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Smoothness in the L_p Minkowski Problem for $p < 1$

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Abstract
We discuss the smoothness and strict convexity of the solution of the L_p-Minkowski problem when $p < 1$ and the given measure has a positive density function.

Keywords L_p Minkowski problem · Monge–Ampère equation

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1 Introduction

Given K in the class $\mathcal{K}_0^n$ of compact convex sets in $\mathbb{R}^n$ that have non-empty interior and contain the origin o, we write h_K and S_K to denote its support function and its surface area measure, respectively, and for $p \in \mathbb{R}$, $S_{K,p}$ to denote its L_p-area measure, where $dS_{K,p} = h_K^{1-p} dS_K$. The L_p-area measure defined by Lutwak [35] is a central notion in convexity, see say Barthe et al. [2], Böröczky et al. [5], Campi and Gronchi [10], Chou [15], Cianchi et al. [17], Gage and Hamilton [19], Haberl and Parapatits [23], Haberl and Schuster [24,25], Haberl et al. [26], He et al. [27], Henk and Linke

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[28], Ludwig [34], Lutwak et al. [37,38], Naor [41], Naor and Romik [42], Paouris [44], Paouris and Werner [45] and Stancu [50].

The L_p Minkowski problem asks for the existence of a convex body $K \in \mathcal{K}_0^n$ whose L_p area measure is a given finite Borel measure ν on S^{n-1}. When $p = 1$, this is the classical Minkowski problem solved by Minkowski [40] for polytopes, and by Alexandrov [1] and Fenchel and Jessen [18] in general. The smoothness of the solution was clarified in a series of papers by Nirenberg [43], Cheng and Yau [14], Pogorelov [46] and Caffarelli [7,8]. For $p > 1$ and $p \neq n$, the L_p Minkowski problem has a unique solution according to Chou and Wang [16], Guan and Lin [22] and Hug, Lutwak, Yang and Zhang [30]. The smoothness of the solution is discussed in Chou and Wang [16], Huang and Lu [29] and Lutwak and Oliker [36]. In addition, the case $p < 1$ has been intensively investigated by Böröczky et al. [4], Böröczky and Trinh [6], Chen [13], Chen et al. [11,12], Ivaki [31], Jiang [32], Lu and Wang [33], Lutwak et al. [39], Stancu [48,49] and Zhu [52–55].

The solution of the L_p-Minkowski problem may not be unique for $p < 1$ according to Chen et al. [12] if $0 < p < 1$, according to Stancu [49] if $p = 0$, and according to Chou and Wang [16] if $p < 0$ small.

In this paper we are interested in this problem when $p < 1$ and ν is a measure with density with respect to the Hausdorff measure $\mathcal{H}^{n-1}$ on S^{n-1}, i.e. in the problem

$$dS_{K,p} = f\, d\mathcal{H}^{n-1} \quad \text{on } S^{n-1}, \tag{1.1}$$

where f is a non-negative Borel function in S^{n-1}.

According to Chou and Wang [16], if $-n < p < 1$ and the Borel function f is bounded from above and below by positive constants, then (1.1) has a solution. More general existence results are provided by the recent works Chen et al. [11] if $p = 0$, Chen et al. [12] if $0 < p < 1$, and Bianchi et al. [3] if $-n < p < 0$. In particular, it is known that (1.1) has a solution if $0 \leq p < 1$ and f is any non-negative function in $L_1(S^{n-1})$ with $\int_{S^{n-1}} f\, d\mathcal{H}^{n-1} > 0$, and if $-n < p < 0$ and f is any non-negative function in $L_{\frac{n}{n+p}}(S^{n-1})$ with $\int_{S^{n-1}} f\, d\mathcal{H}^{n-1} > 0$.

We observe that h is a non-negative positively 1-homogeneous convex function in $\mathbb{R}^n$ which solves the Monge–Ampère equation

$$h^{1-p} \det(\nabla^2 h + hI) = f \quad \text{on} \quad S^{n-1} \tag{1.2}$$

in the sense of measure if and only if h is the support function of a convex body $K \in \mathcal{K}_0^n$ which is the solution of (1.1) (see Sect. 2). Here h is the unknown non-negative (support) function on S^{n-1} to be found, $\nabla^2 h$ denotes the (covariant) Hessian matrix of h with respect to an orthonormal frame on S^{n-1}, and I is the identity matrix. The function h may vanish somewhere even in the case when f is positive and continuous, and when this happens and $p < 1$ the Eq. (1.2) is singular at the zero set of h. Naturally, if h is C^2, then (1.2) is a proper Monge–Ampère equation.

In this paper we study the smoothness and strict convexity of a solution $K \in \mathcal{K}_0^n$ of (1.1) assuming $\tau_2 > f > \tau_1$ for some constants $\tau_2 > \tau_1 > 0$. Concerning these aspects for $p < 1$, we summarise the known results in Theorem 1.1, and the new results in Theorem 1.2.

We say that $x \in \partial K$ is a C^1-smooth point if there is a unique tangent hyperplane to K at x, and observe that ∂K is C^1 if and only if each $x \in \partial K$ is C^1-smooth (see Sect. 2 for all definitions). In addition, we note that h_K is C^1 on S^{n-1} if and only if K is strictly convex, and h_K is strictly convex on any hyperplane avoiding the origin if and only if ∂K is C^1. For $z \in \partial K$, the exterior normal cone at z is denoted by $N(K, z)$, and for $z \in \text{int}\, K$, we set $N(K, z) = \{o\}$. Theorem 1.1(i) and (ii) are essentially due to Caffarelli [7] (see Theorem 3.6), and Theorem 1.1(iii) is due to Chou and Wang [16]. If the function f in (1.1) is C^α for $\alpha > 0$, then Caffarelli [8] proves (iv).

Theorem 1.1 (Caffarelli, Chou, Wang) *If $K \in \mathcal{K}_0^n$ is a solution of* (1.1) *for $n \geq 2$ and $p < 1$, and f is bounded from above and below by positive constants, then the following assertions hold:*

(i) *The set X_0 of the points $x \in \partial K$ with $N(K, x) \subset N(K, o)$ is closed, each point of $X = \partial K \backslash X_0$ is C^1-smooth and X contains no segment.*
(ii) *If $o \in \partial K$ is a C^1-smooth point, then ∂K is C^1.*
(iii) *If $p \leq 2 - n$, then $o \in \text{int}\, K$, and hence K is strictly convex and ∂K is C^1.*
(iv) *If $o \in \text{int}\, K$ and the function f in* (1.1) *is positive and C^α, for some $\alpha > 0$, then ∂K is $C^{2,\alpha}$.*

Concerning strict convexity, assertion (iii) here is optimal because Example 4.2 shows that if $2-n < p < 1$, then it is possible that o belongs to the relative interior of an $(n-1)$-dimensional face of a solution K of (1.1) where f is a positive continuous function. Therefore, the only question left open is the C^1 smoothness of the boundary of the solution if $2 - n < p < 1$.

We note that if $p < 1$ and K is a solution of (1.2) with f positive and $o \in \partial K$, then

$$\dim N(K, o) \leq n - 1. \tag{1.3}$$

Therefore, Theorem 1.1(ii) yields that ∂K is C^1 for the solution K if $n = 2$. In general, we have the following partial results.

Theorem 1.2 *If $K \in \mathcal{K}_0^n$ is a solution of* (1.1) *for $n \geq 2$ and $p < 1$, and f is bounded from above and below by positive constants, then the following assertions hold:*

(i) *If $n = 2$, $n = 3$ or $n > 3$ and $p < 4 - n$, then ∂K is C^1.*
(ii) *If $\mathcal{H}^{n-1}(X_0) = 0$ for the X_0 in Theorem* 1.1(i)*, then ∂K is C^1.*

Our results differ in some cases from the ones in Chou and Wang [16], possibly because [16] considers the equation

$$\det(\nabla^2 h + hI) = f h^{p-1} \quad \text{on } S^{n-1} \tag{1.4}$$

instead of (1.2). In the context of non-negative convex functions, being a solution of this last equation is a priori more restrictive than being a solution of (1.2), even if obviously the two notions coincide when h is positive (see Sect. 2 for more on this point). Chou and Wang [16] proves, under our same assumptions on f, the strict convexity of the solution h of (1.4) on hyperplanes avoiding the origin, and uses this to prove that ∂K is C^1 for the convex body K. We note that if $K \in \mathcal{K}_0^n$ is a solution of

(1.4) for $p < 1$ and f is bounded from below and above by positive constants, then combining Theorem 1.2(ii) with the simple observation (2.11) in Sect. 2 shows that ∂K is C^1, as it was verified by Chou and Wang [16]. In our opinion (1.2) is the right equation to consider and using it we obtain weaker results.

To give an example of how the two equations differ, the support function h of the body K in Example 4.2 (where o belongs to the relative interior of an $(n-1)$-dimensional face) is a solution of (1.2) but not a solution of (1.4).

According to Chou and Wang [16] (see also Lemma 3.1 below), the Monge-Ampère equation (1.2) can be transferred to a Monge-Ampère equation

$$v^{1-p}\det(D^2v) = g \tag{1.5}$$

for a convex function v on $\mathbb{R}^{n-1}$ where g is a given non-negative function and D^2 stands for the Hessian in $\mathbb{R}^{n-1}$.

The proofs of Claims (i) and (ii) in Theorem 1.1 use as an essential tool a result proved by Caffarelli in [7] regarding smoothness and strict convexity of convex solutions of certain Monge–Ampère equation of type (1.5) (see Theorem 3.6). Proving that ∂K is C^1 is equivalent to prove that h_K is strictly convex, and [7] is the key to prove this property in $\{y \in S^{n-1} : h_K(y) > 0\}$.

The proof of Claim (i) in Theorem 1.2 is based on the following result for the singular inequality $v^{1-p}\det D^2v \geq g$.

Proposition 1.3 *Let $\Omega \subset \mathbb{R}^n$ be an open convex set, and let v be a non-negative convex function in Ω with $S = \{x \in \Omega : v(x) = 0\}$. If for $p < 1$ and $\tau > 0$, v is the solution of*

$$v^{1-p}\det D^2v \geq \tau \quad \text{in } \Omega \setminus S \tag{1.6}$$

in the sense of measure, and S is r-dimensional, for $r \geq 1$, then $p \geq -n + 1 + 2r$.

We mention that in Caffarelli [9] a corresponding result for $p = 1$ is established.

The underlying idea behind the proof of this result is the following: On the one hand, the graph of v near S is close to being ruled. Hence, the total variation of the derivative is "small". On the other hand, the total variation of the derivative is "large" because of the Monge-Ampère inequality (1.6).

The inequality $p \geq -n + 1 + 2r$ in this result is close to being optimal, at least when $r = 1$. Indeed, Example 3.2 shows that, for any $p > -n + 3$, there exists a non-negative convex solution of (1.6) in Ω which vanishes on the intersection of Ω with a line. For the version $p = 1$ of Proposition 1.3, Caffarelli [9] proves that $\dim S < n/2$ and that this inequality is optimal.

Proposition 1.3 yields actually somewhat more than Claim (i) in Theorem 1.2; namely, if $r \geq 2$ is an integer, $p < \min\{1, 2r - n\}$ and $K \in \mathcal{K}_0^n$ is a solution of (1.1) with $o \in \partial K$, then $\dim N(K, o) < r$. As a consequence, we have the following technical statements about K, where we also use Theorem 1.2 (ii) for Claim (ii).

Corollary 1.4 *If $p < 1$ and $K \in \mathcal{K}_0^n$, $n \geq 4$, is a solution of* (1.1) *with $o \in \partial K$, then*

(i) $\dim N(K, o) < \frac{n+1}{2}$;

(ii) *if in addition* $n = 4, 5$ *and* ∂K *is not* C^1*, then* $\dim N(K, o) = 2$ *and* $\dim F(K, u) = n - 1$ *for some* $u \in N(K, o)$*.*

In Section 2 we review the notation used in this paper. Section 3 contains results and examples regarding Monge-Ampère equations in $\mathbb{R}^n$, namely Proposition 1.3, Example 3.2 and Proposition 3.4. This last result is the key to prove Theorem 1.2 (ii). In Section 4 we show, for the sake of completeness, how to prove Theorem 1.1 using ideas due to Caffarelli [7,8] and Chou and Wang [16]. Theorem 1.2 and Corollary 1.4 are proved in Section 5.

2 Notation and Preliminaries

As usual, S^{n-1} denotes the unit sphere and o the origin in the Euclidean n-space $\mathbb{R}^n$. The symbol B^n denotes the unit ball in $\mathbb{R}^n$ centred at o and ω_n denotes its volume. If $x, y \in \mathbb{R}^n$, then $\langle x, y\rangle$ is the scalar product of x and y, while $\|x\|$ is the euclidean norm of x. By $[x, y]$ we denote the segment with endpoint x and y.

We write $\mathcal{H}^k$ for k-dimensional Hausdorff measure in $\mathbb{R}^n$.

We denote by ∂E, $\mathrm{int} E$, $\mathrm{cl} E$, and 1_E the *boundary*, *interior*, *closure*, and *characteristic function* of a set E in $\mathbb{R}^n$, respectively. The symbols $\mathrm{aff} E$ and $\mathrm{lin} E$ denote, respectively, the *affine hull* and the *linear hull* of E. The *dimension* $\dim E$ is the dimension of $\mathrm{aff} E$. With the symbol $E \mid L$ we denote the orthogonal projection of E on the linear space L.

Given a function v defined on a subset of $\mathbb{R}^n$, Dv and $D^2 v$ denote its gradient and its Hessian, respectively.

Our next goal is to recall a standard notion of generalised solution of Monge-Ampère equations, usually referred to as *solution in the sense of measure*. Our general reference for notions and facts about Monge-Ampère equations is the survey by Trudinger and Wang [51]. Let v be a convex function defined in an open convex set Ω; the subgradient $\partial v(x)$ of v at $x \in \Omega$ is defined as

$$\partial v(x) = \{z \in \mathbb{R}^n : v(y) \geq v(x) + \langle z, y - x\rangle \text{ for each } y \in \Omega\},$$

which is a non-empty compact convex set. Note that v is differentiable at $x \in \Omega$ if and only if $\partial v(x)$ consists of exactly one vector, which is the gradient of v at x. If $\omega \subset \Omega$ is a Borel set, then we denote by $N_v(\omega)$ the image of ω through the gradient map of v, i.e.

$$N_v(\omega) = \bigcup_{x \in \omega} \partial v(x).$$

Note that as ω is a Borel set, then $N_v(\omega)$ is measurable. Hence, we may define the Monge–Ampère measure associated to v as follows

$$\mu_v(\omega) = \mathcal{H}^n\Big(N_v(\omega)\Big). \tag{2.1}$$

For $p < 1$ and non-negative g on $\mathbb{R}^n$, we say that the non-negative convex function v satisfies the Monge-Ampère equation

$$v^{1-p}\det(D^2 v) = g$$

in the sense of measure (or in the Alexandrov sense) if

$$v^{1-p}\, d\mu_v = g\, d\mathcal{H}^n.$$

Equivalently

$$\int_\omega v^{1-p}(x)\mathrm{d}\mu_v(x) = \int_\omega g(x)\mathrm{d}x$$

for every Borel subset ω of Ω.

A *convex body* in $\mathbb{R}^n$ is a compact convex set with non-empty interior. The treatises Gardner [20], Gruber [21] and Schneider [47] are excellent general references for convex geometry. The function

$$h_K(u) = \max\{\langle u, y\rangle : y \in K\},$$

for $u \in \mathbb{R}^n$, is the *support function* of K. When it is clear the convex body to which we refer we will drop the subscript K from h_K and write simply h. Any convex body K is uniquely determined by its support function. A set $C \subset \mathbb{R}^n$ is a *convex cone* if $\alpha_1 u_1 + \alpha_2 u_2 \in C$ for any $u_1, u_2 \in C$ and $\alpha_1, \alpha_2 \geq 0$.

If S is a convex set in $\mathbb{R}^n$, then $z \in S$ is an extremal point if $z = \alpha x_1 + (1-\alpha)x_2$ for $x_1, x_2 \in S$ and $\alpha \in (0, 1)$ imply $x_1 = x_2 = z$. We note that if S is compact and convex, then S is the convex hull of its extremal points. If C is a convex cone and $u \in C\backslash\{o\}$, we say that $\sigma = \{\lambda u : \lambda \geq 0\}$ is an extremal ray if $\alpha_1 x_1 + \alpha_2 x_2 \in \sigma$ for $x_1, x_2 \in C$ and $\alpha_1, \alpha_2 > 0$ imply $x_1, x_2 \in \sigma$. Now if $C \neq \{o\}$ is a closed convex cone such that the origin is an extremal point of C, then C is the convex hull of its extremal rays.

The *normal cone* of a convex body K at $z \in K$ is defined as

$$N(K, z) = \{u \in \mathbb{R}^n : \langle u, y\rangle \leq \langle u, z\rangle \text{ for all } y \in K\},$$

where $N(K, z) = \{o\}$ if $z \in \mathrm{int}K$ and $\dim N(K, z) \geq 1$ if $z \in \partial K$. This definition can be written also as

$$N(K, z) = \{u \in \mathbb{R}^n : h_K(u) = \langle z, u\rangle\}. \tag{2.2}$$

In particular, $N(K, z)$ is a closed convex cone such that the origin is an extremal point, and

$$h_K(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 h_K(u_1) + \alpha_2 h_K(u_2) \text{ for } u_1, u_2 \in N(K, z) \text{ and } \alpha_1, \alpha_2 > 0. \tag{2.3}$$

A convex body K is C^1-smooth at $p \in \partial K$ if $N(K, p)$ is a ray, and ∂K is C^1 if each $p \in \partial K$ is a C^1-smooth point. Therefore, ∂K is C^1 if and only if the restriction of h_K to any hyperplane not containing o is strictly convex, by (2.3).

We say that a convex body K is *strictly convex* if ∂K contains no segment. The *face* of K with outer normal $u \in \mathbb{R}^n$ is defined as

$$F(K, u) = \{z \in K : h_K(u) = \langle z, u\rangle\},$$

which lies in ∂K if $u \neq o$. Schneider [47, Theorem 1.7.4] proves that

$$\partial h_K(u) = F(K, u). \tag{2.4}$$

Therefore, K is strictly convex if and only if h_K is C^1 on $\mathbb{R}^n \backslash \{o\}$.

A crucial notion for this paper is the one of *surface area measure* S_K of a convex body K, which is a Borel measure on S^{n-1}, defined as follows. For any Borel set $\omega \subset S^{n-1}$:

$$S_K(\omega) = \mathcal{H}^{n-1}\big(\cup_{u\in\omega} F(K, u)\big) = \mathcal{H}^{n-1}\big(\cup_{u\in\omega} \partial h_K(u)\big).$$

Hence, S_K is the analogue of the Monge–Ampère measure for the restriction of h_K to S^{n-1}.

Given a convex body K containing o and $p < 1$, let $S_{K,p}$ denote the L_p area measure of K; namely,

$$\mathrm{d}S_{K,p} = h_K^{1-p}\mathrm{d}S_K. \tag{2.5}$$

Let f be a positive and measurable function on S^{n-1}; we say that h_K is a solution of (1.2) in the sense of measure if

$$\int_\omega h_K(y)^{1-p}\mathrm{d}S_K(y) = \int_\omega f(y)\mathrm{d}\mathcal{H}^{n-1}(y) \tag{2.6}$$

for every Borel subset ω of S^{n-1}.

In what follows we will always assume that f is bounded between two positive constants. Our first remark is that the previous definition is equivalent to the following conditions (a) and (b):

(a) $\dim N(K, o) < n$; or equivalently,

$$\mathcal{H}^{n-1}\big(\{y \in S^{n-1} : h_K(y) = 0\}\big) = \mathcal{H}^{n-1}\big(N(K, o) \cap S^{n-1}\big) = 0, \tag{2.7}$$

(b) for each Borel set $\omega \subset \{y \in S^{n-1} : h_K(y) > 0\}$, we have

$$\int_\omega h_K^{1-p}(y)\,\mathrm{d}S_K(y) = \int_\omega f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y). \tag{2.8}$$

Moreover, condition (b) is in turn equivalent to (b') for each Borel set $\omega \subset \{y \in S^{n-1} : h_K(y) > 0\}$, we have

$$S_K(\omega) = \int_\omega f(y) h_K(y)^{p-1} \, \mathrm{d}\mathcal{H}^{n-1}(y). \tag{2.9}$$

To prove that (b) and (b') are equivalent is a simple exercise (in which one has to take into account the fact that h_K is continuous). Indeed, both claims are in turn equivalent to the following fact: the measure S_K is absolutely continuous with respect to $\mathcal{H}^{n-1}$ on $S^{n-1} \setminus \{y \in S^{n-1} : h_K(y) = 0\}$, and the Radon–Nikodym derivative of S_K with respect to $\mathcal{H}^{n-1}$ is $f h_K^{p-1}$.

Let us prove the equivalence between (2.6) and (a)–(b). To this end, it will be useful the following observation: the set

$$\{x \in \mathbb{R}^n : h_K(x) = 0\}$$

is a closed convex cone. Indeed, it is the set where the non-negative, convex and 1-homogeneous function h_k attains its minimum. For convenience, we set $\omega_0 = \{y \in S^{n-1} : h_K(y) = 0\}$. Assume that (2.6) holds; then (b) follows immediately. If, by contradiction, (a) is false, then ω_0 has non-empty interior so that

$$0 = \int_{\omega_0} h_K(y)^{1-p} \, \mathrm{d}S_K(y) = \int_{\omega_0} f(y) \, \mathrm{d}\mathcal{H}^{n-1}(y) > 0,$$

i.e. a contradiction (in the last inequality we have used the fact that f is bounded from below by a positive constant). Vice versa, assume that (a) and (b) hold. Given a Borel subset ω of S^{n-1} we may write it as the disjoint union of $\omega' = \omega \cap \omega_0$ and $\omega'' = \omega \setminus \omega'$. By (a), $\mathcal{H}^{n-1}(\omega') = 0$, moreover $h_K = 0$ on ω'; hence,

$$\begin{aligned}\int_\omega h_K(y)^{1-p} \, \mathrm{d}S_K(y) &= \int_{\omega''} h_K(y)^{1-p} \, \mathrm{d}S_K(y) \\ &= \int_{\omega''} f(y) \, \mathrm{d}\mathcal{H}^{n-1}(y) \\ &= \int_\omega f(y) \, \mathrm{d}\mathcal{H}^{n-1}(y),\end{aligned}$$

i.e. (2.6).

Our next step is to compare the solutions considered by Chou and Wang [16] with the ones introduced here. In particular, we will show that if h_K is a solution of (1.4), then it verifies conditions (a) and (b) as well [and consequently (2.6)]. Note that being a solution of (1.4) in the sense of measures means that

$$S_K(\omega) = \int_\omega f(y) h_K(y)^{p-1} \, \mathrm{d}\mathcal{H}^{n-1}(y). \tag{2.10}$$

has to hold for every Borel subset of S^{n-1}. In particular (2.9) follows (and then (b)). Moreover, as S_K is finite, $h_K \geq 0$ and f is bounded between two positive constants, the previous relation implies that

$$\int_{S^{n-1}} h_K(y)^{p-1}\, \mathrm{d}\mathcal{H}^{n-1}(y) < +\infty.$$

As $p-1<0$, this yields that the set ω_0 where h_K vanishes on S^{n-1} has zero $(n-1)$-dimensional measure. On the other hand this is the intersection of S^{n-1} with a convex cone. Hence we get condition (a).

In addition, if we now apply (2.10) to ω_0, we get that when h_K is a solution of (1.4) then

$$S_K\big(N(K,o)\cap S^{n-1}\big) = 0. \tag{2.11}$$

Note that (2.11) implies that $\mathcal{H}^{n-1}(X_0) = 0$, in the notation of Theorem 1.2, because $X_0 \subset \cup\{F(K,u) : u \in N(K,o)\cap S^{n-1}\}$ and (2.11) means, by definition,

$$\mathcal{H}^{n-1}\big(\cup_{u\in N(K,o)\cap S^{n-1}} F(K,u)\big) = 0.$$

Hence, applying Theorem 1.2(ii) we deduce that if $K \in \mathcal{K}_0^n$ is a solution of (1.4) for $p<1$ and f is bounded from below and above by positive constants, then ∂K is C^1, as it was verified by Chou and Wang [16].

3 Some Results on Monge–Ampère Equations in Euclidean Space

Lemma 3.1 is the tool to transfer the Monge–Ampère equation (1.2) on S^{n-1} to a Euclidean Monge–Ampère equation on $\mathbb{R}^{n-1}$. For $e \in S^{n-1}$, we consider the restriction of a solution h of (1.2) to the hyperplane tangent to S^{n-1} at e.

Lemma 3.1 *If $e \in S^{n-1}$, h is a convex positively 1-homogeneous non-negative function on $\mathbb{R}^n$ that is a solution of (1.2) for $p<1$ and positive f, and $v(y) = h(y+e)$ holds for $v : e^\perp \to \mathbb{R}$, then v satisfies*

$$v^{1-p}\det(D^2 v) = g \quad \text{on } e^\perp, \tag{3.1}$$

where, for $y \in e^\perp$, we have

$$g(y) = \left(1+\|y\|^2\right)^{-\frac{n+p}{2}} f\left(\frac{e+y}{\sqrt{1+\|y\|^2}}\right).$$

Proof Let $h = h_K$ for $K \in \mathcal{K}_0^n$, and let

$$\widetilde{S} = \{u \in S^{n-1} : h_K(u) = 0\},$$

which is a possibly empty spherically convex compact set whose spherical dimension is at most $n-2$, by (2.7). According to (2.9), the Monge–Ampère equation for h_K can be written in the form

$$dS_K = h_K^{p-1} f \, d\mathcal{H}^{n-1} \quad \text{on } S^{n-1} \setminus \widetilde{S}. \tag{3.2}$$

We consider $\pi : e^\perp \to S^{n-1}$ defined by

$$\pi(x) = (1 + \|x\|^2)^{\frac{-1}{2}} (x + e),$$

which is induced by the radial projection from the tangent hyperplane $e + e^\perp$ to S^{n-1}. Since $\langle \pi(x), e \rangle = (1 + \|x\|^2)^{\frac{-1}{2}}$, the Jacobian of π is

$$\det D\pi(x) = (1 + \|x\|^2)^{\frac{-n}{2}}. \tag{3.3}$$

For $x \in e^\perp$, (2.4) and writing h_K in terms of an orthonormal basis of $\mathbb{R}^n$ containing e yield that v satisfies

$$\partial v(x) = \partial h_K(x + e)|e^\perp = F(K, x + e)|e^\perp = F(K, \pi(x))|e^\perp.$$

Let $S = \pi^{-1}(\widetilde{S})$. For a Borel set $\omega \subset e^\perp \setminus S$, we have

$$\begin{aligned}
\mathcal{H}^{n-1}(N_v(\omega)) &= \mathcal{H}^{n-1}\left(\cup_{x\in\omega} \partial v(x)\right) \\
&= \mathcal{H}^{n-1}\left(\cup_{u \in \pi(\omega)} \left(F(K,u)|e^\perp\right)\right) = \int_{\pi(\omega)} \langle u, e \rangle \, \mathrm{d}S_K(u) \\
&= \int_{\pi(\omega)} \langle u, e \rangle h_K^{p-1}(u) f(u) \, \mathrm{d}\mathcal{H}^{n-1}(u) \\
&= \int_\omega (1 + \|x\|^2)^{\frac{-n-p}{2}} f(\pi(x)) v(x)^{p-1} \, \mathrm{d}\mathcal{H}^{n-1}(x),
\end{aligned}$$

where we used at the last step that

$$v(x) = h_K(x + e) = (1 + \|x\|^2)^{\frac{1}{2}} h_K(\pi(x)).$$

In particular, v satisfies the Monge–Ampère type differential equation

$$\det D^2 v(x) = (1 + \|x\|^2)^{\frac{-n-p}{2}} f(\pi(x)) v(x)^{p-1} \quad \text{on } e^\perp \setminus S.$$

Since $\dim S \le n-2$ by (1.3), v satisfies (3.1) on $e^\perp$. □

Having Lemma 3.1 at hand showing the need to understand related Monge-Ampère equations in Euclidean spaces, we prove Propositions 1.3 and 3.4, and quote Caffarelli's Theorem 3.6.

Proof of Proposition 1.3 Up to changing coordinate system, we may assume, without loss of generality, that $S \subset \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : x_2 = 0\}$ and the origin is contained in the relative interior of S. Therefore, up to restricting Ω, we may also assume that v is continuous on $\mathrm{cl}\,\Omega$, that $\Omega = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : \|x_1\| < s_1, \|x_2\| < s_2\}$ for some constants $s_1, s_2 > 0$ and that $S = \{(x_1, x_2) \in \Omega : x_2 = 0\}$.

Let $\alpha = \max_{\mathrm{cl}\,\Omega} v$ and let us consider the convex body

$$M = \{(x_1, x_2, y) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \times \mathbb{R} : \|x_1\| \le s_1, \|x_2\| \le s_2, v(x_1, x_2) \le y \le \alpha\}.$$

For $t \in (0, s_2/2]$, let

$$\Omega_t = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : \|x_1\| \le s_1/2, \|x_2\| \le t\}.$$

We estimate $\mathcal{H}^n\big(N_v(\Omega_t \setminus S)\big)$. Let $(x_1, x_2) \in \Omega_t \setminus S$ and let $(z_1, z_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$ belong to $\partial v(x_1, x_2)$. We prove that

$$\|z_2\| \le \frac{2\alpha}{s_2} \quad \text{and} \quad \|z_1\| \le \frac{4\alpha}{s_1 s_2} t. \tag{3.4}$$

If $z_2 = 0$ the first inequality in (3.4) holds true. Assume $z_2 \neq 0$. The vector $(z_1, z_2, -1)$ is an exterior normal to M at $p = (x_1, x_2, v(x_1, x_2))$. Since

$$q_1 = \left(x_1, x_2 + \frac{s_2 z_2}{2\|z_2\|}, \alpha\right) \in M$$

(because $\big\|x_2 + s_2 z_2/(2\|z_2\|)\big\| \le \|x_2\| + s_2/2 \le s_2$) then $\langle q_1 - p, (z_1, z_2, -1)\rangle \le 0$. This implies

$$\|z_2\| \le \frac{2}{s_2}(\alpha - v(x_1, x_2))$$

and the first inequality in (3.4). Again, if $z_1 = 0$, then the second inequality (3.4) holds true. Assume $z_1 \neq 0$. We have

$$q_2 = \left(x_1 + \frac{s_1 z_1}{2\|z_1\|}, 0, v(x_1, x_2)\right) \in M,$$

because $\big\|x_1 + s_1 z_1/(2\|z_1\|)\big\| \le s_1$, $(x_1 + s_1 z_1/(2\|z_1\|), 0) \in S$ and therefore $v(x_1, x_2) \ge 0 = v(x_1 + s_1 z_1/(2\|z_1\|), 0)$. The inequality $\langle q_2 - p, (z_1, z_2, -1)\rangle \le 0$ implies the second inequality (3.4).

The inequalities in (3.4) imply

$$\mathcal{H}^n\big(N_v(\Omega_t \setminus S)\big) \le c\, t^r, \tag{3.5}$$

for a suitable constant c independent of t.

Now we estimate $\int_{\Omega_t\setminus S} v(x)^{p-1}\,\mathrm{d}x$. The inclusion of the convex hull of $S\times\{0\}$ and $\{\|x_1\|\le s_1, \|x_2\|\le s_2, y=\alpha\}$ in M implies that $v(x_1,x_2)\le\frac{\alpha}{s_2}\|x_2\|$ for each $(x_1,x_2)\in\Omega_t$ by the convexity of v. Using this estimate it is straightforward to compute that

$$\int_{\Omega_t\setminus S} v(x)^{p-1}\,\mathrm{d}x \ge \mathrm{d}\, t^{n+p-r-1}, \tag{3.6}$$

for a suitable constant d independent on t. The inequalities (3.5) and (3.6) and the differential inequality satisfied by v imply, as $t\to 0^+$,

$$ct^r \ge \mathcal{H}^n\big(N_v(\Omega_t\setminus S)\big) \ge \int_{\Omega_t\setminus S} \tau v(x)^{p-1}\,\mathrm{d}x \ge \tau \mathrm{d}\, t^{n+p-r-1}.$$

This inequality implies $p\ge -n+1+2r$. □

Example 3.2 Let us show that for any $p>-n+3$ there exists a non-negative convex solution of (1.6) in $\Omega=\{(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1} : x_1\in[-1,1], \|x_2\|\le 1\}$ which vanishes on the 1-dimensional space $S=\{(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1} : x_2=0\}$.

To prove this let

$$v(x_1,x_2)=\|x_2\|+f(\|x_2\|)g(x_1),$$

where $f(r)=r^\alpha$, with $\alpha=(p+n-1)/2$, and $g(x_1)=(1+\beta x_1^2)$, with $\beta>0$ sufficiently small. Note that $\alpha>1$ exactly when $p>-n+3$.

The function v is invariant with respect to rotations around the line containing S. To compute $\det D^2v$ at an arbitrary point, it suffices to compute it at $(x_1,0,\ldots,0,r)$, $r\ge 0$. We get

$$\begin{aligned}
v_{x_1x_1} &= f(r)g''(x_1), && \\
v_{x_1x_i} &= 0 && \text{when } 1<i<n,\\
v_{x_1x_n} &= f'(r)g'(x_1), && \\
v_{x_ix_i} &= \frac{1}{r}+\frac{f'(r)}{r}g(x_1) && \text{when } 1<i<n,\\
v_{x_ix_j} &= 0 && \text{when } i\ne j, (i,j)\ne(1,n), (i,j)\ne(n,1),\\
v_{x_nx_n} &= f''(r)g(x_1). &&
\end{aligned}$$

The function v is convex if β is sufficiently small. Indeed, the eigenvalues of D^2v are $\frac{1}{r}+\frac{f'(r)}{r}g(x_1)$, with multiplicity $n-2$, and those of the matrix

$$\begin{pmatrix} fg'' & f'g' \\ f'g' & f''g \end{pmatrix}.$$

The determinant of the latter matrix is

$$2\alpha\beta r^{2(\alpha-1)}\Big(\alpha-1-(1+\alpha)\beta x_1^2\Big),$$

which is positive if $\beta > 0$ is sufficiently small. Thus, all eigenvalues of $D^2 v$ are positive.

We get

$$\det D^2 v = \Big(f'' g f g'' - (f'g')^2\Big)\Big(\frac{1}{r} + \frac{f'}{r} g\Big)^{n-2}$$

which has the same order as $r^{2\alpha-n}$ as $r \to 0^+$. Clearly v has order r, and $v^{1-p}\det D^2 v$ has order $r^{2\alpha-n+1-p}$, which is uniformly bounded from above and below for our choice of α.

The next statement is a slight modification of Lemmas 3.2 and 3.3 from Trudinger and Wang [51]. Its proof closely follows that in [51] and is given here for completeness.

Lemma 3.3 *Let v be a convex function defined on the closure of an open bounded convex set $\Omega \subset \mathbb{R}^n$ satisfying the Monge-Ampère equation*

$$\det D^2 v = \nu$$

for a finite non-negative measure ν on Ω, let $v \equiv 0$ on $\partial\Omega$ and let $tE \subset \Omega \subset E$ for $t > 0$ and an origin centred ellipsoid E.

(i) *If $z \in \Omega$ satisfies $(z + sE) \cap \partial\Omega \neq \emptyset$ for $s > 0$, then*

$$|v(z)| \leq s^{1/n} c_0 \mathcal{H}^n(\Omega)^{1/n} \nu(\Omega)^{1/n}$$

for some $c_0 > 0$ depending on n and t.

(ii) *If $\nu(t\Omega) \geq b\,\nu(\Omega)$ for $b > 0$, then*

$$|v(0)| \geq c_1 \mathcal{H}^n(\Omega)^{1/n} \nu(\Omega)^{1/n} \tag{3.7}$$

for some $c_1 > 0$ depending on n, t and b.

(iii) *If $(z + sE) \cap \partial\Omega \neq \emptyset$ and $\nu(t\Omega) \geq b\,\nu(\Omega)$, then*

$$\frac{|v(z)|}{|v(o)|} \leq \frac{c_1}{c_0} s^{1/n}. \tag{3.8}$$

When $E = B^n$, the number s can be chosen as the distance of z from $\partial\Omega$. In the general case s has the same meaning in the metric induced by the norm whose unitary ball is E.

Proof Let A be a linear transformation such that $B^n = A^{-1}E$, let $\tilde{v}(x) = v(Ax)|\det A|^{-2/n}$, $\widetilde{\Omega} = A^{-1}\Omega$ and let $\tilde{\nu}$ be the measure defined for each Borel set $\omega \subset \widetilde{\Omega}$ as $\tilde{\nu}(\omega) = \nu(A\omega)/|\det A|$. It is known that $\tilde{v}$ solves

$$\det D^2 \tilde{v} = \tilde{\nu} \quad \text{in } \widetilde{\Omega}. \tag{3.9}$$

Moreover, $tB^n \subset \widetilde{\Omega} \subset B^n$. Since $\mathcal{H}^n(\Omega) = |\det A|\mathcal{H}^n(\widetilde{\Omega})$, we have

$$\frac{\mathcal{H}^n(\Omega)}{\omega_n} \le |\det A| \le \frac{\mathcal{H}^n(\Omega)}{\omega_n t^n}. \tag{3.10}$$

Let us prove Claim (i). Let $\tilde{z} = A^{-1}z$. Then $(\tilde{z} + sB^n) \cap \partial\widetilde{\Omega} \neq \emptyset$ and if d denotes the distance of $\tilde{z}$ from $\partial\widetilde{\Omega}$ we have $d \le s$. By choosing proper coordinates we may assume that $\tilde{z} = (0, \dots, 0, d)$, and that $\widetilde{\Omega} \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$. Then

$$\widetilde{\Omega} \subset \widehat{\Omega} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \|(x_1, \dots, x_{n-1})\| < 2, 0 < x_n < 4\}.$$

Let u and w be convex functions such that their graphs are convex cones with vertex at $(\tilde{z}, \tilde{v}(\tilde{z}))$ and bases $\partial\widetilde{\Omega}$ and $\partial\widehat{\Omega}$, respectively. Then

$$N_{\tilde{v}}(\widetilde{\Omega}) \supset N_u(\widetilde{\Omega}) = \partial u(\tilde{z}) \supset \partial w(\tilde{z}). \tag{3.11}$$

Since w is a convex cone over the cylinder $\widehat{\Omega}$, one can easily compute that $\mathcal{H}^n(\partial w(\tilde{z})) \ge c_2|\tilde{v}(\tilde{z})|^n/d$, for a suitable constant $c_2 > 0$. This inequality, (3.9) and (3.11) imply

$$|\tilde{v}(\tilde{z})| \le \left(\frac{d}{c_2}\right)^{1/n} \mathcal{H}^n(N_{\tilde{v}}(\widetilde{\Omega}))^{1/n} = \left(\frac{d}{c_2}\right)^{1/n} \tilde{\nu}(\widetilde{\Omega})^{1/n}.$$

Expressing this inequality in terms of v, Ω and ν and using $d \le s$ and (3.10) concludes the proof of Claim (i).

Let us prove Claim (ii). There exists an unique solution w of $\det D^2 w = \widehat{\nu}$ in $\widetilde{\Omega}$, $w = 0$ in $\partial\widetilde{\Omega}$, where $\widehat{\nu} = \tilde{\nu}$ in $t\widetilde{\Omega}$ and $\widehat{\nu} = 0$ elsewhere (see Theorem 2.1 in [51]). The comparison principle for Monge–Ampère equations (see Lemma 2.4 in [51]) implies $w \ge \tilde{v}$ in $\widetilde{\Omega}$.

Let $z \in t\widetilde{\Omega}$. The distance d of z from $\partial\widetilde{\Omega}$ is larger than or equal to $(1-t)t$ (here we have used the inclusion $tB^n \subset \widetilde{\Omega}$). If $y \in \partial w(z)$ and $l(x) = \langle x, y\rangle + w(z)$, then $l(x) \le w(x)$ for each $x \in \widetilde{\Omega}$, by definition of subgradient. In particular, we have $l(x) \le 0$ for each $x \in \partial\widetilde{\Omega}$. This implies

$$|y| \le \frac{|w(z)|}{d} \le \frac{\sup_{\widetilde{\Omega}} |\tilde{v}|}{t(1-t)}.$$

Therefore,

$$\mathcal{H}^n(N_w(t\widetilde{\Omega})) \le \omega_n \left(\frac{\sup_{\widetilde{\Omega}} |\tilde{v}|}{t(1-t)}\right)^n.$$

This inequality, the equation satisfied by w and the condition $\nu(t\Omega) \geq b\,\nu(\Omega)$ imply

$$\begin{aligned}\sup_{\widetilde{\Omega}} |\tilde{v}| &\geq \frac{t(1-t)}{\omega_n^{1/n}} \mathcal{H}^n(N_w(t\widetilde{\Omega}))^{1/n} = \frac{t(1-t)}{\omega_n^{1/n}} \tilde{\nu}(t\widetilde{\Omega})^{1/n} \\ &\geq \frac{bt(1-t)}{\omega_n^{1/n}} \tilde{\nu}(\widetilde{\Omega})^{1/n}.\end{aligned} \tag{3.12}$$

We claim that

$$|\tilde{v}(o)| \geq \frac{t}{1+t} \sup_{\widetilde{\Omega}} |\tilde{v}|. \tag{3.13}$$

Indeed, let $z \in \widetilde{\Omega}$ be such that $\tilde{v}(z) = \inf_{\widetilde{\Omega}} \tilde{v}$. We may clearly assume $z \neq 0$, since otherwise there is nothing to prove. By choosing proper coordinates we may assume $z = (z_1, 0, \ldots, 0)$ for some $z_1 > 0$. Let l be the linear function defined on the line through o and z and such that $l(o) = \tilde{v}(o)$ and $l(z) = \tilde{v}(z)$. It is $l(s, 0, \ldots 0) = \tilde{v}(o) + s(\inf_{\widetilde{\Omega}} \tilde{v} - \tilde{v}(o))/z_1$. Since $\tilde{v}$ is convex,

$$l(s, 0, \ldots 0) \leq \tilde{v}(s, 0, \ldots 0)$$

for each $s \notin [0, z_1]$ such that $(s, 0, \ldots, 0) \in \widetilde{\Omega}$. When $s = -t$ we obtain $l(-t, 0, \ldots, 0) \leq \tilde{v}(-t, 0, \ldots, 0) \leq 0$. The inequality $l(-t, 0, \ldots, 0) \leq 0$ and the inclusion $\widetilde{\Omega} \subset B^n$ imply (3.13).

The proof of Claim (ii) is concluded by combining (3.12) and (3.13) and expressing the obtained inequality in terms of v, Ω and ν.

Claim (iii) is a consequence of the first two claims. □

The proof of Claim (ii) in Theorem 1.2 is based on the following proposition, which is related to a step in the proof of Theorem E (a) in [16]; however, our proof is substantially different from that in [16].

Proposition 3.4 *Let v be a non-negative convex function defined on the closure of an open convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, such that $S = \{x \in \Omega : v(x) = 0\}$ is non-empty and compact, and v is locally strictly convex on $\Omega \backslash S$. Let $\psi : (0, \infty) \to [0, \infty)$ be monotone decreasing and not identically zero; assume that $\tau_2 > \tau_1 > 0$ and v satisfy*

$$\tau_1 \psi(v) \leq \det D^2 v \leq \tau_2 \psi(v) \tag{3.14}$$

in the sense of measure on $\Omega \backslash S$. If $\dim S \leq n-1$ and $\mu_v(S) = 0$ for the associated Monge-Ampère measure μ_v, then S is a point.

Note that (3.14) means that for each Borel set $\omega \subset \Omega \setminus S$ we have

$$\tau_1 \int_\omega \psi(v(x))\,\mathrm{d}x \leq \mu_v(\omega) \leq \tau_2 \int_\omega \psi(v(x))\,\mathrm{d}x,$$

where μ_v has been defined in (2.1).

Proof We assume, arguing by contradiction, that S is not a point. Choose coordinates so that o is the centre of mass of S. Let $L = \operatorname{lin} S$. By assumption

$$1 \leq \dim L \leq n - 1. \tag{3.15}$$

Let $e = (o, 1) \in \mathbb{R}^n \times \mathbb{R}$. We may assume that Ω is bounded, after possibly substituting it with a bounded open neighbourhood of S. We start by illustrating the idea of the proof.

Sketch of the proof For any small $\varepsilon > 0$, we construct an affine function l_ε such that $l_\varepsilon(x) = \varepsilon$ for $x \in L$, and the convex set $\Omega_\varepsilon = \{v < l_\varepsilon\}$ is well balanced; namely, there exists an ellipsoid E_ε centred at the origin such that $(1/(8n^3))E_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon$ [see (3.19)]. This is the longest part of the argument, and the main idea to construct l_ε is that the graph of l_ε cuts off the smallest volume cap from the graph of v among the hyperplanes in $\mathbb{R}^{n+1}$ containing $L + \varepsilon e$. Subsequently, we apply Lemma 3.3 to Ω_ε and to the function $v - l_\varepsilon$ in the standard way to reach a contradiction. We show that one can choose $z \in S$ so that the corresponding parameter s, as defined in Lemma 3.3, tends to 0 as ε tends to 0. (Equivalently, S contains points whose distance from $\partial\Omega_\varepsilon$, the one induced by the norm whose unit ball is E_ε, tends to 0 as ε tends to 0.) This contradicts (3.8), since $|v(z) - l_\varepsilon(z)|/|v(o) - l_\varepsilon(o)| = \varepsilon/\varepsilon = 1$.

We divide the proof into four steps.

Step 1. Definition of l_ε and of Ω_ε.

Let $\varepsilon_0 = \min_{\partial\Omega} v > 0$ and let us consider the $(n+1)$-dimensional convex body

$$M = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : v(x) \leq y \leq \varepsilon_0\}.$$

For $\varepsilon \in (0, \varepsilon_0)$ define H_ε to be a hyperplane in $\mathbb{R}^{n+1}$

(i) containing $L + \varepsilon e = \{(x, \varepsilon e) \in \mathbb{R}^n \times \mathbb{R} \colon x \in L\}$ and
(ii) cutting off the minimal volume from M (on the side containing the origin) under condition (i).

Let $r > 0$. We claim that there exists $\varepsilon_1 = \varepsilon_1(r)$ so that H_ε is the graph of an affine function l_ε for each $\varepsilon \in (0, \varepsilon_1)$, and, setting

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n : v(x) < l_\varepsilon(x)\},$$

we have

$$\operatorname{cl}\Omega_\varepsilon \subset \Omega, \quad S \subset \Omega_\varepsilon \quad \text{and} \quad \Omega_\varepsilon \cap L \subset (1 + r)S. \tag{3.16}$$

Let $F = \{(x, y) \in M : y = \varepsilon_0\}$ be the upper face of M and let $\mathcal{H}$ be the collection of hyperplanes in $\mathbb{R}^{n+1}$ which intersect both F and $\{(x, y) \in M : y \leq \varepsilon_0/2\}$. Since Ω is bounded and v is locally strictly convex on $\Omega \setminus S$, every hyperplane in $\mathcal{H}$ is not a supporting hyperplane to M. Therefore, by compactness, there exists a constant $\varrho_0 > 0$ such that for every $H \in \mathcal{H}$ both components of $M \backslash H$ are of volume at least ϱ_0. We choose $\varepsilon_1 \in (0, \varepsilon_0/2)$ such that the volume of the cap $\{(x, y) \in M : \ y \leq \varepsilon_1\}$ is less than ϱ_0. This choice implies that the minimum value of the problem which defines H_ε is less than ϱ_0. Therefore, a minimiser H_ε does not belong to $\mathcal{H}$. Since

$H_\varepsilon \cap \{(x,y) \in M : y \le \varepsilon_0/2\} \ne 0$, we have $H_\varepsilon \cap F = \emptyset$. In particular, H_ε is the graph of a affine function defined on $\mathbb{R}^n$ and $\mathrm{cl}\Omega_\varepsilon \subset \Omega$.

The inclusion $S \subset \Omega_\varepsilon$ holds because $v(x) = 0$ and $l_\varepsilon(x) = \varepsilon$ for any $x \in S$.

The origin o, being the centre of mass of S, belongs to the relative interior of S. Since $\dim S > 0$, the relative boundary of $(1+r)S$ does not intersect S. This implies $\inf_{\mathrm{relbd}(1+r)S} v > 0$. Thus, if ε_1 satisfies

$$\varepsilon_1 < \inf_{\mathrm{relbd}(1+r)S} v$$

in addition to the inequalities specified above, then $v(x) > \varepsilon$ and $l_\varepsilon(x) = \varepsilon$ for any $x \in \mathrm{relbd}(1+r)S$ ($l_\varepsilon(x) = \varepsilon$ is a consequence of $(1+r)S \subset L$). This implies $\Omega_\varepsilon \cap L \subset (1+r)S$.

In the rest of the proof we may assume $\varepsilon_1 < \varepsilon_1(1)$ so that

$$\Omega_\varepsilon \cap L \subset 2S. \tag{3.17}$$

Step 2. The centre of mass of Ω_ε is contained in L.

To prove this claim we have to prove that for each $w \in L^\perp \cap \mathbb{R}^n$ we have

$$\int_{\Omega_\varepsilon} \langle x, w\rangle \,\mathrm{d}x = 0. \tag{3.18}$$

Indeed, for $t \in \mathbb{R}$ with $|t|$ small, let

$$F(t) = \int_{\{x\in\Omega: l_\varepsilon(x)+t\langle x,w\rangle - v(x)>0\}} (l_\varepsilon(x) + t\langle x, w\rangle - v(x))\,\mathrm{d}x$$

be the volume cut off by the hyperplane in $\mathbb{R}^{n+1}$ that is the graph of $x \mapsto l_\varepsilon(x)+t\langle x, w\rangle$ from M. By definition of H_ε and l_ε, F has a local minimum at $t = 0$. We have

$$\begin{aligned}\frac{F(t)-F(0)}{t} &= \int_{\{x\in\Omega: l_\varepsilon(x)-v(x)>0\}} \langle x, w\rangle\, dx \\ &\quad + \int_\Omega \left(\frac{l_\varepsilon(x)-v(x)}{t} + \langle x, w\rangle\right) \\ &\quad \times \left(1_{\{x: l_\varepsilon(x)+t\langle x,w\rangle - v(x)>0\}} - 1_{\{x: l_\varepsilon(x)-v(x)>0\}}\right)\mathrm{d}x.\end{aligned}$$

The set where $1_{\{x: l_\varepsilon(x)+t\langle x,w\rangle - v(x)>0\}} - 1_{\{x: l_\varepsilon(x)-v(x)>0\}}$ differs from 0 is contained in

$$A_t = \{x \in \Omega : |l_\varepsilon(x) - v(x)| < |t\langle x, w\rangle|\}$$

and there exists c independent on t such that $\mathcal{H}^n(A_t) < ct$ and $\sup_{A_t} |l_\varepsilon(x) - v(x)| < ct$. As F has a local minimum at $t = 0$, we have

$$0 = \frac{dF}{dt}(0) = \int_{\Omega_\varepsilon} \langle x, w\rangle\,\mathrm{d}x,$$

which proves (3.18).

Step 3. For any $\varepsilon \in (0, \varepsilon_1)$ there exists an ellipsoid E_ε centred at the origin such that

$$\frac{1}{8n^3} E_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon. \tag{3.19}$$

Lemma 2.3.3 in [47] proves that any k-dimensional convex body contains its reflection, with respect to its centre of mass, scaled, with respect to the same centre of mass, by $1/k$. From the fact that the centre of mass of Ω_ε belongs to L, we deduce that

$$-(\Omega_\varepsilon | L^\perp) \subset n(\Omega_\varepsilon | L^\perp). \tag{3.20}$$

According to Loewner's or John's theorems, there exists an ellipsoid $\widetilde{E}$ centred at the origin and $z_1 \in \Omega_\varepsilon$ such that

$$z_1 + \frac{1}{n}\widetilde{E} \subset \Omega_\varepsilon \subset z_1 + \widetilde{E}.$$

It follows from (3.20) that there exists $z_2 \in \Omega_\varepsilon$ such that $z_2 | L^\perp = \frac{-1}{n} z_1 | L^\perp$. In particular, $y_1 = \frac{1}{n+1} z_1 + \frac{n}{n+1} z_2 \in \Omega_\varepsilon$ verifies $y_1 | L^\perp = o$, or in other words, $y_1 \in L \cap \Omega_\varepsilon$. In addition,

$$y_1 + \frac{1}{2n^2}\widetilde{E} \subset \frac{1}{n+1}\left(z_1 + \frac{1}{n}\widetilde{E}\right) + \frac{n}{n+1} z_2 \subset \Omega_\varepsilon.$$

Let $m = \dim L \leq n-1$. Since $y_1 \in L \cap \Omega_\varepsilon$ and (3.17) imply $\frac{1}{2} y_1 \in S$, and since the origin is the centroid of S, we deduce that $y_2 = \frac{-1}{2m} y_1 \in S$. As $2m + 1 < 2n$, we have

$$\frac{1}{4n^3}\widetilde{E} \subset \frac{1}{2m+1}\left(y_1 + \frac{1}{2n^2}\widetilde{E}\right) + \frac{2m}{2m+1} y_2 \subset \Omega_\varepsilon.$$

As $\Omega_\varepsilon \subset 2\widetilde{E}$ follows from $o \in z_1 + \widetilde{E}$, we may choose $E_\varepsilon = 2\widetilde{E}$, proving (3.19).

Step 4. Application of Lemma 3.3 to $v - l_\varepsilon$ and Ω_ε and contradiction.

We observe that

$$v(x) - l_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \partial\Omega_\varepsilon \\ -\varepsilon & \text{if } x \in S. \end{cases} \tag{3.21}$$

Let ν denote the Monge–Ampère measure $\mu_{(v-l_\varepsilon)}$ restricted to Ω_ε. If Ω_0 is an open set such that $\Omega_\varepsilon \subset \Omega_0 \subset \operatorname{cl}\Omega_0 \subset \Omega$, then the set $N_v(\Omega_0)$ is bounded and this implies

$$\nu(\Omega_\varepsilon) = \mathcal{H}^n(N_{(v-l_\varepsilon)}(\Omega_\varepsilon)) \leq \mathcal{H}^n(N_v(\Omega_0)) < \infty.$$

Let $t = 1/(8n^3)$. Formula (3.19) yields $tE_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon$. Let us prove that

$$\nu(t\Omega_\varepsilon) \geq b\nu(\Omega_\varepsilon) \quad \text{for } b = \tau_1 t^n/\tau_2. \tag{3.22}$$

The function v is convex and attains its minimum at o; thus $v(x) \geq v(tx)$ for any $x \in \Omega_\varepsilon$. By this fact, the monotonicity of ψ, (3.14) and the assumptions on S, we deduce that

$$\begin{aligned}
\nu(t\Omega_\varepsilon) = \nu(t(\Omega_\varepsilon \setminus S)) &\geq \tau_1 \int_{t(\Omega_\varepsilon \setminus S)} \psi(v(x))\,\mathrm{d}x \\
&= \tau_1 t^n \int_{\Omega_\varepsilon \setminus S} \psi(v(tz))\,\mathrm{d}z \\
&\geq \tau_1 t^n \int_{\Omega_\varepsilon \setminus S} \psi(v(z))\,\mathrm{d}z \\
&\geq \frac{\tau_1 t^n}{\tau_2}\,\nu(\Omega_\varepsilon \setminus S) = \frac{\tau_1 t^n}{\tau_2}\,\nu(\Omega_\varepsilon)
\end{aligned}$$

proving (3.22).

Let $z \in \operatorname{relbd} S$. We claim that when $\varepsilon \in (0, \varepsilon_1(r))$ then $(z + rE_\varepsilon) \cap \partial\Omega_\varepsilon \neq \emptyset$. This is a consequence of the second and third inclusion in (3.16). Indeed, since $o \in S \subset \Omega_\varepsilon \subset E_\varepsilon$, there exists $q_\varepsilon > 0$ such that $(1 + q_\varepsilon)z \in \partial E_\varepsilon$. The set $z + rE_\varepsilon$ contains the segment $[z, z + r(1 + q_\varepsilon)z]$. Since $q_\varepsilon > 0$, that segment contains the segment $[z, (1 + r)z]$. The second and third inclusion in (3.16) imply $[z, (1 + r)z] \cap \partial\Omega_\varepsilon \neq \emptyset$. This proves the claim.

Lemma 3.3 applies to this situation with $s = r$. Since $v(z) - l_\varepsilon(z) = v(o) - l_\varepsilon(o) = -\varepsilon$ [see (3.21)], (3.8) yields

$$1 = \frac{|v(z) - l_\varepsilon(z)|}{|v(o) - l_\varepsilon(o)|} \leq \frac{c_1}{c_0} r^{1/n}.$$

Since r can be any positive number, we have reached a contradiction. □

We will actually use the following consequence of Proposition 3.4.

Corollary 3.5 *Let $\tau_2 > \tau_1 > 0$, and let g be a function defined on an open convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, such that $\tau_2 > g(x) > \tau_1$ for $x \in \Omega$. For $p < 1$, let v be a non-negative convex solution of*

$$v^{1-p} \det D^2 v = g \quad \text{in } \Omega.$$

If $S = \{x \in \Omega : v(x) = 0\}$ is non-empty, compact and $\mu_v(S) = 0$, and v is locally strictly convex on $\Omega \setminus S$, then S is a point.

Proof All we have to check is that $\dim S \leq n - 1$. It follows from the fact that the left-hand side of the differential equation is zero on S, while the right-hand side is positive. □

The following result by L. Caffarelli (see Theorem 1 and Corollary 1 in [7]) is the key in handling the regularity and strict convexity of the part of the boundary of a convex body K where the support function at some normal vector is positive.

Theorem 3.6 (Caffarelli) *Let $\lambda_2 > \lambda_1 > 0$, and let v be a convex function on an open convex set $\Omega \subset \mathbb{R}^n$ such that*

$$\lambda_1 \le \det D^2 v \le \lambda_2 \tag{3.23}$$

in the sense of measure.

(i) *If v is non-negative and $S = \{x \in \Omega : v(x) = 0\}$ is not a point, then S has no extremal point in Ω.*
(ii) *If v is strictly convex, then v is C^1.*

We recall that (3.23) is equivalent to saying that for each Borel set $\omega \subset \Omega$ we have

$$\lambda_1 \mathcal{H}^n(\omega) \le \mu_v(\omega) \le \lambda_2 \mathcal{H}^n(\omega),$$

where μ_v has been defined in (2.1).

4 Proof of Theorem 1.1

The next lemma provides a tool for the proof of Theorem 1.1(iii). The same result is also proved in Chou and Wang [16]; we present a short argument for the sake of completeness.

Lemma 4.1 *For $n \ge 2$ and $p \le 2 - n$, if $K \in \mathcal{K}_0^n$ and there exists $c > 0$ such that $S_{K,p}(\omega) \ge c\,\mathcal{H}^{n-1}(\omega)$ for any Borel set $\omega \subset S^{n-1}$, then $o \in \operatorname{int} K$.*

Proof We suppose that $o \in \partial K$ and seek a contradiction. We choose $e \in N(K, o) \cap S^{n-1}$ such that $\{\lambda e : \lambda \ge 0\}$ is an extremal ray of $N(K, o)$. Let H^+ be a closed half space containing $\mathbb{R}e$ on the boundary such that $N(K, o) \cap \operatorname{int} H^+ = \emptyset$. Let

$$V_0 = S^{n-1} \cap (e + B^n) \cap \operatorname{int} H^+.$$

It follows by the condition on $S_{K,p}$ that

$$c \int_{V_0} h_K(u)^{p-1}\, \mathrm{d}\mathcal{H}^{n-1} \le \int_{V_0} h_K(u)^{p-1}\, \mathrm{d}S_{K,p} = S_K(V_0) < \infty. \tag{4.1}$$

However, since h_K is convex and $h_K(e) = 0$, there exists $c_0 > 0$ such that

$$h_K(x) \le c_0 \|x - e\| \quad \text{for } x \in e + B^n.$$

We observe that the radial projection of V_0 onto the tangent hyperplane $e + e^\perp$ to S^{n-1} at e is $e + V_0'$ for

$$V_0' = e^\perp \cap (\sqrt{3}\, B^n) \cap \operatorname{int} H^+.$$

If $y \in V_0'$, then $u = (e+y)/\|e+y\|$ verifies $\|u-e\| \geq \|y\|/2$. It follows that

$$\int_{V_0} h_K(u)^{p-1}\, \mathrm{d}\mathcal{H}^{n-1} \geq c_0^{p-1} \int_{V_0} \|u-e\|^{p-1}\, \mathrm{d}\mathcal{H}^{n-1}(u)$$
$$\geq \frac{c_0^{p-1}}{2} \int_{V_0'} \frac{\|y\|^{p-1}}{(1+\|y\|^2)^{n/2}}\, \mathrm{d}\mathcal{H}^{n-1}(y)$$
$$\geq \frac{c_0^{p-1}}{2^{n+1}} \int_{V_0'} \|y\|^{p-1}\, \mathrm{d}\mathcal{H}^{n-1}(y) = \infty$$

as $p \leq 2-n$. This contradicts (4.1), and hence verifies the lemma. □

Proof of Theorem 1.1 *Claim* (i). For $u_0 \in S^{n-1} \backslash N(K, o)$, we choose a spherically convex open neighbourhood Ω_0 of u_0 on S^{n-1} such that for any $u \in \operatorname{cl} \Omega_0$, we have $\langle u, u_0 \rangle > 0$ and $u \notin N(K, o)$. Let $\Omega \subset u_0^{\perp}$ be defined in a way such that $u_0 + \Omega$ is the radial image of Ω_0 into $u_0 + u_0^{\perp}$, and let v be the function on Ω defined as in Lemma 3.1 with $h = h_K$. Since h_K is positive and continuous on $\operatorname{cl} \Omega$, we deduce from Lemma 3.1 that there exist $\lambda_2 > \lambda_1 > 0$ depending on K, u_0 and Ω_0 such that

$$\lambda_1 \leq \det D^2 v \leq \lambda_2 \tag{4.2}$$

on Ω.

First we claim that

$$\text{if } z \in \partial K \text{ and } N(K, z) \not\subset N(K, o), \text{ then } z \text{ is a } C^1\text{-smooth point.} \tag{4.3}$$

We suppose that $\dim N(K, z) \geq 2$, and seek a contradiction. Since $N(K, z)$ is a closed convex cone such that o is an extremal point, the property $N(K, z) \not\subset N(K, o)$ yields an $e \in (N(K, z) \cap S^{n-1}) \backslash N(K, 0)$ generating an extremal ray of $N(K, z)$. We apply the construction above for $u_0 = e$. The convexity of h_K and (2.2) imply $h_K(x) \geq \langle z, x \rangle$ for $x \in \mathbb{R}^n$, with equality if and only if $x \in N(K, z)$. We define $S \subset \Omega$ by $S + e = N(K, z) \cap (\Omega + e)$ and hence o is an extremal point of S. It follows that the function $\tilde{v}$ defined by $\tilde{v}(y) = v(y) - \langle z, y+e \rangle$ is non-negative on Ω, satisfies (4.2), and

$$S = \{y \in \Omega : \tilde{v}(y) = 0\}.$$

These properties contradict Caffarelli's Theorem 3.6(i) as o is an extremal point of S, and in turn we conclude (4.3).

Next we show that

$$h_K \text{ is differentiable at any } u_0 \in S^{n-1} \backslash N(K, o). \tag{4.4}$$

We apply again the construction above for u_0. If $u \in \Omega_0$ and $z \in F(K, u)$, clearly K is C^1-smooth at z (i.e. $N(K, z)$ is a ray) by (4.3). Therefore, by (2.3), v is strictly convex on Ω and Caffarelli's Theorem 3.6(ii) yields that v is C^1 on Ω. In turn, we conclude (4.4).

In addition, $F(K,u)$ is a unique C^1-smooth point for $u \in \Omega_0$ [see (2.4)], yielding that $\Omega_* = \cup\{F(K,u) : u \in \Omega_0\}$ is an open subset of ∂K. Therefore $\Omega_* \subset X$, any point of Ω_* is C^1-smooth [by (2.3)] and Ω_* contains no segment [by (2.4)], completing the proof of Claim (i).

Claim (ii). We suppose that $o \in \partial K$ is C^1-smooth, and there exists $z \in \partial K$ such that K is not C^1-smooth at z. Claim (i) yields that $z \in X_0$, and hence $N(K,z) \subset N(K,o)$, which is a contradiction, verifying Claim (ii).

Claim (iii). This is a consequence of Lemma 4.1 and Claim (i).

Claim (iv). This is a consequence of Lemma 3.1, Claim (i) and Caffarelli [8].

□

Example 4.2 If $n \geq 2$ and $p \in (-n+2, 1)$, then there exists $K \in \mathcal{K}_0^n$ with C^1 boundary such that o lies in the relative interior of a facet of ∂K and $dS_{K,p} = f\, d\mathcal{H}^{n-1}$ for a strictly positive continuous $f : S^{n-1} \to \mathbb{R}$.

Let $q = (p+n-1)/(p+n-2)$. We have $q > 1$. Let

$$g(r) = \begin{cases} (r-1)^q & \text{when } r \geq 1; \\ 0 & \text{when } r \in [0,1); \end{cases}$$

and $\bar{g}(x_1,\dots,x_{n-1}) = g(\|(x_1,\dots,x_{n-1})\|)$. Let $K \in \mathcal{K}_0^n$ be such that $K \cap \{x : x_n \leq 1\} = \{x : 1 \geq x_n \geq \bar{g}(x_1,\dots,x_{n-1})\}$ and $\partial K \cap \{x : x_n > 0\}$ is a C^2 surface with Gauss curvature positive at every point. Clearly $K \cap \{x : x_n = 0\}$ is a $(n-1)$-dimensional face of K which contains o in its relative interior and has unit outer normal $(0,\dots,0,-1)$.

To prove that $dS_{K,p} = f\, d\mathcal{H}^{n-1}$ for a positive continuous $f : S^{n-1} \to \mathbb{R}$, it suffices to prove that there is a neighbourhood of the South pole where $dS_{K,p}/d\mathcal{H}^{n-1}$ is continuous and bounded from above and below by positive constants. Let h be the support function of K and, for $y \in \mathbb{R}^{n-1}$, let $v(y) = h(y,-1)$ be the restriction of h to the hyperplane tangent to S^{n-1} at the South pole. It suffices to prove that in a neighbourhood U of o, v satisfies the equation $v^{1-p} \det D^2 v = G$ with a function G which is bounded from above and below by positive constants.

If $y \in U \setminus \{o\}$ we have

$$v(y) = h(y,-1) = \langle (x', \bar{g}(x')), (y,-1) \rangle \quad \text{where} \quad D\bar{g}(x') = y. \tag{4.5}$$

If U is sufficiently small, then $v(y)$ depends only on $\|y\|$. Let $y = (z,0,\dots,0)$, with $z > 0$ small and let $r = 1 + (z/q)^{1/(q-1)}$. We have

$$D\bar{g}(r,0,\dots,0)) = (z,0,\dots,0)$$

and (4.5) gives

$$\begin{aligned} v(z,0,\dots,0) &= rq(r-1)^{q-1}-(r-1)^q \\ &= z+\frac{q-1}{q^{n-1+p}}z^{n-1+p}. \end{aligned}$$

(Note that $n-1+p>1$.) Clearly $v(0,\dots,0)=h(0,\dots,0,-1)=0$. When $z>0$, we have

$$\begin{aligned} v_{y_1y_1} &= \frac{q-1}{q^{n-1+p}}(n-1-p)(n-2-p)z^{n-3+p} \\ v_{y_iy_i} &= \frac{1}{z}+\frac{q-1}{q^{n-1+p}}(n-1-p)z^{n-3+p} \qquad \text{when } i\neq 1 \\ v_{y_iy_j} &= 0 \qquad \text{when } i\neq j, \end{aligned}$$

and, as $z\to 0^+$

$$v(z,0,\dots,0)^{1-p}\det D^2v(z,0,\dots,0)=c+o(1),$$

for a suitable constant $c>0$. This implies the existence of a function G positive and continuous on U such that

$$\mathcal{H}^{n-1}\big(N_v(\omega\cap\{v>0\})\big)=\int_{\omega\cap\{v>0\}} nG(y)v(y)^{p-1}\,\mathrm{d}y$$

for any Borel set $\omega\subset U$. To conclude the proof that v is a solution in the sense of Alexandrov of $v^{1-p}\det D^2v=G$ in U it remains to prove that $\mathcal{H}^{n-1}\big(\{y\in U: v(y)=0\}\big)=0$, but this is obvious since $\{y\in U: v(y)=0\}=\{o\}$.

We remark that h is not a solution of (1.4) because (2.11) fails.

5 Proofs of Theorem 1.2 and Corollary 1.4

Proof of Theorem 1.2 We may assume that $o\in\partial K$ since otherwise ∂K is C^1 by Theorem 1.1. Let $e\in N(K,o)\cap S^{n-1}$ be such that $\langle u,e\rangle>0$ for any $u\in N(K,o)\cap S^{n-1}$. Let v be defined on $\Omega=e^\perp$ as in Lemma 3.1 with $h=h_K$ and let $S=\{x\in e^\perp : v(x)=0\}$. We have

$$S+e=N(K,o)\cap(e^\perp+e), \tag{5.1}$$

by (2.2). If K is not C^1-smooth at o, then $\dim S\geq 1$ and, by Proposition 1.3, $p\geq n-4$ (note that here the dimension of the ambient space is $n-1$). This proves Theorem 1.2(i).

To prove Theorem 1.2(ii) we observe that

$$N_{h_K}(e+S)=\bigcup_{u\in N(K,o)} F(K,u)=X_0,$$

where X_0 is defined as in Theorem 1.1(i). The equality on the left in this formula follows by (2.4) and the equality on the right follows by Theorem 1.1(i). Thus,

$$N_v(S) = X_0|e^\perp,$$

and if $\mathcal{H}^{n-1}(X_0) = 0$, then $\mu_v(S) = 0$. We observe that S is compact, by (5.1), that v is locally strictly convex, by Theorem 1.1(i), and that $\dim S \le n-2$, by (1.3). Hence, Theorem 1.2(ii) follows by Corollary 3.5 and (5.1). □

Proof of Corollary 1.4 Claim (i) is an immediate consequence of (2.2), Proposition 1.3 and Lemma 3.1. This claim implies that when $n = 4$ or $n = 5$ and ∂K is not C^1 then $\dim N(K, o) = 2$. In this case $N(K, o) \cap S^{n-1}$ is a closed arc: let e_1 and e_2 be its endpoints. If $u \in N(K, o) \cap S^{n-1}$, $u \ne e_1$, $u \ne e_2$, then $F(K, u)$ is contained in the intersection of the two supporting hyperplanes $\{x \in \mathbb{R}^n : \langle x, e_i\rangle = h_K(e_i)\}$, $i = 1, 2$. Thus,

$$\mathcal{H}^{n-1}\Big(\bigcup\{F(K,u) : u \in N(K,o) \cap S^{n-1}, u \ne e_1, u \ne e_2\}\Big) = 0.$$

Therefore $\dim F(K, e_1) = n - 1$ or $\dim F(K, e_2) = n - 1$, because otherwise

$$\bigcup\{F(K,u) : u \in N(K,o) \cap S^{n-1}\},$$

which coincides with X_0 by Theorem 1.1 (i), has $(n - 1)$-dimensional Hausdorff measure equal to zero and ∂K is C^1 by Theorem 1.2 (ii). □

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