THE L_p DUAL MINKOWSKI PROBLEM FOR $p>1$ AND $q>0$

KÁROLY J. BÖRÖCZKY AND FERENC FODOR

Abstract. General L_p dual curvature measures have recently been introduced by Lutwak, Yang and Zhang [24]. These new measures unify several other geometric measures of the Brunn-Minkowski theory and the dual Brunn-Minkowski theory. L_p dual curvature measures arise from qth dual intrinsic volumes by means of Alexandrov-type variational formulas. Lutwak, Yang and Zhang [24] formulated the L_p dual Minkowski problem, which concerns the characterization of L_p dual curvature measures. In this paper, we solve the existence part of the L_p dual Minkowski problem for $p>1$ and $q>0$, and we also discuss the regularity of the solution.

1. Introduction

In this paper we solve the existence part of the Minkowski problem for L_p dual curvature measures with parameters $p>1$ and $q>0$. It is, in part, aiming at solving the Monge-Ampère equation

$$\det(\nabla^2 h + h\,\mathrm{Id}) = h^{p-1}(\|\nabla h\|^2+h^2)^{\frac{n-q}{2}}\cdot f$$

on S^{n-1}, where the unknown h is the support function of a convex body, ∇h and $\nabla^2 h$ are the gradient and the Hessian of h with respect to a moving orthonormal frame.

The L_p dual curvature measures emerged recently [24] as a family of geometric measures which unify several important families of measures in the Brunn-Minkowski theory and its dual theory of convex bodies.

Our setting is the Euclidean n-space $\mathbb{R}^n$ with $n\geq 2$. We write o to denote the origin in $\mathbb{R}^n$, $\langle\cdot,\cdot\rangle$ for the standard inner product, and $\|\cdot\|$ for its induced norm. We denote the unit ball by $B^n=\{x\in\mathbb{R}^n:\|x\|\leq 1\}$, the unit sphere by $S^{n-1}=\partial B^n$. $\mathcal{H}^k(\cdot)$ stands for the k-dimensional Hausdorff measure, and for the n-dimensional volume (Lebesgue measure) we use the notation $V(\cdot)$. In particular, the volume of the unit ball is $\kappa_n=V(B^n)=\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ and its surface area is $\mathcal{H}^{n-1}(S^{n-1})=n\kappa_n$, where Γ is Euler's gamma function (cf. Artin [3]). We call a compact convex set $K\subset\mathbb{R}^n$ with non-empty interior a convex body. We use the symbol $\mathcal{K}^n_o$ to denote the family of compact convex sets in $\mathbb{R}^n$ containing the origin, and $\mathcal{K}^n_{(o)}$ to denote the family of all convex bodies K which contain o in their interior, that is, $o\in\operatorname{int}K$. For detailed information on the theory of convex bodies we refer to the recent books by Gruber [14] and Schneider [27].

For a convex compact set $K\subset\mathbb{R}^n$, the support function $h_K(u):S^{n-1}\to\mathbb{R}$ is defined as $h_K(u)=\max\{\langle x,u\rangle : x\in K\}$. For $u\in S^{n-1}$, the face of K with exterior unit normal u is $F(K,u)=\{x\in K:\langle x,u\rangle=h_K(u)\}$. For $x\in\partial K$, let the spherical image of x be defined as $\boldsymbol{\nu}_K(\{x\})=\{u\in S^{n-1}:h_K(u)=\langle x,u\rangle\}$. For a Borel set $\eta\subset S^{n-1}$, the reverse spherical image is

$$\boldsymbol{\nu}_K^{-1}(\eta)=\{x\in\partial K:\boldsymbol{\nu}_K(x)\cap\eta\neq\emptyset\}=\cup_{u\in\eta}F(K,u).$$

If K has a unique supporting hyperplane at x, then we say that K is smooth at x, and in this case $\boldsymbol{\nu}_K(\{x\})$ contains exactly one element that we denote by $\nu_K(x)$ and call it the exterior unit normal of K at x.

2010 *Mathematics Subject Classification*: 52A40.
Keywords: L_p dual Minkowski problem; Monge-Ampère equation.

The classical Minkowski problem in the Brunn-Minkowski theory of convex bodies is concerned with the characterization of the so-called surface area measure. The surface area measure of a convex body can be defined in a direct way as follows. Let $\partial' K$ denote the subset of the boundary of K where there is a unique outer unit normal vector. It is well-known that $\partial K\backslash\partial' K$ is the countable union of compact sets of finite $\mathcal{H}^{n-2}$-measure (see Schneider [27, Theorem 2.2.5]), and hence $\partial' K$ is Borel and $\mathcal{H}^{n-1}(\partial K\setminus\partial' K)=0$. Then $\nu_K:\partial' K\to S^{n-1}$ is a function that is usually called the spherical Gauss map, and ν_K is continuous on $\partial' K$. The surface area measure of K, denoted by $S(K,\cdot)$, is a Borel measure on S^{n-1} such that for any Borel set $\eta\subset S^{n-1}$, we have $S(K,\eta)=\mathcal{H}^{n-1}(\boldsymbol{\nu}_K^{-1}(\eta))$. It is an important property of the surface area measure that it satisfies Minkowski's variational formula

$$\lim_{\varepsilon\to 0^+}\frac{V(K+\varepsilon L)-V(K)}{\varepsilon}=\int_{S^{n-1}}h_L\,dS(K,\cdot) \tag{1}$$

for any convex body $L\subset\mathbb{R}^n$.

The classical Minkowski problem asks for necessary and sufficient conditions for a Borel measure on S^{n-1} to be the surface area measure of a convex body. A particularly important case of the Minkowski problem is for discrete measures. Let $P\subset\mathbb{R}^n$ be a polytope, which is defined as the convex hull of a finite number of points in $\mathbb{R}^n$ provided $\mathrm{int}P\neq\emptyset$. Those faces whose dimension is $n-1$ are called facets. A polytope P has a finite number of facets and the union of facets covers the boundary of P. Let $u_1,\ldots,u_k\in S^{n-1}$ be the exterior unit normal vectors of the facets of P. Then $S(P,\cdot)$ is a discrete measure on S^{n-1} concentrated on the set $\{u_1,\ldots,u_k\}$, and $S(P,\{u_i\})=\mathcal{H}^{n-1}(F(P,u_i))$, $i=1,\ldots,k$. The Minkowski problem asks the following: let μ be a discrete positive Borel measure on S^{n-1}. Under what conditions does there exist a polytope P such that $\mu=S(P,\cdot)$? Furthermore, if such a P exists, is it unique? This polytopal version, along with the case when the surface area measure of K is absolutely continuous with respect to the spherical Lebesgue measure, was solved by Minkowski [25, 26]. He also proved the uniqueness of the solution. For general measures the problem was solved by Alexandrov [1, 2] and independently by Fenchel and Jensen. The argument for existence uses the Alexandrov variational formula of the surface area measure, and the uniqueness employs the Minkowski inequality for mixed volumes. In summary, the necessary and sufficient conditions for the existence of the solution of the Minkowski problem for μ are that for any linear subspace $L\leq\mathbb{R}^n$ with $\dim L\leq n-1$, $\mu(L\cap S^{n-1})<\mu(S^{n-1})$, and that the centre of mass of μ is at the origin, that is, $\int_{S^{n-1}}u\,\mu(du)=0$.

Similar questions have been posed for $K\in\mathcal{K}_o^n$, and at least partially solved, for other measures associated with convex bodies in the Brunn-Minkowski theory, for example, the integral curvature measure $J(K,\cdot)$ of Alexandrov (see (5) below), or the L_p surface area measure $dS_p(K,\cdot)=h_K^{1-p}dS(K,\cdot)$ for $p\in\mathbb{R}$ introduced by Lutwak [23], where $S_1(K,\cdot)=S(K,\cdot)$ $(p=1)$ is the classical surface area measure, and $S_0(K,\cdot)$ $(p=0)$ is the cone volume measure (logarithmic Minkowski problem). Here some care is needed if $p>1$, when we only consider the case $o\in\partial K$ if the resulting L_p surface area measure $S_p(K,\cdot)$ is finite. For a detailed overview of these measures and their associated Minkowski problems and further references see, for example, Schneider [27], and Huang, Lutwak, Yang and Zhang [17].

Lutwak built the dual Brunn-Minkowski theory in the 1970s as a "dual" counterpart of the classical theory. Although there is no formal duality between the classical and dual theories, one can say roughly that in the dual theory the radial function plays a similar role as the support function in the classical theory. The dual Brunn-Minkowski theory concerns the class of compact star shaped sets of $\mathbb{R}^n$. A compact set $S\subset\mathbb{R}^n$ is star shaped with respect to a point $p\in S$ if for all $s\in S$, the segment $[p,s]$ is contained in S. We denote the class of compact sets in $\mathbb{R}^n$ that are star shaped with respect to o by $\mathcal{S}_o^n$, and the set of those elements of $\mathcal{S}_o^n$ that contain o in their interiors are denoted by $\mathcal{S}^n_{(o)}$. Clearly, $\mathcal{K}^n_o \subset \mathcal{S}^n_o$ and $\mathcal{K}^n_{(o)} \subset \mathcal{S}^n_{(o)}$. For a star shaped set $S \in \mathcal{S}^n_o$, we define the radial function of S as $\varrho_S(u) = \max\{t \geq 0 : tu \in S\}$ for $u \in S^{n-1}$.

Dual intrinsic volumes for convex bodies $K \in \mathcal{K}^n_{(o)}$ were defined by Lutwak [22] whose definition works for all $q \in \mathbb{R}$. For $q > 0$, we extend Lutwak's definition of the qth dual intrinsic volume $\widetilde{V}_q(\cdot)$ to a compact convex set $K \in \mathcal{K}^n_o$ as

$$\widetilde{V}_q(K) = \frac{1}{n} \int_{S^{n-1}} \varrho^q_K(u)\, \mathcal{H}^{n-1}(du), \tag{2}$$

which is normalized in such a way that $\widetilde{V}_n(K) = V(K)$. We note that ϱ_K is continuous at all $u \in S^{n-1}$ but a compact set of $\mathcal{H}^{n-1}$-measure zero (see Lemma 2.1). We observe that $\widetilde{V}_q(K) = 0$ if $\dim K \leq n-1$, and $\widetilde{V}_q(K) > 0$ if K is full dimensional. We note that dual intrinsic volumes for $q = 0, \ldots, d$ are the coefficients of the dual Steiner polynomial for star shaped compact sets, where the radial sum replaces the Minkowski sum. The qth dual intrinsic volumes, which arise as coefficients naturally satisfy (19), and this provides the possibility to extend their definition for arbitrary $q \in \mathbb{R}$ in the case when $o \in \mathrm{int} K$ and for $q > 0$ when $o \in K$.

Extending the definition of Huang, Lutwak, Yang, Zhang [17] and Lutwak, Yang, Zhang [24] for $K \in \mathcal{K}^n_{(o)}$, if $K \in \mathcal{K}^n_o$ and $\eta \subset S^{n-1}$ is a Borel set, then the reverse radial Gauss image of η is

$$\boldsymbol{\alpha}^*_K(\eta) = \{u \in S^{n-1} : \varrho_K(u)u \in F(K, v) \text{ for some } v \in \eta\} = \{u \in S^{n-1} : \varrho_K(u)u \in \boldsymbol{\nu}^{-1}_K(\eta)\},$$

which is Lebesgue measurable according to Lemma 2.3. For the measurability of $\boldsymbol{\alpha}^*_K(\eta)$ in the case $K \in \mathcal{K}^n_{(0)}$, see [27, Lemma 2.2.4]. For a convex body $K \in \mathcal{K}^n_o$ and $q \in \mathbb{R}$, the qth dual curvature measure $\widetilde{C}_q(K, \cdot)$ is a Borel measure on S^{n-1} defined in [17] as

$$\widetilde{C}_q(K, \eta) = \frac{1}{n} \int_{\boldsymbol{\alpha}^*_K(\eta)} \varrho^q_K(u) \mathcal{H}^{n-1}(du). \tag{3}$$

Similar to the case of qth dual intrinsic volumes, the notion of qth dual curvature measures can be extended to compact convex sets $K \in \mathcal{K}^n_o$ when $q > 0$ using (3). Here if $\dim K \leq n-1$, then $\widetilde{C}_q(K, \cdot)$ is the trivial measure. We note that the so-called cone volume measure $V(K, \cdot) = \frac{1}{n} S_0(K, \cdot) = \frac{1}{n} h_K S(K, \cdot)$, and Alexandrov's integral curvature measure $J(K, \cdot)$ can both be represented as dual curvature measures as

$$V(K, \cdot) = \tfrac{1}{n} S_0(K, \cdot) = \widetilde{C}_n(K, \cdot) \tag{4}$$

$$J(K^*, \cdot) = \widetilde{C}_0(K, \cdot) \text{ provided } o \in \mathrm{int} K. \tag{5}$$

Based on Alexandrov's integral curvature measure, the L_p Alexandrov integral curvature measure

$$dJ_p(K, \cdot) = \varrho^p_K \, dJ(K, \cdot)$$

was introduced by Huang, Lutwak, Yang, Zhang [18] for $p \in \mathbb{R}$ and $K \in \mathcal{K}^n_{(o)}$.

We note that the qth dual curvature measure is a natural extension of the cone volume measure $V(K, \cdot) = \frac{1}{n} h_K S(K, \cdot)$ also in the variational sense, Corollary 4.8 of Huang, Lutwak, Yang, Zhang [17] states the following generalization of Minkowski's formula (1). For arbitrary convex bodies $K, L \in \mathcal{K}^n_{(o)}$, we have

$$\lim_{\varepsilon \to 0^+} \frac{\widetilde{V}_q(K + \varepsilon L) - \widetilde{V}_q(K)}{\varepsilon} = \int_{S^{n-1}} \frac{h_L}{h_K} \, d\widetilde{C}_q(K, \cdot). \tag{6}$$

In this paper, we actually do not use (6), but use Lemma 3.3, which is a variational formula in the sense of Alexandrov for dual curvature measures of polytopes.

For integers $q = 0, \ldots, n$, dual curvature measures arise in a similar way as in the Brunn-Minkowski theory by means of localized dual Steiner polynomials. These measures satisfy (3), and hence their definition can be extended for $q \in \mathbb{R}$. Huang, Lutwak, Yang and Zhang [17] proved that the qth dual curvature measure of a convex body $K \in \mathcal{K}^n_{(o)}$ can also be obtained from the qth dual intrinsic volume by means of an Alexandrov-type variational formula.

Lutwak, Yang, Zhang [24] introduced a more general version of the dual curvature measure where a star shaped set $Q \in \mathcal{S}^n_{(o)}$ is also involved; namely, for a Borel set $\eta \subset S^{n-1}$, $q \in \mathbb{R}$ and $K \in \mathcal{K}^n_{(o)}$, we have

$$\widetilde{C}_q(K,Q,\eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(\eta)} \varrho_K^q(u)\varrho_Q^{n-q}(u)\mathcal{H}^{n-1}(du) \tag{7}$$

and the associated qth dual intrinsic volume with parameter body Q is

$$\widetilde{V}_q(K,Q) = \widetilde{C}_q(K,Q,S^{n-1}) = \frac{1}{n}\int_{S^{n-1}} \varrho_K^q(u)\varrho_Q^{n-q}(u)\mathcal{H}^{n-1}(du). \tag{8}$$

According to Lemma 5.1 in [24], if $q \neq 0$ and the Borel function $g:\, S^{n-1} \to \mathbb{R}$ is bounded, then

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_q(K,Q,u) = \frac{1}{n}\int_{\partial' K} g(\nu_K(x))\langle \nu_K(x), x\rangle \|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x) \tag{9}$$

where $\|x\|_Q = \min\{\lambda \geq 0:\, \lambda x \in Q\}$ is a continuous, even and 1-homogeneous function satisfying $\|x\|_Q > 0$ for $x \neq o$. The advantage of introducing the star body Q is apparent in the equiaffine invariant formula (see Theorem 6.8 in [24]) stating that if $\varphi \in \mathrm{SL}(n,\mathbb{R})$, then

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_q(\varphi K, \varphi Q, u) = \int_{S^{n-1}} g\left(\frac{\varphi^{-t}u}{\|\varphi^{-t}u\|}\right) d\widetilde{C}_q(K,Q,u), \tag{10}$$

where φ^{-t} denotes the transpose of the inverse of φ.

For $q > 0$, we extend these notions and fundamental observations to any convex body containing the origin on its boundary. In particular, for $q > 0$, $K \in \mathcal{K}^n_o$ and $Q \in \mathcal{S}^n_{(o)}$, we can define the associated curvature measure by (7) and the associated dual intrinsic volume by (8), where $\widetilde{C}_q(K,Q,\cdot)$ is a finite Borel measure on S^{n-1}, and $\widetilde{V}_q(K,Q,\cdot)$ is finite. In addition, for $q > 0$, we extend (9) in Lemma 6.1 and (10) in Lemma 6.5 to any $K \in \mathcal{K}^n_o$.

L_p dual curvature measures were also introduced by Lutwak, Yang and Zhang [24]. They provide a common framework that unifies several other geometric measures of the (L_p) Brunn-Minkowski theory and the dual theory: L_p surface area measures, L_p integral curvature measures, and dual curvature measures, cf. [24]. For $q \in \mathbb{R}$, $Q \in \mathcal{S}^n_{(o)}$, $p \in \mathbb{R}$ and $K \in \mathcal{K}^n_{(o)}$, we define the L_p qth dual curvature measure $\widetilde{C}_{p,q}(K,Q,\cdot)$ of K with respect to Q by the formula

$$d\widetilde{C}_{p,q}(K,Q,\cdot) = h_K^{-p} d\widetilde{C}_q(K,Q,\cdot). \tag{11}$$

While we also discuss the measures $\widetilde{C}_{p,q}(K,Q,\cdot)$ involving a $Q \in \mathcal{S}^n_{(o)}$, we concentrate on $\widetilde{C}_{p,q}(K,\cdot)$ in this paper, which represents many fundamental measures associated to a $K \in \mathcal{K}^n_{(o)}$. Basic examples are

$$\begin{aligned} \widetilde{C}_{p,n}(K,\cdot) &= S_p(K,\cdot)\\ \widetilde{C}_{0,q}(K,\cdot) &= \widetilde{C}_q(K,\cdot)\\ \widetilde{C}_{p,0}(K,\cdot) &= J_p(K^*,\cdot). \end{aligned}$$

Alexandrov-type variational formulas for the dual intrinsic volumes were proved by Lutwak, Yang and Zhang, cf. Theorem 6.5 in [24], which produce the L_p dual curvature measures $\widetilde{C}_{p,q}(K,Q,\cdot)$. In this paper we will use a simpler variational formula, cf. Lemma 3.3 for the qth dual intrinsic volumes that we specialize for our particular setting.

In Problem 8.1 in [24] the authors introduced the L_p dual Minkowski existence problem: Find necessary and sufficient conditions that for fixed $p,q\in\mathbb{R}$ and $Q\in\mathcal{S}^n_{(o)}$ and a given Borel measure μ on S^{n-1} there exists a convex body $K\in\mathcal{K}^n_{(o)}$ such that $\mu=\widetilde{C}_{p,q}(K,Q,\cdot)$. As they note in [24], this version of the Minkowski problem includes earlier considered other variants (L_p Minkowski problem, dual Minkowski problem, L_p Aleksandrov problem) for special choices of the parameters. For $Q=B^n$ and an absolutely continuous measure μ with density function f, the L_p dual Minkowski problem constitutes solving the Monge-Ampère equation

$$\det(\nabla^2 h + h\,\mathrm{Id}) = \tfrac{1}{n}\, h^{p-1}\cdot(\|\nabla h\|^2+h^2)^{\frac{n-q}{2}}\cdot f \tag{12}$$

for the non-negative L_1 Borel function f with $\int_{S^{n-1}} f\,d\mathcal{H}^{n-1}>0$ (see (93) in Section 7). Actually, if $Q\in\mathcal{S}^n_{(o)}$, then the related Monge-Ampère equation is (see (94) in Section 7)

$$\det(\nabla^2 h(u) + h(u)\,\mathrm{Id}) = \tfrac{1}{n}\, h(u)^{p-1}\|\nabla h(u)+h(u)\,u\|_Q^{n-q}\cdot f(u). \tag{13}$$

The case of the L_p dual Minkowski problem for even measures has received much attention but is not discussed here, see Böröczky, Lutwak, Yang, Zhang [5] concerning the L_p surface area $S_p(K,\cdot)$, Böröczky, Lutwak, Yang, Zhang, Zhao [6], Jiang Wu [20] and Henk, Pollehn [15], Zhao [30] concerning the qth dual curvature measure $\widetilde{C}_q(K,\cdot)$, and Huang, Zhao [19] concerning the L_p dual curvature measure for detailed discussion of history and recent results.

Let us indicate the known solutions of the L_p dual Minkowski problem when only mild conditions are imposed on the given measure μ or on the function f in (12). We do not state the exact conditions, rather aim at a general overview. For any finite Borel measure μ on S^{n-1} such that the measure of any open hemi-sphere is positive, we have that

- if $p>0$ and $p\neq 1,n$, then $\mu=S_p(K,\cdot)=n\widetilde{C}_{p,n}(K,\cdot)$ for some $K\in\mathcal{K}^n_o$, where the case $p>1$ and $p\neq n$ is due to Chou, Wang [11] and Hug, Lutwak, Yang, Zhang [16], while the case $0<p<1$ is due to Chen, Li, Zhu [9];
- if $p\geq 0$ and $q<0$, then $\mu=\widetilde{C}_{p,q}(K,\cdot)$ for some $K\in\mathcal{K}^n_o$ where the case $p=0$ ($\mu=\widetilde{C}_q(K,\cdot)$) is due to Zhao [29] (see also Li, Sheng, Wang [21]), and the case $p>0$ is due to Huang, Zhao [19] and Gardner, Hug, Xing, Ye, Weil [13].

In addition, if $p>q$ and f is C^α for $\alpha\in(0,1]$, then (12) has a unique positive $C^{2,\alpha}$ solution according to Huang, Zhao [19].

Naturally, the L_p dual Monge-Ampère equation (12) has a solution in the above cases for any non-negative L_1 function f whose integral on any open hemi-sphere is positive. In addition, if $-n<p\leq 0$ and f is any non-negative $L_{\frac{n}{n+p}}$ function on S^{n-1} such that $\int_{S^{n-1}} f\,d\mu>0$, then (12) has a solution, where the case $p=0$ is due to Chen, Li, Zhu [10], and the case $p\in(-n,0)$ is due to Bianchi, Böröczky, Colesanti, Yang [4].

We also note that if $p\leq 0$ and μ is discrete such that any n elements of suppμ are independent vectors, then $\mu=S_p(K,\cdot)=n\cdot\widetilde{C}_{p,n}(P,\cdot)$ for some polytope $P\in\mathcal{K}^n_{(o)}$ according to Zhu [31, 32].

In this paper, we first solve the discrete L_p dual Minkowski problem if $p>1$ and $q>0$.

Theorem 1.1. *Let $Q\in\mathcal{S}^n_{(o)}$, $p>1$ and $q>0$ with $p\neq q$, and let μ be a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere. Then there exists a polytope $P\in\mathcal{K}^n_{(o)}$ such that $\widetilde{C}_{p,q}(P,Q,\cdot)=\mu$.*

Remark If $p>q$, then the solution is unique according to Theorem 8.3 in Lutwak, Yang and Zhang [24].

We note that, in fact, we prove the existence of a polytope $P_0\in\mathcal{K}^n_{(o)}$ satisfying

$$\widetilde{V}_q(P_0,Q)^{-1}\widetilde{C}_{p,q}(P_0,Q,\cdot)=\mu,$$

which P_0 exists even if $p = q$ (see Theorem 3.1).

Let us turn to a general, possibly non-discrete Borel measure μ on S^{n-1}. As the example at the end of the paper by Hug, Lutwak, Yang, Zhang [16] shows, even if μ has a positive continuous density function with respect to the Hausdorff measure on S^{n-1}, for $q = n$ and $1 < p < n$, it may happen that the only solution K has the origin on its boundary. In this case, h_K has some zero on S^{n-1} even if it occurs with negative exponent in $\widetilde{C}_{p,q}(K,\cdot)$. Therefore if $Q \in \mathcal{S}^n_{(o)}$, $p > 1$ and $q > 0$, the natural form the L_p dual Minkowski problem is the following (see Chou, Wang [11] and Hug, Lutwak, Yang, Zhang [16] if $q = n$). For a given non-trivial finite Borel measure μ, find a convex body $K \in \mathcal{K}^n_o$ such that

$$d\widetilde{C}_q(K, Q, \cdot) = h_K^p d\mu. \tag{14}$$

It is natural to assume that $\mathcal{H}^{n-1}(\Xi_K) = 0$ in (14) for

$$\Xi_K = \{x \in \partial K : \text{there exists exterior normal } u \in S^{n-1} \text{ at } x \text{ with } h_K(u) = 0\}, \tag{15}$$

which property ensures that the surface area measure $S(K,\cdot)$ is absolutely continuous with respect to $\widetilde{C}_q(K, Q, \cdot)$ (see Corollary 6.2). Actually, if $q = n$ and $Q = B^n$, then $d\widetilde{C}_n(K,\cdot) = \frac{1}{n}\, h_K\, dS(K,\cdot)$, and [11] and [16] consider the problem

$$dS(K,\cdot) = nh_K^{p-1} d\mu, \tag{16}$$

where the results of [16] about (16) yield the uniqueness of the solution in (16) for $q = n$, $p > 1$ and $Q = B^n$ only under the condition $\mathcal{H}^{n-1}(\Xi_K) = 0$ (see Section 4 for more detailed discussion).

Theorem 1.2. *Let $Q \in \mathcal{S}^n_{(o)}$, $p > 1$ and $q > 0$ with $p \neq q$, and let μ be a finite Borel measure on S^{n-1} that is not concentrated on any closed hemisphere. Then there exists a $K \in \mathcal{K}^n_o$ with $\mathcal{H}^{n-1}(\Xi_K) = 0$ and $\mathrm{int}K \neq \emptyset$ such that $d\widetilde{C}_q(K, Q, \cdot) = h_K^p d\mu$, where $K \in \mathcal{K}^n_{(o)}$ provided $p > q$.*

The solution in Theorem 1.2 is known to be unique in some cases:

- if $p > q$ and μ is discrete (K is a polytope) according to Lutwak, Yang and Zhang [24],
- if $p > q$, Q is a ball and μ has a C^α density function f for $\alpha \in (0,1]$ according to Huang, Zhao [19],
- if $p > 1$, Q is a ball and $q = n$ according to Hug, Lutwak, Yang, Zhang [16].

For Theorem 1.2, in fact, we prove the existence of a convex body $K_0 \in \mathcal{K}^n_o$ such that

$$\widetilde{V}_q(K_0, Q)^{-1}\, d\widetilde{C}_q(K_0, Q, \cdot) = h_K^p\, d\mu,$$

which K_0 exists even if $p = q$ (see Theorem 5.2).

Concerning regularity, we prove the following statements based on Caffarelli [7,8] (see Section 7). We note that if ∂Q is C^2_+ for $Q \in \mathcal{S}^n_{(o)}$, then Q is convex.

Theorem 1.3. *Let $p > 1$, $q > 0$, $Q \in \mathcal{S}^n_{(o)}$, $0 < c_1 < c_2$ and let $K \in \mathcal{K}^n_o$ with $\mathcal{H}^{n-1}(\Xi_K) = 0$ and $\mathrm{int}K \neq \emptyset$ be such that*

$$d\widetilde{C}_q(K, Q, \cdot) = h_K^p f\, d\mathcal{H}^{n-1}$$

for some Borel function f on S^{n-1} satisfying $c_1 \leq f \leq c_2$.

- **(i):** *$\partial K \backslash \Xi_K = \{z \in \partial K : h_K(u) > 0$ for all $u \in N(K, z)\}$ and $\partial K \backslash \Xi_K$ is C^1 and contains no segment, moreover h_K is C^1 on $\mathbb{R}^n \backslash N(K, o)$.*
- **(ii):** *If f is continuous, then each $u \in S^{n-1} \backslash N(K, o)$ has a neighbourhood U on S^{n-1} such that the restriction of h_K to U is $C^{1,\alpha}$ for any $\alpha \in (0,1)$.*
- **(iii):** *If f is in $C^\alpha(S^{n-1})$ for some $\alpha \in (0,1)$, and ∂Q is C^2_+, then $\partial K \backslash \Xi_K$ is C^2_+, and each $u \in S^{n-1} \backslash N(K, o)$ has a neighbourhood where the restriction of h_K is $C^{2,\alpha}$.*

We note that in Theorem 1.3 (ii), the same neighbourhood U of u works for every $\alpha \in (0,1)$. In addition, Theorem 1.3 (i) yields that for any convex $W \subset \mathbb{R}^n \backslash N(K,o)$, $h_K(u+v) < h_K(u)+h_K(v)$ for independent $u, v \in W$. For the case $o \in \operatorname{int} K$ in Theorem 1.3, see the more appealing statements in Theorem 1.5.

We recall that according to Theorem 1.2, if $p > q > 0$ and $p > 1$, then $K \in \mathcal{K}^n_{(o)}$ holds for the solution K of the L_p dual Minkowski problem. On the other hand, Example 7.2 shows that if $1 < p < q$, then the solution K of the L_p dual Minkowski problem provided by Theorem 1.2 may satisfy that $o \in \partial K$ and o is not a smooth point. Next we show that K is still strictly convex in this case, at least if $q \leq n$.

Theorem 1.4. *If $1 < p < q \leq n$, $Q \in \mathcal{S}^n_{(o)}$, $0 < c_1 < c_2$ and $K \in \mathcal{K}^n_o$ with $\mathcal{H}^{n-1}(\Xi_K) = 0$ and $\operatorname{int} K \neq \emptyset$ be such that*

$$d\widetilde{C}_q(K,Q,\cdot) = h_K^p f \, d\mathcal{H}^{n-1}$$

for some Borel function f on S^{n-1} satisfying $c_1 \leq f \leq c_2$, then K is strictly convex; or equivalently, h_K is C^1 on $\mathbb{R}^n \backslash o$.

If $q = n$, then Theorems 1.3 and 1.4 are due to Chou, Wang [11]. We do not know whether Theorem 1.4 holds if $q > n$ (see the comments at the end of Section 7).

We note that even if $Q = B^n$ in Theorem 1.4, the equiaffine invariant formula (10) for $K \in \mathcal{K}^n_o$ simplifies the proof of Theorems 1.4; namely, we use dual curvature measures with a parameter body different from Euclidean balls in the argument for Lemma 7.8. The reason is that if $o \in \partial K$ and $N(K,o)$ contains a pair of vectors with obtuse angle, then we need to transform K via a linear transform $\varphi \in \mathrm{SL}(n,\mathbb{R})$ in such a way that any two vectors in $N(\varphi K, o)$ make an acute angle.

We note that if $o \in \operatorname{int} K$, then the ideas leading to Theorem 1.3 work for any $p, q \in \mathbb{R}$.

Theorem 1.5. *Let $p, q \in \mathbb{R}$, $Q \in \mathcal{S}^n_{(o)}$, $0 < c_1 < c_2$ and let $K \in \mathcal{K}^n_{(o)}$ be such that*

$$d\widetilde{C}_{p,q}(K,Q,\cdot) = f \, d\mathcal{H}^{n-1}$$

for some Borel function f on S^{n-1} satisfying $c_1 \leq f \leq c_2$. We have that

- **(i):** *K is smooth and strictly convex, and h_K is C^1 on $\mathbb{R}^n \backslash \{o\}$;*
- **(ii):** *if f is continuous, then the restriction of h_K to S^{n-1} is in $C^{1,\alpha}$ for any $\alpha \in (0,1)$;*
- **(iii):** *if $f \in C^\alpha(S^{n-1})$ for $\alpha \in (0,1)$, and ∂Q is C^2_+, then ∂K is C^2_+, and h_K is $C^{2,\alpha}$ on S^{n-1}.*

The rest of the paper is organized as follows. Up to Section 5, we assume $Q = B^n$ in order to simplify formulae. We discuss properties of the dual curvature measure in Section 2, and prove Theorem 1.1 in Section 3. Fundamental properties of L_p the dual curvature measures are considered in Section 4, and we use all these results to prove Theorem 1.2 in Section 5. Section 6 presents the way how to extend the arguments to the case when Q is any star body. The regularity of the solution is discussed in Section 7.

2. On the dual curvature measure

The goal of this section is for $q > 0$, to extend the results of Huang, Lutwak, Yang and Zhang [17] about the dual curvature measure $\widetilde{C}_q(K,\cdot)$ when $K \in \mathcal{K}^n_{(o)}$ to the case when $K \in \mathcal{K}^n_o$. For any measure, we take the measure of the empty set to be zero.

For any compact convex set K in $\mathbb{R}^n$ and $z \in \partial K$, we write $N(K,z)$ to denote the normal cone at z; namely,

$$N(K,z) = \{y \in \mathbb{R}^n : \langle y, x - z \rangle \leq 0 \text{ for } x \in K\}.$$

If $z \in \text{int}K$, then simply $N(K,z) = \{o\}$. For compact, convex sets $K, L \subset \mathbb{R}^n$, we define their Hausdorff distance as

$$\delta_H(K,L) := \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

It is a metric on the space of compact convex sets, and the induced metric space is locally compact according to the Blaschke selection theorem. For basic properties of Hausdorff distance we refer to Schneider [27], and also to Gruber [14].

First we extend Lemma 3.3 in [17]. Let $K \in \mathcal{K}_o^n$ with $\text{int}K \neq \emptyset$. We recall that the so-called singular points $z \in \partial K$ where $\dim N(K,z) \geq 2$ form a Borel set of zero $\mathcal{H}^{n-1}$ measure, and hence its complement $\partial' K$ of smooth points is also a Borel set. For $z \in \partial' K$, we write $\nu_K(z)$ to denote the unique exterior normal at z. In addition, for any $z \in \partial K$, we define $\boldsymbol{\nu}_K(z) = N(K,z) \cap S^{n-1}$, and hence $\boldsymbol{\nu}_K^{-1}(\eta) = \cup_{u \in \eta} F(K,u)$ is the total inverse Gauss image of a Borel set $\eta \subset S^{n-1}$; namely, the set of all $z \in \partial K$ with $N(K,z) \cap \eta \neq \emptyset$. In particular, if $o \in \partial K$, then we have

$$\Xi_K = \boldsymbol{\nu}_K^{-1}\left(N(K,o) \cap S^{n-1}\right). \tag{17}$$

If $o \in \text{int}K$, then $\Xi_K = \emptyset$. We also observe that the dual of $N(K,o)$ is

$$N(K,o)^* = \{y \in \mathbb{R}^n : \langle y, x \rangle \leq 0 \text{ for } x \in N(K,o)\} = \text{cl}\{\lambda x : \lambda \geq 0 \text{ and } x \in K\},$$

and hence

$$\Xi_K = K \cap \partial N(K,o)^*. \tag{18}$$

If $o \in \text{int}K$, then simply $N(K,o)^* = \mathbb{R}^n$. The following properties of ϱ_K readily follow from the definition.

Lemma 2.1. *If $K \in \mathcal{K}_o^n$, then ϱ_K is upper semicontinuous. In addition, if $\dim K \leq n-1$, then $\varrho_K(u) = 0$ for $u \in S^{n-1} \backslash \text{lin}\, K$, and if $\text{int}\, K \neq \emptyset$, then ϱ_K is continuous on $S^{n-1} \backslash \partial N(K,o)^*$ and $\varrho_K(u) = 0$ for $u \in S^{n-1} \backslash N(K,o)^*$.*

For $q > 0$, we extend Lutwak's definition of the qth dual intrinsic volume $\widetilde{V}_q(\cdot)$ to a compact convex set $K \in \mathcal{K}_o^n$ as

$$\widetilde{V}_q(K) = \frac{1}{n} \int_{S^{n-1}} \varrho_K^q(u)\, \mathcal{H}^{n-1}(du), \tag{19}$$

and hence $\widetilde{V}_n(K) = V(K)$. It follows from Lemma 2.1 that $\widetilde{V}_q(K)$ is well-defined and $\widetilde{V}_q(K) = 0$ if $\dim K \leq n-1$.

For a $K \in \mathcal{K}_o^n$ with $\text{int}K \neq \emptyset$ and a Borel set $\eta \subset S^{n-1}$, let

$$\boldsymbol{\alpha}_K^*(\eta) = \{u \in S^{n-1} : \varrho_K(u)u \in F(K,v) \text{ for some } v \in \eta\} = \{u \in S^{n-1} : \varrho_K(u)u \in \boldsymbol{\nu}_K^{-1}(\eta)\}.$$

Following Huang, Lutwak, Yang, Zhang [17] and Lutwak, Yang, Zhang [24], the set $\boldsymbol{\alpha}_K^*(\eta)$ is called the reverse radial Gauss image of η.

Lemma 2.2. *If $K \in \mathcal{K}_o^n$ with $\text{int}K \neq \emptyset$, then*

$$S^{n-1} \cap (\text{int}N(K,o)^*) \subset \boldsymbol{\alpha}_K^*\left(S^{n-1} \backslash N(K,o)\right) \subset S^{n-1} \cap N(K,o)^*, \tag{20}$$

$$\boldsymbol{\alpha}_K^*\left(S^{n-1} \cap N(K,o)\right) = S^{n-1} \backslash \left(\text{int}N(K,o)^*\right). \tag{21}$$

Proof. If $o \in \text{int}\, K$, then $N(K,o) = \{o\}$, and hence the statements are trivial. Therefore we assume that $o \in \partial K$.

It follows from (18) that

$$(\text{int}\, N(K,o)^*) \cap \partial K = \{x \in \partial K : h_K(v) > 0 \text{ for all } v \in \boldsymbol{\nu}_K(x)\}. \tag{22}$$

Now (22) yields directly the first containment relation of (20), and $K\subset N(K,o)^*$ implies the second containment relation.

To prove (21), let $u,v\in S^{n-1}$ be such that $\varrho_K(u)u\in F(K,v)$. If $v\in N(K,o)\cap S^{n-1}$, then $F(K,v)\subset \Xi_K$, thus (22) yields that $u\not\in \operatorname{int} N(K,o)^*$. On the other hand, if $u\not\in \operatorname{int} N(K,o)^*$, then either $u\not\in N(K,o)^*$, and hence $\varrho_K(u)=0$, or $u\in\partial N(K,o)^*$, therefore $\varrho_K(u)u\in\Xi_K$ in both cases. We conclude $v\in N(K,o)$, and in turn (21). □

We note that the radial map $\widetilde{\pi}:\mathbb{R}^n\backslash\{o\}\to S^{n-1}$, $\tilde{\pi}(x)=x/\|x\|$ is locally Lipschitz. We write $\tilde{\pi}_K$ to denote the restriction of $\tilde{\pi}$ onto the $(n-1)$-dimensional Lipschitz manifold $(\partial K)\backslash\Xi_K=(\partial K)\cap \operatorname{int}N(K,o)^*$. For any $z\in(\partial' K)\backslash\Xi_K$, the Jacobian of $\tilde{\pi}_K$ at z is

$$\langle\nu_K(z),\tilde{\pi}_K(z)\rangle\|z\|^{-(n-1)}=\langle\nu_K(z),z\rangle\|z\|^{-n}. \tag{23}$$

Lemma 2.3. *If $K\in\mathcal{K}^n_o$ with $\operatorname{int}K\neq\emptyset$ and $\eta\subset S^{n-1}$ is a Borel set, then $\boldsymbol{\alpha}^*_K(\eta)\subset S^{n-1}$ is Lebesgue measurable.*

Proof. Since $\boldsymbol{\alpha}^*_K(\eta\cap N(K,o))\cap\boldsymbol{\alpha}^*_K(\eta\backslash N(K,o))\subset\partial N(K,o)^*\cap S^{n-1}$ by Lemma 2.2, and $\mathcal{H}^{n-1}(\partial N(K,o)^*\cap S^{n-1})=0$, it is equivalent to prove that both $\boldsymbol{\alpha}^*_K(\eta\cap N(K,o))$ and $\boldsymbol{\alpha}^*_K(\eta\backslash N(K,o))$ are Lebesgue measurable.

If $\eta\cap N(K,o)\neq\emptyset$, then we claim that

$$S^{n-1}\backslash N(K,o)^*\subset\boldsymbol{\alpha}^*_K(\eta\cap N(K,o))\subset S^{n-1}\backslash\operatorname{int}N(K,o)^*. \tag{24}$$

The second containment relation follows from Lemma 2.2. For the first containment relation in (24), let $v\in\eta\cap N(K,o)$. Since $o\in F(K,v)$ and $\varrho_K(u)=0$ for $u\in S^{n-1}\backslash N(K,o)^*$, it follows that $S^{n-1}\backslash N(K,o)^*\subset\boldsymbol{\alpha}^*_K(\{v\})$. Thus we have (24), and in turn $\eta\cap N(K,o)$ is Lebesgue measurable.

Next we consider $\eta\backslash N(K,o)$. Since $\partial' K$ is Borel, we have that $\sigma_K=\partial' K\cap\operatorname{int}N(K,o)^*$ is Borel, as well. We write $\tilde{\nu}_K:\sigma_K\to S^{n-1}\backslash N(K,o)$ to denote the restriction of ν_K to σ_K. As $\tilde{\nu}_K$ is continuous on σ_K, we deduce that $\tilde{\nu}_K^{-1}(\eta\backslash N(K,o))$ is Borel. In addition, $\tilde{\pi}_K$ is also continuous on $\partial K\cap\operatorname{int}N(K,o)^*$, thus $\tilde{\pi}_K\circ\tilde{\nu}_K^{-1}(\eta\backslash N(K,o))$ is also Borel. Since

$$\tilde{\pi}_K\circ\tilde{\nu}_K^{-1}(\eta\backslash N(K,o))\subset\boldsymbol{\alpha}^*_K(\eta\backslash N(K,o))\subset\tilde{\pi}_K\circ\tilde{\nu}_K^{-1}(\eta\backslash N(K,o))\cup\tilde{\pi}_K\Big((\partial K\cap\operatorname{int}N(K,o)^*)\backslash\partial' K\Big).$$

Here $\mathcal{H}^{n-1}\Big((\partial K\cap\operatorname{int}N(K,o)^*)\backslash\partial' K\Big)=0$ and $\tilde{\pi}_K$ is locally Lipschitz, therefore $\boldsymbol{\alpha}^*_K(\eta\backslash N(K,o))$ is Lebesgue measurable, as well. □

Extending the definition in Huang, Lutwak, Yang, Zhang [17], for a convex compact set $K\in\mathcal{K}^n_o$ and $q>0$, the qth dual curvature measure $\widetilde{C}_q(K,\cdot)$ is a Borel measure on S^{n-1} defined in a way such that if $\eta\subset S^{n-1}$ is Borel, then

$$\widetilde{C}_q(K,\eta)\;=\;0\;\text{ if }\dim K\leq n-1,\text{ and} \tag{25}$$

$$\widetilde{C}_q(K,\eta)\;=\;\frac{1}{n}\int_{\boldsymbol{\alpha}^*_K(\eta)}\varrho_K^q(u)\,d\mathcal{H}^{n-1}(u)\;\text{ if }\operatorname{int}K\neq\emptyset. \tag{26}$$

Here, if $\operatorname{int}K\neq\emptyset$, then ϱ_K is continuous on $S^{n-1}\backslash\partial N(K,o)^*$, therefore $\widetilde{C}_q(K,\cdot)$ is well-defined by Lemma 2.3.

Since $\varrho_K(u)=0$ for $u\in S^{n-1}\backslash N(K,o)^*$, and $\mathcal{H}^{n-1}(S^{n-1}\cap\partial N(K,o)^*)=0$, we deduce from (21) that if $q>0$, then

$$\widetilde{C}_q\left(K,N(K,o)\cap S^{n-1}\right)=\widetilde{C}_q\left(K,\{u\in S^{n-1}:\,h_K(u)=0\}\right)=0. \tag{27}$$

For $u\in S^{n-1}$, we write $r_K(u)=\varrho_K(u)u\in\partial K$. Since $\tilde{\pi}_K$ is locally Lipschitz, $\mathcal{H}^{n-1}$ almost all points of $S^{n-1}\cap(\operatorname{int}N(K,o)^*)$ are in the image of $(\partial' K)\cap(\operatorname{int}N(K,o)^*)$ by $\tilde{\pi}_K$. Therefore for $\mathcal{H}^{n-1}$ almost all points $u\in S^{n-1}\cap(\operatorname{int}N(K,o)^*)$, there is a unique exterior unit normal

$\alpha_K(u)$ at $r_K(u) \in \partial K$. Here α_K is the so-called reverse radial Gauss map. For the other points $u \in S^{n-1} \cap (\mathrm{int}N(K,o)^*)$, we just choose an exterior unit normal $\alpha_K(u)$ at $r_K(u) \in \partial K$. The extensions of Lemma 3.3 and Lemma 3.4 in [17] to the case when the origin may lie on the boundary of convex bodies are the following.

Lemma 2.4. *If $q > 0$, $K \in \mathcal{K}_o^n$ with $\mathrm{int}K \neq \emptyset$, and $g : S^{n-1} \to [0,\infty)$ is Borel measurable, then*

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_q(K,u) = \frac{1}{n}\int_{S^{n-1}\cap(\mathrm{int}N(K,o)^*)} g(\alpha_K(u))\varrho_K(u)^q\, d\mathcal{H}^{n-1}(u) \tag{28}$$

$$= \frac{1}{n}\int_{\partial' K\setminus \Xi_K} g(\nu_K(x))\langle \nu_K(x), x\rangle \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x), \tag{29}$$

$$= \frac{1}{n}\int_{\partial' K} g(\nu_K(x))\langle \nu_K(x), x\rangle \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x) \tag{30}$$

Proof. To prove (28), the integral of g can be approximated by integrals of finite linear combinations of characteristic functions of Borel sets of S^{n-1}, and hence we may assume that $g = \mathbf{1}_\eta$ for a Borel set $\eta \subset S^{n-1}$. In this case,

$$\int_{S^{n-1}\cap N(K,o)} \mathbf{1}_\eta\, d\widetilde{C}_q(K,\cdot) = 0$$

by (27), and

$$\int_{S^{n-1}\setminus N(K,o)} \mathbf{1}_\eta\, d\widetilde{C}_q(K,\cdot) = \widetilde{C}_q(K,\eta\setminus N(K,o)) = \int_{S^{n-1}\cap(\mathrm{int}N(K,o)^*)} \mathbf{1}_\eta(\alpha_K(u))\varrho_K(u)^q\, d\mathcal{H}^{n-1}(u)$$

by (20) and the definition of $\widetilde{C}_q(K,\cdot)$, verifying (28).

In turn, (28) yields (29) by (23). For (30), we observe that if $x \in \Xi_K \cap \partial' K$, then $\langle \nu_K(x), x\rangle = 0$. ☐

Now we prove that the qth dual curvature measure is continuous on $\mathcal{K}_o^n$ for $q > 0$.

Lemma 2.5. *For $q > 0$, $\widetilde{V}_q(K)$ is a continuous function of $K \in \mathcal{K}_o^n$ with respect to the Hausdorff distance.*

Proof. Let $R > 0$ be such that $K \subset \mathrm{int}R\,B^n$. Let $K_m \in \mathcal{K}_o^n$ be a sequence of compact convex sets tending to K with respect to Hausdorff distance. In particular, we may assume that $K_m \subset RB^n$ for all K_m.

If $\dim K \le n-1$, then there exists $v \in S^{n-1}$ such that $K \subset v^\perp$, where $v^\perp$ denotes the orthogonal (linear) complement of v. For $t \in [0,1)$, we write

$$\Psi(v,t) = \{x \in \mathbb{R}^n : |\langle v, x\rangle| \le t\}$$

to denote the closed region of width $2t$ between two hyperplanes orthogonal to v and symmetric to 0.

There exists a $t_0 \in (0,1)$ such that for any $t \in (0,t_0)$ and $v \in S^{n-1}$ it holds that $\mathcal{H}^{n-1}(S^{n-1} \cap \Psi(v,t)) < 3t(n-1)\kappa_{n-1}$.

Let $\varepsilon \in (0,t_0)$. We claim that there exists an m_ε such that for all $m > m_\varepsilon$ and for any $u \in S^{n-1}\setminus\Psi(v,\varepsilon)$, we have

$$\varrho_{K_m}(u) \le \varepsilon. \tag{31}$$

Since $K_m \to K$ in the Hausdorff metric, there exists an index m_ε such that for all $m > m_\varepsilon$ it holds that $K_m \subset K + \varepsilon^2 B^n \subset \Psi(v,\varepsilon^2)$. Then if $u \in S^{n-1}\setminus\Psi(v,\varepsilon)$, then

$$\varepsilon^2 \ge \langle v, \varrho_{K_m}(u)u\rangle = \varrho_{K_m}(u)\langle v, u\rangle \ge \varrho_{K_m}(u)\cdot\varepsilon,$$

yielding (31). We deduce from (31) and $K_m \subset RB^n$ that for any $\varepsilon \in (0, t_0)$, if $m > m_\varepsilon$, then

$$\begin{aligned}\widetilde{V}_q(K_m) &\leq \int_{S^{n-1}\setminus\Psi(v,\varepsilon)} \varepsilon^q \, d\mathcal{H}^{n-1}(u) + \int_{S^{n-1}\cap\Psi(v,\varepsilon)} R^q \, d\mathcal{H}^{n-1}(u) \\ &\leq n\kappa_n\varepsilon^q + 3\varepsilon(n-1)\kappa_{n-1}R^q,\end{aligned}$$

therefore $\lim_{m\to\infty} \widetilde{V}_q(K_m) = 0 = \widetilde{V}_q(K)$.

Next, let $\operatorname{int} K \neq \emptyset$ such that $o \in \partial K$. Since the functions $\varrho_{K_m}(u)$, $m = 1, \ldots$ are uniformly bounded, by Lebesgue's dominated convergence theorem it is sufficient to prove that

$$\lim_{m\to\infty} \varrho_{K_m}(u) = \varrho_K(u) \quad \text{for } u \in S^{n-1}\setminus\partial N(K,o)^*, \tag{32}$$

as $\mathcal{H}^{n-1}(S^{n-1} \cap \partial N(K,o)^*) = 0$. Now, let $\varepsilon \in [0, 1)$.

Case 1. Let $u \in S^{n-1} \cap \operatorname{int} N(K,o)^*$.

Then $\varrho_K(u) > 0$, and $(1-\varepsilon)\varrho_K(u)u \in \operatorname{int} K$ and $(1+\varepsilon)\varrho_K(u)u \notin K$. Thus, there exists an index $m(u,\varepsilon) > 0$ such that for all $m > m(u,\varepsilon)$ it holds that $(1-\varepsilon)\varrho_K(u)u \in K_m$ and $(1+\varepsilon)\varrho_K(u)u \notin K_m$, or in other words,

$$(1-\varepsilon)\varrho_K(u) \leq \varrho_{K_m}(u) \leq (1+\varepsilon)\varrho_K(u),$$

which in turn yields (32) in this case.

Case 2. Let $u \in S^{n-1}\setminus N(K,o)^*$.

Then $\varrho_K(u) = 0$, and there exists $v \in S^{n-1} \cap \operatorname{int} N(K,o)$ such that $\langle u, v\rangle > 0$. As $K_m \to K$, there exists an index $m(u,v,\varepsilon) > 0$ such that for all $m > m(u,v,\varepsilon)$ it holds that $K_m \subset K + \varepsilon\langle u,v\rangle B^n$, and thus $h_{K_m}(v) < \varepsilon\langle u,v\rangle$. Therefore, for all $m > m(u,v,\varepsilon)$,

$$\varepsilon\langle u,v\rangle > h_{K_m}(v) \geq \langle \varrho_{K_m}(u)u, v\rangle = \varrho_{K_m}(u)\langle u,v\rangle.$$

This yields that $\varrho_{K_m}(u) < \varepsilon$ for all $m > m(u,v,\varepsilon)$, and thus (32) holds by $\varrho_K(u) = 0$.

Finally, let $\operatorname{int} K \neq \emptyset$ and $o \in \operatorname{int} K$. The argument for this case is analogous to the one used above in Case 1. □

The following Proposition 2.6 extends Lemma 3.6 from Huang, Lutwak, Yang, Zhang [17] about $K \in \mathcal{K}^n_{(o)}$ to the case when $K \in \mathcal{K}^n_o$.

Proposition 2.6. *If $q > 0$, and $\{K_m\}$, $m \in \mathbb{N}$, tends to K for $K_m, K \in \mathcal{K}^n_o$, then $\widetilde{C}_q(K_m, \cdot)$ tends weakly to $\widetilde{C}_q(K, \cdot)$.*

Proof. Since any element of $\mathcal{K}^n_o$ can be approximated by elements of $\mathcal{K}^n_{(o)}$, we may assume that each $K_m \in \mathcal{K}^n_{(o)}$. We fix $R > 0$ such that $K \subset \operatorname{int} R\, B^n$, and hence we may also assume that $K_m \subset RB^n$ for all K_m. We need to prove that if $g : S^{n-1} \to \mathbb{R}$ is continuous, then

$$\lim_{m\to\infty} \int_{S^{n-1}} g(u)\, d\widetilde{C}_q(K_m, u) = \int_{S^{n-1}} g(u)\, d\widetilde{C}_q(K, u) \tag{33}$$

First we assume that $o \in \partial K$. If $\dim K \leq n-1$, then $\widetilde{C}_q(K, \cdot)$ is the constant zero measure by (25). Since $\widetilde{C}_q(K_m, S^{n-1}) = \widetilde{V}_q(K_m)$ tends to zero according to Lemma 2.5, we conclude (33) in this case.

Therefore we may assume that $\operatorname{int} K \neq \emptyset$ and $o \in \partial K$. To simplify notation, we set

$$\sigma = N(K,o)^*.$$

According to Lemma 2.4, (33) is equivalent to

$$\lim_{m\to\infty} \int_{S^{n-1}} g(\alpha_{K_m}(u))\varrho_{K_m}(u)^q \, d\mathcal{H}^{n-1}(u) = \int_{S^{n-1}\cap(\operatorname{int}\sigma)} g(\alpha_K(u))\varrho_K(u)^q \, d\mathcal{H}^{n-1}(u). \tag{34}$$

Since $\tilde{\pi}_K$ is Lipschitz and $\mathcal{H}^{n-1}(S^{n-1}\cap(\partial\sigma))=0$, to verify (34), and in turn (33), it is sufficient to prove

$$\lim_{m\to\infty}\int_{\tilde{\pi}_K((\mathrm{int}\,\sigma)\cap\partial' K)} g(\alpha_{K_m}(u))\varrho_{K_m}(u)^q\,d\mathcal{H}^{n-1}(u) = \int_{\tilde{\pi}_K((\mathrm{int}\,\sigma)\cap\partial' K)} g(\alpha_K(u))\varrho_K(u)^q\,d\mathcal{H}^{n-1}(u) \tag{35}$$

$$\lim_{m\to\infty}\int_{S^{n-1}\setminus\sigma} g(\alpha_{K_m}(u))\varrho_{K_m}(u)^q\,d\mathcal{H}^{n-1}(u) = 0. \tag{36}$$

To prove (35) and (36), it follows from $K_m\subset RB^n$, the continuity of g and Lemma 2.5 that there exists $M>0$ such that

$$\begin{aligned} |\varrho_{K_m}(u)| &\leq R \quad \text{for } u\in S^{n-1},\\ |g(u)| &\leq M \quad \text{for } u\in S^{n-1},\\ \widetilde{C}_q(K_m,S^{n-1}) &\leq M \quad \text{for } m\in\mathbb{N} \end{aligned} \tag{37}$$

We deduce from (37) that Lebesgue's Dominated Convergence Theorem applies both in (35) and (36). For (35), let $u\in\tilde{\pi}_K((\mathrm{int}\,\sigma)\cap\partial' K)$. Readily, $\lim_{m\to\infty}\varrho_{K_m}(u)^q=\varrho_K(u)^q$. Since $\alpha_K(u)$ is the unique normal at $\varrho_K(u)u\in\partial' K$, we have $\lim_{m\to\infty}\alpha_{K_m}(u)=\alpha_K(u)$, and hence $\lim_{m\to\infty}g(\alpha_{K_m}(u))=g(\alpha_K(u))$ by the continuity of g. In turn, we conclude (35) by Lebesgue's Dominated Convergence Theorem.

Turning to (36), it follows from Lebesgue's Dominated Convergence Theorem, $q>0$ and (37) that it is sufficient to prove that if $\varepsilon>0$ and $u\in S^{n-1}\setminus\sigma$, then

$$\varrho_{K_m}(u)\leq\varepsilon \tag{38}$$

for $m\geq m_0$ where m_0 depends on $u,\{K_m\},\varepsilon$. Since $u\notin\sigma=N(K,o)^*$, there exists $v\in N(K,o)$ such that $\langle v,u\rangle=\delta>0$. As $h_K(v)=0$ and K_m tends to K, there exists m_0 such that $h_{K_m}(v)\leq\delta\varepsilon$ if $m\geq m_0$. In particular, if $m\geq m_0$, then

$$\varepsilon\delta\geq h_{K_m}(v)\geq\langle v,\varrho_K(u)u\rangle=\varrho_K(u)\delta,$$

yielding (38), and in turn (36).

Finally, the argument leading to (35) implies (33) also in the case when $o\in\mathrm{int}K$, completing the proof of Proposition 2.6. □

3. Proof of Theorem 1.1 for $Q=B^n$

To verify Theorem 1.1, we prove the following statement, which also holds if $p=q$.

Theorem 3.1. *Let $p>1$ and $q>0$, and let μ be a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere. Then there exists a polytope $P\in\mathcal{K}^n_{(o)}$ such that $\widetilde{V}_q(P)^{-1}\widetilde{C}_{p,q}(P,\cdot)=\mu$.*

We recall that $\tilde{\pi}:\mathbb{R}^n\setminus\{o\}\to S^{n-1}$ is the radial projection, and for a convex body K in $\mathbb{R}^n$ and $u\in S^{n-1}$, the face of K with exterior unit normal u is the set

$$F(K,u)=\{x\in K:\langle x,u\rangle=h_K(u)\}.$$

We observe that if $P\in\mathcal{K}^n_o$ is a polytope with $\mathrm{int}\,P\neq\emptyset$, and $v_1,\ldots,v_l\in S^{n-1}$ are the exterior normals of the facets of P not containing the origin, then

$$\begin{aligned} \mathrm{supp}\,\widetilde{C}_q(P,\cdot) &= \{v_1,\ldots,v_l\}, \text{ and}\\ \widetilde{C}_q(P,\{v_i\}) &= \frac{1}{n}\int_{\tilde{\pi}(F(P,v_i))}\varrho_P^q(u)\,d\mathcal{H}^{n-1}(u) \quad \text{for } i=1,\ldots,l. \end{aligned} \tag{39}$$

Let $p>1$, $q>0$ and μ be a discrete measure on S^{n-1} that is not concentrated on any closed hemi-sphere. Let $\operatorname{supp}\mu=\{u_1,\dots,u_k\}$, and let $\mu(\{u_i\})=\alpha_i>0$, $i=1,\dots,k$. For any $z=(t_1,\dots,t_k)\in(\mathbb{R}_{\geq 0})^k$, we define

$$\begin{aligned}\Phi(z) &= \sum_{i=1}^{k}\alpha_i t_i^p,\\ P(z) &= \{x\in\mathbb{R}^n:\,\langle x,u_i\rangle\leq t_i\,\forall i=1,\dots,k\},\\ \Psi(z) &= \widetilde{V}_q(P(z)).\end{aligned}\tag{40}$$

Since $\alpha_i>0$ for $i=1,\dots,k$, the set $Z=\{z\in(\mathbb{R}_{\geq 0})^k:\,\Phi(z)=1\}$ is compact, and hence Lemma 2.5 yields the existence of $z_0\in Z$ such that

$$\Psi(z_0)=\max\{\Psi(z):\,z\in Z\}.$$

We prove that $o\in\operatorname{int}P(z_0)$ and there exists $\lambda_0>0$ such that

$$\widetilde{V}_q(\lambda_0P(z_0))^{-1}\widetilde{C}_{p,q}(\lambda_0P(z_0),\cdot)=\mu.$$

Lemma 3.2. *If $p>1$ and $q>0$, then $o\in\operatorname{int}P(z_0)$.*

Proof. It is clear from the construction that $o\in P(z_0)$. We assume that $o\in\partial P(z_0)$, and seek a contradiction. Without loss of generality, we may assume that $z_0=(t_1,\dots,t_k)\in(\mathbb{R}_{\geq 0})^k$, where there exists $1\leq m<k$ such that $t_1=\ldots=t_m=0$ and $t_{m+1},\dots,t_k>0$. For sufficiently small $t>0$, we define

$$\begin{aligned}\tilde{z}_t &= \Big(\overbrace{0,\dots,0}^{m},(t_{m+1}^p-\alpha t^p)^{\frac{1}{p}},\dots,(t_k^p-\alpha t^p)^{\frac{1}{p}}\Big)\quad\text{for } \alpha=\tfrac{\alpha_1+\ldots+\alpha_m}{\alpha_{m+1}+\ldots+\alpha_k},\text{ and}\\ z_t &= \Big(\overbrace{t,\dots,t}^{m},(t_{m+1}^p-\alpha t^p)^{\frac{1}{p}},\dots,(t_k^p-\alpha t^p)^{\frac{1}{p}}\Big).\end{aligned}$$

Simple substitution shows that $\Phi(z_t)=1$, so $z_t\in Z$.

We prove that there exist $\tilde{t}_0,\tilde{c}_1,\tilde{c}_2>0$ depending on p, q, μ and z_0 such that if $t\in(0,\tilde{t}_0]$, then

$$\Psi(\tilde{z}_t)\;\geq\;\Psi(z_0)-\tilde{c}_1t^p,\tag{41}$$

$$\Psi(z_t)\;\geq\;\Psi(\tilde{z}_t)+\tilde{c}_2t,\tag{42}$$

therefore

$$\Psi(z_t)\geq\Psi(z_0)-\tilde{c}_1t^p+\tilde{c}_2t.\tag{43}$$

We choose $R>0$ such that $P(z_0)\subset\operatorname{int}RB^n$ and $R\geq\max\{t_{m+1},\dots,t_k\}$.

We start with proving (41), and set $\varrho_0=\min\{t_{m+1},\dots,t_k\}$. We frequently use the following form of Bernoulli's inequality that says that if $\tau\in(0,1)$ and $\eta>0$, then

$$(1-\tau)^\eta\geq 1-\max\{1,\eta\}\cdot\tau.\tag{44}$$

It follows from (44) and $\varrho_0\leq t_i\leq R$, $i=m+1,\dots,k$, that there exist $s_0,c_0>0$, depending on z_0, μ and p such that if $t\in(0,s_0)$, then

$$(t_i^p-\alpha t^p)^{\frac{1}{p}}>t_i-c_0t^p>\varrho_0/2\;\text{ for } i=m+1,\dots,k.\tag{45}$$

We consider the cone $N(P(z_0),o)^*=\{x\in\mathbb{R}^n:\,\langle x,u_i\rangle\leq 0\;\forall i=1,\dots,m\}$ that satisfies that $\varrho_{P(z_0)}(u)>0$ for $u\in S^{n-1}$ if and only if $u\in N(P(z_0),o)^*$. It follows from (45) that $\varrho_{P(\tilde{z}_t)}(u)>0$ for $u\in S^{n-1}$ also if and only if $u\in N(P(z_0),o)^*$, and even $\varrho_{P(\tilde{z}_t)}(u)>\varrho_0/2$ in this case.

Let $u \in N(P(z_0),o)^* \cap S^{n-1}$, and hence $\varrho_{P(\tilde{z}_t)}(u)\cdot u$ lies in a facet $F(P(\tilde{z}_t),u_i)$ for an $i \in \{m+1,\ldots,k\}$, thus

$$\langle \varrho_{P(\tilde{z}_t)}(u)u,u_i\rangle = (t_i^p-\alpha t^p)^{\frac{1}{p}} > t_i - c_0t^p > \varrho_0/2.$$

Combining the last estimate with $\varrho_{P(\tilde{z}_t)}(u)\le R$, we deduce that $\langle u,u_i\rangle \ge \frac{\varrho_0}{2R}$. Let $s>0$ be defined by $\langle su,u_i\rangle = t_i$. Then $s \ge \varrho_{P(z_0)(u)}$, and hence

$$s-\varrho_{P(\tilde{z}_t)}(u) = \frac{\langle su,u_i\rangle - \langle \varrho_{P(\tilde{z}_t)}(u)u,u_i\rangle}{\langle u,u_i\rangle} \le \frac{t_i-(t_i-c_0t^p)}{\langle u,u_i\rangle} \le \frac{2Rc_0}{\varrho_0}\cdot t^p,$$

thus $\varrho_{P(\tilde{z}_t)}(u) \ge \varrho_{P(z_0)}(u) - \frac{2Rc_0}{\varrho_0}\cdot t^p$. We choose $t_0>0$ with $t_0\le s_0$ depending on z_0 and p such that $\frac{2Rc_0}{\varrho_0}\cdot t_0^p < \varrho_0/2$. Since $\varrho_0 \le \varrho_{P(z_0)}(u)\le R$, we deduce from (44) that there exists $c_1>0$ depending on μ, z_0, q and p that if $t\in(0,t_0)$ and $u\in C\cap S^{n-1}$, then

$$\varrho_{P(\tilde{z}_t)}(u)^q \ge \left(\varrho_{P(z_0)}(u) - \frac{2Rc_0}{\varrho_0}\cdot t^p\right)^q \ge \varrho_{P(z_0)}(u)^q - c_1\cdot t^p,$$

which yields (41) by (19) and by taking into account that $N(P(\tilde{z}_t),o)^* = N(P(z_0),o)^*$.

The main idea of the proof of (42) is that we construct a set $\widetilde{G}_t\subset S^{n-1}$ for sufficiently small $t>0$ whose $\mathcal{H}^{n-1}$ measure is of order t, and if $u\in\widetilde{G}_t$, then $\varrho_{P(z_t)}(u)\ge r$ for a suitable constant $r>0$ while $\varrho_{P(\tilde{z}_t)}(u)=0$. In order to show that the constants involved really depend only on p, q μ and $P(z_0)$, we start to set them with respect to $P(z_0)$.

We may assume, possibly after reindexing $u_1,\ldots,u_m$, that $\dim F(P(z_0),u_1)=n-1$. In particular, there exist $r>0$ and $y_0\in F(P(z_0),u_1)\backslash\{o\}$ such that

$$\langle y_0,u_i\rangle \le h_{P(z_0)}(u_i)-8r \quad \text{for } i=2,\ldots,k.$$

For $v=y_0/\|y_0\|\in S^{n-1}\cap u_1^\perp$, we consider $y=y_0+4rv$, and hence $4r\le\|y\|\le R$, and

$$\langle y,u_i\rangle \le h_{P(z_0)}(u_i)-4r \quad \text{for } i=2,\ldots,k.$$

Note that $P(\tilde{z}_t)\to P(z_0)$ as $t\to 0^+$ and also $P(\tilde{z}_t)\subset P(z_0)$ for $t>0$. Therefore there exists a positive $t_1\le\min\{r,t_0\}$, depending only on p, q, μ and z_0 such that if $t\in(0,t_1]$, then

$$\langle y,u_i\rangle \le h_{P(\tilde{z}_t)}(u_i)-2r \quad \text{for } i=2,\ldots,k \quad \text{and} \quad P(z_t)\subset RB^n. \tag{46}$$

For two vectors $a,b\in\mathbb{R}^n$, we denote by $[a,b]$ $((a,b))$ the closed (open) segment with endpoints a and b. Let the $(n-2)$-dimensional unit ball G be defined as

$$G=u_1^\perp\cap v^\perp\cap B^n.$$

Then we have that $y+rG\subset F(P(z_0),u_1)$ and $(y+rG)+r[o,u_1]\subset y+2rB^n$. Let $G_t = (y+rG)+t(o,u_1]$ be the $(n-1)$-dimensional right spherical cylinder of height $t<\min\{t_1,r\}$, whose base $y+rG$ does not belong to G_t. We deduce from (46) and $h_{P(z_t)}(u_1)=t$ that $G_t\subset P(z_t)\backslash N(P(z_0),o)^*\subset P(z_t)\backslash P(\tilde{z}_t)$.

Let $\widetilde{G}_t$ be the the radial projection of G_t to S^{n-1}. For $x\in G_t$, we have $\langle x,v\rangle=\|y\|\ge 4r$ and $\|x\|\le R$, therefore

$$\mathcal{H}^{n-1}(\widetilde{G}_t)=\int_{G_t}\left\langle\frac{x}{\|x\|},v\right\rangle\|x\|^{-(n-1)}\,d\mathcal{H}^{n-1}(x)\ge\frac{4r\mathcal{H}^{n-1}(G_t)}{R^n}=\frac{4r\cdot r^{n-2}\kappa_{n-2}}{R^n}\cdot t=\frac{4r^{n-1}\kappa_{n-2}}{R^n}\cdot t.$$

Since $\varrho_{P(\tilde z_t)}(u)\le\varrho_{P(z_t)}(u)$ for all $u\in S^{n-1}$, and if $u\in\widetilde G_t$, then $\varrho_{P(z_t)}(u)\ge\|y\|\ge 4r$ and $\varrho_{P(\tilde z_t)}(u)=0$, we deduce that

$$\begin{aligned}\Psi(z_t)&=\frac1n\int_{S^{n-1}}\varrho^q_{P(z_t)}(u)d\mathcal H^{n-1}(u)\\&=\frac1n\int_{S^{n-1}\setminus\widetilde G_t}\varrho^q_{P(z_t)}(u)d\mathcal H^{n-1}(u)+\frac1n\int_{\widetilde G_t}\varrho^q_{P(z_t)}(u)d\mathcal H^{n-1}(u)\\&\ge\frac1n\int_{S^{n-1}}\varrho^q_{P(\tilde z_t)}(u)d\mathcal H^{n-1}(u)+\frac1n\int_{\widetilde G_t}\varrho^q_{P(z_t)}(u)d\mathcal H^{n-1}(u)\\&\ge\Psi(\tilde z_t)+\frac{(4r)^q\cdot 4r^{n-1}\kappa_{n-2}}{nR^n}\cdot t,\end{aligned}$$

which proves (42). Combining (41) and (42), we obtain (43).

Finally, we deduce from $p>1$ and (43) that if $t>0$ is sufficiently small, then $\Psi(P(z_t))>\Psi(P(z_0))$, which contradicts the optimality of z_0, and yields Lemma 3.2. □

As we already know that $o\in\operatorname{int}P(z_0)$ by Lemma 3.2, we can freely decrease $h_{P(z_0)}(u_i)$ for $i=1,\dots,k$, and increase it if $\dim F(P(z_0),u_i)=n-1$. To control what happens to $\Psi(z)$ when we perturb $P(z_0)$, we use Lemma 3.3, which is a consequence of Theorem 4.4 in [17]. Let $\mathbb R_+$ denote set of the positive real numbers.

Lemma 3.3 (Huang, Lutwak, Yang, Zhang, [17])**.** *If $q\ne0$, $\eta\in(0,1)$ and $z_t=(z_1(t),\dots,z_k(t))\in\mathbb R^k_+$ for $t\in(-\eta,\eta)$ are such that $\lim_{t\to0^+}\frac{z_i(t)-z_i(0)}{t}=z_i'(0)\in\mathbb R$ for $i=1,\dots,k$ exists, then the $P(z_t)$ defined in* (86) *satisfies that*

$$\lim_{t\to0^+}\frac{\widetilde V_q(P(z_t))-\widetilde V_q(P(z_0))}{t}=q\sum_{i=1}^k\frac{z_i'(0)}{h_{P(z_0)}(u_i)}\cdot\widetilde C_q(P(z_0),\{u_i\}).$$

For the sake of completeness, in Section 6 we prove a general version of Lemma 3.3 about the variation of $\widetilde V_q(P(z(t)),Q)$ in the case when Q is an arbitrary star body, cf. Lemma 6.7.

We note that $\operatorname{supp}C_q(P(z_0),\cdot)\subset\{u_1,\dots,u_k\}$, where $\widetilde C_q(P(z_0),\{u_i\})>0$ if and only if $\dim F(P(z_0),u_i)=n-1$.

Lemma 3.4. *If $p>1$ and $q>0$, then* $\dim F(P(z_0),u_i)=n-1$ *for* $i=1,\dots,k$.

Proof. We suppose that $\dim F(P(z_0),u_1)<n-1$, and seek a contradiction. We may assume that $\dim F(P(z_0),u_k)=n-1$. For small $t\ge0$, we consider

$$\tilde z(t)=(t_1-t,t_2,\dots,t_k),$$

and $\theta(t)=\Phi(P(\tilde z(t))$. In particular, $\theta(0)=1$ and $\theta'(0)=-p\alpha_1t_1^{p-1}$, and hence

$$z(t)=\theta(t)^{-1/p}\tilde z(t)=(z_1(t),\dots,z_k(t))\in Z$$

satisfies $\frac{d}{dt}\theta(t)^{-1/p}|_{t=0^+}=\alpha_1t_1^{p-1}$ and $z_i'(0)=\alpha_1t_1^{p-1}t_i>0$ for $i=2,\dots,k$. We deduce from Lemma 3.3 and $\widetilde C_q(P(z_0),\{u_1\})=0$ that

$$\lim_{t\to0^+}\frac{\widetilde V_q(P(z(t)))-\widetilde V_q(P(z_0))}{t}=q\sum_{i=2}^k\frac{z_i'(0)}{h_{P(z_0)}(u_i)}\cdot\widetilde C_q(P(z_0),\{u_i\})\ge\frac{q\,z_k'(0)}{h_{P(z_0)}(u_k)}\cdot\widetilde C_q(P(z_0),\{u_k\})>0,$$

therefore $\widetilde V_q(P(z(t)))>\widetilde V_q(P(z_0))$ for small $t>0$. This contradicts the optimality of z_0, and proves Lemma 3.4. □

Proof of Theorem 3.1 According to Lemmas 3.2 and 3.4,

we have $\dim F(P(z_0),u_i) = n-1$ for $i=1,\ldots,k$, $o \in \mathrm{int}\, P(z_0)$ and $h_{P(z_0)}(u_i) = t_i$ for $i = 1,\ldots,k$. Let $(g_1,\ldots,g_k) \in \mathbb{R}^k$ satisfying $\sum_{i=1}^k g_i\alpha_i t_i^{p-1} = 0$ such that not all g_i are zero. If $t \in (-\varepsilon,\varepsilon)$ for small $\varepsilon > 0$, then consider

$$\tilde{z}(t) = (t_1 + g_1 t, \ldots, t_k + g_k t),$$

and $\theta(t) = \Phi(P(\tilde{z}(t))$. In particular, $\theta(0) = 1$ and

$$\theta'(0) = p\sum_{i=1}^k g_i\alpha_i t_i^{p-1} = 0.$$

Therefore

$$z(t) = \theta(t)^{-1/p}\tilde{z}(t) = (z_1(t),\ldots,z_k(t)) \in Z$$

satisfies $\frac{d}{dt}\theta(t)^{-1/p}|_{t=0} = 0$ and $z_i'(0) = g_i$ for $i = 1,\ldots,k$. We deduce from Lemma 3.3 and $h_{P(z_0)}(u_i) = t_i$ for $i = 1,\ldots,k$ that

$$\lim_{t\to 0}\frac{\widetilde{V}_q(P(z(t))) - \widetilde{V}_q(P(z_0))}{t} = q\sum_{i=1}^k \frac{g_i}{t_i}\cdot \widetilde{C}_q(P(z_0),\{u_i\}).$$

Since $\widetilde{V}_q(P(z(t)))$ attains its maximum at $t=0$ by the optimality of z_0, we have

$$\sum_{i=1}^k \frac{g_i}{t_i}\cdot \widetilde{C}_q(P(z_0),\{u_i\}) = 0. \tag{47}$$

In particular, (47) holds whenever $(g_1,\ldots,g_k) \in \mathbb{R}^k\backslash\{o\}$ satisfies $\sum_{i=1}^k g_i\alpha_i t_i^{p-1} = 0$, or in other words, there exists a $\lambda \in \mathbb{R}$ such that

$$\lambda\cdot\frac{\widetilde{C}_q(P(z_0),\{u_i\})}{t_i} = \alpha_i t_i^{p-1} \text{ for } i = 1,\ldots,k.$$

Since $\lambda > 0$ and $p > 1$, there exists a $\lambda_0 > 0$ such that $\lambda = \lambda_0^{-p}\widetilde{V}_q(P(z_0))$, and hence

$$\alpha_i = \widetilde{V}_q(\lambda_0 P(z_0))^{-1} h_{\lambda_0 P(z_0)}(u_i)^{-p}\widetilde{C}_q(\lambda_0 P(z_0),\{u_i\}) \text{ for } i = 1,\ldots,k.$$

In other words,

$$\mu = \widetilde{V}_q(\lambda_0 P(z_0))^{-1} h_{\lambda_0 P(z_0)}(u_i)^{-p}\widetilde{C}_q(\lambda_0 P(z_0),\cdot).$$

This finishes the proof of Theorem 3.1. □

Proof of Theorem 1.1 in the case of $Q = B^n$ We have $p \neq q$. According to Theorem 3.1, there exists a polytope $P_0 \in \mathcal{K}^n_{(o)}$ such that $\widetilde{V}_q(P_0)^{-1}\widetilde{C}_{p,q}(P_0,\cdot) = \mu$. For $\lambda = \widetilde{V}_q(P_0)^{\frac{-1}{q-p}}$ and $P = \lambda P_0$, we have

$$\widetilde{C}_{p,q}(P,\cdot) = \lambda^{q-p}\widetilde{C}_{p,q}(P_0,\cdot) = \widetilde{V}_q(P_0)^{-1}\widetilde{C}_{p,q}(P_0,\cdot) = \mu.$$

□

4. On the L_p dual curvature measures

According to Lemma 5.1 in Lutwak, Yang, Zhang [24], if $K \in \mathcal{K}^n_{(o)}$, $p \in \mathbb{R}$ and $q > 0$, then for any Borel function $g : S^{n-1} \to \mathbb{R}$, we have that

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_{p,q}(K,u) = \frac{1}{n}\int_{\partial' K} g(\nu_K(x))\langle \nu_K(x), x\rangle^{1-p}\|x\|^{q-n}\, d\mathcal{H}^{n-1}(x). \tag{48}$$

As a simple consequence of Lemma 2.4, we can partially extend (48) to allow $o \in \partial K$.

Corollary 4.1. *If $p>1$, $q>0$, $K\in\mathcal{K}_o^n$ with $\mathrm{int}K\neq\emptyset$, $\widetilde{C}_{p,q}(K,S^{n-1})<\infty$ and $\mathcal{H}^{n-1}(\Xi_K)=0$, and the Borel function $g:\,S^{n-1}\to\mathbb{R}$ is bounded, then*

$$\int_{S^{n-1}} g(u)\,d\widetilde{C}_{p,q}(K,u)=\frac{1}{n}\int_{\partial' K} g(\nu_K(x))\langle \nu_K(x),x\rangle^{1-p}\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x).$$

Proof. Knowing that $\widetilde{C}_{p,q}(K,S^{n-1})<\infty$, it follows from Lemma 2.4 and $\mathcal{H}^{n-1}(\Xi_K)=0$ that

$$\begin{aligned}\int_{S^{n-1}} g(u)\,d\widetilde{C}_{p,q}(K,u) &= \int_{S^{n-1}} g(u)h_K(u)^{-p}\,d\widetilde{C}_q(K,u)\\ &= \frac{1}{n}\int_{\partial' K\setminus\Xi_K} g(\nu_K(x))h_K(\nu_K(x))^{-p}\langle \nu_K(x),x\rangle\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x)\\ &= \frac{1}{n}\int_{\partial' K} g(\nu_K(x))\langle \nu_K(x),x\rangle^{1-p}\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x).\end{aligned}$$

□

Next, we prove a basic estimate on the inradius of K in terms of its total L_p dual curvature measure. For a convex body $K\in\mathcal{K}^n_{(o)}$, we write $r(K)$ to denote the maximal radius of balls contained in K. Since $o\in K$, Steinhagen's theorem yields the existence of $w\in S^{n-1}$ such that

$$|\langle x,w\rangle|\leq 2nr(K)\quad\text{for } x\in K. \tag{49}$$

Lemma 4.2. *For $n\geq 2$, $p>1$ and $q>0$, there exists a constant $c>0$ depending only on p,q,n such that if $K\in\mathcal{K}^n_{(o)}$, then*

$$\widetilde{C}_{p,q}(K,S^{n-1})\geq c\cdot r(K)^{-p}\cdot\widetilde{V}_q(K).$$

Proof. We may assume that $r(K)=1$, and hence (49) yields the existence of $w\in S^{n-1}$ such that

$$|\langle x,w\rangle|\leq 2n\quad\text{for } x\in K. \tag{50}$$

Let $\widetilde{K}=K|w^\perp$ be the orthogonal projection of K to the hyperplane $w^\perp$, and hence the radial function $\varrho_{\widetilde{K}}$ is positive and continuous on $w^\perp\cap S^{n-1}$. We consider the concave function f and the convex function g on $\widetilde{K}=K|w^\perp$ such that

$$K=\left\{y+tw:\,y\in\widetilde{K}\text{ and } g(y)\leq t\leq f(y)\right\}.$$

We divide $w^\perp\cap S^{n-1}$ into pairwise disjoint Borel sets $\widetilde{\Omega}_1,\ldots,\widetilde{\Omega}_m$ of positive $\mathcal{H}^{n-2}$ measure such that for each $\widetilde{\Omega}_i$, there exists a $\varrho_i>0$ satisfying

$$\varrho_i/2\leq\varrho_{\widetilde{K}}(u)\leq\varrho_i\quad\text{for } u\in\widetilde{\Omega}_i. \tag{51}$$

For any $i=1,\ldots,m$, we consider

$$\begin{aligned}\Omega_i &= \left\{u\cos\alpha+w\sin\alpha:\,u\in\widetilde{\Omega}_i\text{ and }\alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right\}\subset S^{n-1},\\ \Psi_i &= \{\varrho_K(u)u:\,u\in\Omega_i\}\subset\partial K.\end{aligned}$$

It follows that $S^{n-1}\backslash\{w,-w\}$ is divided into the pairwise disjoint Borel sets $\Omega_1,\ldots,\Omega_m$, and $\partial K\backslash\{f(o)w,g(o)w\}$ is divided into the pairwise disjoint Borel sets $\Psi_1,\ldots,\Psi_m$.

According to (48) and Lemma 2.4, to verify Lemma 4.2, it is sufficient to prove that there exists a constant $c>0$ depending only on n,p,q such that if $i=1,\ldots,m$, then

$$\int_{\partial' K\cap\Psi_i}\langle\nu_K(x),x\rangle^{1-p}\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x)\geq c\int_{\partial' K\cap\Psi_i}\langle\nu_K(x),x\rangle\|x\|^{q-n}\,d\mathcal{H}^{n-1}(x). \tag{52}$$

We define

$$R = 4(2n)^2. \tag{53}$$

Case 1. $\varrho_i \leq R$

If $\varrho_i \leq R$ and $x \in \partial' K \cap \Psi_i$, then (50) yields that

$$\langle \nu_K(x), x\rangle \leq \|x\| \leq R + 2n \quad \text{for } x \in \Psi_i,$$

and hence $\langle \nu_K(x), x\rangle^{1-p} \geq \langle \nu_K(x), x\rangle (R+2n)^{-p}$. Therefore we may choose $c = (R+2n)^{-p}$ in (52).

Case 2. $\varrho_i > R$

If $\varrho_i > R$, then consider the set

$$\Phi_i = \left\{ tu : u \in \widetilde{\Omega}_i \text{ and } 0 < t \leq \varrho_i/4 \right\} \subset \Psi_i | w^\perp,$$

and subdivide Ψ_i into

$$\begin{aligned} \Psi_i^0 &= \{y + f(y)w : y \in \Phi_i\} \cup \{y + g(y)w : y \in \Phi_i\} \subset \Psi_i \cap \left(\tfrac{\varrho_i}{4} + 2n\right) B^n, \text{ and} \\ \Psi_i^1 &= \Psi_i \backslash \Psi_i^0 \subset \Psi_i \backslash \left(\tfrac{\varrho_i}{4} B^n\right). \end{aligned}$$

We claim that

$$\langle \nu_K(x), x\rangle \leq 6n \quad \text{for } x \in \partial' K \cap \Psi_i^0. \tag{54}$$

We observe that $x = y + tw$ for some $y \in \Phi_i$ and $t \in [-2n, 2n]$, and $s = f(2y)$ satisfies $s \in [-2n, 2n]$ and $2y + sw \in \Psi_i$. It follows that

$$\langle \nu_K(x), 2y + sw\rangle \leq \langle \nu_K(x), x\rangle = \langle \nu_K(x), y + tw\rangle,$$

and hence

$$\langle \nu_K(x), y\rangle \leq \langle \nu_K(x), tw\rangle - \langle \nu_K(x), sw\rangle \leq 4n.$$

We conclude that $\langle \nu_K(x), y + tw\rangle = \langle \nu_K(x), y\rangle + \langle \nu_K(x), tw\rangle \leq 6n$, in accordance with (54).

In turn, (54) yields that $\langle \nu_K(x), x\rangle^{1-p} \geq \langle \nu_K(x), x\rangle (6n)^{-p}$ for $x \in \partial' K \cap \Psi_i^0$, and hence

$$\int_{\partial' K \cap \Psi_i^0} \langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n} \, d\mathcal{H}^{n-1}(x) \geq (6n)^{-p} \int_{\partial' K \cap \Psi_i^0} \langle \nu_K(x), x\rangle \|x\|^{q-n} \, d\mathcal{H}^{n-1}(x). \tag{55}$$

Next, we prove the existence of $\gamma_1 > 0$ depending on n, p, q such that

$$\int_{\partial' K \cap \Psi_i^0} \langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n} \, d\mathcal{H}^{n-1}(x) \geq \begin{cases} \gamma_1 \mathcal{H}^{n-2}(\widetilde{\Omega}_i) \varrho_i^{q-1} & \text{if } \quad q > 1 \\ \gamma_1 \mathcal{H}^{n-2}(\widetilde{\Omega}_i) & \text{if } \quad q \in (0, 1] \end{cases}. \tag{56}$$

Let us consider $x = y + f(y)w \in \Psi_i^0 \cap \partial' K$ for some $y \in \Phi_i \backslash (2nB^n)$. Since $\|y\| \leq \|x\| \leq 2\|y\|$ by (50), it follows from (54) that

$$\langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n} \geq (6n)^{1-p} \min\{1, 2^{q-n}\} \, \|y\|^{q-n}.$$

Therefore there exists $\gamma_2 > 0$ depending on n, p, q such that

$$\begin{aligned} \int_{\partial' K \cap \Psi_i^0} \langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n} \, d\mathcal{H}^{n-1}(x) &\geq \gamma_2 \int_{\Phi_i \backslash (2nB^n)} \|y\|^{q-n} \, d\mathcal{H}^{n-1}(x) \\ &= \gamma_2 \mathcal{H}^{n-2}(\widetilde{\Omega}_i) \int_{2n}^{\varrho_i/4} t^{q-n} t^{n-2} \, dt \\ &= \gamma_2 \mathcal{H}^{n-2}(\widetilde{\Omega}_i) \int_{2n}^{\varrho_i/4} t^{q-2} \, dt, \end{aligned}$$

and in turn we conclude (56).

The final part of the argument is the estimate

$$\int_{\partial' K\cap\Psi_i^1} \langle \nu_K(x), x\rangle \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x) \le 2^q 16n \cdot \mathcal{H}^{n-2}(\widetilde{\Omega}_i)\cdot \varrho_i^{q-1}. \tag{57}$$

Let $\Omega_i^1 = \pi_K(\Psi_i^1)$. If $x = y + sw \in \Psi_i^1$ for $y \in (\Psi_i|w^\perp)\setminus\Phi_i$, then $y \in (\Psi_i|w^\perp)\setminus(\frac{\varrho_i}{4}B^n)$ and $|s| \le 2n$. It follows that $|\tan\alpha| \le \frac{2n}{\varrho_i/4} = \frac{8n}{\varrho_i}$ for the angle α of x and y. In particular,

$$\Omega_i^1 \subset \pi_K\left(\widetilde{\Omega}_i + \left[\frac{-8n}{\varrho_i}, \frac{8n}{\varrho_i}\right]\cdot w\right)$$

which, in turn, yields that

$$\mathcal{H}^{n-1}(\Omega_i^1) \le \frac{16n}{\varrho_i}\,\mathcal{H}^{n-2}(\widetilde{\Omega}_i).$$

We deduce from (23) and from the fact that $\|x\| \le \varrho_i + 2n \le 2\varrho_i$ for $x \in \Psi_i^1$ that

$$\int_{\partial' K\cap\Psi_i^1} \langle \nu_K(x), x\rangle \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x) = \int_{\Omega_i^1} \varrho_K(u)^q\, d\mathcal{H}^{n-1}(u) \le \frac{16n}{\varrho_i}\,\mathcal{H}^{n-2}(\widetilde{\Omega}_i)\cdot(2\varrho_i)^q,$$

yielding (57).

We deduce from (56) and (57) the existence of $\gamma_3 > 0$ depending on n, p, q such that

$$\int_{\partial' K\cap\Psi_i^0} \langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x) \ge \gamma_3 \int_{\partial' K\cap\Psi_i^1} \langle \nu_K(x), x\rangle \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x). \tag{58}$$

Combining (55) and (58) implies (52) if $\varrho_i > R$, as well, completing the proof of Lemma 4.2. □

Next we investigate the limit of convex bodies with bounded L_p dual curvature measure in Lemmas 4.3 and 4.4.

Lemma 4.3. *If $p > 1$, $0 < q \le p$ and $K_m \in \mathcal{K}^n_{(o)}$ for $m \in \mathbb{N}$ tend to $K \in \mathcal{K}^n_o$ with* $\mathrm{int}K \neq \emptyset$ *such that $\widetilde{C}_{p,q}(K_m, S^{n-1})$ stays bounded, then $K \in \mathcal{K}^n_{(o)}$.*

Proof. Let us suppose that $o \in \partial K$, and seek a contradiction. We claim that there exists a vector $w \in \mathrm{int}N(K,o)^*$ such that $-w \in N(K,o)\cap S^{n-1}$. If this property fails, then $(-N(K,o))\cap \mathrm{int}N(K,o)^* = \emptyset$, and hence the Hahn-Banach theorem yields the existence of a vector $v \in S^{n-1}$ such that $\langle v, u\rangle \le 0$ if $u \in N(K,o)^*$, and $\langle v, u\rangle \ge 0$ if $u \in -N(K,o)$, and hence $v \in N(K,o)^*$. Since $\langle v, v\rangle = 1 > 0$ contradicts $\langle v, u\rangle \le 0$ if $u \in N(K,o)^*$, we conclude the existence of the required w.

To simplify notation, we set $B(r) = w^\perp \cap (rB^n)$ for $r > 0$. The conditions in Lemma 4.3 and (48) yield the existence of some $M > 0$ such that for each K_m, we have that

$$\begin{aligned} M &> \widetilde{C}_{p,q}(K_m, S^{n-1}) = \frac{1}{n}\int_{\partial' K_m} \langle \nu(K_m, x), x\rangle^{1-p}\|x\|^{q-n}\, d\mathcal{H}^{n-1}(x) \\ &\ge \frac{1}{n}\int_{\partial' K_m\cap B^n} \|x\|^{1-n+q-p}\, d\mathcal{H}^{n-1}(x) \ge \frac{1}{n}\int_{\partial' K_m\cap B^n} \|x\|^{1-n}\, d\mathcal{H}^{n-1}(x). \end{aligned} \tag{59}$$

We note that since $K_m \to K$ and $o \in \partial K$, for sufficiently large m, $\partial' K_m \cap B^n \neq \emptyset$ and the right-hand side of (59) is greater than zero. As $w \in \mathrm{int}N(K,o)^*$ and $w \in N(K,o)$, there exist a $\varrho \in (0,1)$ and a non-negative convex function f on $B(2\varrho)$ with $f(o) = 0$ such that

$$U = \{z + f(z)w : z \in B(2\varrho)\} \subset \partial K.$$

In particular, there exist an $\eta > 0$ such that

$$\|x|w^\perp\| \ge 2\eta\|x\| \quad \text{for } x \in U. \tag{60}$$

We may assume that $\varrho \in (0,1)$ is small enough to ensure that $U \subset \mathrm{int}B^n$.

Since $\int_{B(\varrho)} \|z\|^{1-n} d\mathcal{H}^{n-1}(z) = \infty$, there exists some $\delta \in (0, \varrho)$ such that

$$\frac{1}{n} \int_{B(\varrho)\backslash B(\delta)} \left(\frac{\|z\|}{\eta}\right)^{1-n} d\mathcal{H}^{n-1}(z) > M. \tag{61}$$

There exist and an m_0 such that if $m > m_0$, then for some convex function f_m on $B(\varrho)$, we have

$$U_m = \{z + f_m(z)w : z \in B(\varrho)\backslash B(\delta)\} \subset (\partial K_m) \cap (\text{int} B^n), \tag{62}$$

and (compare (60))

$$\|z\| \geq \eta \|z + f_m(z)w\| \ \text{ for } z \in B(\varrho)\backslash B(\delta). \tag{63}$$

We deduce from (59), (62) and (63), and finally from (61) that

$$\begin{aligned} M &> \frac{1}{n} \int_{U_m} \|x\|^{1-n} \, d\mathcal{H}^{n-1}(x) \geq \frac{1}{n} \int_{B(\varrho)\backslash B(\delta)} \|z + f_m(z)w\|^{1-n} d\mathcal{H}^{n-1}(z) \\ &\geq \frac{1}{n} \int_{B(\varrho)\backslash B(\delta)} \left(\frac{\|z\|}{\eta}\right)^{1-n} d\mathcal{H}^{n-1}(z) > M. \end{aligned}$$

This is a contradiction, and in turn we conclude Lemma 4.3. □

Lemma 4.4. *If $p > 1$, $q > 0$ and $K_m \in \mathcal{K}^n_{(o)}$ for $m \in \mathbb{N}$ tend to $K \in \mathcal{K}^n_o$ with $\text{int} K \neq \emptyset$ such that $\widetilde{C}_{p,q}(K_m, S^{n-1})$ stays bounded, then $\mathcal{H}^{n-1}(\Xi_K) = 0$.*

Proof. We fix a point $z \in \text{int} K$, and for any bounded $X \subset \mathbb{R}^n \backslash \{z\}$, we define the set

$$\sigma(X) = \{z + \lambda(x - z) : x \in X \text{ and } \lambda > 0\}.$$

We observe that $\sigma(X)$ is open if $X \subset \partial K$ is relatively open, and $\sigma(X) \cup \{o\}$ is closed if X is compact.

We will use the weak continuity of the $(n-1)$th curvature measure. In particular, according to Theorem 4.2.1 and Theorem 4.2.3 in Schneider [27], if $\beta \subset \mathbb{R}^n$ is open, then

$$\liminf_{m \to \infty} \mathcal{H}^{n-1}(\beta \cap \partial K_m) \geq \mathcal{H}^{n-1}(\beta \cap \partial K). \tag{64}$$

Let us suppose, on the contrary, that $\mathcal{H}^{n-1}(\Xi_K) > 0$, and hence $o \in \partial K$, and seek a contradiction. Choose some large $M, R > 0$, and a compact set $\widetilde{\Xi} \subset \Xi_K \backslash \{o\}$ such that

$$\begin{aligned} K_m &\subset RB^n, \\ \widetilde{C}_{p,q}(K_m, S^{n-1}) &\leq M \ \text{ for } m \in \mathbb{N}, \\ \mathcal{H}^{n-1}(\widetilde{\Xi}) &= \omega > 0. \end{aligned}$$

Now there exists some $\eta > 0$ such that

(i): $(\eta B^n) \cap \sigma(\widetilde{\Xi} + \eta B^n) = \emptyset$.

Since $p > 1$, we may choose $\varepsilon > 0$ such that

$$\frac{(2\varepsilon)^{1-p}}{n} \cdot \min\{\eta^{q-n}, R^{q-n}\} \cdot (\omega/2) > M. \tag{65}$$

We have $\mathcal{H}^{n-1}(\widetilde{\Xi} \cap \partial' K) = \omega$. For any $x \in \widetilde{\Xi} \cap \partial' K$, there exists $r_x \in (0, \eta)$ such that

$$h_K(u) \leq \varepsilon \ \text{ if } u \in S^{n-1} \text{ is exterior normal at } y \in \partial K \cap (x + r_x B^n), \tag{66}$$

and we define $B_x = \text{int}(x + r_x B^n)$. Let

$$\mathcal{U} = \bigcup_{x \in \widetilde{\Xi} \cap \partial' K} (B_x \cap \partial K),$$

which is a relatively open subset of ∂K satisfying

(a): $(\eta B^n) \cap \sigma(\mathcal{U}) = \emptyset$,
(b): $\mathcal{H}^{n-1}(\mathcal{U}) \geq \omega$,
(c): $h_K(u) \leq \varepsilon$ if $u \in S^{n-1}$ is exterior normal at $x \in \mathrm{cl}\,\mathcal{U}$.

It follows that (applying (64) in the case (b')) that there exists m_0 such that if $m \geq m_0$, then
(a'): $\|x\| \geq \eta$ if $x \in \sigma(\mathcal{U}) \cap \partial K_m$,
(b'): $\mathcal{H}^{n-1}(\sigma(\mathcal{U}) \cap \partial K_m) \geq \omega/2$,
(c'): $h_K(u) \leq 2\varepsilon$ if $u \in S^{n-1}$ is exterior normal at $x \in \sigma(\mathcal{U}) \cap \partial K_m$.

For any $x \in \sigma(\mathcal{U}) \cap \partial K_m$, (a') and $K_m \subset RB^n$ yield that

$$\|x\|^{q-n} \geq \min\{\eta^{q-n}, R^{q-n}\}.$$

It follows first by (48), then by (b'), (c') and (65), that

$$M \geq \widetilde{C}_{p,q}(K_m, S^{n-1}) \geq \frac{1}{n} \int_{\sigma(\mathcal{U}) \cap \partial' K_m} \langle \nu_K(x), x\rangle^{1-p} \|x\|^{q-n}\, d\mathcal{H}^{n-1}(x) > M.$$

This contradiction proves Lemma 4.4. □

5. Theorem 1.2 for general convex bodies if $Q = B^n$

For $w \in S^{n-1}$ and $\alpha \in (-1, 1)$, we write

$$\Omega(w, \alpha) = \{u \in S^{n-1} : \langle u, w\rangle > \alpha\}.$$

The following is a simple but useful observation.

Lemma 5.1. *For a finite Borel measure μ on S^{n-1} not concentrated on a closed hemi-sphere, there exists $t \in (0, 1)$ such that for any $w \in S^{n-1}$, we have $\mu(\Omega(w, t)) > t$.*

First we prove the following variant of Theorem 1.2 involving the dual intrinsic volume.

Theorem 5.2. *For $p > 1$ and $q > 0$, and finite Borel measure μ on S^{n-1} not concentrated on a closed hemi-sphere, there exists a convex body $K \in \mathcal{K}^n_o$ with $\mathrm{int}K \neq \emptyset$ and $\mathcal{H}^{n-1}(\Xi_K) = 0$ such that*

$$\widetilde{V}_q(K) h_K^p d\mu = d\widetilde{C}_q(K, \cdot),$$

and in addition, $K \in \mathcal{K}^n_{(o)}$ if $p \geq q$.

Proof. We choose a sequence of discrete measures μ_m tending to μ that are not concentrated on any closed hemispheres. It follows from Theorem 3.1, that there exists polytope $P_m \in \mathcal{K}^n_{(o)}$ such that

$$d\mu_m = \frac{1}{\widetilde{V}_q(P_m)}\, d\widetilde{C}_{p,q}(P_m, \cdot) = \frac{h_{P_m}^{-p}}{\widetilde{V}_q(P_m)}\, d\widetilde{C}_q(P_m, \cdot) \tag{67}$$

for each m, and hence we may assume that

$$\frac{\widetilde{C}_{p,q}(P_m, S^{n-1})}{\widetilde{V}_q(P_m)} < 2\mu(S^{n-1}). \tag{68}$$

We claim that there exists $R > 0$ such that

$$P_m \subset RB^n. \tag{69}$$

We prove (69) by contradiction, thus we suppose that $R_m = \max_{x \in P_m} \|x\|$ tends to infinity. We choose $v_m \in S^{n-1}$ such that $R_m v_m \in P_m$, and we may assume by possibly taking a subsequence that v_m tends to $v \in S^{n-1}$. We deduce from Lemma 5.1 that there exist $s, t > 0$ such that $\mu(\Omega(v, 2t)) > 2s$. As v_m tends to $v \in S^{n-1}$ and μ_m tends weakly to μ, we may also assume that $\Omega(v,2t)\subset\Omega(v_m,t)$ and $\mu_m(\Omega(v,2t))>s$, therefore $\mu_m(\Omega(v_m,t))>s$ for each m. Since $h_{P_m}(u)\geq\langle R_m v_m,u\rangle\geq R_m t$ for $u\in\Omega(v_m,t)$, we deduce from (67) that

$$s<\mu_m(\Omega(v_m,t))=\int_{\Omega(v_m,t)}\frac{h_{P_m}^{-p}(u)}{\widetilde{V}_q(P_m)}\,d\widetilde{C}_q(P_m,u)\leq R_m^{-p}t^{-p}\frac{\widetilde{C}_q(P_m,S^{n-1})}{\widetilde{V}_q(P_m)}\leq R_m^{-p}t^{-p}.$$

In particular, $R_m^p\leq s^{-1}t^{-p}$, contradicting the fact that R_m tends to infinity, and in turn proving (69).

It follows from (69) that P_m tends to a compact convex set $K\in\mathcal{K}_o^n$ with $K\subset RB^n$. We deduce from (68) and Lemma 4.2 that $r(K)>0$.

We observe that $h_{P_m}^p$ tends uniformly to h_K^p, and hence also $\widetilde{V}_q(P_m)h_{P_m}^p$ tends uniformly to $\widetilde{V}_q(K)h_K^p$ by Lemma 2.5. Therefore given any continuous function f, we have

$$\lim_{m\to\infty}\int_{S^{n-1}}f(u)\widetilde{V}_q(P_m)h_{P_m}^p(u)\,d\mu_m=\int_{S^{n-1}}f(u)\widetilde{V}_q(K)h_K^p(u)\,d\mu.$$

It follows from Proposition 2.6 that the dual curvature measure $\widetilde{C}_q(P_m,\cdot)$ tends weakly to $\widetilde{C}_q(K,\cdot)$, thus (67) yields

$$\int_{S^{n-1}}f(u)\widetilde{V}_q(K)h_K^p(u)\,d\mu=\int_{S^{n-1}}f(u)\,d\widetilde{C}_q(K,u).$$

Since the last property holds for all continuous function f, we conclude that

$$\widetilde{V}_q(K)h_K^p\,d\mu=d\widetilde{C}_q(K,\cdot),$$

as it is required.

Having (68) at hand, Lemma 4.4 yields that $\mathcal{H}^{n-1}(\Xi_K)=0$, and Lemma 4.3 implies that if $p\geq q$, then $K\in\mathcal{K}_{(o)}^n$. □

Proof of Theorem 1.2 in the case of $Q=B^n$ Let $p>1$, $q>0$ and $p\neq q$. According to Theorem 5.2, there exists a $K_0\in\mathcal{K}_{(o)}^n$ with $\mathrm{int}K_0\neq\emptyset$ and $\mathcal{H}^{n-1}(\Xi_{K_0})=0$ such that $\widetilde{V}_q(K_0)^{-1}\widetilde{C}_{p,q}(K_0,\cdot)=\mu$. For $\lambda=\widetilde{V}_q(K_0)^{\frac{-1}{q-p}}$ and $K=\lambda K_0$, we have

$$\widetilde{C}_{p,q}(K,\cdot)=\lambda^{q-p}\widetilde{C}_{p,q}(K_0,\cdot)=\widetilde{V}_q(K_0)^{-1}\widetilde{C}_{p,q}(K_0,\cdot)=\mu.$$

It follows from Theorem 5.2 that $o\in\mathrm{int}K$ if $p>q$. □

6. The L_p dual curvature measure involving the star body Q

In this section, we discuss how to extend the results of Sections 2 to 5 about dual curvature measures $\widetilde{C}_q(K,\cdot)$ and L_p dual curvature measures $\widetilde{C}_{p,q}(K,\cdot)$ to $\widetilde{C}_q(K,Q,\cdot)$ and $\widetilde{C}_{p,q}(K,Q,\cdot)$, where Q is a star body. We recall that if $q>0$, $Q\in\mathcal{S}_{(o)}^n$ and $K\in\mathcal{K}_o^n$, then

$$\widetilde{V}_q(K,Q)=\frac{1}{n}\int_{S^{n-1}}\varrho_K^q(u)\varrho_Q^{n-q}(u)\,d\mathcal{H}^{n-1}(u),\tag{70}$$

and if, in addition, $\eta\subset S^{n-1}$ is a Borel set, then

$$\widetilde{C}_q(K,Q,\eta)=\frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)}\varrho_K^q(u)\varrho_Q^{n-q}(u)\,d\mathcal{H}^{n-1}(u).\tag{71}$$

Since for $Q\in\mathcal{S}_{(o)}^n$, ϱ_Q is a positive continuous function on S^{n-1}, essentially the same arguments as in Section 2 yield the analogues Lemmas 6.1, 6.3 and Proposition 6.4 of Lemmas 2.4, 2.5 and 2.6. We note that

$$\widetilde{C}_q(K,Q,S^{n-1}\cap N(K,o))=0\tag{72}$$

as $\varrho_K(u) = 0$ if $u \in \boldsymbol{\alpha}^*_K(\text{int}N(K,o))$, and

$$\boldsymbol{\alpha}^*_K(S^{n-1} \cap N(K,o)) \backslash \boldsymbol{\alpha}^*_K(S^{n-1} \cap \text{int}N(K,o)) \subset S^{n-1} \cap \partial N(K,o)^*$$

and $\mathcal{H}^{n-1}(S^{n-1} \cap \partial N(K,o)^*) = 0$.

For Lemma 2.4, the only additional observation needed is that if $u \in S^{n-1}$ and $x = \varrho_K(u)u \in \partial K$, then $\|x\|_Q = \varrho_K(u)/\varrho_Q(u)$.

Lemma 6.1. *If $q > 0$, $Q \in \mathcal{S}^n_{(0)}$, $K \in \mathcal{K}^n_o$ with $\text{int}K \neq \emptyset$, and the Borel function $g : S^{n-1} \to \mathbb{R}$ is bounded, then*

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_q(K,Q,u) = \frac{1}{n} \int_{S^{n-1} \cap (\text{int}N(K,o)^*)} g(\alpha_K(u)) \varrho_K(u)^q \varrho_Q(u)^{n-q}\, d\mathcal{H}^{n-1}(u) \tag{73}$$

$$= \frac{1}{n} \int_{\partial' K \backslash \Xi_K} g(\nu_K(x)) \langle \nu_K(x), x \rangle \|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x), \tag{74}$$

$$= \frac{1}{n} \int_{\partial' K} g(\nu_K(x)) \langle \nu_K(x), x \rangle \|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x) \tag{75}$$

From (75) we deduce the following.

Corollary 6.2. *If $q > 0$, $Q \in \mathcal{S}^n_{(0)}$, $K \in \mathcal{K}^n_o$ with $\text{int}K \neq \emptyset$ and $\mathcal{H}^{n-1}(\Xi_K) = 0$, then the surface area measure $S(K,\cdot)$ is absolutely continuous with respect to $\widetilde{C}_q(K,Q,\cdot)$.*

Corollary 6.2 will be useful for the differential equation representing the L_p dual Minkowski problem in Section 7.

Now, arguments verifying Lemma 6.3 and Proposition 6.4 use (73) in a similar way as the proofs of Lemmas 2.5 and 2.6 are based on (28).

Lemma 6.3. *For $q > 0$ and $Q \in \mathcal{S}^n_{(0)}$, $\widetilde{V}_q(K)$ is a continuous function of $K \in \mathcal{K}^n_o$ with respect to the Hausdorff distance.*

Proposition 6.4. *If $q > 0$, $Q \in \mathcal{S}^n_{(0)}$ and $\{K_m\}$, $m \in \mathbb{N}$, tends to K for $K_m, K \in \mathcal{K}^n_o$, then $\widetilde{C}_q(K_m,Q,\cdot)$ tends weakly to $\widetilde{C}_q(K,Q,\cdot)$.*

For $q > 0$, we extend Theorem 6.8 in [24] (see (10)) to any convex body containing the origin on its boundary. For $Q \in \mathcal{S}^n_{(o)}$, we observe that if $P \in \mathcal{K}^n_o$ is a polytope with $\text{int}\,P \neq \emptyset$ and $v_1, \ldots, v_l \in S^{n-1}$ are the exterior normals of the facets of P not containing the origin, then Lemma 6.1 yields

$$\begin{aligned} \text{supp}\, \widetilde{C}_q(P,Q,\cdot) &= \{v_1, \ldots, v_l\}, \qquad \text{and} \\ \widetilde{C}_q(P,Q,\{v_i\}) &= \frac{1}{n} \int_{\tilde{\pi}(F(P,v_i))} \varrho_P^q(u) \varrho_Q^{n-q}(u)\, d\mathcal{H}^{n-1}(u) \\ &= \frac{1}{n} \int_{F(P,v_i))} h_P(v_i) \|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x) \qquad \text{for } i = 1, \ldots, l. \end{aligned} \tag{76}$$

Lemma 6.5. *If $q > 0$, $K \in \mathcal{K}^n_o$, $Q \in \mathcal{S}^n_{(o)}$, g is a bounded real Borel function on S^{n-1} and $\varphi \in \text{SL}(n,\mathbb{R})$, then*

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_q(\varphi K, \varphi Q, u) = \int_{S^{n-1}} g\left(\frac{\varphi^{-t}u}{\|\varphi^{-t}u\|}\right) d\widetilde{C}_q(K,Q,u).$$

Proof. It is sufficient to prove Lemma 6.5 for the case when g is continuous. Therefore, it follows from Proposition 6.4 and polytopal approximation that we may assume that K is an n-dimensional polytope. We write $u_1, \ldots, u_k$ to denote the exterior unit normals of K, and set $F_i = F(K,u_i)$ for

$i=1,\dots,k$. It follows that the exterior unit normal at the facet φF_i of φK is $v_i=\frac{\varphi^{-t}u_i}{\|\varphi^{-t}u_i\|}$ for $i=1,\dots,k$.

For any $i=1,\dots,k$, $\det\varphi=1$ yields that the volumes of the cones over the facets do not change, and hence $\frac{1}{n}h_{\varphi K}(v_i)\cdot\mathcal{H}^{n-1}(\varphi F_i)=\frac{1}{n}h_K(u_i)\cdot\mathcal{H}^{n-1}(F_i)$, which in turn implies that

$$\det\left(\varphi|_{u_i^\perp}\right)=\frac{h_K(u_i)}{h_{\varphi K}(v_i)}. \tag{77}$$

We note that the linearity of φ yields $\|\varphi y\|_{\varphi Q}=\|y\|_Q$ for any $y\in\mathbb{R}^n$. We deduce first from (76) and later from (77) that

$$\begin{aligned}\int_{S^{n-1}}g(u)\,d\widetilde{C}_q(\varphi K,\varphi Q,u) &= \frac{1}{n}\sum_{i=1}^k\int_{\varphi F_i}g(v_i)\|x\|_{\varphi Q}^{q-n}h_{\varphi K}(v_i)\,d\mathcal{H}^{n-1}(x)\\ &= \frac{1}{n}\sum_{i=1}^k\int_{F_i}g(v_i)\|y\|_Q^{q-n}h_{\varphi K}(v_i)\det\left(\varphi|_{u_i^\perp}\right)\,d\mathcal{H}^{n-1}(y)\\ &= \frac{1}{n}\sum_{i=1}^k\int_{F_i}g\left(\frac{\varphi^{-t}u_i}{\|\varphi^{-t}u_i\|}\right)\|y\|_Q^{q-n}h_K(u_i)\,d\mathcal{H}^{n-1}(y),\end{aligned}$$

which in turn implies Lemma 6.5 by (76). □

For $w\in S^{n-1}$ and $\alpha\in(-1,1)$, we define

$$\Gamma(w,\alpha)=\{u\in S^{n-1}:|\langle u,w\rangle|<\alpha\}.$$

Since the restriction of the radial projection $\tilde{\pi}$ satisfies that $\|\tilde{\pi}(x_1)-\tilde{\pi}(x_2)\|\le\|x_1-x_2\|$ for $x_1,x_2\in(w^\perp\cap S^{n-1})+\operatorname{lin}w$, we have

Lemma 6.6. *If $w\in S^{n-1}$ and $\alpha\in(-1,1)$, then*

$$\mathcal{H}^{n-1}(\Gamma(w,\alpha))\le(n-2)\kappa_{n-2}\cdot\frac{2\alpha}{\sqrt{1-\alpha^2}}.$$

For Lemma 6.7, we start with $u_1,\dots,u_k\in S^{n-1}$ that are not contained in a closed hemi-sphere. For $z=(z_1,\dots,z_k)\in\mathbb{R}^k_+$, we define

$$P(z)=\{x\in\mathbb{R}^n:\langle x,u_i\rangle\le z_i\text{ for }i=1,\dots,k\}. \tag{78}$$

We observe that $P(z)$ is an n-dimensional polytope with $o\in\operatorname{int}P(z)$, and any facet exterior unit normal is among $u_1,\dots,u_k$. The following is the special case of polytopes of Theorem 6.2 in Lutwak, Yang, Zhang [24]. For the sake of completeness, we provide the proof in this special case.

Lemma 6.7 (Lutwak, Yang, Zhang [24])**.** *If $q\neq0$, $Q\in\mathcal{S}^n_{(o)}$, $\eta\in(0,1)$ and $z_t=(z_1(t),\dots,z_k(t))\in\mathbb{R}^k_+$ for $t\in(-\eta,\eta)$ are such that $\lim_{t\to0^+}\frac{z_i(t)-z(0)}{t}=z_i'(0)\in\mathbb{R}$ for $i=1,\dots,k$ exists, then the $P(z_t)$ defined in (78) satisfies that*

$$\lim_{t\to0^+}\frac{\widetilde{V}_q(P(z_t),Q)-\widetilde{V}_q(P(z_0),Q)}{t}=q\sum_{i=1}^k\frac{z_i'(0)}{h_{P(z_0)}(u_i)}\cdot\widetilde{C}_q(P(z_0),Q,\{u_i\}).$$

Proof. We set $P_0=P(z_0)$. We may assume that $F(P_0,u_i)$ is an $(n-1)$-dimensional facet of P_0 if and only if $i\le l$ where $l\le k$.

For a point $x\in\mathbb{R}^n$ and affine d-plane A, $1\le d\le n-1$, we write $\delta(x,A)$ for the distance of x from A. For $i=1,\dots,k$, let H_i be the hyperplane $\{x\in\mathbb{R}^n:\langle u_i,x\rangle=z_i(0)\}$, and for $i,j\in\{1,\dots,k\}$ with $u_i\neq\pm u_j$, let $A_{ij}=H_i\cap H_j$, which is an affine $(n-2)$-plane not containing the origin. Therefore $\operatorname{lin}A_{ij}$ is $(n-1)$-dimensional, and let $w_{ij}\in S^{n-1}$ be orthogonal to $\operatorname{lin}A_{ij}$.

Choosing the number Δ in such a way that for any $i,j\in\{1,\dots,k\}$ with $u_i\neq\pm u_j$, we have $(1-\langle u_i,u_j\rangle^2)^{-1/2}\leq\Delta$, we deduce that if $s>0$ and $i,j\in\{1,\dots,k\}$ with $u_i\neq\pm u_j$, then

$$y\in H_i \text{ and } d(y,H_j)\leq s \text{ yield } d(y,A_{ij})\leq\Delta s. \tag{79}$$

Possibly decreasing $\eta>0$, we may assume that there exist $r,R,Z>0$ such that if $t\in(-\eta,\eta)$, then

$$\begin{array}{rcll} rB^n\subset & P(z_t) & \subset RB^n, & \text{and}\\ |z_i(t)-z_i(0)| & \leq & Z|t| & \text{for } i=1,\dots,k.\end{array}$$

If $u\in\tilde\pi(F(P(z_t),u_i)$ for $i\in\{1,\dots,l\}$ and $t\in(-\eta,\eta)$, then $\langle\varrho_{P(z_t)}(u)\,u,u_i\rangle=z_i(t)\geq r$, therefore

$$\langle u,u_i\rangle\geq\frac{r}{R}. \tag{80}$$

In addition, $\varrho_{P(z_t)}(u)\,u\in P(z_t)$, thus

$$\varrho_{P(z_t)}(u)\leq\frac{z_i(t)}{\langle u_i,u\rangle}. \tag{81}$$

Now if $u\in\tilde\pi(F(P_0,u_i))$ for $i\in\{1,\dots,l\}$ and $\varrho_{P(z_t)}(u)<\frac{z_i(t)}{\langle u_i,u\rangle}$, then there exists $j\in\{1,\dots,k\}$ with $u_j\neq\pm u_i$ satisfying $\varrho_{P(z_t)}(u)\,u\in F(P(z_t),u_j)$, or in other words,

$$\varrho_{P(z_t)}(u)=\frac{z_j(t)}{\langle u_j,u\rangle};$$

and we claim that

$$u\in\ \Gamma(w_{ij},c_1\cdot|t|),\quad \text{where } c_1=\tfrac{\Delta RZ}{r^2},\quad\text{and} \tag{82}$$

$$|\varrho_{P(z_t)}(u)-\varrho_{P_0}(u)|\leq\ c_2\cdot|t|,\quad \text{where } c_2=\tfrac{R^2Z}{r^2} \tag{83}$$

On the one hand, (80) yields that

$$\left\|\varrho_{P_0}(u)\,u-\frac{z_i(t)}{\langle u_i,u\rangle}\,u\right\|=\frac{|z_i(0)-z_i(t)|}{\langle u_i,u\rangle}\leq\frac{RZ}{r}\cdot|t|. \tag{84}$$

On the other hand, since $\frac{z_i(t)}{\langle u_i,u\rangle}\,u\notin P(z_t)$, there exists $j\in\{1,\dots,k\}$ with $u_j\neq\pm u_i$ such that

$$\left\langle u_j,\frac{z_i(t)}{\langle u_i,u\rangle}\,u\right\rangle>z_j(t), \tag{85}$$

and hence $u_j\neq\pm u_i$. In turn it follows from (84) that

$$d(\varrho_{P_0}(u)\,u,H_j)=z_j(t)-\langle u_j,\varrho_{P_0}(u)\,u\rangle<\left\langle u_j,\frac{z_i(t)}{\langle u_i,u\rangle}\,u\right\rangle-\langle u_j,\varrho_{P_0}(u)\,u\rangle\leq\frac{RZ}{r}\cdot|t|.$$

We deduce from (79) that $d(\varrho_{P_0}(u)u,A_{ij})\leq\frac{\Delta RZ}{r}\cdot|t|$, and hence

$$|\langle w_{ij},\varrho_{P_0}(u)\,u\rangle|\leq\frac{\Delta RZ}{r}\cdot|t|.$$

Finally, $\varrho_{P_0}(u)\geq r$ yields (82).

For (83), we deduce from $\varrho_{P(z_t)}(u)=\frac{z_j(t)}{\langle u_j,u\rangle}$, (84) and (85) that

$$\langle u_j,\varrho_{P_0}(u)\,u\rangle>z_j(t)-\frac{RZ}{r}\cdot|t|=\varrho_{P(z_t)}(u)\langle u_j,u\rangle-\frac{RZ}{r}\cdot|t|.$$

On the other hand, since $\varrho_{P_0}(u)\,u\in P_0$, we have

$$\langle u_j,\varrho_{P_0}(u)\,u\rangle\leq z_j(0)\leq z_j(t)+Z|t|\leq\varrho_{P(z_t)}(u)\langle u_j,u\rangle+Z|t|,$$

which in turn yields

$$\langle u_j, u\rangle|\varrho_{P(z_t)}(u) - \varrho_{P_0}(u)| \leq \frac{2RZ}{r} \cdot |t|.$$

Since $\langle u_j, u\rangle \geq \frac{r}{R}$ according to (80) for j instead of i, we conclude (83).

For $i = 1, \ldots, k$, we write X_i to denote the set of all $u \in \tilde{\pi}(F(P_0, u_i))$ such that $u \in \Gamma(w_{ij}, c_1 \cdot |t|)$ for some $j \in \{1, \ldots, k\}$ with $u_j \neq \pm u_i$. Using $\varrho_{P_0}(u) = \frac{z_i(0)}{\langle u_i, u\rangle}$ for $i = 1, \ldots, l$ and $u \in \tilde{\pi}(F(P_0, u_i))$, (81) and (82), it follows that $F(t) = \frac{\widetilde{V}_q(P(z_t),Q) - \widetilde{V}_q(P_0,Q)}{t}$ satisfies

$$\begin{aligned} F(t) &= \frac{1}{n}\sum_{i=1}^{l}\int_{\tilde{\pi}(F(P_0,u_i))} \frac{\varrho_{P(z_t)}(u)^q - \varrho_{P_0}(u)^q}{t} \cdot \varrho_Q(u)^{n-q}\, d\mathcal{H}^{n-1}(u) \\ &= \frac{1}{n}\sum_{i=1}^{k}\int_{X_i} \left(\frac{\varrho_{P(z_t)}(u)^q - \varrho_{P_0}(u)^q}{t} + \frac{z_i(0)^q - z_i(t)^q}{\langle u, u_i\rangle^q t}\right) \cdot \varrho_Q(u)^{n-q}\, d\mathcal{H}^{n-1}(u) + \\ &\quad + \frac{1}{n}\sum_{i=1}^{l}\int_{\tilde{\pi}(F(P_0,u_i))} \frac{z_i(t)^q - z_i(0)^q}{\langle u, u_i\rangle^q t} \cdot \varrho_Q(u)^{n-q}\, d\mathcal{H}^{n-1}(u) \end{aligned}$$

We deduce from (80), (83) and $|z_i(t) - z_i(0)| \leq Z|t|$ that

$$\frac{\varrho_{P(z_t)}(u)^q - \varrho_{P_0}(u)^q}{t} + \frac{z_i(0)^q - z_i(t)^q}{\langle u, u_i\rangle^q t}$$

is uniformly bounded on $\tilde{\pi}(F(P_0, u_i))$ as t tends to 0. Since $h_{P_0}(u_i) = z_i(0)$ for $i = 1, \ldots, l$ and $\mathcal{H}^{n-1}(X_i) = O(t)$ according to Lemma 6.6, we deduce

$$\lim_{t\to 0} F(t) = \frac{q}{n}\sum_{i=1}^{l}\int_{\tilde{\pi}(F(P_0,u_i))} \frac{z_i(0)^{q-1} z_i'(0)}{\langle u, u_i\rangle^q t} \cdot \varrho_Q(u)^{n-q}\, d\mathcal{H}^{n-1}(u) = q\sum_{i=1}^{l} \frac{z_i'(0)}{h_{P_0}(u_i)} \cdot \widetilde{C}_q(P_0, Q, \{u_i\}).$$

As $\widetilde{C}_q(P_0, Q, \{u_i\}) = 0$ for $i > l$, we conclude Lemma 6.7. □

Now we sketch the necessary changes needed to extend Theorem 3.1 to the case when Q is a star body.

Theorem 6.8. *Let $p > 1$, $q > 0$ and $Q \in \mathcal{S}^n_{(o)}$, and let μ be a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere. Then there exists a polytope $P \in \mathcal{K}^n_{(o)}$ such that $\widetilde{V}_q(P, Q)^{-1}\widetilde{C}_{p,q}(P, Q, \cdot) = \mu$.*

Sketch of the proof Theorem 6.8. Let $p > 1$, $q > 0$ and μ a discrete measure on S^{n-1} that is not concentrated on any closed hemisphere. Let $\operatorname{supp}\mu = \{u_1, \ldots, u_k\}$, and let $\mu(\{u_i\}) = \alpha_i > 0$, $i = 1, \ldots, k$. For any $z = (t_1, \ldots, t_k) \in (\mathbb{R}_{\geq 0})^k$, we define

$$\begin{aligned} \Phi(z) &= \sum_{i=1}^{k} \alpha_i t_i^p, \\ P(z) &= \{x \in \mathbb{R}^n : \langle x, u_i\rangle \leq t_i \; \forall i = 1, \ldots, k\}, \\ \Psi(z) &= \widetilde{V}_q(P(z), Q). \end{aligned} \tag{86}$$

Since $\alpha_i > 0$ for $i = 1, \ldots, k$, the set $Z = \{z \in (\mathbb{R}_{\geq 0})^k : \Phi(z) = 1\}$ is compact, and hence Lemma 6.3 yields the existence of $z_0 \in Z$ such that

$$\Psi(z_0) = \max\{\Psi(z) : z \in Z\}.$$

Now, similarly to the proof of Theorem 3.1, only using Lemma 6.7 in place of Lemma 3.3, we prove that $o \in \operatorname{int} P(z_0)$ and that there exists a $\lambda_0 > 0$ such that

$$\widetilde{V}_q(\lambda_0 P(z_0), Q)^{-1} \widetilde{C}_{p,q}(\lambda_0 P(z_0), Q, \cdot) = \mu.$$

Therefore we can choose $P = \lambda_0 P(z_0)$ in Theorem 6.8. □

Proof of Theorem 1.1. Theorem 6.8 yields Theorem 1.1 using the same argument as the one at the end of Section 3. □

Next, we extend the results of Section 4 on the L_p dual curvature measures to the case when a star body $Q \in \mathcal{S}^n_{(o)}$ is involved. The first of these extensions can be obtained as Corollary 4.1.

Corollary 6.9. *If $p > 1$, $q > 0$, $Q \in \mathcal{S}^n_{(o)}$, $K \in \mathcal{K}^n_o$ with $\operatorname{int} K \neq \emptyset$, $\widetilde{C}_{p,q}(K, S^{n-1}) < \infty$ and $\mathcal{H}^{n-1}(\Xi_K) = 0$, and the Borel function $g : S^{n-1} \to \mathbb{R}$ is bounded, then*

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_{p,q}(K, Q, u) = \frac{1}{n} \int_{\partial' K} g(\nu_K(x)) \langle \nu_K(x), x \rangle^{1-p} \|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x).$$

Lemmas 6.10 and 6.11 can be proved essentially the same way as Lemmas 4.2 and 4.3.

Lemma 6.10. *For $n \geq 2$, $p > 1$, $q > 0$ and $Q \in \mathcal{S}^n_{(o)}$, there exists constant $c > 0$ depending only on p, q, n, Q such that if $K \in \mathcal{K}^n_{(o)}$, then*

$$\widetilde{C}_{p,q}(K, Q, S^{n-1}) \geq c \cdot r(K)^{-p} \cdot \widetilde{V}_q(K, Q).$$

Lemma 6.11. *If $p > 1$, $0 < q \leq p$, $Q \in \mathcal{S}^n_{(o)}$ and $K_m \in \mathcal{K}^n_{(o)}$ for $m \in \mathbb{N}$ tend to $K \in \mathcal{K}^n_o$ with $\operatorname{int} K \neq \emptyset$ such that $\widetilde{C}_{p,q}(K_m, Q, S^{n-1})$ stays bounded, then $K \in \mathcal{K}^n_{(o)}$.*

Since the sequence $\{\widetilde{C}_{p,q}(K_m, Q, S^{n-1})\}$ in Lemma 6.12 is bounded if and only if $\{\widetilde{C}_{p,q}(K_m, S^{n-1})\}$ is bounded, Lemma 4.4 directly yields Lemma 6.12.

Lemma 6.12. *If $p > 1$, $q > 0$, $Q \in \mathcal{S}^n_{(o)}$ and $K_m \in \mathcal{K}^n_{(o)}$ for $m \in \mathbb{N}$ tend to $K \in \mathcal{K}^n_o$ with $\operatorname{int} K \neq \emptyset$ such that $\widetilde{C}_{p,q}(K_m, Q, S^{n-1})$ stays bounded, then $\mathcal{H}^{n-1}(\Xi_K) = 0$.*

Using Theorem 6.8, Proposition 6.4 and Lemmas 6.10, 6.11 and 6.12, an argument similar to the one leading to Theorem 5.2 implies

Theorem 6.13. *For $p > 1$, $q > 0$, $Q \in \mathcal{S}^n_{(o)}$ and a finite Borel measure μ on S^{n-1} not concentrated on a closed hemisphere, there exists a convex body $K \in \mathcal{K}^n_o$ with $\operatorname{int} K \neq 0$ and $\mathcal{H}^{n-1}(\Xi_K) = 0$ such that*

$$\widetilde{V}_q(K, Q) h_K^p d\mu = d\widetilde{C}_q(K, Q, \cdot),$$

and, in addition, $K \in \mathcal{K}^n_{(o)}$ if $p \geq q$.

Proof of Theorem 1.2. Theorem 6.13 yields Theorem 1.2 using essentially the same argument as the one at the end of Section 5. □

7. The regularity of the solution

Given $p > 1$, $q > 0$, and a finite non-trivial Borel measure μ on S^{n-1} not concentrated on any closed hemisphere, the L_p dual Minkowski problem asks for a convex body $K \in \mathcal{K}^n_o$ such that $\mathcal{H}^{n-1}(\Xi_K) = 0$ and

$$h_K^p\, d\mu = d\widetilde{C}_q(K, \cdot). \tag{87}$$

First we discuss why the condition $\mathcal{H}^{n-1}(\Xi_K) = 0$ is natural.

Example 7.1. *For $p>1$ and $q>0$ with $p\neq q$, there exists a discrete measure μ on S^{n-1} and polytopes P_0 and P such that*

$$h_P^p d\mu = d\widetilde{C}_q(P,\cdot) \quad \text{and} \quad h_{P_0}^p d\mu = d\widetilde{C}_q(P_0,\cdot)$$

with $o\in \mathrm{int}P$ and $\mathcal{H}^{n-1}(\Xi_{P_0})>0$.

Proof. Let P_0 be any polytope in $\mathbb{R}^n$ such that $u_0,\dots,u_k$ denote the exterior unit normals to its facets, $h_{P_0}(u_0)=0$, $h_{P_0}(u_i)>0$ for $i=1,\dots,k$, and no closed hemisphere contains $u_1,\dots,u_k$. Let $\mathrm{supp}\,\mu=\{u_1,\dots,u_k\}$, and let $\mu(\{u_i\})=\widetilde{C}_{p,q}(P_0,\{u_i\})$ for $i=1,\dots,k$. According to Theorem 1.1, there exists a polytope $P\in\mathcal{K}^n_{(o)}$ such that $\widetilde{C}_{p,q}(P,\cdot)=\mu$. $\square$

We recall that according to Hug, Lutwak, Yang, Zhang [16], if $p>1$ and $q=n$, then there is a unique solution P to the L_p dual Minkowski problem (87) for any measure μ on S^{n-1} not concentrated on any closed hemisphere with $\mathcal{H}^{n-1}(\Xi_p)=0$; namely, $P\in\mathcal{K}^n_{(o)}$.

We now turn to measures on S^{n-1} that are absolutely continuous with respect to the spherical Lebesgue measure. We write D and D^2 to denote the derivative and the Hessian of real functions on Euclidean spaces, and ∇ and ∇^2 to denote the gradient and the Hessian of real functions on S^{n-1} with respect to a moving orthonormal frame on S^{n-1}.

First, let us discuss some relation between the support function and the boundary of a convex body. Let $C\in\mathcal{K}^n_{(o)}$. If $y\in\mathbb{R}^n\backslash\{o\}$, then it is well-known (see Schneider [27]) that the face with exterior normal y is the set of derivatives of the support function h_C at y; namely,

$$F(C,y)=\partial h_C(y)=\{z\in\mathbb{R}^n : h_C(x)\geq h_C(y)+\langle z,x-y\rangle \text{ for each } x\in\mathbb{R}^n\}. \tag{88}$$

We note that h_C is differentiable at $\mathcal{H}^n$ almost all points of $\mathbb{R}^n$ being convex, and $\mathcal{H}^{n-1}$ almost all points of S^{n-1} being, in addition, 1-homogeneous. It follows that whenever h_C is differentiable at $u\in S^{n-1}$ (and hence for $\mathcal{H}^{n-1}$ almost every $u\in S^{n-1}$), we have

$$Dh_C(u) = x \quad \text{where } u \text{ is an exterior normal at } x\in\partial C; \tag{89}$$

$$\langle Dh_C(u),u\rangle = h_C(u). \tag{90}$$

In addition, (90) yields

$$Dh_C(u) = \nabla h_K(u)+h_K(u)\,u, \quad \text{and} \tag{91}$$

$$\|x\| = \|Dh_C(u)\|=\sqrt{h(u)^2+\|\nabla h_C(u)\|^2}. \tag{92}$$

According to Corollary 6.2, if $q>0$ and $\mathcal{H}^{n-1}(\Xi_K)=0$ for $K\in\mathcal{K}^n_o$, then the surface area measure $S(K,\cdot)$ is absolutely continuous with respect to $\widetilde{C}_q(K,\cdot)$. We deduce from Lemma 6.1, (89) and (92) that if $d\widetilde{C}_{p,q}(K,\cdot)=f\,d\mathcal{H}^{n-1}$ for a non-negative L_1 function f on S^{n-1}, then the Monge-Ampère equation for the L_p dual curvature measure is

$$\det(\nabla^2 h_K(u)+h_K(u)\,\mathrm{Id}) = n\,h_K^{p-1}(u)\left(\|\nabla h_K(u)\|^2+h_K(u)^2\right)^{\frac{n-q}{2}}\cdot f(u). \tag{93}$$

In the case when a star body $Q\in\mathcal{S}^n_{(o)}$ is involved, we deduce from Lemma 6.1, (89) and (91) that the Monge-Ampère equation for the L_p dual curvature measure is

$$\det(\nabla^2 h_K(u)+h_K(u)\,\mathrm{Id}) = n\,h_K^{p-1}(u)\,\|\nabla h_K(u)+h_K(u)\,u\|_Q^{n-q}\cdot f(u). \tag{94}$$

In the rest of this section, we consider solutions to (94) in the case when there exist $c_2>c_1>0$ satisfying

$$c_1<f(u)<c_2 \quad \text{for } u\in S^{n-1}. \tag{95}$$

Example 7.2. *Given $q > p > 1$, there exists a $K \in \mathcal{K}_o^n$ such that $\operatorname{int} K \neq \emptyset$, $o \in \partial K$ is not a smooth point, $\Xi_K = \{o\}$ and h_K satisfies both (93) and (95) in the sense of measure, and f is positive and continuous on S^{n-1}.*

Proof. For positive functions g_1 and g_2 on B^{n-1}, we write

$$g_1 \approx g_2 \ \text{ if } \alpha_1 g_1(x) \leq g_2(x) \leq \alpha_2 g_1(x) \text{ for } x \in B^{n-1}\backslash\{o\},$$

where $\alpha_2 > \alpha_1 > 0$ are constants depending only on n, p, q.

We define $g : \mathbb{R}^{n-1} \to \mathbb{R}$ by the formula

$$g(x) = \|x\| + \|x\|^\theta \ \text{ for } \theta = q/p > 1,$$

and we consider a convex body $K \in \mathcal{K}_o^n$ such that the graph of g above B^{n-1} is part of ∂K and $\partial K\backslash\{o\}$ is C_+^2. We observe that

$$N(K, o) = \{(x, t) : \, x \in \mathbb{R}^{n-1} \text{ and } t \leq -\|x\|\}.$$

For $x \in B^n\backslash\{o\}$,

$$\begin{aligned} Dg(x) &= x\left(\|x\|^{-1} + \theta\|x\|^{\theta-2}\right), \\ \|Dg(x)\| &= 1 + \theta\|x\|^{\theta-1} \approx 1. \end{aligned}$$

For $y = (x, g(x)) \in \partial K$ and $x \in B^{n-1} \setminus \{o\}$ we have

$$\begin{aligned} \nu_K(y) &= (1 + \|Dg(x)\|^2)^{-1}(Dg(x), -1), \\ \langle \nu_K(y), y\rangle &= (1 + \|Dg(x)\|^2)^{-1}\langle (Dg(x), -1), (x, g(x))\rangle = (\theta - 1)\|x\|^\theta \approx \|x\|^\theta, \\ \|y\| &= \sqrt{\|x\|^2 + (\|x\| + \|x\|^\theta)^2} = \|x\|\sqrt{2 + 2\|x\|^{\theta-1} + \|x\|^{2\theta-2}} \approx \|x\|. \end{aligned}$$

At $x \in B^{n-1}\backslash\{o\}$, we have

$$\det D^2 g(x) = \theta(\theta - 1)\|x\|^{\theta-2}(\|x\|^{-1} + \theta\|x\|^{\theta-2})^{n-2} \approx \|x\|^{\theta-2}\|x\|^{-n+2} = \|x\|^{\theta-n}.$$

Setting $u = \nu_K(y)$ and writing $\kappa(y)$ to denote the Gaussian curvature at y, we have

$$\det\left(\nabla^2 h_K(u) + h_K(u)\,\mathrm{Id}\right) = \kappa(y)^{-1} = \frac{(1 + \|Dg(x)\|^2)^{\frac{1+n}{2}}}{\det D^2 g(x)} \approx \|x\|^{n-\theta}.$$

Let us consider the spherically open set

$$\mathcal{U} = \{\nu_K(y) : \, y = (x, g(x)) \text{ and } x \in \operatorname{int} B^{n-1}\backslash\{o\}\}.$$

Since $\partial K\backslash\{o\}$ is C_+^2, we deduce that there exists some continuous function f on $S^{n-1}\backslash N(K, o)$ such that $d\widetilde{C}_{p,q}(K, \cdot) = f\, d\mathcal{H}^{n-1}$ on $S^{n-1}\backslash N(K, o)$. It follows from (93) and the considerations above that if $u \in \mathcal{U}$ with $u = \nu_K(y)$ and $y = (x, g(x))$ for $x \in \operatorname{int} B^{n-1}\backslash\{o\}$, then $\|y\|^2 = \|\nabla h_K(u)\|^2 + h_K(u)^2$ and

$$f(u) = \frac{1}{n}\det\left(\nabla^2 h_K(u) + h_K(u)\,\mathrm{Id}\right) h_K(u)^{1-p}\left(\|\nabla h_K(u)\|^2 + h_K(u)^2\right)^{\frac{q-n}{2}}$$

$$= \frac{1}{n}\frac{(1 + \|Dg(x)\|^2)^{\frac{1+n}{2}}}{\det D^2 g(x)}\left[(\theta - 1)\|x\|^\theta\right]^{1-p}\left(\|x\|^2 + (\|x\| + \|x\|^\theta)^2\right)^{\frac{q-n}{2}} \tag{96}$$

$$\approx \|x\|^{n-\theta}\|x\|^{\theta(1-p)}\|x\|^{q-n} = \|x\|^{q-\theta p} = 1. \tag{97}$$

The expression (96) has some limit $F > 0$ as $x \in B^{n-1}\backslash\{o\}$ tends to o according to the formulas above, therefore defining $f(u) = F$ for $u \in N(K, o) \cap S^{n-1}$, (97) yields that f is a positive continuous function on S^{n-1} satisfying (93) in the sense of measure. □

Let us recall some fundamental properties of Monge-Ampère equations based on the survey Trudinger, Wang [28]. Given a convex function v defined in an open convex set Ω of $\mathbb{R}^n$, Dv and D^2v denote its gradient and its Hessian, respectively. When v is a convex function defined in an open convex set $\Omega \subset \mathbb{R}^n$, the subgradient $\partial v(x)$ of v at $x \in \Omega$ is defined as

$$\partial v(x) = \{z \in \mathbb{R}^n : v(y) \geq v(x) + \langle z, y - x\rangle \text{ for each } y \in \Omega\},$$

which is a compact convex set. If $\omega \subset \Omega$ is a Borel set, then we denote by $N_v(\omega)$ the image of ω via the gradient map of v, i.e.

$$N_v(\omega) = \bigcup_{x\in\omega} \partial v(x).$$

The associated Monge-Ampère measure is defined by

$$\mu_v(\omega) = \mathcal{H}^n\Big(N_v(\omega)\Big). \tag{98}$$

We observe that if v is C^2, then

$$\mu_v(\omega) = \int_\omega \det(D^2 v\ )\, d\mathcal{H}^n.$$

We say that a convex function v is the solution of a Monge-Ampère equation in the sense of measure (or in the Alexandrov sense), if it solves the corresponding integral formula for μ_v instead of the original formula for $\det(D^2v)$.

If K is any convex body in $\mathbb{R}^n$, then

$$\partial h_K(u) = F(K, u), \tag{99}$$

see Schneider [27, Thm. 1.7.4]. In particular, for any Borel $\omega \subset S^{n-1}$, the surface area measure S_K satisfies

$$S_K(\omega) = \mathcal{H}^{n-1}\big(\cup_{u\in\omega} F(K, u)\big) = \mathcal{H}^{n-1}\big(\cup_{u\in\omega} \partial h_K(u)\big),$$

and hence S_K is the analogue of the Monge-Ampère measure for the restriction of h_K to S^{n-1}.

We use Lemma 7.3 to transfer the L_p dual Minkowski Monge-Ampère equation (94) on S^{n-1} to a Euclidean Monge-Ampère equation on $\mathbb{R}^{n-1}$. For $e \in S^{n-1}$, we consider the restriction of a solution h of (94) to the hyperplane tangent to S^{n-1} at e.

Lemma 7.3. *For $p > 1$, $q > 0$, $Q \in \mathcal{S}^n_{(o)}$, $e \in S^{n-1}$ and $K \in \mathcal{K}^n_o$ with $\mathcal{H}^{n-1}(\Xi_K) = 0$, if $h = h_K$ is a solution of (94) for a non-negative function f, and $v(y) = h_K(y + e)$ holds for $v : e^\perp \to \mathbb{R}$, then v satisfies*

$$\det(D^2v(y)) = nv(y)^{p-1} \left\|Dv(y) + (\langle Dv(y), y\rangle - v(y)) \cdot e\right\|_Q^{n-q} g(y) \quad \text{ on } e^\perp \tag{100}$$

in the sense of measure, where for $y \in e^\perp$, we have

$$g(y) = \frac{1}{n\,(1 + \|y\|^2)^{\frac{n+p}{2}}} \cdot f\left(\frac{e + y}{\sqrt{1 + \|y\|^2}}\right).$$

Remark. $\|Dv(y)+(\langle Dv(y), y\rangle - v(y))\cdot e\|_Q^{n-q} = (\|Dv(y)\|^2+(\langle Dv(y), y\rangle - v(y))^2)^{\frac{n-q}{2}}$ if $Q = B^n$.

Proof. Using Corollary 6.9 and (90), the Monge-Ampère equation for h_K can be written in the form

$$dS_K = n\, h_K^{p-1} \|Dh_K\|_Q^{n-q} f\, d\mathcal{H}^{n-1} \quad \text{ on } S^{n-1}. \tag{101}$$

We consider $\pi : e^\perp \to S^{n-1}$ defined by

$$\pi(y) = (1 + \|y\|^2)^{\frac{-1}{2}}(y + e), \tag{102}$$

which is induced by the radial projection from the tangent hyperplane $e+e^\perp$ to S^{n-1}. Since $\langle \pi(x), e\rangle = (1+\|x\|^2)^{\frac{-1}{2}}$, the Jacobian of π is

$$\det D\pi(y) = (1+\|y\|^2)^{\frac{-n+1}{2}}. \tag{103}$$

For $y\in e^\perp$, (99) and writing h_K in terms of an orthonormal basis of $\mathbb{R}^n$ containing e, yields that v satisfies

$$\partial v(y) = \partial h_K(y+e)|e^\perp = F(K, y+e)|e^\perp = F(K,\pi(y))|e^\perp. \tag{104}$$

Let $u=\pi(y)$ for some $y\in e^\perp$, where v is differentiable. As h_K is homogeneous of degree one, we have $Dh_K(y+e) = Dh_K(u)$, and therefore

$$Dv(y) = Dh_K(y+e)|e^\perp = Dh_K(u)|e^\perp,$$

and hence $Dh_K(u) = Dv(y) - te$ for some $t\in\mathbb{R}$. Now $\langle Dh_K(u), u\rangle = h_K(u)$ according to (90), which in turn yields by $u=(1+\|y\|^2)^{-\frac{1}{2}}(y+e)$ and $h_K(u) = (1+\|x\|^2)^{-\frac{1}{2}}v(y)$ that $t=\langle Dv(y),y\rangle - v(y)$. In other words, if v is differentiable at $y\in e^\perp$ and $u=\pi(y)$, then

$$Dh_K(u) = Dh_K(e+y) = Dv(y) + (\langle Dv(y),y\rangle - v(y))\cdot e. \tag{105}$$

For a bounded Borel set $\omega\subset e^\perp$, we have using (105) that

$$\begin{aligned}
\mathcal{H}^{n-1}(N_v(\omega)) &= \mathcal{H}^{n-1}\left(\cup_{y\in\omega}\partial v(y)\right)\\
&= \mathcal{H}^{n-1}\left(\cup_{u\in\pi(\omega)}\left(F(K,u)|e^\perp\right)\right) = \int_{\pi(\omega)}\langle u,e\rangle\, dS_K(u)\\
&= n\int_{\pi(\omega)}\langle u,e\rangle h_K^{p-1}(u)\|Dh_K(u)\|^{n-q} f(u)\, d\mathcal{H}^{n-1}(u)\\
&= n\int_\omega (1+\|y\|^2)^{\frac{-n-p}{2}} v(y)^{p-1}\left\|Dv(y) + (\langle Dv(y),y\rangle - v(y))\cdot e\right\|^{n-q} f(\pi(y))\, d\mathcal{H}^{n-1}(y),
\end{aligned}$$

where we used in the last step that

$$v(y) = h_K(y+e) = (1+\|y\|^2)^{\frac{1}{2}} h_K(\pi(y)).$$

In particular, v satisfies the Monge-Ampère type differential equation

$$\det D^2 v(y) = \tfrac{1}{n}(1+\|y\|^2)^{-\frac{n+p}{2}} v(y)^{p-1}\left\|Dv(y) + (\langle Dv(y),y\rangle - v(y))\cdot e\right\|^{n-q} f(\pi(y)) \quad \text{on } e^\perp,$$

or in other words, v satisfies (100) on $e^\perp$. □

The following results by Caffarelli (see Theorem 1 and Corollary 1 in [7] for (i) and (ii), and for (iii)) are at the core of the discussion regarding the part of the boundary of a convex body K, where the support function at some normal vector is positive.

Theorem 7.4 (Caffarelli)**.** *Let $\lambda_2>\lambda_1>0$, and let v be a convex function on an open bounded convex set $\Omega\subset\mathbb{R}^n$ such that*

$$\lambda_1 \le \det D^2 v \le \lambda_2 \tag{106}$$

in the sense of measure.

- **(i):** *If v is non-negative and $S=\{y\in\Omega:\, v(y)=0\}$ is not a point, then S has no extremal point in Ω.*
- **(ii):** *If v is strictly convex, then v is C^1.*

We recall that (106) is equivalent to saying that for each bounded Borel set $\omega \subset \Omega$, we have

$$\lambda_1 \mathcal{H}^n(\omega) \leq \mu_v(\omega) \leq \lambda_2 \mathcal{H}^n(\omega),$$

where μ_v has been defined in (98).

Caffarelli [8] strengthened Theorem 7.4 to have some estimates on Hölder continuity under additional assumptions on v.

Theorem 7.5 (Caffarelli)**.** *For real functions v and f on an open bounded convex set $\Omega \subset \mathbb{R}^n$, let v be strictly convex, and let f be positive and continuous such that*

$$\det D^2 v = f \tag{107}$$

in the sense of measure.

- **(i):** *Each $z \in \Omega$ has an open ball $B \subset \Omega$ around z such that the restriction of v to B is in $C^{1,\alpha}(B)$ for any $\alpha \in (0,1)$.*
- **(ii):** *If f is in $C^\alpha(\Omega)$ for some $\alpha \in (0,1)$, then each $z \in \Omega$ has an open ball $B \subset \Omega$ around z such that the restriction of v to B is in $C^{2,\alpha}(B)$.*

Proof. For (i), it is the direct consequence of Caffarelli [8] Theorem 1 that if v is strictly convex, and f is positive and continuous, then each $z \in \Omega$ has an open ball $B \subset \Omega$ around z such that the restriction of v to B is in the Sobolev space $W^{2,l}(B)$ for any $l > 1$. However, the Sobolev Embedding Theorem (see Demengel, Demengel [12]) yields that if $l > n$ is chosen in such a way that $\frac{n}{l} = 1 - \alpha$, then $W^{2,l}(B) \subset C^{1,\alpha}(B)$.

Finally, (ii) is just Theorem 2 of Caffarelli [8]. $\square$

We will use, in the rest of the section, that there exists an $\omega \in (0,1)$ depending on $Q \in \mathcal{K}^n_{(o)}$ such that

$$\omega\|x\| \leq \|x\|_Q \leq \omega^{-1}\|x\| \text{ for } x \in \mathbb{R}^n. \tag{108}$$

Proof of Theorem 1.3. We recall that it is assumed that for some $c_2 > c_1 > 0$ it holds that

$$c_1 < f(u) < c_2 \text{ for } u \in S^{n-1}$$

in (93).

First, we analyse Lemma 7.3 for a fixed $e \in S^{n-1} \backslash N(K,o)$. Since h_K is continuous, there exist $\alpha(e) \in (0,1)$ and $\delta(e) > 0$ depending on e and K such that $h_K(u) \geq \delta(e)$ for $u \in \mathrm{cl}\Omega(e,\alpha(e))$, and hence $\mathrm{cl}\Omega(e,\alpha(e)) \cap N(K,o) = \emptyset$, where $\Omega(e,\alpha(e))$ is the spherical cap defined in the first paragraph of Section 5. It also follows that there exists $\xi(e) \in (0,1)$ depending on e and K such that if some $u \in \mathrm{cl}\Omega(e,\alpha(e))$ is the exterior normal at $x \in \partial K$, then $\xi(e) \leq \|x\| \leq 1/\xi(e)$. Let us consider the open $(n-1)$-ball $\Omega_e = \pi^{-1}(\Omega(e,\alpha(e)))$ for π defined in (102), and let v be the same function as in Lemma 7.3 on $e^\perp$. We deduce from (89), (105) and (108) that there exists $\tilde{\xi}(e) \in (0,1)$ depending on e and K such that if v is differentiable at $y \in \Omega_e$, then

$$\tilde{\xi}(e) \leq \|Dv(y) + (\langle Dv(y), y\rangle - v(y)) \cdot e\|_Q \leq 1/\tilde{\xi}(e). \tag{109}$$

Since v is bounded on $\mathrm{cl}\Omega_e$ with an upper bound depending on e and K and $v(y) = h_K(y+e) \geq \delta(e)$ for $y \in \mathrm{cl}\Omega_e$, we deduce from (109) and Lemma 7.3 that there exist $\lambda_2(e) > \lambda_1(e) > 0$ depending on e and K such that

$$\lambda_1(e) \leq \det D^2 v(y) \leq \lambda_2(e) \text{ for } y \in \Omega_e. \tag{110}$$

In order to prove that $K \backslash \Xi_K$ is C^1, we claim that for any $z \in \partial K$,

$$\dim N(K,z) \geq 2 \text{ yields } N(K,z) \subset N(K,o). \tag{111}$$

We assume, on the contrary that there exists $z \in \partial K$ such that

$$\dim N(K,z) \geq 2 \text{ and } N(K,z) \not\subset N(K,o),$$

and seek a contradiction. It follows that there exists an extremal vector e of $N(K,z)\cap S^{n-1}$ such that $h_K(e) > 0$.

We observe that $v(y) = h_K(y+e) \geq \langle y+e, z\rangle$ for $y \in \Omega$ with equality if and only if $y \in S = \pi^{-1}(N(K,z)\cap \Omega(e,\alpha(e)))$, therefore the first degree polynomial $l(y) = \langle y+e, z\rangle$ satisfies

$$v(y) - l(y) \begin{cases} = & 0 \quad \text{if } y \in S \\ > & 0 \quad \text{if } y \in \Omega_e\backslash S. \end{cases}$$

We have $\lambda_1(e) \leq \det D^2(v-l) \leq \lambda_2(e)$ on Ω_e by (110). Since $\dim S \geq 1$ for $S = \{y \in e^\perp \cap \Omega_e : v(y) - l(y) = 0\}$ and o is an extremal point of S by the choice of e, we have contradicted Caffarelli's Theorem 7.4 (i), completing the proof of (111).

In turn, (111) yields that

$$\begin{array}{lcl} \partial K\backslash\Xi_K & = & \{z \in \partial K : h_K(u) > 0 \text{ for all } u \in N(K,z)\}, \quad \textit{and} \\ \partial K\backslash\Xi_K & \text{is} & C^1. \end{array} \tag{112}$$

Next we prove that v is strictly convex on $\mathrm{cl}\Omega_e$ for $e \in S^{n-1}\backslash N(K,o)$; or equivalently,

$$v\left(\frac{y_1+y_2}{2}\right) < \frac{v(y_1)+v(y_2)}{2} \quad \text{for } y_1, y_2 \in \Omega_e \text{ with } y_1 \neq y_2. \tag{113}$$

Let $e + \frac{1}{2}(y_1+y_2)$ be an exterior normal at $z \in \partial K$. Since $\Omega_e \cap N(K,o) = \emptyset$, it follows from (112) that $z \in \partial K\backslash\Xi_K$ and z is a smooth point. For $i = 1, 2$, $e + y_i$ and $e + \frac{1}{2}(y_1+y_2)$ are independent, therefore

$$v(y_i) = h_K(e+y_i) > \langle z, e+y_i\rangle.$$

We conclude that

$$\frac{v(y_1)+v(y_2)}{2} > \left\langle z, e + \frac{y_1+y_2}{2}\right\rangle = h_K\left(e + \frac{y_1+y_2}{2}\right) = v\left(\frac{y_1+y_2}{2}\right),$$

proving (113).

We deduce from (110), the strict convexity (113) of v, and from Caffarelli's Theorem 7.4 (ii) that v is C^1 on $\mathrm{cl}\Omega_e$ for any $e \in S^{n-1}\backslash N(K,o)$. We deduce that h_K is C^1 on $\mathbb{R}^n\backslash N(K,o)$, and hence $\partial K\backslash\Xi_K$ contains no segment, completing the proof of Theorem 1.3 (i).

Next, we turn to Theorem 1.3 (ii) and (iii), and hence we assume that f is continuous. Let $e \in S^{n-1}\backslash N(K,o)$, and we apply again Lemma 7.3 for this e. Since v is C^1 on $\mathrm{cl}\Omega_e$, we deduce that the right hand side of (100) is continuous. As v is strictly convex on $\mathrm{cl}\Omega_e$ by (113), Theorem 7.5 (i) applies, and hence there exists an open ball B of $e^\perp$ centred at o such that v is $C^{1,\alpha}$ on B for any $\alpha \in (0,1)$. In turn, we deduce Theorem 1.3 (ii).

Finally, let us assume that f is C^α on S^{n-1}. As v is $C^{1,\alpha}$ on B, it follows that the right hand side of (100) is C^α on B, as well. Therefore Theorem 7.5 (ii) yields that v is $C^{2,\alpha}$ on an open ball $B_0 \subset B$ of $e^\perp$ centred at o. We deduce from (110) that $\det D^2 v(0) > 0$, concluding the proof of Theorem 1.3. □

Next, we discuss how the ideas leading to Theorem 1.3 work for any $p, q \in \mathbb{R}$ provided that $o \in \mathrm{int}K$. First of all, the following is the version of Lemma 7.3 for the case when $K \in \mathcal{K}^n_{(o)}$, which can be proved just like Lemma 7.3.

Lemma 7.6. *For $p, q \in \mathbb{R}$, $Q \in \mathcal{S}^n_{(o)}$, $e \in S^{n-1}$ and $K \in \mathcal{K}^n_{(o)}$, if $h = h_K$ is a solution of (94) for a non-negative function f, and $v(y) = h(y+e)$ holds for $v : e^\perp \to \mathbb{R}$, then v satisfies*

$$\det(D^2 v(y)) = nv(y)^{p-1}\left\|Dv(y) + (\langle Dv(y), y\rangle - v(y))\cdot e\right\|_Q^{n-q} g(y) \quad \textit{on } e^\perp$$

in the sense of measure, where for $y \in e^\perp$, we have that

$$g(y) = \frac{1}{n\,(1+\|y\|^2)^{\frac{n+p}{2}}} \cdot f\left(\frac{e+y}{\sqrt{1+\|y\|^2}}\right).$$

Proof of Theorem 1.5. Instead of Corollary 6.9, we use that according to Lemma 5.1 in Lutwak, Yang and Zhang [24], it holds that

$$\int_{S^{n-1}} g(u)\, d\widetilde{C}_{p,q}(K,Q,u) = \frac{1}{n}\int_{\partial' K} g(\nu_K(x))\langle \nu_K(x), x\rangle^{1-p}\|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x)$$

for any bounded Borel function $g:\, S^{n-1} \to \mathbb{R}$. Therefore using (89), the Monge-Ampère equation for h_K can be written in the form

$$dS_K = n\, h_K^{p-1}\|Dh_K\|_Q^{n-q} f\, d\mathcal{H}^{n-1} \quad \text{on } S^{n-1}.$$

Now the same argument as for Theorem 1.3 yields Theorem 1.5 (i), and the versions of Theorem 1.5 (ii) and (iii), where h_K is locally $C^{1,\alpha}$ on S^{n-1} in Theorem 1.5 (ii), and h_K is locally $C^{2,\alpha}$ on S^{n-1} in Theorem 1.5 (iii). However, S^{n-1} is compact, therefore h_K is globally $C^{1,\alpha}$ on S^{n-1} in Theorem 1.5 (ii), and h_K is globally $C^{2,\alpha}$ on S^{n-1} in Theorem 1.5 (iii). ☐

Finally, we start our preparations for proving Theorem 1.4. The following Lemma 7.7 is essentially proved in Lemma 3.2 and Lemma 3.3 in [28] (see the remarks after Lemma 7.7).

Lemma 7.7. *Let v be a convex function defined on the closure of an open bounded convex set $\Omega \subset \mathbb{R}^n$ such that the Monge-Ampère measure μ_v is finite on Ω and $v \equiv 0$ on $\partial\Omega$, and let $z_0 + tE \subset \Omega \subset z_0 + E$ for $t > 0$ and $z_0 \in \Omega$ and an origin centred ellipsoid E.*

(i): *If $z \in \Omega$ satisfies $(z + s\,E) \cap \partial\Omega \neq \emptyset$ for $s > 0$, then*

$$|v(z)| \leq s^{1/n}\tau_0\mathcal{H}^n(\Omega)^{1/n}\mu_v(\Omega)^{1/n}$$

for some $\tau_0 > 0$ depending on n, t.

(ii): *If $\mu_v(z_0 + tE) \geq b\,\mu_v(\Omega)$ for $b > 0$, then*

$$|v(z_0)| \geq \tau_1\mathcal{H}^n(\Omega)^{1/n}\mu_v(\Omega)^{1/n} \tag{114}$$

for some $\tau_1 > 0$ depending on n, t and b.

We remark that Lemma 3.2 in [28] proves (114) with $\sup_\Omega |v|$ instead of $|v(z_0)|$ with a different $\tau_1^* > 0$. The inequality (114) follows from that and the claim that if Ω is an open bounded convex set in $\mathbb{R}^n$ and v is a convex function on clΩ, that vanishes on $\partial\Omega$, and $z_0 + tE \subset \Omega \subset z_0 + E$ for an origin centred ellipsoid E, then

$$|v(z_0)| \geq t/(t+1)\sup_\Omega |v|. \tag{115}$$

To prove (115), we note that v is non-positive, and choose $z_1 \in \text{cl}\Omega$ where v attains its minimum. Since $z_2 = z_0 - t(z_1 - z_0) \in \text{cl}\Omega$ and $z_0 = \frac{t}{1+t}\,z_1 + \frac{1}{1+t}\,z_2$, we have

$$v(z_0) \leq \frac{t}{1+t}\,v(z_1) + \frac{1}{1+t}\,v(z_2) \leq \frac{t}{1+t}\,v(z_1),$$

and since $t < 1$, this verifies (115) with $\tau_1 = \frac{t+1}{t}\tau_1^*$.

Now we show that Theorem 1.4 is invariant under volume preserving linear transformations.

Lemma 7.8. *Let $1 < p < q$, $Q \in \mathcal{S}^n_{(o)}$, $\varphi \in \text{SL}(n,\mathbb{R})$ and let $K \in \mathcal{K}^n_o$ with $\mathcal{H}^{n-1}(\Xi_K) = 0$ and* int$K \neq \emptyset$*. If*

$$d\widetilde{C}_q(K,Q,\cdot) = h_K^p f\, d\mathcal{H}^{n-1}$$

for $c_2>c_1>0$ and for a Borel function f on S^{n-1} satisfying $c_1\leq f\leq c_2$, then

$$d\widetilde{C}_q(\varphi K,\cdot)=h_{\varphi K}^p\tilde{f}\,d\mathcal{H}^{n-1}$$

for $\tilde{c}_2>\tilde{c}_1>0$ and for a Borel function $\tilde{f}$ on S^{n-1} satisfying $\tilde{c}_1\leq\tilde{f}\leq\tilde{c}_2$ where $\mathcal{H}^{n-1}(\Xi_{\varphi K})=0$.

Proof. Since φ is Lipschitz, we deduce that $\mathcal{H}^{n-1}(\Xi_{\varphi K})=0$.

As a first step to analyse the density function of $\widetilde{C}_q(\varphi K,\cdot)$ with respect to $\mathcal{H}^{n-1}$, we prove that

$$d\widetilde{C}_q(\varphi K,\varphi Q,\cdot)=h_{\varphi K}^p f^*\,d\mathcal{H}^{n-1} \tag{116}$$

for $c_2^*>c_1^*>0$ and for a Borel function f^* on S^{n-1} satisfying $c_1^*\leq f^*\leq c_2^*$. For $\eta\subset S^{n-1}$, $\mathbf{1}_\eta$ denotes the characteristic function of η. We note that (116) is equivalent to prove that if $\eta\subset S^{n-1}\backslash N(\varphi K,o)$ is Borel, then

$$c_1^*\mathcal{H}^{n-1}(\eta)\leq\widetilde{C}_{p,q}(\varphi K,\varphi Q,\eta)=\int_{S^{n-1}}\mathbf{1}_\eta h_{\varphi K}^{-p}d\widetilde{C}_q(\varphi K,\varphi Q,\cdot)\leq c_2^*\mathcal{H}^{n-1}(\eta). \tag{117}$$

We consider the C^1 diffeomorphism $\tilde{\varphi}:\,S^{n-1}\to S^{n-1}$ defined by

$$\tilde{\varphi}(u)=\frac{\varphi^t u}{\|\varphi^t u\|},$$

which satisfies that if $\eta\subset S^{n-1}$, then

$$\mathbf{1}_{\tilde{\varphi}\eta}(u)=\mathbf{1}_\eta\left(\frac{\varphi^{-t}u}{\|\varphi^{-t}u\|}\right)$$

There exist $\aleph_1,\aleph_2\in(0,1)$ depending on φ such that

$$\begin{array}{rcccll}\aleph_1\|u\| & \leq & \|\varphi^{-t}(u)\| & \leq & \aleph_1^{-1}\|u\| & \text{for } u\in S^{n-1};\\ \aleph_2\mathcal{H}^{n-1}(\eta) & \leq & \mathcal{H}^{n-1}(\tilde{\varphi}(\eta)) & \leq & \aleph_2^{-1}\mathcal{H}^{n-1}(\eta) & \text{for any Borel set } \eta\subset S^{n-1}.\end{array} \tag{118}$$

We also note if $u\in S^{n-1}$ is an exterior normal at $z\in\partial K$, then $\varphi^{-t}u$ is an exterior normal at $\varphi z\in\partial\varphi K$, and hence

$$h_{\varphi K}(\varphi^{-t}u)=\langle\varphi^{-t}u,\varphi z\rangle=h_K(u).$$

It also follows that $\varphi^t N(\varphi K,o)=N(K,o)$. Therefore, if $\eta\subset S^{n-1}\backslash N(\varphi K,o)$ is Borel, then $\tilde{\varphi}\eta\subset S^{n-1}\backslash N(K,o)$, and we deduce from Lemma 6.5 that

$$\begin{aligned}\widetilde{C}_{p,q}(\varphi K,\varphi Q,\eta) &= \int_{S^{n-1}}\mathbf{1}_\eta h_{\varphi K}^{-p}d\widetilde{C}_q(\varphi K,\varphi Q,\cdot)\\ &= \int_{S^{n-1}}\mathbf{1}_\eta\left(\frac{\varphi^{-t}u}{\|\varphi^{-t}u\|}\right)h_{\varphi K}\left(\frac{\varphi^{-t}u}{\|\varphi^{-t}u\|}\right)^{-p}d\widetilde{C}_q(K,Q,u)\\ &= \int_{S^{n-1}}\mathbf{1}_{\tilde{\varphi}\eta}(u)\|\varphi^{-t}u\|^p h_K(u)^{-p}d\widetilde{C}_q(K,Q,u).\end{aligned}$$

We deduce from (118) and the condition on $\widetilde{C}_{p,q}(K,Q,\cdot)$ that

$$c_1\aleph_1^p\mathcal{H}^{n-1}(\tilde{\varphi}\eta)\leq\widetilde{C}_{p,q}(\varphi K,\varphi Q,\eta)\leq c_2\aleph_1^{-p}\mathcal{H}^{n-1}(\tilde{\varphi}\eta).$$

Therefore applying the estimate of (118) on $\mathcal{H}^{n-1}(\tilde{\varphi}\eta)$ yields (117).

According to Corollary 6.9, we have that

$$\widetilde{C}_{p,q}(\varphi K,\varphi Q,\eta)=\frac{1}{n}\int_{\partial'\varphi K}\mathbf{1}_\eta(\nu_{\varphi K}(x))\langle\nu_{\varphi K}(x),x\rangle^{1-p}\|x\|_{\varphi Q}^{q-n}\,d\mathcal{H}^{n-1}(x).$$

There exists an $\aleph_3\in(0,1)$ depending on φ and Q such that

$$\aleph_3\|x\|\leq\|x\|_{\varphi Q}\leq\aleph_3^{-1}\|x\|\;\;\text{for } x\in\mathbb{R}^n.$$

In particular, the last estimate, Corollary 6.9 and (117) imply

$$\tilde{c}_1\mathcal{H}^{n-1}(\eta) \leq \widetilde{C}_{p,q}(\varphi K, \eta) \leq \tilde{c}_2\mathcal{H}^{n-1}(\eta)$$

holds for any Borel set $\eta \subset S^{n-1}\backslash N(\varphi K, o)$, where $\tilde{c}_1 = c_1^* \min\{\aleph_3^{q-n}, \aleph_3^{n-q}\}$ and $\tilde{c}_2 = 1/\tilde{c}_1$, completing the proof of Lemma 7.8. □

We use Lemma 7.8 as follows. For any convex body $K \in \mathcal{K}_o^n$ such that $o \in \partial K$ and $\mathrm{int}K \neq \emptyset$, there exist $a \in S^{n-1}$, $\beta \in (0,1)$ and $r_0 > 0$ such that

$$\{x \in r_0 B^n : \langle x, a\rangle \geq \beta \|x\|\} \subset K.$$

Therefore, there exists $\varphi \in \mathrm{SL}(n,\mathbb{R})$ such that $\varphi\, a = \lambda a$ for $\lambda > 0$ and $\varphi(x) = (\sqrt{3}\beta/\sqrt{1-\beta^2})^{1/n}\, x$ for $x \in a^\perp$, thus for some $r_1 > 0$, we have

$$\left\{x \in r_1 B^n : \langle x, a\rangle \geq \frac{1}{2}\|x\|\right\} \subset \varphi K.$$

In particular, for this $\varphi \in \mathrm{SL}(n,\mathbb{R})$, we have $\langle x, a\rangle \leq -\frac{\sqrt{3}}{2}\|x\|$ for any $x \in N(\varphi K, o)$, thus

$$\langle x_1, x_2\rangle \geq \frac{1}{2}\|x_1\|\,\|x_2\| \;\text{ for } x_1, x_2 \in N(\varphi K, o). \tag{119}$$

Proof of Theorem 1.4. If $o \in \mathrm{int}\, K$, then Theorem 1.5 (i) yields Theorem 1.4. Therefore, we assume that $o \in \partial K$ and $\mathrm{int}K \neq \emptyset$ for $K \in \mathcal{K}_o^n$. We may also assume by Lemma 7.8 and (119) that on the one hand, we have

$$\langle x_1, x_2\rangle \geq \frac{1}{2}\|x_1\|\,\|x_2\| \;\text{ for } x_1, x_2 \in N(K, o), \tag{120}$$

and on the other hand, using (93) that there exist $c_2' > c_1' > 0$ and a real Borel function f on S^{n-1} with $c_1' < f < c_2'$ such that

$$\det(\nabla^2 h_K + h_K\,\mathrm{Id}) = n\, h_K^{p-1}\left(\|\nabla h_K\|^2 + h_K^2\right)^{\frac{n-q}{2}} \cdot f. \tag{121}$$

We assume, on the contrary, that h_K is not differentiable at some point of S^{n-1}, or equivalently, that ∂K contains an at least one dimensional face according to (99), and seek a contradiction using Lemma 7.7. It follows from Theorem 1.3 (i) that any at least one dimensional face of K contains the origin o.

For $\Xi_K = \cup\{F(K,u) : u \in S^{n-1} \text{ and } h_K(u) = 0\}$, we define $\gamma > 0$ and $w \in S^{n-1}$ such that

$$\gamma = \max\{\|z\| : z \in \Xi_K\} > 0 \;\text{ and }\; \gamma w \in \Xi_K. \tag{122}$$

Let $e \in S^{n-1}$ be an exterior normal at $(\gamma/2)w \in \Xi_K$, therefore (88) yields

$$\partial h_K(e) = F(K,e) \text{ and } o, \gamma w \in F(K,e). \tag{123}$$

We may choose a closed convex cone C_0 with apex o such that

$$\begin{array}{rcl} N(K,o)\backslash\{o\} & \subset & \mathrm{int}C_0, \\ \|x\| & < & 2\gamma \text{ for any } x \in \partial K \text{ with } \boldsymbol{\nu}_K(x) \cap C_0 \neq \emptyset. \end{array} \tag{124}$$

We choose $\delta > 0$ such that

$$\{u \in S^{n-1} : h_K(u) \leq \delta\} \subset \mathrm{int}\, C_0. \tag{125}$$

Let v be the function of Lemma 7.3 associated to e and h_K on $e^\perp$, and hence (121) yields that

$$\det(D^2 v(y)) \;\geq\; \frac{c_1}{(1+\|y\|^2)^{\frac{n+p}{2}}}\, v(y)^{p-1}(\|Dv(y)\|^2 + (\langle Dv(y), y\rangle - v(y))^2)^{\frac{n-q}{2}}, \tag{126}$$

$$\det(D^2 v(y)) \;\leq\; \frac{c_2}{(1+\|y\|^2)^{\frac{n+p}{2}}}\, v(y)^{p-1}(\|Dv(y)\|^2 + (\langle Dv(y), y\rangle - v(y))^2)^{\frac{n-q}{2}} \tag{127}$$

in the sense of measure, and $c_i = c_i' n$ for $i = 1, 2$.

It follows from (122) and (123) that

$$\begin{aligned} \partial v(o) &= \partial h_K(e)|e^\perp = F(K,e);\\ \gamma &= \max\{\|z\| : z\in \partial v(o)\} > 0;\\ \gamma w &\in \partial v(o), \quad \text{where } w\in S^{n-1}\cap e^\perp. \end{aligned}$$

Since v is convex, we have that

$$v(y) \geq \max\{0, \gamma\langle w, y\rangle\} \ \text{ for any } y\in e^\perp, \tag{128}$$

and if $t>0$ tends to zero, then

$$v(tw) = \gamma\, t + o(t). \tag{129}$$

For small $\varepsilon>0$, let us consider the first degree polynomial l_ε on $e^\perp$ defined by

$$l_\varepsilon(y) = (\gamma - \sqrt{\varepsilon})\langle w, y\rangle + \varepsilon,$$

whose graph passes through εe and $\sqrt{\varepsilon}\, w + \gamma\sqrt{\varepsilon}\, e$.

We define

$$\begin{aligned} \Omega_\varepsilon &= \{y\in e^\perp :\, v(y) < l_\varepsilon(y)\},\\ \widetilde{\Omega}_\varepsilon &= \{y\in e^\perp :\, v(y) \leq l_\varepsilon(y)\} = \mathrm{cl}\,\Omega_\varepsilon, \end{aligned}$$

where $\widetilde{\Omega}_\varepsilon$ is a closed convex set, and Ω_ε is its relative interior with respect to $e^\perp$. We have $o\in\Omega_\varepsilon$, and since $v(y)\geq(\gamma-\sqrt{\varepsilon})\langle w,y\rangle$ for $y\in e^\perp$ by (128), we also have

$$\max\{l_\varepsilon(y) - v(y) :\, y\in\widetilde{\Omega}_\varepsilon\} = \max\{l_\varepsilon(y) - v(y) :\, y\in\Omega_\varepsilon\} = l_\varepsilon(o) - v(o) = \varepsilon. \tag{130}$$

We observe that $l_\varepsilon(y)\geq\gamma\langle w,y\rangle$ for $y\in e^\perp$ if and only if $\langle w,y\rangle\leq\sqrt{\varepsilon}$. It follows that, provided $\varepsilon>0$ is small enough to satisfy $\sqrt{\varepsilon}<\gamma$, if $y\in\widetilde{\Omega}_\varepsilon$, then we have

$$\begin{aligned} \tfrac{-2\varepsilon}{\gamma} \ < \ \langle w, y\rangle &\leq \sqrt{\varepsilon},\\ v(y) &\leq \gamma\sqrt{\varepsilon}. \end{aligned} \tag{131}$$

We observe that if $t\in(0,\sqrt{\varepsilon}/2)$, then $l_\varepsilon(tw)-\gamma t\geq\varepsilon/2$, and hence (129) yields the existence of $\theta_\varepsilon\in(0,\sqrt{\varepsilon}]$ such that

$$\theta_\varepsilon\, w\in\Omega_\varepsilon \ \text{ and } \ \lim_{\varepsilon\to 0^+}\varepsilon/\theta_\varepsilon = 0. \tag{132}$$

We consider the set

$$\mathcal{U} = \big((e+e^\perp)\cap \mathrm{int}C_0\big) - e \subset e^\perp,$$

that is open in the topology of $e^\perp$. If $v(y)\leq\delta$ for $y\in e^\perp$, then $h_K(u)\leq\delta$ for $u=(y+e)/\|y+e\|\in S^{n-1}$, therefore (125) yields

$$\{y\in e^\perp :\, v(y)\leq\delta\}\subset\mathcal{U}.$$

In particular, we deduce from (89), (105) and (124) that if $v(y)\leq\delta$ at some $y\in e^\perp$ where v is differentiable, then

$$\|Dv(y) + (\langle Dv(y), y\rangle - v(y))\cdot e\| \leq 2\gamma. \tag{133}$$

Let $L = e^\perp\cap w^\perp$, and let us consider the closed convex set

$$Y = \{y\in e^\perp :\, v(y) = 0\} = \big(N(K,o)\cap(e+e^\perp)\big) - e,$$

and hence (120) and (131) imply

$$o\in Y\subset\sqrt{3}B^n \ \text{ and } \ Y = \cap_{\varepsilon>0}\Omega_\varepsilon.$$

Therefore (131) and (133) yield the existence of some $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then

$$\Omega_\varepsilon \;\subset\; 2B^n; \tag{134}$$

$$\left(\|Dv(y)\|^2 + (\langle Dv(y), y\rangle - v(y))^2\right)^{\frac{1}{2}} \;\leq\; 2\gamma \;\text{ provided } v \text{ is differentiable at } y \in \Omega_\varepsilon. \tag{135}$$

Using (134) and (135), we deduce that (126) and (127) yield the existence of $\tilde{c}_1, \tilde{c}_2 > 0$ depending on K and e and independent of ε such that if $\varepsilon \in (0, \varepsilon_0)$, then

$$\tilde{c}_1\, v(y)^{p-1}\|Dv(y)\|^{n-q} \leq \det(D^2 v(y)) \leq \tilde{c}_2 v(y)^{p-1}\|Dv(y)\|^{n-q} \tag{136}$$

hold on Ω_ε in the sense of measure.

We deduce from (132) that we may also assume that if $\varepsilon \in (0, \varepsilon_0)$, then we have (compare (131))

$$\frac{\theta_\varepsilon}{16n} \geq \frac{2\varepsilon}{\gamma}. \tag{137}$$

In the following, we assume $\varepsilon \in (0, \varepsilon_0)$.

As Ω_ε is bounded by (134), Loewner's (or John's) theorem provides an $(n-1)$-dimensional ellipsoid $E_\varepsilon \subset e^\perp$ centred at the origin and a $z_\varepsilon \in \Omega_\varepsilon$ such that

$$z_\varepsilon + \tfrac{1}{n}\, E_\varepsilon \subset \Omega_\varepsilon \subset z_\varepsilon + E_\varepsilon. \tag{138}$$

Let $h_{E_\varepsilon}(w) = h_\varepsilon$, and let $a_\varepsilon \in E_\varepsilon$ satisfy $\langle a_\varepsilon, w\rangle = h_\varepsilon$. It follows from (132) and (138) that $z_\varepsilon + E_\varepsilon$ contains a segment of length θ_ε, therefore

$$h_\varepsilon \geq \theta_\varepsilon/2 \geq \frac{16n\varepsilon}{\gamma} \;\text{ and }\; \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{h_\varepsilon} = 0. \tag{139}$$

On the one hand, $o \in \Omega_\varepsilon \subset z_\varepsilon + E_\varepsilon$ yields $\langle z_\varepsilon, w\rangle \leq h_\varepsilon$, and on the other hand, we deduce from (131), (137) and (138) that

$$\langle z_\varepsilon, w\rangle - \frac{h_\varepsilon}{n} = \left\langle z_\varepsilon - \frac{a_\varepsilon}{n}, w\right\rangle \geq \frac{-2\varepsilon}{\gamma} \geq \frac{-h_\varepsilon}{8n},$$

therefore

$$\frac{7h_\varepsilon}{8n} \leq \langle z_\varepsilon, w\rangle \leq h_\varepsilon. \tag{140}$$

If $y \in \Omega_\varepsilon \subset z_\varepsilon + E_\varepsilon$, then $\langle w, y\rangle \leq 2h_\varepsilon$ by (140), thus the definition of l_ε and (139) imply

$$v(y) \leq l_\varepsilon(y) \leq \gamma\langle w, y\rangle + \varepsilon \leq \left(2\gamma + \frac{\gamma}{16n}\right) h_\varepsilon.$$

We write $\aleph_1, \aleph_2, \ldots$ to denote constants that depend on n, p, q, γ, K, e and are independent of ε. We deduce from (136) that

$$\mu_v(\Omega_\varepsilon) \leq \int_{\Omega_\varepsilon} \tilde{c}_2\, v(y)^{p-1}\, d\mathcal{H}^{n-1}(y) \leq \aleph_1 h_\varepsilon^{p-1}\mathcal{H}^{n-1}(\Omega_\varepsilon) \leq \aleph_1 h_\varepsilon^{p-1}\mathcal{H}^{n-1}(E_\varepsilon). \tag{141}$$

In order to apply Lemma 7.7, we prove

$$\mu_v(z_\varepsilon + \tfrac{1}{2n}\, E_\varepsilon) \geq \aleph_2 h_\varepsilon^{p-1}\mathcal{H}^{n-1}(E_\varepsilon). \tag{142}$$

We note that

$$y - \tfrac{1}{2}\, a_\varepsilon \in E_\varepsilon \;\text{ for } y \in \tfrac{1}{2}\, E_\varepsilon. \tag{143}$$

Let $Z_\varepsilon = z_\varepsilon + \frac{1}{2n}\, E_\varepsilon$, and hence

$$\mathcal{H}^{n-1}(Z_\varepsilon) = \frac{1}{2^{n-1}n^{n-1}}\, \mathcal{H}^{n-1}(E_\varepsilon). \tag{144}$$

It follows from (138) and (143) that if $y \in Z_\varepsilon$, then

$$y - \tfrac{1}{2n}\, a_\varepsilon \in z_\varepsilon + \tfrac{1}{n}\, E_\varepsilon \subset \Omega_\varepsilon.$$

In turn, we deduce from (131) and (139) that if $y \in Z_\varepsilon$, then

$$\langle y, w\rangle - \frac{h_\varepsilon}{2n} = \left\langle y - \tfrac{1}{2n}\, a_\varepsilon, w\right\rangle \geq \frac{-2\varepsilon}{\gamma} \geq \frac{-h_\varepsilon}{8n},$$

therefore

$$\langle y - \tfrac{3h_\varepsilon}{8n}\, w, w\rangle \geq 0. \tag{145}$$

On the one hand, it follows from (128) and (145) that if $y \in Z_\varepsilon$, then

$$v(y) \geq \gamma\langle y, w\rangle \geq \gamma \cdot \frac{3h_\varepsilon}{8n}. \tag{146}$$

On the other hand, it follows from (145) and the convexity of v, and finally by (139) that if v is differentiable at $y \in Z_\varepsilon$, then

$$\begin{aligned}\gamma\langle y, w\rangle - \langle Dv(y), \tfrac{3h_\varepsilon}{8n}\, w\rangle &\leq v(y) - \langle Dv(y), \tfrac{3h_\varepsilon}{8n} \cdot w\rangle \leq v\left(y - \tfrac{3h_\varepsilon}{8n}\, w\right) \leq l_\varepsilon\left(y - \tfrac{3h_\varepsilon}{8n}\, w\right) \\ &\leq \gamma\left\langle y - \tfrac{3h_\varepsilon}{8n}\, w, w\right\rangle + \varepsilon \leq \gamma\langle y, w\rangle - \gamma \cdot \tfrac{3h_\varepsilon}{8n} + \gamma \cdot \tfrac{h_\varepsilon}{16n} = \gamma\langle y, w\rangle - \gamma \cdot \tfrac{5h_\varepsilon}{16n}.\end{aligned}$$

In particular, if v is differentiable at $y \in Z_\varepsilon$, then $\langle Dv(y), w\rangle \geq \frac{5}{6}\,\gamma$, which, in turn, yields that

$$\|Dv(y)\| \geq \tfrac{5}{6}\,\gamma. \tag{147}$$

Since (136) implies

$$\mu_v(z_\varepsilon + \tfrac{1}{2n}\, E_\varepsilon) \geq \int_{Z_\varepsilon} c_1 v(y)^{p-1} \|Dv(y)\|^{n-q}\, d\mathcal{H}^{n-1}(y),$$

we conclude (142) from (144), (146) and (147).

We deduce from combining (141) and (142) that

$$\mu_v(z_\varepsilon + \tfrac{1}{2n}\, E_\varepsilon) \geq \aleph_3 \mu_v(\Omega_\varepsilon) \tag{148}$$

for $\aleph_3 = \aleph_1/\aleph_2$.

We define $\tilde{v} = v - l_\varepsilon$, which also satisfies (148). In particular, $\tilde{v}$ satisfies the conditions of Lemma 7.7 with $\Omega = \Omega_\varepsilon$, $E = E_\varepsilon$, $t = \frac{1}{2n}$, $b = \aleph_3$, $z = o$ and $z_0 = z_\varepsilon$. In addition, we deduce from (131) that we can use

$$s = \frac{2\varepsilon}{\gamma h_\varepsilon}$$

in Lemma 7.7. We conclude from Lemma 7.7 that

$$\frac{|\tilde{v}(o)|}{|\tilde{v}(z_\varepsilon)|} \leq \aleph_4 s^{1/n}. \tag{149}$$

However, $\tilde{v}(o) = -\varepsilon$, and (115) yield that

$$|\tilde{v}(z_\varepsilon)| \geq \frac{t}{t+1} \cdot \sup_{\Omega_\varepsilon} |\tilde{v}| \geq \frac{\varepsilon}{4n}.$$

We deduce from (149) that if $\varepsilon \in (0, \varepsilon_0)$, then

$$2n \leq \aleph_4 \left(\frac{2\varepsilon}{\gamma h_\varepsilon}\right)^{1/n}. \tag{150}$$

Here $\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{h_\varepsilon} = 0$ according to (139), which fact clearly contradicts (150). Finally, this contradiction proves Theorem 1.4. □

Remark. The reason that our method of proof does not work if $q > n$ is that in that case $\|Dv(y)\|^{n-q}$ can be arbitrarily large if $v(y) > 0$ and is very small.

8. Acknowledgements

The authors are indebted to Lászlo Székelyhidi Jr. for helpful discussions. Both authors are supported by grant NKFIH K 116451. First named author is also supported by grants NKFIH ANN 121649, NKFIH K 109789 and NKFIH KH 129630.

References

[1] A.D. Aleksandrov, *On the theory of mixed volumes. III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies* **3** (1938), 27–46.

[2] A.D. Aleksandrov, *On the surface area measure of convex bodies*, Mat. Sbornik N.S. **6** (1939), 167–174.

[3] Emil Artin, *The gamma function*, Translated by Michael Butler. Athena Series: Selected Topics in Mathematics, Holt, Rinehart and Winston, New York-Toronto-London, 1964.

[4] Gabriele Bianchi, Károly J. Böröczky, Andrea Colesanti, and Deane Yang, *The L_p-Minkowski problem for $-n < p < 1$ according to Chou-Wang*, Advances in Math. **accepted**, arXiv:1710.04401.

[5] Károly J. Böröczky, Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *The logarithmic Minkowski problem*, J. Amer. Math. Soc. **26** (2013), 831–852.

[6] Károly J. Böröczky, Erwin Lutwak, Deane Yang, Gaoyong Zhang, and Yiming Zhao, *The dual Minkowski problem for symmetric convex bodies*, arXiv:1703.06259.

[7] Luis Caffarelli, *A localization property of viscosity solutions to Monge-Ampère equation and their strict convexity*, Ann. Math. **131** (1990), 129-134.

[8] Luis Caffarelli, *Interior $W^{2,p}$-estimates for solutions of the Monge-Ampère equation*, Ann. Math. **131** (1990), 135-150.

[9] Shibing Chen, Qi-rui Li, and Guangxian Zhu, *On the L_p Monge-Ampère equation*, J. Differential Equations **263** (2017), no. 8, 4997–5011.

[10] Shibing Chen, Qi-rui Li, and Guangxian Zhu, *The Logarithmic Minkowski Problem for non-symmetric measures*, Trans. AMS, accepted.

[11] Kai-Seng Chou and Xu-Jia Wang, *The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry*, Adv. Math. **205** (2006), 33-83.

[12] Francoise Demengel and Gilbert Demengel, *Functional Spaces for the Theory of Elliptic Partial Differential Equations*, Vol. 205, Springer, Berlin, 2012.

[13] Richard J. Gardner, Daniel Hug, Sudan Xing, Deping Ye, and Wolfgang Weil, *General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski Problem I*, arxiv:1802.09572.

[14] Peter M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 336, Springer, Berlin, 2007.

[15] Martin Henk and Hannes Pollehn, *Necessary subspace concentration conditions for the even dual Minkowski problem*, Adv. Math. **323** (2018), 114–141.

[16] Daniel Hug, Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *On the L_p Minkowski problem for polytopes*, Discrete Comput. Geom. **33** (2005), no. 4, 699–715.

[17] Yong Huang, Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Math. **216** (2016), no. 2, 325–388.

[18] Yong Huang, Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *The L_p Alexandrov problem for the L_p integral curvature*, J. Differential Geom. **110** (2018), 1–29.

[19] Yong Huang and Yiming Zhao, *On the L_p dual Minkowski problem*, Adv. Math. **332** (2018), 57–84.

[20] Yongsheng Jiang and Yonghong Wu, *On the 2-dimensional dual Minkowski problem*, Journal of Differential Equations **263** (2017), 3230-3243.

[21] Qi-Rui Li, Weimin Sheng, and Xu-Jia Wang, *Flow by Gauss curvature to the Alexandrov and Minkowski problems*, JEMS, accepted.

[22] Erwin Lutwak, *Dual mixed volumes*, Pacific J. Math. **58** (1975), no. 2, 531–538.

[23] Erwin Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem.* J. Differential Geom. **38** (1993), 131-150.

[24] Erwin Lutwak, Deane Yang, and Gaoyong Zhang, *L_p-dual curvature measures*, Adv. Math. **329** (2018), 85-132.

[25] Hermann Minkowski, *Allgemeine Lehrsätze über die konvexen Polyeder*, Nachr. Ges. Wiess. Göttingen (1897), 189–219 (German).

[26] Hermann Minkowski, *Volumen und Oberfläche*, Math. Ann. **57** (1903), no. 4, 447–495 (German).

[27] Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014.

[28] Neil S. Trudinger and Xu-Jia Wang, *The Monge-Ampère equation and its geometric applications*, in: Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, (2008), 467-524.
[29] Yiming Zhao, *The dual Minkowski problem for negative indices*, Calc. Var. Partial Differential Equations **56(2)** (2017), Art. 18, 16.
[30] Yiming Zhao, *Existence of solutions to the even dual Minkowski problem*, J. of Diff. Geom. **accepted**.
[31] Guangxian Zhu, *The centro-affine Minkowski problem for polytopes*, J. Differ. Geom. **101** (2015), 159-174.
[32] Guangxian Zhu, *The L_p Minkowski problem for polytopes for $p < 0$*, Indiana Univ. Math. J. **66** (2017), 1333-1350.

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13-15, H-1053 Budapest, Hungary, and Department of Mathematics, Central European University, Nádor u. 9, H-1051, Budapest, Hungary
E-mail address: boroczky.karoly.j@renyi.mta.hu

Department of Geometry, Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary
E-mail address: fodorf@math.u-szeged.hu