Number of facets of Gaussian polytopes

Károly J. Böröczky
Alfréd Rényi Institute of Mathematics, Budapest
UPC, BarcelonaTech

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joint with Gábor Lugosi, Daniel Hug, Matthias Reitzner

Gaussian polytopes

- $X_1, \ldots, X_n$ independent standard Gaussian samples from $\mathbb{R}^d$
- $\varphi(x) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \exp\left(-\frac{1}{2}\|x\|^2\right)$
- n tends to infinity
- $\Pi_n = [X_1, \ldots, X_n]$ - Gaussian polytope
- $Q_n = [\pm X_1, \ldots, \pm X_n]$ - symmetric Gaussian polytope
- $f_k(P) =$number of k-faces of a polytope P

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The symmetric Gaussian polytope $Q_n = [\pm X_1, \dots, \pm X_n]$ is equivalent for us with AC^n, where C^n is the standard crosspolytope $[T^{n-1}, -T^{n-1}]$

Small n

There exists threshold $\varrho(\lambda)$ for $\lambda > 1$

- $\varrho(\lambda) \to 1$ if $\lambda \to 1$
- $\varrho(\lambda) \sim [2 \log \lambda]^{-1}$ if $\lambda \to \infty$

Theorem (Vershik-Sporyshev, 1992)

If $n/d \to \lambda$ and $k < \varrho(\lambda) \cdot d$, then

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Donoho-Tanner If $k < \varrho_s(n/d) \cdot d$ then $f_k(\Pi_n) = \binom{n}{k+1}$ with probability tending to one.

Very large n

- $\beta(T^k, T^{d-1})$ - inner angle of T^{d-1} at k-faces

Theorem

If d is fixed and n tends to infinity, then for any $k \leq d-1$, we have

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- Raynaud (1970) - $k = d-1$ (facets)
- Affentranger, Schneider (1992) - Random projection
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Böröczky-Henk (2003)

$$\beta(T^k, T^{d-1}) = \frac{(k+1)^{\frac{d-k-2}{2}} e^{\frac{d-3k-3}{2}}}{\sqrt{2}^{d-k} \sqrt{\pi}^{d-k-1} d^{\frac{d-k-2}{2}}} \cdot \left(1 + O\left(\frac{k^2+1}{d}\right)\right)$$

What does happen in between?

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There exist absolute constants $c_1, c_2 > 0$, such that

$$\left(c_1 \log\left(\frac{2n}{d}\right)\right)^{\frac{d-1}{2}} < \mathbb{E} f_{d-1}(Q_n) < \left(c_2 \log\left(\frac{2n}{d}\right)\right)^{\frac{d-1}{2}}$$

Facets

Theorem

If d tends to infinity, and $n \geq d^{(\log d)^3}$, then

$$\begin{aligned}\mathbb{E} f_{d-1}(\Pi_n) &= d^{\frac{-1}{2}} 2^d \pi^{\frac{d-1}{2}} (\log n)^{\frac{d-1}{2} - \frac{d}{4 \log n}} \\ &\quad \times \exp\left[\frac{d \log d}{2 \log n} - \frac{d \log(2\sqrt{\pi}/e)}{2 \log n} + O\left(\frac{d}{(\log n)^{3/2}}\right)\right]\end{aligned}$$

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- In the case $n = d^{f(d)d}$ where $f(d)$ tends to infinity with d, the asymptotic formula simplifies to

$$\mathbb{E} f_{d-1}(\Pi_n) \sim d^{\frac{-1}{2}} 2^d \pi^{\frac{d-1}{2}} (\log n)^{\frac{d-1}{2}}$$

Idea of the argument

$$\Phi(z) := \int_{-\infty}^{z} \varphi(dt) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}\|t\|^2}\, dt \quad \text{for } z \in \mathbb{R}$$

Lemma

If $n > 30d \log d$, then

$$\mathbb{E} f_{d-1}(\Pi_n) \sim \binom{n}{d} \frac{n-d}{\pi\sqrt{d}} \int_{1}^{\infty} \Phi(z)^{n-d-1} z^{-1} e^{-\frac{d+1}{2} z^2}\, dz$$

Main estimate

Lemma

There exists λ_0 *and* d_0 *such that if* $d \geq d_0$ *and* $\lambda \geq \lambda_0$ *for* $\lambda = \log \frac{p}{d}$, *then*

$$\int_1^\infty \Phi(z)^p z^{-1} e^{-\frac{d+1}{2} z^2} \, dz =$$

$$\left(1 + O(e^{-\sqrt{\log d}} + \tfrac{\log \lambda}{\lambda})\right) \cdot 2^d \pi^{\frac{d+1}{2}} \lambda^{\frac{d-1}{2}}$$

$$\times \exp\left[-\frac{d \log \lambda}{4\lambda} - \frac{d \log(2\sqrt{\pi}/e)}{2\lambda} + O\left(\frac{d}{\lambda^{3/2}}\right)\right]$$

$$\times \frac{p!d!}{(p+d+1)!}.$$

where $\frac{\log(2\sqrt{\pi}/e)}{2} = 0.1327\ldots$.