exponent of ε is of the correct order. In addition if the error in Corollary 1.4 could be reduced to ε , then we would have the stability version of the Blaschke-Santaló inequality of the correct order.

We note that the exponent of ε is $1/(6n)$ in the stability version of the Blaschke-Santaló inequality proved in K.J. Böröczky [10].

Proof of Theorem 2.2: Let C be the o -symmetric convex body provided by Lemma 2.1. Moreover let u be a unit vector, and let $\alpha > 0$ such that $\alpha u \in \partial C$ and a section $C \cap (u^\perp + tu)$ for $t \in (-\alpha, \alpha)$ is an $(n-1)$ -ball of radius $\varphi(t)$, and of area $F(t) = \varphi(t)^{n-1} \kappa_{n-1}$. In turn $\alpha^{-1}u \in \partial C^o$, and if $t \in (-\alpha^{-1}, \alpha^{-1})$, then $C^o \cap (u^\perp + tu)$ is an $(n-1)$ -ball of radius $\psi(t)$, and of area $G(t) = \psi(t)^{n-1} \kappa_{n-1}$. We observe that for $t \in (-1, 1)$, $B^n \cap (u^\perp + tu)$ is an $(n-1)$ -ball of radius $(1-t^2)^{\frac{1}{2}}$ and of area $M(t) = (1-t^2)^{\frac{n-1}{2}} \kappa_{n-1}$. We define $F(t) = 0$, $G(t) = 0$ and $M(t) = 0$ if $t \geq \alpha$, $t \geq \alpha^{-1}$, and $t \geq 1$, respectively. For $v \in u^\perp$, $r \in (-\alpha, \alpha)$ and $s \in (-\alpha^{-1}, \alpha^{-1})$, we have

$$\varphi(r) \cdot \psi(s) = \langle ru + \varphi(r)v, su + \psi(s)v \rangle - rs \leq 1 - rs.$$

In particular $M(\sqrt{rs}) \geq \sqrt{F(r)G(s)}$, and

$$\left(\int_0^\infty M \right)^2 = \frac{\kappa_n^2}{4} \leq \frac{(1+\varepsilon)|C| \cdot |C^o|}{4} = (1+\varepsilon) \left(\int_0^\infty F \right) \left(\int_0^\infty G \right).$$

Therefore we may apply Corollary 1.4, and deduce that there exist $a, b > 0$ such that

$$\int_0^\infty |a F(bt) - M(t)| dt \leq c \cdot \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}} \cdot \kappa_n. \quad (2)$$

Let Φ be the linear transform such that $\Phi u = b^{-1}u$, and if $v \in u^\perp$ then $\Phi v = a^{\frac{1}{n-1}}v$. Therefore $\tilde{C} = \Phi C$ is an o -symmetric convex body with axial

symmetry around $\mathbb{R}u$ such that

$$\delta_{\text{BM}}(\tilde{C}, B^n) = \delta_{\text{BM}}(C, B^n) \geq \tilde{\gamma} \delta_{\text{BM}}(K, B^n)^2, \quad (3)$$

and the area of $\tilde{C} \cap (u^\perp + tu)$ is $a F(bt)$ for any $t \in (0, b^{-1}\alpha)$. In particular

$$|\tilde{C} \Delta B^n| \leq 2c \cdot \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}} \cdot \kappa_n, \quad (4)$$

where Δ denotes the symmetric difference. Let us show that (4) forces $\partial\tilde{C}$ to be contained in a thin spherical shell. If a cap of B^n of depth h is disjoint from \tilde{C} , then its volume is less than $|\tilde{C} \Delta B^n|$. Since the cap contains a cone of height h whose base is an $(n-1)$ -ball of radius \sqrt{h} , we have

$$(1 - \gamma_1 |\tilde{C} \Delta B^n|^{\frac{2}{n+1}}) B^n \subset \tilde{C}.$$

for some $\gamma_1 > 0$ depending only on n .

Next we set $s = \gamma_1 |\tilde{C} \Delta B^n|^{\frac{2}{n+1}}$. Let us assume that a point $p \in \tilde{C}$ is of distance t from B^n where $s < t < \frac{1}{4}$. The points where the tangent lines from p touch $(1-s)B^n$ determine a cap of $(1-s)B^n$ whose depth is between $t/2$ and $2t$. It follows that $\tilde{C} \setminus B^n$ contains a cone of height t , whose base is an $(n-1)$ -ball of radius $\sqrt{t}/8$. Therefore there exists $\gamma_2 > 0$ depending only on n , such that

$$\tilde{C} \subset (1 + \gamma_2 |\tilde{C} \Delta B^n|^{\frac{2}{n+1}}) B^n.$$

Finally we conclude Theorem 2.2 by (3) and (4). \square

As it is explained in K.J. Böröczky [10], the stability version Theorem 2.2 yields stability versions with the same order of the error term for two basic affine invariant inequalities. The first is the affine isoperimetric inequality of W. Blaschke [5] or [7] (see L.A. Santaló [32] for $n \geq 4$), and the other is the isoperimetric inequality for the geominimal surface area by C.M. Petty [27]. The monograph K. Leichtweiß [22], and the survey paper E. Lutwak [24] provide introductions into the by now classical theory of these notions.

3 The stability version of the Brunn-Minkowski inequality due to Figalli, Maggi, Pratelli

For any $\alpha, \beta > 0$, and measurable sets $X, Y, Z \subset \mathbb{R}^n$ with

$$\alpha X + \beta Y \subset Z,$$

the Brunn-Minkowski inequality says that

$$|Z|^{\frac{1}{n}} \geq \alpha|X|^{\frac{1}{n}} + \beta|Y|^{\frac{1}{n}}.$$

Obviously the case $\alpha = \beta = 1$ yields the general case.

Since the days of H. Minkowski, there are stability versions of the Brunn-Minkowski inequality if X and Y are convex bodies, mostly in terms of the so called Hausdorff metric, see the survey paper H. Groemer [18]. In higher dimensions, the best estimates are due to V.I. Diskant [12] and H. Groemer [17].

Recently A. Figalli, F. Maggi, A. Pratelli [14] and [15] obtained an optimal stability version of the Brunn-Minkowski inequality in terms of the volume difference. To define the “homothetic distance” $A(K, C)$ of convex bodies K and C , let $\alpha = |K|^{\frac{-1}{n}}$ and $\beta = |C|^{\frac{-1}{n}}$, and let

$$A(K, C) = \min \{ |\alpha K \Delta (x + \beta C)| : x \in \mathbb{R}^n \}.$$

We observe that $|\alpha K \cap (x + \beta C)|^{\frac{1}{n}}$ is a concave function of $x \in \alpha K - \beta C$ by the Brunn-Minkowski inequality. Therefore if both K and C are o -symmetric, and $|C| = |K|$, then

$$A(K, C) = |K \Delta C|/|K|. \quad (5)$$

Next let

$$\sigma(K, C) = \max \left\{ \frac{|C|}{|K|}, \frac{|K|}{|C|} \right\}.$$

THEOREM 3.1 (Figalli, Maggi, Pratelli) For $\gamma^* = \left(\frac{(2-2^{\frac{n-1}{n}})^{\frac{3}{2}}}{122n^7} \right)^2$, and any convex bodies K and C in \mathbb{R}^n ,

$$|K + C|^{\frac{1}{n}} \geq (|K|^{\frac{1}{n}} + |C|^{\frac{1}{n}}) \left[1 + \frac{\gamma^*}{\sigma(K, C)^{\frac{1}{n}}} \cdot A(K, C)^2 \right].$$

We will need the product form of the Brunn-Minkowski inequality. Since

$$\begin{aligned} \frac{1}{2} \left(|K|^{\frac{1}{n}} + |C|^{\frac{1}{n}} \right) &= |K|^{\frac{1}{2n}} |C|^{\frac{1}{2n}} \left[1 + \frac{1}{2} \left(\sigma(K, C)^{\frac{1}{4n}} - \sigma(K, C)^{\frac{-1}{4n}} \right)^2 \right] \\ &\geq |K|^{\frac{1}{2n}} |C|^{\frac{1}{2n}} \left[1 + \frac{(\sigma(K, C) - 1)^2}{32n^2 \sigma(K, C)^{\frac{4n-1}{2n}}} \right], \end{aligned}$$

we conclude with $\sigma = \sigma(K, C)$ that

$$|\frac{1}{2}(K + C)| \geq \sqrt{|K| \cdot |C|} \left[1 + \frac{(\sigma - 1)^2}{32n\sigma^2} + \frac{n\gamma^*}{\sigma^{\frac{1}{n}}} \cdot A(K, C)^2 \right]. \quad (6)$$

4 Prékopa-Leindler inequality in higher dimensions for even functions

Let $f, g, m : \mathbb{R}^n \rightarrow [0, \infty]$ be such that $m(\frac{x+y}{2}) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$, and for $t > 0$, let

$$\begin{aligned}\Phi_t &= \{x \in \mathbb{R}^n : f(x) \geq t\} \quad \text{and} \quad F(t) = |\Phi_t| \\ \Psi_t &= \{x \in \mathbb{R}^n : g(x) \geq t\} \quad \text{and} \quad G(t) = |\Psi_t| \\ \Omega_t &= \{x \in \mathbb{R}^n : m(x) \geq t\} \quad \text{and} \quad M(t) = |\Omega_t|.\end{aligned}$$

As it was observed in K.M. Ball [1], the condition on f, g, m yields that if $\Phi_r, \Psi_s \neq \emptyset$ for $r, s > 0$, then

$$\frac{1}{2}(\Phi_r + \Psi_s) \subset \Omega_{\sqrt{rs}}. \quad (7)$$

Therefore the Brunn-Minkowski inequality yields that

$$M(\sqrt{rs}) \geq \left(\frac{F(r)^{\frac{1}{n}} + G(s)^{\frac{1}{n}}}{2} \right)^n \geq \sqrt{F(r) \cdot G(s)} \quad (8)$$

for all $r, s > 0$. In particular we deduce the Prékopa-Leindler inequality by Theorem 1.3, as

$$\int_{\mathbb{R}^n} m = \int_0^\infty M(t) dt \geq \sqrt{\int_0^\infty F(t) dt \cdot \int_0^\infty G(t) dt} = \sqrt{\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g}.$$

The main goal of this section is to prove a stability version of the Prékopa-Leindler inequality at least for even functions. First let

$$\omega(\varepsilon) = \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}},$$

which is the error estimate in Theorem 1.2 (and hence the error estimate in Corollary 1.4). If φ and ψ are real functions, then we write $\varphi \ll \psi$ if there exists a $\gamma > 0$ depending only on n such that $|\varphi| \leq \gamma \cdot \psi$.

THEOREM 4.1 *If $m, f, g : \mathbb{R}^n \rightarrow [0, \infty]$ are even and integrable such that m is log-concave, $m(\frac{x+y}{2}) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$, and*

$$\int_{\mathbb{R}^n} m \leq (1 + \varepsilon) \sqrt{\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g}$$

for $\varepsilon > 0$, then there exists $a > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^n} |a f(x) - m(x)| dx &\ll \sqrt{\omega(\varepsilon)} \int_{\mathbb{R}^n} m \\ \int_{\mathbb{R}^n} |a^{-1} g(x) - m(x)| dx &\ll \sqrt{\omega(\varepsilon)} \int_{\mathbb{R}^n} m. \end{aligned}$$

Proof: As in the one dimensional case (see K.M. Ball, K.J. Böröczky [3], Section 6), we may assume that $f, g : \mathbb{R}^n \rightarrow [0, \infty]$ are even and log-concave probability distributions. We may also assume that $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0 \in (0, 1)$ is chosen in a suitable way and depends only on n .

We define Φ_t, Ψ_t, Ω_t and $F(t), G(t), M(t)$ analogously as at the beginning of the section. We observe that Φ_t, Ψ_t, Ω_t are o -symmetric convex bodies, and $F(t), G(t), M(t)$ are decreasing and log-concave, and F, G are probability distributions on $[0, \infty]$. Since $\int_0^\infty M = \int_{\mathbb{R}^n} m \leq (1 + \varepsilon)$, it follows from Corollary 1.4 that there exists some $b > 0$ such that

$$\begin{aligned} \int_0^\infty |bF(bt) - M(t)| dt &\ll \omega(\varepsilon) \\ \int_0^\infty |b^{-1}G(b^{-1}t) - M(t)| dt &\ll \omega(\varepsilon). \end{aligned}$$

We may assume that $b \geq 1$. For $x \in \mathbb{R}^n$, we define

$$\begin{aligned} \tilde{f}(x) &= b^{-1} f(b^{\frac{1}{n}} x) \\ \tilde{g}(x) &= b g(b^{\frac{1}{n}} x). \end{aligned}$$

The main strategy of the proof is as follows. First we verify

$$\int_{\mathbb{R}^n} |\tilde{f}(x) - \tilde{g}(x)| dx \ll \sqrt{\omega(\varepsilon)}. \quad (9)$$

Along the way, we establish $b - 1 \ll \sqrt{\omega(\varepsilon)}$, which in turn yields

$$\int_{\mathbb{R}^n} |f(x) - g(x)| dx \ll \sqrt{\omega(\varepsilon)}. \quad (10)$$

Finally we conclude Theorem 4.1 from (10) and (7).

For $t > 0$, let

$$\begin{aligned} \tilde{\Phi}_t &= \{x \in \mathbb{R}^n : \tilde{f}(x) \geq t\} \quad \text{where } \tilde{\Phi}_t = b^{\frac{1}{n}} \Phi_{bt} \text{ if } \tilde{\Phi}_t \neq \emptyset \\ \tilde{\Psi}_t &= \{x \in \mathbb{R}^n : \tilde{g}(x) \geq t\} \quad \text{where } \tilde{\Psi}_t = b^{-\frac{1}{n}} \Psi_{b^{-1}t} \text{ if } \tilde{\Psi}_t \neq \emptyset. \end{aligned}$$

These sets satisfy

$$\begin{aligned}\int_0^\infty ||\tilde{\Phi}_t| - M(t)| dt &\ll \omega(\varepsilon) \\ \int_0^\infty ||\tilde{\Psi}_t| - M(t)| dt &\ll \omega(\varepsilon),\end{aligned}$$

and (7) yields that if $\tilde{\Phi}_t \neq \emptyset$ and $\tilde{\Psi}_t \neq \emptyset$ for $t > 0$, then

$$\frac{1}{2}(b^{\frac{-1}{n}}\tilde{\Phi}_t + b^{\frac{1}{n}}\tilde{\Psi}_t) \subset \Omega_t. \quad (11)$$

The main task is to estimate the L_1 distance of \tilde{f} and \tilde{g} using

$$\int_{\mathbb{R}^n} |\tilde{f}(x) - \tilde{g}(x)| dx = \int_0^\infty |\tilde{\Phi}_t \Delta \tilde{\Psi}_t| dt.$$

We dissect $[0, \infty)$ into I and J , where $t \in I$, if $\frac{3}{4}M(t) < |\tilde{\Phi}_t| < \frac{5}{4}M(t)$ and $\frac{3}{4}M(t) < |\tilde{\Psi}_t| < \frac{5}{4}M(t)$, and $t \in J$ otherwise. If $t \in J$, then

$$|\tilde{\Phi}_t \Delta \tilde{\Psi}_t| \leq |\tilde{\Phi}_t| + |\tilde{\Psi}_t| \leq 10 \left(||\tilde{\Phi}_t| - M(t)| + ||\tilde{\Psi}_t| - M(t)| \right).$$

Therefore

$$\int_J |\tilde{\Phi}_t \Delta \tilde{\Psi}_t| dt \ll \omega(\varepsilon). \quad (12)$$

In addition if ε_0 is small enough, then

$$\int_J M(t) dt \leq 4 \int_J \left(||\tilde{\Phi}_t| - M(t)| + ||\tilde{\Psi}_t| - M(t)| \right) dt \ll \omega(\varepsilon) < \frac{1}{2}. \quad (13)$$

Turning to I , it follows from the Prékopa-Leindler inequality and (13) that

$$\int_I M(t) dt \geq 1 - \int_J M(t) dt > \frac{1}{2}. \quad (14)$$

For $t \in I$, we define $\alpha(t) = |\tilde{\Phi}_t|/M(t)$ and $\beta(t) = |\tilde{\Psi}_t|/M(t)$, and hence $\frac{3}{4} < \alpha(t), \beta(t) < \frac{5}{4}$, and

$$\int_0^\infty M(t) \cdot (|\alpha(t) - 1| + |\beta(t) - 1|) dt \ll \omega(\varepsilon). \quad (15)$$

In addition let

$$\begin{aligned}\sigma(t) &= \sigma\left(b^{\frac{-1}{n}}\tilde{\Phi}_t, b^{\frac{1}{n}}\tilde{\Psi}_t\right) = \max\left\{\frac{b^2\beta(t)}{\alpha(t)}, \frac{\alpha(t)}{b^2\beta(t)}\right\} \\ \eta(t) &= \frac{(\sigma(t) - 1)^2}{32n\sigma(t)^2} + \frac{n\gamma^*}{\sigma(t)^{\frac{1}{n}}} \cdot A(\tilde{\Phi}_t, \tilde{\Psi}_t)^2,\end{aligned}$$

where γ^* comes from Theorem 3.1. It follows from $\alpha(t), \beta(t) > \frac{3}{4}$, (6) and (11) that

$$\begin{aligned}M(t) &\geq M(t) \cdot \sqrt{\alpha(t) \cdot \beta(t)}(1 + \eta(t)) \\ &\geq M(t) \cdot (1 - \max\{0, 1 - \alpha(t)\} - \max\{0, 1 - \beta(t)\})(1 + \eta(t)) \\ &\geq M(t) \cdot (1 - |\alpha(t) - 1| - |\beta(t) - 1| + \tfrac{1}{2}\eta(t)).\end{aligned}$$

In particular (15) yields

$$\int_I M(t) \cdot \eta(t) dt \ll \omega(\varepsilon). \quad (16)$$

Next we estimate b . Let $t \in I$. If $\alpha(t) \geq b\beta(t)$ then

$$|\alpha(t) - 1| + |\beta(t) - 1| \geq \frac{\sqrt{b} - 1}{\sqrt{b}} \geq \frac{b - 1}{2b} \geq \frac{(b - 1)^2}{32nb^2}.$$

If $\alpha(t) < b\beta(t)$ then $\sigma(t) > b$, and

$$\eta(t) > \frac{(b - 1)^2}{32nb^2}.$$

We deduce by (14), (15) and (16) that

$$\frac{(b - 1)^2}{64nb^2} \leq \int_I M(t) \cdot \frac{(b - 1)^2}{32nb^2} dt \leq \int_0^\infty M(t) \cdot (\eta(t) + |\alpha(t) - 1| + |\beta(t) - 1|) dt \ll \omega(\varepsilon).$$

Since $\frac{b-1}{b} > \frac{1}{2}$ if $b > 2$, we deduce that

$$b - 1 \ll \sqrt{\omega(\varepsilon)}. \quad (17)$$

It also follows that $\sigma(t) < 2$ if ε_0 is small enough, and hence $A(\tilde{\Phi}_t, \tilde{\Psi}_t)^2 \ll \eta(t)$.

For $t \in I$, we deduce using (5) that

$$\begin{aligned} |\tilde{\Phi}_t \Delta \tilde{\Psi}_t|^2 &\leq 3 \left[|\alpha(t)^{\frac{-1}{n}} \tilde{\Phi}_t \Delta \beta(t)^{\frac{-1}{n}} \tilde{\Psi}_t|^2 + |\alpha(t)^{\frac{-1}{n}} \tilde{\Phi}_t \Delta \tilde{\Phi}_t|^2 + |\beta(t)^{\frac{-1}{n}} \tilde{\Psi}_t \Delta \tilde{\Psi}_t|^2 \right] \\ &= 3 \left[A(\tilde{\Phi}_t, \tilde{\Psi}_t)^2 + |\alpha(t) - 1|^2 + |\beta(t) - 1|^2 \right] \cdot M(t)^2. \end{aligned}$$

In turn we have

$$\begin{aligned} \left(\int_I |\tilde{\Phi}_t \Delta \tilde{\Psi}_t| dt \right)^2 &\leq \int_I \frac{|\tilde{\Phi}_t \Delta \tilde{\Psi}_t|^2}{M(t)} dt \int_I M(t) dt \\ &\ll \int_I A(\tilde{\Phi}_t, \tilde{\Psi}_t)^2 \cdot M(t) dt + \omega(\varepsilon) \\ &\ll \int_I \eta(t) \cdot M(t) dt + \omega(\varepsilon) \ll \omega(\varepsilon). \end{aligned}$$

Combining this estimate with (12) yields

$$\int_{\mathbb{R}^n} |\tilde{f}(x) - \tilde{g}(x)| dx = \int_0^\infty |\tilde{\Phi}_t \Delta \tilde{\Psi}_t| dt \ll \sqrt{\omega(\varepsilon)}. \quad (18)$$

Turning to the L_1 distance of f and g , $f(b^{\frac{-1}{n}}x) \geq f(x)$ for $x \in \mathbb{R}^n$ and the estimate (17) on b yield

$$\begin{aligned} \int_{\mathbb{R}^n} |f - \tilde{f}| &\leq \int_{\mathbb{R}^n} (f - b^{-1}f) + \int_{\mathbb{R}^n} [b^{-1}f(b^{\frac{-1}{n}}x) - b^{-1}f(x)] dx \\ &= 2(1 - b^{-1}) \ll \sqrt{\omega(\varepsilon)}. \end{aligned}$$

Similarly $\int_{\mathbb{R}^n} |g(x) - \tilde{g}(x)| dx \ll \sqrt{\omega(\varepsilon)}$, therefore (18) implies

$$\int_{\mathbb{R}^n} |f(x) - g(x)| dx = \int_0^\infty |\Phi_t \Delta \Psi_t| dt \leq \gamma_1 \sqrt{\omega(\varepsilon)} \quad (19)$$

for a $\gamma_1 > 0$ depending only on n .

Finally, we compare f and m . It follows by (7) that if $\Phi_t, \Psi_t \neq \emptyset$ for $t > 0$, then

$$\Phi_t \cap \Psi_t \subset \Omega_t. \quad (20)$$

Using the γ_1 of (19), we have

$$\begin{aligned} 1 - \gamma_1 \sqrt{\omega(\varepsilon)} &\leq \int_0^\infty |\Phi_t| dt - \int_0^\infty |\Phi_t \Delta \Psi_t| dt \\ &\leq \int_0^\infty |\Phi_t \cap \Psi_t| dt \leq \int_0^\infty |\Omega_t| dt \leq 1 + \varepsilon. \end{aligned}$$

Since (20) yields that

$$|\Phi_t \Delta \Omega_t| \leq |\Phi_t \Delta \Psi_t| + |\Omega_t| - |\Phi_t \cap \Psi_t|,$$

we conclude that

$$\int_{\mathbb{R}^n} |f(x) - m(x)| dx = \int_0^\infty |\Phi_t \Delta \Omega_t| dt \ll \sqrt{\omega(\varepsilon)}.$$

Similarly we have $\int_{\mathbb{R}^n} |g(x) - m(x)| dx \ll \sqrt{\omega(\varepsilon)}$. \square

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