

Towards a logarithmic Brunn-Minkowski theory

K. B., Erwin Lutwak, Deane Yang, Gaoyong Zhang
Alfréd Rényi Institute of Mathematics and NYU-Poly

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Minkowski problem

- ▶ K, C - convex bodies in \mathbb{R}^n
- ▶ h_K - support function of K
- ▶ S_K - surface area measure of K on S^{n-1}
- ▶ L - linear subspace, $L \neq \{o\}, \mathbb{R}^n$
- ▶ μ - non-trivial Borel measure on S^{n-1}

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"Minkowski problem"

$\mu = S_K$ for some **unique** convex body K (up to translation) iff

1. $\mu(L \cap S^{n-1}) < \mu(S^{n-1})$ for any L with $\dim L = n - 1$
2. $\int_{S^{n-1}} u d\mu(u) = o$

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If μ is even, then the first condition is enough

Brunn-Minkowski inequality

$$\alpha, \beta > 0$$

$$\alpha K + \beta C = \{x \in \mathbb{R}^n : \langle u, x \rangle \leq \alpha h_K(u) + \beta h_C(u) \forall u \in S^{n-1}\}$$

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Brunn-Minkowski inequality $\alpha \in (0, 1)$

$$V(\alpha K + (1 - \alpha)C) \geq V(K)^\alpha V(C)^{1-\alpha}$$

with equality iff K and C are homothetic

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Minkowski inequality If $V(K) = V(C)$, then

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To solve the [Minkowski problem](#),

- ▶ Minimize $\int_{S^{n-1}} h_C d\mu$ under the condition $V(C) = 1$
- ▶ Uniqueness comes from the Minkowski inequality

Logarithmic Minkowski problem - Cone volume measure

$dV_K = \frac{1}{n} h_K dS_K$ - cone volume measure on S^{n-1} if $o \in \text{int}K$

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Theorem (Even logarithmic Minkowski problem, BLYZ)

Let μ be an even Borel measure on S^{n-1} .

$\mu = V_K$ for some o -symmetric convex body K iff

- (i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \mu(S^{n-1})$ for any $L \neq \{o\}, \mathbb{R}^n$
- (ii) If equality holds for some L , then $\text{supp } \mu \subset L \cup L'$ for some complementary L'

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Idea: Minimize $\int_{S^{n-1}} \log h_C d\mu$ under the condition $V(C) = 1$

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Conjecture (Uniqueness)

$V_K = V_C$ for o -symmetric convex bodies K and C with

$V(K) = V(C)$ iff K and C have dilated direct summands; namely,

$K = K_1 \oplus \dots \oplus K_m$ and $C = C_1 \oplus \dots \oplus C_m$ with $K_j = \lambda_j C_j$ for

$\lambda_1, \dots, \lambda_m > 0$.

Isotropic position of a measure on S^{n-1}

Theorem (BLYZ)

Let μ be a Borel probability measure on S^{n-1} . There exists $A \in \text{GL}(n)$ such that

$$\int_{S^{n-1}} \frac{Au}{\|Au\|} \otimes \frac{Au}{\|Au\|} d\mu(u) = \frac{1}{n} \text{Id}_n$$

iff

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- ▶ Sufficiency if $\mu(L \cap S^{n-1}) < \frac{\dim L}{n}$ is due to Klartag (supergaussian marginals of probability measures on \mathbb{R}^n)
- ▶ Discrete case is due to Carlen-Lieb-Loss (extremals for the Brascamp-Lieb inequality). See also Benneth-Carbery-Christ-Tao, Carlen&Cordero-Erausquin

Logarithmic Brunn-Minkowski inequality

$\alpha \in [0, 1]$, $o \in \text{int}K, \text{int}C$

$$\alpha K +_o (1 - \alpha)C = \{x \in \mathbb{R}^n : \langle u, x \rangle \leq h_K(u)^\alpha h_C(u)^{1-\alpha} \forall u \in S^{n-1}\}$$

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Conjecture (Logarithmic Brunn-Minkowski conjecture)

$\alpha \in (0, 1)$, K, C are o -symmetric

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with equality iff K and C have dilated direct summands.

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Conjecture (Logarithmic Minkowski conjecture)

For o -symmetric K, C , if $V(K) = V(C)$, then

$$\int_{S^{n-1}} \log h_C dV_K \geq \int_{S^{n-1}} \log h_K dV_K,$$

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- ▶ Interesting for any log-concave measure instead of volume
- ▶ $n = 2$ for volume (BLYZ)
- ▶ K and C are unconditional for any log-concave measure (Bollobás&Leader and Cordero-Erausquin&Frédérizi&Maurey on coordinatewise product)
- ▶ K and C are dilates for the gaussian measure (Cordero-Erausquin&Frédérizi&Maurey on B -conjecture)