Stability of some interrelated geometric and functional inequalities

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November 30, 2009

Brunn-Minkowski, Prékopa-Leindler, Blaschke-Santaló
known cases of Mahler's conjecture

Notation

- c - positive absolute constant
- o - origin in $\mathbb{R}^n$
- $\langle x, y \rangle$ - scalar product in $\mathbb{R}^n$
- $|x|$ - l_2 norm in $\mathbb{R}^n$
- B^n - l_2 unit ball in $\mathbb{R}^n$
- K and M are convex bodies in $\mathbb{R}^n$
- $|K|$ - volume of K

The Brunn-Minkowski inequality

$|K| = |M| = 1$, $0 < \lambda \leq \frac{1}{2}$, o is the centroid of K and M

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$|\lambda K + (1 - \lambda)M)| \geq 1$, equality iff $K = M$.

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Figalli, Maggi, Pratelli

$$|\lambda K + (1-\lambda)M)| \geq 1 + cn^{-7}\lambda \cdot |K\Delta M|^2$$

The Prékopa-Leindler inequality

Theorem (Prékopa-Leindler)

If m, f, g are non-negative integrable functions on $\mathbb{R}^n$ satisfying $m(\frac{1}{2}(x+y)) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} m \geq \sqrt{\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} g}.$$

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Equality case (Dubuc)

There exist a log-concave h, and $a > 0$, $b \in \mathbb{R}^n$ with positive integral on $\mathbb{R}$ such that for a.e. $x \in \mathbb{R}^n$, we have

$$\begin{aligned} m(x) &= h(x) \\ f(x) &= a \cdot h(x+b) \\ g(x) &= a^{-1} \cdot h(x-b). \end{aligned}$$

The stability version of the Prékopa-Leindler inequality

Conditions

f, g log-concave, $m(\frac{1}{2}(x+y)) \geq \sqrt{f(x)g(y)}$ for $x, y \in \mathbb{R}^n$

$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1$

$\int_{\mathbb{R}^n} xf(x)\, dx = \int_{\mathbb{R}^n} xg(x)\, dx = o$

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L_1 distance $\delta(f,g) = \int_{\mathbb{R}^n} |f-g|$

Conjecture

$$\int_{\mathbb{R}^n} m \geq 1 + \gamma(n) \cdot \delta(f,g)^2$$

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Known cases (Ball, Böröczky)

If either $n = 1$, or f, g even, then

$$\int_{\mathbb{R}^n} m \geq 1 + \gamma(n) \cdot \delta(f,g)^8$$

The Blaschke-Santaló inequality

Polar $K^o = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for any } y \in K\}$.

Theorem (Blaschke-Santaló inequality)

If o is the centroid (or Santaló point) of K, then

$$|K| \cdot |K^o| \leq |B^n|^2$$

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$$\delta_{\mathrm{BM}}(K, M) = \min \{\ln \lambda : K - x \subset \Phi(M - y) \subset \lambda(K - x) \text{ for } \Phi \in \mathrm{GL}(n), x, y \in \mathbb{R}^n\}.$$

Stability (Böröczky) $\delta_{\mathrm{BM}}(K, B^n) = \varepsilon$

$$|K| \cdot |K^o| \leq (1 - \gamma(n)\varepsilon^{5n}) \cdot |B^n|^2$$

Optimal exponent probably $(n + 1)/2$ instead of $5n$

Generalized functional BS inequality

Theorem (Ball, Artstein-Klartag-Milman, Fradelizi-Meyer, Lehec)

For any measurable $f : \mathbb{R}^n \to \mathbb{R}_+$ *with positive integral there exists* $z \in \mathbb{R}^n$ *such that if measurable* $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $g : \mathbb{R}^n \to \mathbb{R}_+$ *with positive integrals satisfy*

$$f(x)g(y) \leq \varrho(\langle x - z, y - z \rangle)^2$$

for every $x, y \in \mathbb{R}^n$ *with* $\langle x - z, y - z \rangle > 0$*, then*

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(x)\,dx \leq \left(\int_{\mathbb{R}^n} \varrho(|x|^2)\,dx \right)^2 .$$

Equality is known, then functions are essentially log-concave.

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for every $x, y \in \mathbb{R}^n$ with $\langle x - z, y - z\rangle > 0$, then

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- If f is even then a suitable z is the origin.
- If f is log-concave, then choose z such that $K_{f,z} = \{x \in \mathbb{R}^n : \int_0^\infty r^{n-1} f(z + rx)\, dx \geq 1\}$ has the origin as its center of mass.

Stability of the generalized functional BS inequality

Theorem (Böröczky, Fradelizi)

Let $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$, $f, g : \mathbb{R}^n \to \mathbb{R}_+$ log-concave with positive integrals, ϱ is non-increasing, the center of mass of $K_{f,z}$ is the origin for $z \in \mathbb{R}^n$, and $f(x)g(y) \leq \varrho(\langle x - z, y - z \rangle)^2$ for every $x, y \in \mathbb{R}^n$ with $\langle x - z, y - z \rangle > 0$, moreover for $\varepsilon \in (0, 1)$,

$$(1+\varepsilon) \int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(x)\,dx \geq \left(\int_{\mathbb{R}^n} \varrho(|x|^2)\,dx \right)^2,$$

then there exist $d > 0$ and a positive definite matrix $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \left| \varrho(|x|^2) - d f(Tx) \right| dx \quad < \quad \gamma \varepsilon^{\frac{1}{32n^2}} \cdot \int_{\mathbb{R}_+} r^{n-1} \varrho(r^2)\,dr$$

$$\int_{\mathbb{R}^n} \left| \varrho(|x|^2) - d^{-1} g(T^{-1}x) \right| dx \quad < \quad \gamma \varepsilon^{\frac{1}{32n^2}} \cdot \int_{\mathbb{R}_+} r^{n-1} \varrho(r^2)\,dr,$$

where γ depends on n, and is polynomial in n.

Mahler's conjecture

$W = [-1,1]^n$, T =simplex with o is the centroid

- If $K = -K$, then $|K| \cdot |K^o| \geq |W| \cdot |W^o|$
- If $o \in \operatorname{int} K$, then $|K| \cdot |K^o| \geq |T| \cdot |T^o|$

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Theorem (Nazarov,Petrov, Ryabogin, Zvavitch)

If $K = -K$, and $\delta_{\mathrm{BM}}(K, W) = \varepsilon \leq \omega$, then

$$|K| \cdot |K^o| \geq [1 + \gamma\varepsilon] \cdot |W| \cdot |W^o|,$$

where $\gamma, \omega > 0$ depend on n.

Theorem (Reisner,Kim,?????)

If $\delta_{\mathrm{BM}}(K, T) = \varepsilon \leq \omega$, then

$$|K| \cdot |K^o| \geq [1 + \gamma\varepsilon] \cdot |T| \cdot |T^o|,$$

where $\gamma, \omega > 0$ depend on n.

Zonoids and unconditional functions

Zonoids

- (Reisner) Mahler conjecture for zonoids
- (Böröczky-Hug) Stability

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Unconditional bodies

- (Saint Raymond, Meyer, Reisner) Mahler conjecture for unconditional convex bodies.
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Unconditional functions

- (Fradelizi, Gordon, Meyer, Reisner) Analogue of Mahler conjecture for log-concave unconditional functions, equality characterized.
- Stability not known