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Finite Packing and Covering by Congruent Convex Domains*

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Abstract. For a convex domain K, let $H(K)$ be a circumscribed polygon with at most six sides whose area is minimal, and let $\tilde{H}(K)$ be an inscribed hexagon with at most six sides whose area is maximal. According to the celebrated result by L. Fejes Tóth [6], if a hexagon contains n non-overlapping congruent copies of K, then its area is at least $n \cdot A(H(K))$, and if n pairwise non-crossing congruent copies of K cover a hexagon, then its area is at most $n \cdot A(\tilde{H}(K))$. Here two convex domains C_1 and C_2 are non-crossing if there exist complementary half-planes l^- and l^+ such that $l^- \cap C_1 \subset C_2$ and $l^+ \cap C_2 \subset C_1$. In this paper we generalize the results of L. Fejes Tóth to packings inside or coverings of any convex domain provided that the number of copies is high enough. In the case of packings of centrally symmetric domains, our results are optimal. Finally, let K be centrally symmetric, and let D_n be the convex domain with minimal area containing n non-overlapping congruent copies of K. Then we show that $R(D_n)/r(D_n)$ stays bounded as n tends to infinity.

1. Introduction

For a given convex domain K, let $H(K)$ be a circumscribed polygon with at most six sides whose area is minimal. It is well known (see [6]) that if a polygon P having at most six sides contains n non-overlapping congruent copies of K, then

$$A(P) \geq n \cdot A(H(K)). \tag{1}$$

In this paper we provide a stronger version of this theorem:

Theorem 1. *If a convex domain D contains n non-overlapping congruent copies of a convex domain K, then*

$$A(D) \geq n \cdot A(H(K))$$

provided that $n \geq N$ where N depends only on K.

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We show in Section 3 that the N in Theorem 1 does depend on K (even for centrally symmetric domains), and cannot be chosen to be an absolute constant. This fact was already known to L. Fejes Tóth.

The constant $A(H(K))$ is not optimal in Theorem 1 for the typical convex domain K (in the sense of the Baire category, see [7] for typical properties of convex domains). This can be shown by extending the method of G. Fejes Tóth [4]. On the other hand, if K is centrally symmetric, then one can choose $H(K)$ to be centrally symmetric according to Dowker [2]. Thus there exists a lattice tiling by translates of $H(K)$, and hence taking large domains as D shows that the constant $A(H(K))$ is optimal in Theorem 1 in this case. The proof of Theorem 1 also yields

Corollary 2. *For a centrally symmetric convex domain K that is not a parallelogram, let D_n be a convex domain with minimal area that contains n non-overlapping congruent copies of K. Then*

$$c_1 \cdot \sqrt{n} < r(D_n) < R(D_n) < c_2 \cdot \sqrt{n},$$

where the positive constants c_1 and c_2 depend on K.

Just as L. Fejes Tóth [6], we need some extra assumptions in case of coverings: We call two convex domains C_1 and C_2 *non-crossing* if there exist complementary half-planes l^- and l^+ such that $l^- \cap C_1 \subset C_2$ and $l^+ \cap C_2 \subset C_1$. We note that translates are always non-crossing. For a convex domain K, let $\tilde{H}(K)$ be an inscribed hexagon with at most six sides whose area is maximal.

Theorem 3. *If a convex domain D is covered by n non-crossing congruent copies of a convex domain K, then*

$$A(D) \leq n \cdot A(\tilde{H}(K))$$

provided that $n \geq \tilde{N}$ where $\tilde{N}$ depends only on K.

If K is centrally symmetric, then one can choose $\tilde{H}(K)$ to be centrally symmetric according to Dowker [2]. Thus there exists a lattice tiling by translates of $\tilde{H}(K)$, and hence taking large domains as D shows that the constant $A(\tilde{H}(K))$ is optimal in Theorem 3. On the other hand, the constant $A(\tilde{H}(K))$ is not optimal in Theorem 3 for the typical convex domain K. This can be shown by extending the method of G. Fejes Tóth and Zamfirescu [5].

The $\tilde{N}$ in Theorem 3 can be most probably chosen to be an absolute constant. For translative coverings, $\tilde{N} = 26$ works according to G. Fejes Tóth [3].

It is widely believed that Theorem 3 holds without the assumption that the copies are non-crossing. This conjecture seems to be rather obvious to believe, yet it may happen that a convex domain D can be covered by n congruent copies of a convex domain K, but it cannot be done in a pairwise non-crossing manner. We recall the following example due to Heppes: Let D be a unit square, and let K be the convex hull of the midpoints of the sides of D and two opposite vertices. Then K and a rotated image by $\pi/2$ cover D. On the other hand, it is not hard to see that two non-crossing copies of K cannot cover D.

The arguments in this paper are variations of the original proofs in [6]. In particular, they depend on the Dowker theorems (see [2]).

2. Imitating a Cell Decomposition

One of the fundamental observations in [6] was that the average side for a cell decomposition of a convex domain is at most six. First we improve on this bound if the number of the edges on the boundary are large.

A planar topological cell complex Σ is a collection of finitely many two-cells, edges (the one-cells) and vertices (the zero-cells). The edges are Jordan arcs (that are convex arcs in this paper), and the endpoints of the edges are among the vertices. In addition, the two-cells are regions that are bounded by finitely many edges. The fundamental property that makes Σ a cell complex is that the intersection of any two of the cells is a cell itself. We write $f_i(\Sigma)$ to denote the number of i-cells of Σ. Then the Euler formula yields

Proposition 4. *Let Σ be topological cell complex such that* supp Σ *is a convex domain, and each vertex is of degree at least three. We write $k_1, \ldots, k_{f_2(\Sigma)}$ to denote the numbers of sides of the two-cells, and b to denote the number of edges of Σ that are contained in exactly one two-cell. Then*

$$\sum_{i=1}^{f_2(\Sigma)} (6 - k_i) \geq b + 6.$$

Proof. Since $3 \cdot f_0(\Sigma) \leq 2 \cdot f_1(\Sigma)$, we deduce by the Euler formula $f_0(\Sigma) - f_1(\Sigma) + f_2(\Sigma) = 1$ that

$$6 \cdot f_2(\Sigma) \geq 2 \cdot f_1(\Sigma) + 6.$$

On the other hand, counting the number of sides of each two-cell shows that $\sum_{i=1}^{f_2(\Sigma)} k_i = 2 \cdot f_1(\Sigma) - b$. □

The first step in order to prove Theorem 1 would be to define a cell decomposition of D into convex cells such that each cell contains exactly one of the congruent copies of K. The only little problem is that such a cell decomposition may not exist, therefore we save the essential properties of a cell decomposition following the ideas in [6].

Lemma 5. *Let D be a convex domain that contains the non-overlapping convex domains $K_1, \ldots, K_n$. Then there exist non-overlapping convex domains $\Pi_1, \ldots, \Pi_n \subset D$ satisfying the following properties*:

(i) $K_i \subset \Pi_i$.
(ii) $\Pi_1, \ldots, \Pi_n$ *cover* ∂D.
(iii) Π_i *is bounded by* $k_i \geq 2$ *convex arcs that we call edges. The edges intersecting* int D *are segments, and the rest of the edges are the maximal convex arcs of* $\partial D \cap \Pi_i$.

(iv) *The number b of edges contained in ∂D satisfy*

$$\sum_{i=1}^{n}(6-k_i) \geq b+6.$$

Proof. Let $\Pi_1, \ldots, \Pi_n$ be non-overlapping convex domains such that $K_i \subset \Pi_i \subset D$, and the total area covered by the convex domains $\Pi_1, \ldots, \Pi_n$ is maximal under these conditions. Since two non-overlapping convex sets can be separated by a line, each Π_i is the intersection of a polygon P_i and D. Now $\operatorname{int} P_i \cap \partial D$ consists of finitely many convex arcs whose closures we call edges of Π_i. The set of vertices of Π_i consists of the endpoints of its edges in ∂D, and all the vertices of some P_j that are contained in $\Pi_i \cap \operatorname{int} D$. These vertices divide $\partial \Pi_i \cap \operatorname{int} D$ into finitely many segments whose closures form the rest of the edges of Π_i.

Now $\Pi_1, \ldots, \Pi_n$ may not cover D, and the closure of a connected component of $\operatorname{int} D \setminus \bigcup_{i=1}^{n} \Pi_i$ is called a hole. The maximality of $A(\Pi_i)$ yields that each hole Q is a possibly non-convex domain that is bounded by finitely many convex arcs $s_1, \ldots, s_k$ with the following properties: s_{j-1} and s_j share a common endpoint for $j = 1, \ldots, k$ (where $s_0 = s_k$) that is a vertex of some Π_i, and no other intersection occurs among $s_1, \ldots, s_k$, and no other vertices of any Π_i are contained in Q. We note that each s_j is contained either in the boundary of D or in a straight edge of Π_i.

We may assume that s_1 intersects $\operatorname{int} D$, and is contained in the edge e_1 of Π_{i_1}. Since $A(P_1)$ is maximal, we deduce that one endpoint v_2 of s_1 lies in the relative interior of e_1. We may assume that v_2 is the common endpoint of s_1 and s_2. Then s_2 is contained in an edge e_2 of some Π_{i_2} where v_2 is an endpoint of e_2, and the other endpoint v_3 of s_2 lies in the relative interior of e_2. Now we obtain by induction that s_j is contained in an edge e_j of some Π_{i_j}, and, for $j = 2, \ldots, k$, the common endpoint v_j of s_{j-1} and s_j is a vertex of e_j, and lies in the relative interior of e_{j-1}. We also deduce that the common endpoint v_1 of s_k and s_1 is a vertex of e_1, and lies in the relative interior of e_k, and hence Q is a convex polygon that is contained in the interior of D. In particular, we conclude that $\Pi_1, \ldots, \Pi_n$ cover ∂D.

Finally, in order to estimate the average number of sides of $\Pi_1, \ldots, \Pi_n$, we construct a related topological cell decomposition Σ of D. If there exists no hole, then $\Pi_1, \ldots, \Pi_n$ defines a cell decomposition, and (iv) follows directly from Proposition 4. Otherwise let $\{Q_1, \ldots, Q_m\}$ be the set of holes, and let $q_j \in \operatorname{int} Q_j$. The idea for defining Σ is to shrink each Q_j to q_j. The two-cells of Σ are $\Pi_1^*, \ldots, \Pi_n^*$, where Π_i^* is the union of Π_i and all triangles of the form $\operatorname{conv}\{q_j, s\}$ such that s is a side of Q_j and $e \subset \Pi_i$. We note that Π_i^* and Π_j^* do not overlap for $i \neq j$. Now an edge of Σ is either an edge of some Π_i contained in ∂D, or of the form $\Pi_i^* \cap \Pi_j^*$ for $i \neq j$ if it contains a segment.

Let us assume that the intersection of Π_i^* and Π_j^*, $i \neq j$, contains a segment. Then $\Pi_i^* \cap \Pi_j^*$ is the union of $\Pi_i \cap \Pi_j$ and any segment $q_k v$ where the hole Q_k has one–one sides in Π_i and Π_j, and these two sides meet at v. We call the vertex v of some Π_i a dead vertex if it is the common vertex for two holes, and not the vertex for any other Π_j. Therefore the family of vertices for Σ is the union of the vertices of $\Pi_1, \ldots, \Pi_n$ except for the dead vertices, and Σ is actually a topological cell complex. Now Π_i^* has at least as many edges as Π_i has, and hence (iv) is a consequence of Proposition 4. $\square$

3. Packing Congruent Domains

For a convex domain K and $n \geq 3$, we write $t_K(n)$ to denote the minimal area of polygons with at most n sides containing K. In particular, $t_K(6) = A(H(K))$. According to the Dowker theorem for circumscribed polygons (see [2]), $t_K(n)$ is a convex function of n. We define $t_K(2) = 2 \cdot t_K(3)$. Since $t_K(3) \leq 2 \cdot t_K(4)$, the function $t_K(n)$ stays convex even for $n \geq 2$.

In order to prove Theorem 1, we may assume that K is not a polygon with at most five sides, and hence $t_K(5) > t_K(6)$ holds. In this case we verify that there exist positive constants γ_1 and γ_2 depending on K such that

$$A(D) > n \cdot A(H(K)) + \gamma_1 \cdot P(D) - \gamma_2. \tag{2}$$

We write $K_1, \ldots, K_n$ to denote the non-overlapping congruent copies of K, and we may assume that D is the convex hull of these domains. Let $\Pi_1, \ldots, \Pi_n$ be the convex domains associated to $K_1, \ldots, K_n$ by Lemma 5. In this proof, σ always denotes an edge of some Π_i that is contained in ∂D. Let $\sigma \subset \Pi_i$. We write x_i to denote the centre of a circle with radius $r(K)$ inscribed into K_i, and $C(\sigma, x_i)$ to denote the union of all segments connecting x_i to the points of σ. Now any tangent lines to ∂D at the points of σ avoid K_i, which in turn yields that

$$A\left(C(\sigma, x_i)\right) \geq \tfrac{1}{2} \cdot r(K) \cdot |\sigma|,$$

where $|\cdot|$ stands for the arc length. We deduce that there exist positive constants λ and c_1 such that if $|\sigma| > \lambda$, then

$$A\left(C(\sigma, x_i)\right) \geq t_K(2) + c_1 \cdot |\sigma|. \tag{3}$$

Next we assume that $|\sigma| \leq \lambda$, and let p and q denote the endpoints of σ. The total curvature $\alpha(\sigma)$ is defined to be the variation of the angle of the tangent from p to q along σ (it might be larger than π). If $\alpha(\sigma) > 0$, then σ intersects Π_i because D is the convex hull of $K_1, \ldots, K_n$. Therefore there exist positive $\alpha < \pi/6$ and β depending on λ and K satisfying the following property: if $\alpha(\sigma) < \alpha$, then the other edge of Π_i at p encloses an angle larger than β with the segment pq, and a similar property holds for q. Let $Q(\sigma)$ be the quadrilateral that is bounded by the line pq, the lines of the other edges of Π_i at p and q, and the tangent of σ that is parallel to pq. Then there exists a constant c_2 depending on λ, α and β such that if $\alpha(\sigma) < \alpha$, then

$$A\left(Q(\sigma)\right) \leq c_2 \cdot \alpha(\sigma). \tag{4}$$

We deduce by (3) and (4) that

$$A(\Pi_i) \geq t_K(k_i) + \sum_{\substack{\sigma\subset\Pi_i \\ |\sigma|>\lambda}} c_1 \cdot |\sigma| - \sum_{\substack{\sigma\subset\Pi_i \\ |\sigma|\leq\lambda \\ \alpha(\sigma)<\alpha}} c_2 \cdot \alpha(\sigma) - \sum_{\substack{\sigma\subset\Pi_i \\ |\sigma|\leq\lambda \\ \alpha(\sigma)\geq\alpha}} t_K(2).$$

Since the total curvature of ∂D is 2π, it follows that

$$A(D) \geq \sum_{i=1}^{n} t_K(k_i) + \sum_{|\sigma|>\lambda} c_1 \cdot |\sigma| - c_2 \cdot 2\pi - \frac{2\pi}{\alpha} \cdot t_K(2). \tag{5}$$

Now the concavity of $t_K(n)$ yields that

$$t_K(k_i) \geq A(H(K)) + (t_K(5) - t_K(6)) \cdot (6 - k_i).$$

We write b to denote the number of edges of $\Pi_1, \dots, \Pi_n$ that are contained in ∂D. Then Lemma 5(iv) yields that

$$\sum_{i=1}^{n} t_K(k_i) \geq n \cdot A(H(K)) + (t_K(5) - t_K(6)) \cdot b$$
$$\geq n \cdot A(H(K)) + \sum_{|\sigma| \leq \lambda} \frac{t_K(5) - t_K(6)}{\lambda} \cdot |\sigma|.$$

In turn, we conclude (2) by (5). Now $A(D) \geq n \cdot A(K)$ and the isoperimetric inequality yield that $P(D) > 2\sqrt{A(K)/\pi} \cdot \sqrt{n}$, therefore $A(D) > n \cdot A(H(K))$ holds for large n. □

One may hope that the N in Theorem 1 can be chosen to be an absolute constant. We now present an example showing that this is not the case, not even if K is centrally symmetric: Let $\varepsilon > 0$ be small, and consider in a coordinate system the convex hull of the points $p = (1, 0)$, $q = (0, 1)$, and the hyperbole arc with the equation $x \cdot y = \varepsilon$ in the positive corner. We write γ to denote the arc between p and q on the boundary. We define K to be the centrally symmetric convex domain whose boundary consists of γ and three other arcs congruent with γ. If s is the segment with length $4(n-1)$ parallel to the first coordinate axis, then $s + K$ contains n non-overlapping translates of K, and

$$A(s + K) = (n-1) \cdot 4 + A(K) = n \cdot 4 - \varepsilon \cdot \ln \frac{1}{4\varepsilon} - 2\varepsilon.$$

On the other hand, any tangent to the hyperbole arc and the coordinate axes enclose a triangle of area 2ε, and hence $A(H(K)) \geq 4 - 12\varepsilon$. Therefore the N of Theorem 1 satisfies

$$N \geq \frac{1}{12} \cdot \ln \frac{1}{4\varepsilon}.$$

Finally, we investigate the shape of the optimal packing. Let K be a centrally symmetric convex domain that is not a parallelogram, and let D_n be a convex domain with minimal area that contains n non-overlapping congruent copies of K.

We may assume that $H(K)$ is centrally symmetric according to Dowker [2], and hence there exits a tiling of the plane by translates of $H(K)$. If B_n is a circle with minimal radius that contains n tiles, then

$$A(D_n) \leq A(B_n) \leq n \cdot A(H(K)) + \gamma \cdot \sqrt{n}, \tag{6}$$

where γ depends on K. We deduce by (2) that the perimeter of D_n satisfies $P(D_n) \leq \gamma' \cdot \sqrt{n}$ for large n and for some constant γ' depending on K. Therefore $r(D_n) \cdot P(D_n) > A(D_n)$ yields that $r(D_n) > A(K)/\gamma' \cdot \sqrt{n}$. We conclude $R(D_n) = O(\sqrt{n})$, which fact completes the proof of Corollary 2.

4. Coverings by Non-Crossing Congruent Domains

For any finite non-crossing covering of a convex domain D, there exists an associated cell decomposition of D. This fact was observed by L. Fejes Tóth (see [6]) but his argument seemed to have some gaps. A rather long detailed proof was provided in [1]. Here we present an argument for the sake of completeness.

Lemma 6. *Let D be a convex domain that is covered by the pairwise non-crossing convex domains $K_1, \dots, K_n$ such that no $(n-1)$ copies out of $K_1, \dots, K_n$ cover D. Then there exists a cell decomposition of D into the two-cells $\Pi_1, \dots, \Pi_n \subset D$, satisfying the following properties*:

(i) $\Pi_i \subset K_i$.

(ii) *We write k_i to denote the number of edges of Π_i, and b to denote the total number of edges contained in ∂D. Then*

$$\sum_{i=1}^{n}(6-k_i) \geq b+6.$$

Proof. There exists a covering $\Pi_1, \dots, \Pi_n$ of D by convex, compact sets such that each $\Pi_i \subset K_i$, the sets $\Pi_1, \dots, \Pi_n$ are pairwise non-crossing, and $\sum_i A(\Pi_i)$ is minimal under the previous two conditions. Now each Π_i is a convex domain because each K_i is needed in order to cover D.

We suppose that there exist some Π_i and Π_j, $i \neq j$, that overlap, and seek a contradiction. The idea is that we can cut off a small part of certain Π_i in a way that the resulting family still forms a non-crossing cover of D. The difficulty is to ensure that the resulting family is pairwise non-crossing.

For $i < j$, we write l_{ij} to denote some line that witnesses that Π_i and Π_j are non-crossing, and l_{ij}^+ and l_{ij}^- to denote the half-planes such that $l_{ij}^+ \cap \Pi_j$ contains $l_{ij}^+ \cap \Pi_i$, and $l_{ij}^- \cap \Pi_i$ contains $l_{ij}^- \cap \Pi_j$, respectively. We may assume that Π_2 overlaps l_{12}^-, and hence there exists a supporting line l to Π_2 that intersects Π_2 in a single point p where $p \in \operatorname{int} l_{12}^-$.

After possibly renumbering the domains $\Pi_3, \dots, \Pi_n$, let $\Pi_2, \dots, \Pi_m$, $2 \leq m \leq n$, form the family of Π_i, $i \geq 2$, such that $p \in \Pi_i$, and p has a neighbourhood U_i such that $\Pi_i \cap U_i \subset \Pi_2 \cap U_i$. Then there exists a half-plane $\tilde{l}^+$ such that the line $\tilde{l}$ bounding $\tilde{l}^+$ is parallel to l, the interior of $\tilde{l}^+$ contains p, and $\tilde{l}^+ \cap \Pi_i \subset \tilde{l}^+ \cap \Pi_2$ holds for $i = 2, \dots, m$. In addition we may assume that if a domain Π_k or a line l_{ij} does not contain p, then it does not intersect the cap $\tilde{l}^+ \cap \Pi_2$.

We define Π_i' to be the closure of $\Pi_i \backslash \tilde{l}^+$ if $i = 2, \dots, m$, and $\Pi_i' = \Pi_i$ otherwise. Then $\Pi_1', \dots, \Pi_n'$ cover D because $\tilde{l}^+ \cap \Pi_2$ lies in $l_{12}^- \cap \Pi_1$. It also follows that each Π_i' is a convex domain and $\Pi_i' \subset K_i$. Now l_{ij} witnesses that Π_i' and Π_j' are non-crossing for any $i < j$ unless $2 \leq i \leq m$, $j > m$, $p \in l_{ij}$ and $\Pi_j' = \Pi_j$ overlaps l_{ij}^-. In this case we distinguish two possibilities: If Π_i' and Π_j' do not overlap, then they are readily non-crossing. Otherwise, the half-line $\tilde{l} \cap l_{ij}^-$ intersects $\partial \Pi_j$ in a point $p' \in \operatorname{int} l_{ij}^-$. Let $q \neq p$ be the other endpoint of $l_{ij} \cap \Pi_j$, and hence the line $p'q$ witnesses that Π_i' and Π_j' are non-crossing. Since the total area of $\Pi_1', \dots, \Pi_n'$ is less than the total area of $\Pi_1, \ldots, \Pi_n$, we have arrived at a contradiction. Therefore $\Pi_1, \ldots, \Pi_n$ determine a topological cell decomposition of D, and hence (ii) is a consequence of Proposition 4. □

For a convex domain K, we write $\tilde{t}_K(n)$ to denote the maximal area of an inscribed polygon with at most n sides. In particular, $A(\tilde{H}(K)) = \tilde{t}_K(6)$. We also define $\tilde{t}_K(2) = 0$. Since $\tilde{t}_K(3) \geq \frac{1}{2}\, t_K(4)$, the function $\tilde{t}_K(n)$ is concave for $n \geq 2$ according to the Dowker theorem for inscribed polygons (see [2]).

In order to prove Theorem 3, we may assume that D is not a polygon with at most five sides, and hence $\tilde{t}_K(6) > \tilde{t}_K(5)$. We may also assume that no $n-1$ out of the n congruent copies cover D. Let $\Pi_1, \ldots, \Pi_n$ form the cell decomposition of Lemma 6. We write Π'_i to denote the convex hull of the k_i vertices of Π where Π'_i might be a segment. Then $A(\Pi'_i) \leq \tilde{t}_K(k_i)$ holds by definition, and hence the concavity of $\tilde{t}_K(n)$ yields that

$$A(\Pi'_i) \leq \tilde{t}_K(6) + (\tilde{t}_K(6) - \tilde{t}_K(5)) \cdot (k_i - 6).$$

We conclude by Lemma 6(ii) that

$$\sum_{i=1}^{n} A(\Pi'_i) \leq n \cdot A(\tilde{H}(K)) - (\tilde{t}_K(6) - \tilde{t}_K(5)) \cdot (b + 6). \tag{7}$$

Let σ be an edge of Σ that is contained in ∂D, and write $\alpha(\sigma)$ to denote the total curvature of σ. Since σ is contained in a translate of K, it is not hard to see that

$$A(\operatorname{conv} \sigma) \leq c \cdot \alpha(\sigma)$$

holds for some c depending on K. Now the total curvature of ∂D is 2π, and hence

$$A(D) \leq \sum_{i=1}^{n} A(\Pi'_i) + c \cdot 2\pi.$$

Thus Theorem 3 follows by (7) if $b \geq c \cdot 2\pi/(\tilde{t}_K(6) - \tilde{t}_K(5))$. Finally let $b < c \cdot 2\pi/(\tilde{t}_K(6) - \tilde{t}_K(5))$. Since the perimeter of D is at most $b \cdot P(K)$, the isoperimetric inequality yields $A(D) < A_0$ where A_0 depends only on K. Therefore we may choose $\tilde{N} = A_0/A(\tilde{H}(K))$.

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