CONE-VOLUME MEASURE AND STABILITY

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ABSTRACT. We prove stability results for two central inequalities involving the cone-volume measure of a centered convex body: the subspace concentration conditions and the U-functional/volume inequality.

1. INTRODUCTION

Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ having non-empty interiors, i.e., $K \in \mathcal{K}^n$ is a convex compact subset of the n-dimensional Euclidean space $\mathbb{R}^n$ with $\operatorname{int}(K) \neq \emptyset$. As usual, we denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^n \times \mathbb{R}^n$ with associated Euclidean norm $\| \cdot \|$, and $S^{n-1} \subset \mathbb{R}^n$ denotes the $(n-1)$-dimensional unit sphere, i.e., $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

For $K \in \mathcal{K}^n$ we write $S_K(\cdot)$ and $h_K(\cdot)$ to denote its surface area measure and support function, respectively, and ν_K to denote the Gauß map assigning the outer unit normal $\nu_K(x)$ to an $x \in \partial_* K$, where $\partial_* K$ consists of all points in the boundary ∂K of K having a unique outer normal vector. If the origin o lies in $K \in \mathcal{K}^n$, the *cone-volume measure of K* on S^{n-1} is given by

$$(1.1) \qquad \mathrm{V}_K(\omega) = \int_\omega \frac{h_K(u)}{n} \, \mathrm{d}S_K(u) = \int_{\nu_K^{-1}(\omega)} \frac{\langle x, \nu_K(x) \rangle}{n} \, \mathrm{d}\mathcal{H}_{n-1}(x),$$

where $\omega \subseteq S^{n-1}$ is a Borel set and, in general, $\mathcal{H}_k(x)$ denotes the k-dimensional Hausdorff measure. Instead of $\mathcal{H}_n(\cdot)$, we also write $\mathrm{V}(\cdot)$ for the n-dimensional volume.

The name cone-volume measure stems from the fact that if K is a polytope with facets $F_1, \ldots, F_m$ and corresponding outer unit normals $u_1, \ldots, u_m$, then

$$(1.2) \qquad \mathrm{V}_K(\omega) = \sum_{i=1}^m \mathrm{V}([o, F_i]) \delta_{u_i}(\omega).$$

Here δ_{u_i} is the Dirac delta measure on S^{n-1} concentrated at u_i, and for $x_1, \ldots, x_m \in \mathbb{R}^n$ and subsets $S_1, \ldots, S_l \subseteq \mathbb{R}^n$ we denote the convex hull of the set $\{x_1, \ldots, x_m, S_1, \ldots, S_l\}$ by $[x_1, \ldots, x_m, S_1, \ldots, S_l]$. With this notation $[o, F_i]$ is the cone with apex o and basis F_i.

In recent years, cone-volume measures have appeared and were studied in various contexts, see, e.g., F. Barthe, O. Guedon, S. Mendelson and A. Naor [3], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [8, 9], R. Gardner, D. Hug and W. Weil [20], M. Gromov and V.D. Milman [22], M. Ludwig [35], M. Ludwig and M. Reitzner [36], E. Lutwak, D. Yang and G. Zhang [41], A. Naor [44], A. Naor and D. Romik [45], G. Paouris and E. Werner [46], A. Stancu [51], G. Zhu [54, 55], K.J. Böröczky and P. Hegedűs [5], L. Ma [43], Y. Huang, E. Lutwak, D. Yang and G. Zhang [32].

In particular, cone-volume measures are the subject of the *logarithmic Minkowski problem*. This is the particular interesting case $p = 0$ of the general L_p-Minkowski problem which is at the core of the L_p-Brunn-Minkowski theory, one of the cornerstones of modern convex geometry. The L_p-Minkowski problem asks for a characterization of the L_p surface area measure

$$S_K^{(p)}(\omega) = \int_\omega h_K(u)^{1-p} \mathrm{d}S_K(u),$$

of a convex body K containing o in its interior among the finite Borel measures on the sphere. Here $\omega \subseteq S^{n-1}$ is a Borel set and $p \geq 0$. We refer to [31, 33, 37, 38, 40, 42, 48, 56] and the reference within for detailed information about the problem. Here we just mention that for $p = 1$, the L_1-Minkowski problem is the classical Minkowski problem, which was solved for polytopes and particular convex bodies by Minkowski, and in full generality by Aleksandrov, and Fenchel and Jessen: μ is the surface area measure of a convex body if and only if μ is positive on each open hemisphere and

$$\int_{S^{n-1}} u \, \mathrm{d}\mu(u) = o.$$

Moreover, such a convex body is unique up to translations.

Roughly speaking, for $p > 1$ and $p \neq n$ or $p > 0$ and even measures, the only restriction on a measure being the L_p surface area measure of a convex body is that it has to be positive on each open hemisphere. The case $p = 0$, corresponding to the $\mathrm{SL}(n)$ invariant cone-volume measure $V_K(\omega) = (1/n)S_K^{(0)}(\omega)$, is different. Here one needs, as in the classical case $p = 1$, additional restrictions on the measure. Of particular and central importance is here the *subspace concentration condition* which was introduced K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang and by which they gave a complete solution of the logarithmic Minkowski problem for even measures [9].

We say that a finite Borel measure μ on S^{n-1} satisfies the *subspace concentration condition* if for any linear subspace $L \subseteq \mathbb{R}^n$

$$(1.3) \qquad \mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \mu(S^{n-1}),$$

and equality in (1.3) for some L implies the existence of a complementary linear subspace $\widetilde{L}$ such that

$$(1.4) \qquad \mu(\widetilde{L} \cap S^{n-1}) = \frac{\dim \widetilde{L}}{n} \mu(S^{n-1}),$$

and hence $\operatorname{supp} \mu \subseteq L \cup \widetilde{L}$, i.e., the support of the measure "lives" in $L \cup \widetilde{L}$.

Via this condition, cone-volume measures of origin-symmetric convex bodies have been completely characterized by K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang.

Theorem I ([9, Theorem 1.1]). *A non-zero finite even Borel measure on S^{n-1} is the cone-volume measure of an origin-symmetric convex body if and only if it satisfies the subspace concentration condition.*

In the planar case, this result was proved earlier for discrete measures, i.e., for polygons, by A. Stancu [49, 50]. For cone-volume measures of origin-symmetric polytopes, the necessity of (1.3) was independently shown by M. Henk, A. Schürmann and J.M.Wills [29] and B. He, G. Leng and K. Li [28].

We recall that the centroid of a k-dimensional convex compact set $M \subset \mathbb{R}^n$ is defined as

$$\operatorname{cen}(M) = \mathcal{H}_k(M)^{-1} \int_M x \, \mathrm{d}\mathcal{H}_k(x),$$

and a convex body will be called *centered* if $\operatorname{cen}(K) = o$.

Centered bodies seem to be the right and natural class of convex bodies in order to extend Theorem I to general convex bodies, and in a recent paper the authors proved the necessity of the subsapce concentration condition for this class.

Theorem II ([7, Theorem 1.1]). *Let $K \in \mathcal{K}^n$ be centered. Then its cone-volume measure V_K satisfies the subspace concentration condition.*

For polytopes this result was proved by M. Henk and E. Linke [30]. If K is not centered, then the subspace concentration condition may not hold any more. In fact, it was recently shown by G. Zhu [54] that for $u_1, \ldots, u_m \in S^{n-1}$ in general position, $m \geq n + 1$, and arbitrary positive numbers $\gamma_1, \ldots, \gamma_m$ there always exists a (not necessarily centered) polytope $P \in \mathcal{K}^n$ with outer unit normals u_i and $V_P(\{u_i\}) = \gamma_i$, $1 \leq i \leq m$. In other words, Zhu settled the logarithmic Minkowski problem for discrete measures whose support is in general position. In [6] this result was unified with the sufficiency part of the subspace concentration condition in the even discrete case by introducing the notation of *essential* subspaces. For a given finite Borel measure μ on S^{n-1} a subspace L, $1 \leq \dim L \leq n - 1$, is called essential if $L \cap \operatorname{supp}\mu$ is not concentrated on any closed hemisphere of $L \cap \operatorname{supp}\mu$. K.J. Böröczky, P. Hegedűs and G. Zhu [6] proved that every finite discrete measure on S^{n-1} which satisfies the subspace concentration condition with respect to essential subspaces is the cone-volume measure of a polytope. In the case $n = 2$, this result was obtained before by A. Stancu [50]. In general,

however, the centroid of such a polytope P is not the origin, and the characterization of cone-volume measures of general polytopes or convex bodies is still a challenging and important problem.

For a convex body K containing the origin in its interior, E. Lutwak, D. Yang and G. Zhang [39] defined the $\mathrm{SL}(n)$ invariant quantity $\mathrm{U}(K)$ as an integral over subsets $(u_1, \ldots, u_n) \in S^{n-1} \times \cdots \times S^{n-1}$, by

$$\mathrm{U}(K) = \left(\int_{u_1 \wedge \ldots \wedge u_n \neq 0} \mathrm{d}V_K(u_1) \cdots \mathrm{d}V_K(u_n) \right)^{\frac{1}{n}},$$

where $u_1 \wedge \ldots \wedge u_n \neq 0$ means that the vectors $u_1, \ldots, u_n$ are linearly independent. The U-functional has been proved useful in obtaining strong inequalities for the volume of projection bodies [39]. For information on projection bodies we refer to [18, 27, 48], and for more information on the importance of centro-affine functionals we refer to [25, 26, 34, 36] and the references within.

We readily have $\mathrm{U}(K) \leq V(K)$, and equality holds if and only if $V_K(L \cap S^{n-1}) = 0$ for any non-trivial subspace of $\mathbb{R}^n$ according to K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [10]. As a consequence of Theorem II it was shown in [7] that

Theorem III ([7, Corollary 1.3]). *Let $K \in \mathcal{K}^n$ be centered. Then*

$$U(K) \geq \frac{(n!)^{1/n}}{n} V(K),$$

with equality if and only if K is a parallelepiped.

The statement was conjectured in [10]. It was proved for polytopes in [30], where the special cases if K is an origin-symmetric polytope, or if $n = 2, 3$ were verified by B. He, G. Leng and K. Li [28], and G. Xiong [53], respectively.

Here we present stronger stability versions of Theorem II and Theorem III. Stability results are an important issue in many areas of mathematics since they provide a quantitative characterization of the extremal solution of inequalities. Prominent examples are, e.g., isoperimetric inequalities ([14, 16]), the Brunn-Minkowski inequality ([15, 21]), the Orlicz-Petty projection inequality ([4]), Sobolev ([12, 17]) and Gagliardo-Nirenberg ([11]) inequalities.

In order to present our stability results we need two notions of distance between the "shapes" of two convex bodies. Let $K, M \in \mathcal{K}^n$, and let $K' = K - c(K)$, $M' = M - c(M)$ be their translates whose centroids are the origin. Then we define

$$(1.5) \qquad \delta_{\mathrm{hom}}(K, M) = \min\{\lambda \geq 0 : \exists t > 0, \ M' \subset t \, K' \subset \mathrm{e}^\lambda M'\},$$

$$(1.6) \qquad \delta_{\mathrm{vol}}(K, M) = \mathrm{V}\left(\left[\mathrm{V}(M)^{-1/n} M' \right] \Delta \left[\mathrm{V}(K)^{-1/n} K' \right] \right),$$

where $A \Delta B$ denotes the symmetric difference of two sets, i.e., $A \Delta B = A \setminus B \cup B \setminus A$. Then both distances δ_{hom} and δ_{vol} are metrics on the space

of centered convex bodies in $\mathbb{R}^n$ whose volumes are 1. We remark that the equality case in the subspace concentration condition for the cone-volume measure V_K of a convex body K, i.e.,

$$\mathrm{V}_K(L \cap S^{n-1}) = \frac{\dim L}{n} \mathrm{V}(K) \text{ and } \mathrm{V}_K(\widetilde{L} \cap S^{n-1}) = \frac{\dim \widetilde{L}}{n} \mathrm{V}(K)$$

for complentary proper subspaces $L, \widetilde{L}$ is equivalent to a representation of K as $K = M_1 + M_2$ where $M_1 \subset L^\perp$, $M_2 \subset \widetilde{L}^\perp$ are complementary convex bodies, i.e., they are contained in complementary linear spaces. Here, as usual, $L^\perp$ denotes the orthogonal complement a linear subspace.

Theorem 1.1. *There exist constants $\varepsilon_0, \gamma_h, \gamma_v > 0$ depending only on the dimension n, such that, if $K \in \mathcal{K}^n$ is centered and*

$$\mathrm{V}_K(L \cap S^{n-1}) > \frac{d - \varepsilon}{n} \mathrm{V}(K)$$

for a proper linear subspace L with $\dim L = d$ and $\varepsilon \in (0, \varepsilon_0)$, then there exist $(n-d)$-dimensional compact convex set $C \subset L^\perp$, and complementary d-dimensional compact convex set M such that

$$\delta_{\mathrm{hom}}(K, C + M) \leq \gamma_h \varepsilon^{1/(5n)} \quad and \quad \delta_{\mathrm{vol}}(K, C + M) \leq \gamma_v \varepsilon^{1/5}.$$

Observe that the range of ε, i.e., ε_0, in Theorem 1.1 has to depend on the dimension. For if, let $K \in \mathcal{K}^n$ be a centered simplex and let L be generated by d outer normals of the simplex, $d \in \{1, \ldots, n-1\}$. Then we have $\mathrm{V}_K(L \cap S^{n-1}) = \frac{d}{n+1} \mathrm{V}(K)$.

Actually, if L is 1-dimensional, then a more precise version of Theorem 1.1 holds.

Theorem 1.2. *There exist constants $\tilde{\varepsilon}_0, \tilde{\gamma}_h, \tilde{\gamma}_v > 0$ depending only on the dimension n, such that, if $K \in \mathcal{K}^n$ is centered and*

$$\mathrm{V}_K(L \cap S^{n-1}) > \frac{1 - \varepsilon}{n} \mathrm{V}(K)$$

for a linear subspace L with $\dim L = 1$ and $\varepsilon \in (0, \tilde{\varepsilon}_0)$, then there exist an $(n-1)$-dimensional compact convex set $C \subset L^\perp$ with $c(C) = o$, and $x, y \in \partial K$ such that $y = -e^s x$ where $|s| < \tilde{\gamma}_v \varepsilon^{\frac{1}{6}}$, $[x, y] + C \subset K$, and

$$K \subset [x, y] + (1 + \tilde{\gamma}_h \varepsilon^{\frac{1}{6n}})C \quad and \quad \mathrm{V}(K) \leq (1 + \tilde{\gamma}_v \varepsilon^{\frac{1}{6}})\mathrm{V}([x, y] + C).$$

We use this theorem in order to deduce the following stability version of Theorem III.

Theorem 1.3. *There exist constants $\varepsilon_*, \gamma_*, \overline{\gamma}, \widetilde{\gamma} > 0$ depending only on n with the following property: for each $\varepsilon \in (0, \varepsilon_*)$ and centered $K \in \mathcal{K}^n$ with*

$$\mathrm{U}(K) \leq (1 + \varepsilon) \frac{(n!)^{1/n}}{n} \mathrm{V}(K)$$

there exists a parallelepiped P such that

(1) $(1 - \overline{\gamma}\varepsilon^{\frac{1}{6n}})P \subset K \subset P$;

(2) $\mathrm{V}(P \backslash K) \le \widetilde{\gamma} \varepsilon^{\frac{1}{6}} \mathrm{V}(K)$;

(3) If F is a facet of P then $\mathcal{H}_{n-1}(F \cap K) \ge (1 - \gamma_* \varepsilon^{\frac{1}{6}}) \mathcal{H}_{n-1}(F)$.

The paper is organized as follows. In the next section we collect some basic facts and notations from convexity and we also state some of the main lemmas of the proof of Theorem II which we need for the proof of the stability version. The proofs of Theorem 1.1, 1.2 are given in Section 6 and are prepared in sections 3-5. In Section 3 we state properties of the symmetric volume difference, in Section 4 we study consequences of the stability of the Brunn-Minkowski inequality and in Section 5 some more properties of a certain log-concave functional which is of central interest in our investigations are presented. Finally, in Section 7 we prove Theorem 1.3.

Acknowledgements. We are grateful to Rolf Schneider for various ideas shaping this paper. We also acknowledge fruitful discussions with Daniel Hug and David Preiss about the Gauß-Green theorem.

2. Preliminaries

Good general references for the theory of convex bodies are provided by the books of Gardner[18], Gruber[23], Schneider[48] and Thompson[52].

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of convex body $K \in \mathcal{K}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{\langle x, y \rangle : y \in K\}.$$

A boundary point $x \in \partial K$ is said to have a unit outer normal (vector) $u \in S^{n-1}$ provided $\langle x, u \rangle = h_K(u)$. A point $x \in \partial K$ is called singular if it has more than one unit outer normal, and $\partial_* K$ is the set of all non-singular boundary points. It is well known that the set of singular boundary points of a convex body has 0 $\mathcal{H}_{n-1}$ measure. For each Borel set $\omega \subset S^{n-1}$, the inverse spherical image of ω is the set of all points of ∂K which have an outer unit normal belonging to ω. Since the inverse spherical image of ω differs from $\nu_K^{-1}(\omega) \subseteq \partial_* K$ by a 0 $\mathcal{H}_{n-1}$ measure set, we will often make no distinction between the two sets.

For $K \in \mathcal{K}^n$ the Borel measure S_K on S^{n-1} given by

$$S_K(\omega) = \mathcal{H}_{n-1}(\nu_K^{-1}(\omega))$$

is called the (Aleksandrov-Fenchel-Jessen) surface area measure. Observe that

$$\mathrm{V}(K) = \mathrm{V}_K(S^{n-1}) = \int_{S^{n-1}} \frac{h_K(u)}{n} d\, S_K(u).$$

As usual, for two subsets $C, D \subseteq \mathbb{R}^n$ and reals $\nu, \mu \ge 0$ the Minkowski combination is defined by

$$\nu\, C + \mu\, D = \{\nu\, c + \mu\, d : c \in C,\, d \in D\}.$$

By the celebrated Brunn-Minkowski inequality we know that the n-th root of the volume of the Minkowski combination is a concave function. More

precisely, for two convex compact sets $K_0, K_1 \subset \mathbb{R}^n$ and for $\lambda \in [0, 1]$ we have

$$(2.1) \qquad V((1 - \lambda) K_0 + \lambda K_1)^{1/n} \geq (1 - \lambda) V(K_0)^{1/n} + \lambda V(K_1)^{1/n}$$

with equality for some $0 < \lambda < 1$ if and only if K_0 and K_1 lie in parallel hyperplanes or are homothetic, i.e., there exist $t \in \mathbb{R}^n$ and $\mu \geq 0$ such that $K_1 = t + \mu K_0$ (see also [19]).

Let $f : C \to \mathbb{R}_{>0}$ be a positive function on an open convex subset $C \subset \mathbb{R}^n$ with the property that there exists a $k \in \mathbb{N}$ such that $f^{1/k}$ is concave. Then by the (weighted) arithmetic-geometric mean inequality

$$\begin{aligned} f((1 - \lambda) x + \lambda y) &= \left(f^{1/k}((1 - \lambda) x + \lambda y) \right)^k \\ &\geq \left((1 - \lambda) f^{1/k}(x) + \lambda f^{1/k}(y) \right)^k \\ &\geq f^{1-\lambda}(x) \cdot f^\lambda(y). \end{aligned}$$

This means that f belongs to the class of log-concave functions which by the positivity of f is equivalent to

$$\ln f((1 - \lambda) x + \lambda y) \geq (1 - \lambda) \ln f(x) + \lambda \ln f(y)$$

for $\lambda \in [0, 1]$. Hence, for all $x, y \in C$ there exists a subgradient $g(y) \in \mathbb{R}^n$ such that (cf., e.g., [47, Sect. 23])

$$(2.2) \qquad \ln f(x) - \ln f(y) \leq \langle g(y), x - y \rangle.$$

If f is differentiable at y, the subgradient is the gradient of $\ln f$ at y, i.e., $g(y) = \nabla \ln f = \frac{1}{f(y)} \nabla f(y)$.

For a subspace $L \subseteq \mathbb{R}^n$, let $L^\perp$ be its orthogonal complement subspace, and for $X \subseteq \mathbb{R}^n$ we denote by $X|L$ its orthogonal projection onto L, i.e., the image of X under the linear map forgetting the part of X belonging to $L^\perp$.

Here, for a convex body $K \in \mathcal{K}^n$ and a d-dimensional subspace L, $1 \leq d \leq n - 1$, we are interested in the function measuring the volume of K intersected with planes parallel to $L^\perp$, i.e., in the function

$$(2.3) \qquad f_{K,L} : L \to \mathbb{R}_{\geq 0} \text{ with } x \mapsto \mathcal{H}_k(K \cap (x + L^\perp)),$$

where $k = n - d$ is the dimension of $L^\perp$. By the Brunn-Minkowski inequality and the remark above, $f_{K,L}$ is a log-concave on function on $K|L$ which is positive at least in the relative interior of $K|L$ (cf. [1]). $f_{K,L}$ is also called the k-dimensional X-ray of K parallel to $L^\perp$ (cf. [18]). By well-known properties of (log-)concave functions we have (see, e.g., [47, 48])

Proposition 2.1.

i) $f_{K,L}$ is continuous on $\text{int}\,(K)|L$. Moreover, $f_{K,L}$ is Lipschitzian on any compact subset of $(\text{int}\,K)|L$.

ii) $f_{K,L}$ is on $\text{int}\,(K)|L$ almost everywhere differentiable, i.e., there exists a dense subset $D \subseteq \text{int}\,(K)|L$, where $\nabla f_{K,L}$ exists.

Now let $K \in \mathcal{K}^n$ be centered and L be a d-dimensional linear subspace. In view of Fubini's theorem we have

$$(2.4) \qquad o = o|L = \int_K (x|L) \, \mathrm{d}\mathcal{H}_n(x) = \int_{K|L} f_{K,L}(\hat{x}) \, \hat{x} \, \mathrm{d}\mathcal{H}_d(\hat{x}),$$

which means that $f_{K,L}$ is a centered function. The core ingredients for the proof of Theorem II are the next two lemmas.

Lemma 2.2 ([7, Lemma 3.3]). *Let $K \in \mathcal{K}^n$ with $o \in \operatorname{int} K$ and let L be a d-dimensional linear subspace, then*

$$n \, V_K(L \cap S^{n-1}) = d \, V(K) + \int_{K|L} \langle \nabla f_{K,L}(x), x \rangle \, d\mathcal{H}_d(x).$$

Lemma 2.3 ([7, Lemma 3.4]). *Let $K \in \mathcal{K}^n$ be centered and let L be a d-dimensional linear subspace, then*

$$\int_{K|L} \langle \nabla f_{K,L}(x), x \rangle \, d\mathcal{H}_d(x) \leq 0,$$

with equality if and only if $f_{K,L}$ is constant on $K|L$.

3. Some properties of the symmetric volume distance

First we show that the distance δ_{hom} can be estimated in terms of δ_{vol}. These types of estimates have been around, only we were not able to locate them in the form we need.

Lemma 3.1. *Let $K \in \mathcal{K}^n$ with $c(K) = o$.*
   (i) *If $Q \subset K$ is a convex body with $V(K \backslash Q) \leq t \, V(K)$ for $t \in (0, \frac{1}{e})$, then $(1 - (et)^{1/n})K \subset Q$.*
   (ii) *If Q is a convex body with $V(K \Delta Q) \leq t V(K)$ for $t \in (0, \frac{1}{4^n e})$, then $(1 - (et)^{1/n})K \subset Q \subset (1 + 4(et)^{1/n})K$.*

Proof. The main tool is the following result due to B. Grünbaum [24]. If $M \in \mathcal{K}^n$, and H^+ is a half space containing $c(M)$, then

$$(3.1) \qquad\qquad V(M \cap H^+) \geq V(M)/e.$$

To prove (i), let $\lambda = 0$ if $o \notin \operatorname{int} Q$, and let $\lambda > 0$ be maximal with the property that $\lambda K \subset Q$ otherwise. In addition, let $x = o$ if $o \notin \operatorname{int} Q$, and let x be a common boundary point of Q and λK otherwise. Therefore, there exists a half space H_1^+ such that x lies on its boundary, and $H_1^+ \cap \operatorname{int} Q = \emptyset$. Now there exists a $y \in K$ such that $x = \lambda y$, and hence x is the centroid of $x + (1 - \lambda)K = \lambda y + (1 - \lambda)K \subset K$. It follows from (3.1) that

$$t V(K) \geq V(H_1^+ \cap K) \geq V(H_1^+ \cap (x + (1 - \lambda)K)) \geq V((1 - \lambda)K)/e,$$

and thus $t \geq \frac{(1-\lambda)^n}{e}$.

To prove (ii), we observe that $\lambda K \subset Q$ for $\lambda = 1 - (et)^{1/n}$ by (i). We may assume that $Q \backslash K \neq \emptyset$, and let $\mu > 1$ be minimal with the property that

$Q \subset \mu K$. For a common boundary point z of Q and μK, let $w \in K$ such that $z = \mu w$. In particular, w is the centroid of

$$w + \frac{\lambda(\mu - 1)}{\mu} K \subset \frac{1}{\mu} z + \frac{\mu - 1}{\mu} Q \subset Q.$$

In addition there exists a half space H_2^+ such that w lies on its boundary, and $H_2^+ \cap \mathrm{int}\, K = \emptyset$. We deduce again from (3.1) that

$$t \mathrm{V}(K) \geq \mathrm{V}(H_2^+ \cap Q) \geq V\left(H_2^+ \cap \left(w + \frac{\lambda(\mu - 1)}{\mu} K\right)\right) \geq \frac{\lambda^n(\mu - 1)^n}{\mu^n e} \mathrm{V}(K).$$

Now $t < \frac{1}{4^n e}$ yields that $\lambda > \frac{1}{2}$ and $2(e\,t)^{1/n} < \frac{1}{2}$, which in turn implies that $\mu \leq (1 - 2(et)^{1/n})^{-1} < 1 + 4(et)^{1/n}$. $\qquad\square$

Corollary 3.2. *Let $K, Q \in \mathcal{K}^n$. Then*

$$\begin{aligned} \delta_{\mathrm{hom}}(K, Q) &\leq 12\, \delta_{\mathrm{vol}}(K, Q)^{1/n} & \text{if } \delta_{\mathrm{vol}}(K, Q) < \tfrac{1}{4^n e}; \\ \delta_{\mathrm{vol}}(K, Q) &\leq 3n\, \delta_{\mathrm{hom}}(K, Q) & \text{if } \delta_{\mathrm{hom}}(K, Q) < \tfrac{1}{2n}. \end{aligned}$$

Proof. We will use the fact that $1 + s < e^s < 1 + 2s$ and $1 - s < e^{-s} < 1 - \frac{s}{2}$ for $s \in (0, 1)$.

Due to the translation and scaling invariance of the distances $\delta_{\mathrm{vol}}(\cdot, \cdot), \delta_{\mathrm{hom}}(\cdot, \cdot)$ we may assume that $c(K) = c(Q) = o$, and $\mathrm{V}(K) = \mathrm{V}(Q) = 1$. In particular, $\mathrm{V}(K \Delta Q) = \delta_{\mathrm{vol}}(K, Q)$, and hence the estimates for the exponential function and Lemma 3.1 yield with $s = \delta_{\mathrm{vol}}(K, Q)$ that

$$e^{-2e^{1/n}s^{1/n}} K \subset (1 - (se)^{\frac{1}{n}})K \subset Q \cap K \subset Q.$$

Using the analogous formula $e^{-2e^{1/n}s^{1/n}} Q \subset K$, we conclude the first estimate.

For the second estimate, let $t = \delta_{\mathrm{hom}}(K, Q)$. It follows that $e^{-t} K \subset Q \subset e^t K$, thus $\mathrm{V}(K \Delta Q) \leq e^{nt} - e^{-nt} < 3nt$. $\qquad\square$

Our next goal is Lemma 3.4 stating that one does not need to insist on the common centroid in the definition of δ_{vol}. We prepare the argument by the following observation for which we denote by $\|x\|_{K-K} = \min\{\rho \geq 0 : x \in \rho(K - K)\}$ the norm induced by the difference body $K - K$.

Lemma 3.3. *Let $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. Then*

$$\mathrm{V}(K \Delta (x + K)) \leq 2n\|x\|_{K-K}\mathrm{V}(K).$$

Proof. We may assume that $x \neq o$. Let $y, z \in K$ such that $x = \|x\|_{K-K}(y - z)$, and hence

$$\|x\|_{K-K} = \|x\|/\|y - z\|.$$

Applying Steiner symmetrization with respect to the hyperplane $x^\perp$ shows that

$$\mathrm{V}(K) \geq \frac{\|y - z\|}{n} \mathcal{H}_{n-1}(K|x^\perp).$$

We deduce by Fubini's theorem that

$$V(K\Delta(x+K)) \le 2\|x\|\mathcal{H}_{n-1}(K|x^\perp) \le 2n\|x\|_{K-K}V(K).$$

$\square$

Lemma 3.4. *Let $K, Q \in \mathcal{K}^n$ with $V(K\Delta Q) \le tV(K)$ for $t \in (0, \frac{1}{4^n e})$. Then*

$$\|c(Q) - c(K)\|_{K-K} \le 4nt \quad \text{and} \quad \delta_{\text{vol}}(K,Q) \le 9n^2 t.$$

Proof. We may assume that $V(K) = 1$, $c(K) = o$ and that the Löwner ellipsoid E, i.e., the minimal volume ellipsoid containing $K-K$, is a ball (see, e.g., [23]). In particular, $n^{-1/2}E \subset K - K \subset E$, and the Brunn-Minkowski and Rogers-Shephard theorems yield that $2^n \le V(K-K) \le \binom{2n}{n}$ (cf. [48, Theorem 10.4]). Since the volume of a centrally symmetric convex body over the volume of its Löwner ellipsoid is at least $2^n/(n!V(B^n))$ according to K. Ball [2], we have

$$2^n \le V(E) \le \binom{2n}{n}\frac{n!}{2^n}V(B^n) < \sqrt{3} \cdot \frac{2^n n^n}{e^n}V(B^n).$$

It follows that

$$(3.2) \qquad \frac{2}{\sqrt{e\pi}}B^n \subset K - K \subset nB^n \quad \text{and thus} \quad \frac{1}{n}\|x\| \le \|x\|_{K-K} \le 2\|x\|.$$

Therefore, to prove Lemma 3.4, it is sufficient to verify the corresponding estimate for $\|c(Q)\|$.

If $c(Q) = o$, then we are done, otherwise let $u = c(Q)/\|c(Q)\|$. We have $Q \subset 2K \subset 2nB^n$ by Lemma 3.1 and (3.2), and $V(Q) \ge 1 - t$ implies $V(Q)^{-1} < 2$. By (3.2) we also have

$$\|c(Q)\|_{K-K} \le 2\|c(Q)\| = 2V(Q)^{-1}\langle u, c(Q)\rangle = 2V(Q)^{-1}\left\|\int_Q \langle u, x\rangle\, dx\right\|,$$

and since $c(K) = o$ we get

$$\|c(Q)\|_{K-K} \le 2V(Q)^{-1}\left\|\int_Q \langle u, x\rangle\, dx\right\|$$

$$(3.3) \qquad\qquad = 2V(Q)^{-1}\left\|\int_{Q\setminus K} \langle u, x\rangle\, dx - \int_{K\setminus Q} \langle u, x\rangle\, dx\right\|$$

$$= 4\int_{K\Delta Q} |\langle u, x\rangle|\, dx \le 4nt.$$

Let $K' = K + c(Q)$, thus Lemma 3.3 and (3.3) imply that $V(K\Delta K') \le 8n^2 t$. We observe that $Q' = c(Q) + V(Q)^{-1/n}(Q - c(Q))$ satisfies $c(Q') = c(Q)$, $V(Q') = 1$, and $V(Q'\Delta Q) \le t$ by $1 - t \le V(Q) \le 1 + t$ (cf. Lemma 3.1). Therefore

$$\delta_{\text{vol}}(K,Q) = V(K'\Delta Q') \le V(K'\Delta K) + V(K\Delta Q) + V(Q\Delta Q') < 9n^2 t.$$

$\square$

4. Some consequences of the stability of the Brunn-Minkowski inequality

Concerning the Brunn-Minkowski theory, including the properties of mixed volumes, the main reference is R. Schneider [48]. We use the Brunn-Minkowski theory in $L^\perp$ in the terminology of Theorem 1.1, whose dimension is $k = n - d$. For $k, m \geq 1$, let

$$I_m^k = \{(i_1, \ldots, i_m) : i_j \in \mathbb{N}, \ j = 1, \ldots, m \text{ and } i_1 + \ldots + i_m = k\}.$$

For compact convex sets $C_1, \ldots, C_m$ in $\mathbb{R}^k$ and $(i_1, \ldots, i_m) \in I_m^k$, the non-negative mixed volumes $V(C_1, i_1; \ldots; C_m, i_m)$ were defined by H. Minkowski in a way such that if $\alpha_1, \ldots, \alpha_m \geq 0$, then

$$(4.1) \quad \mathcal{H}_k\left(\sum_{j=1}^m \alpha_j C_j\right) = \sum_{(i_1,\ldots,i_m) \in I_m^k} \frac{k!}{i_1, \ldots, i_k}$$
$$V(C_1, i_1; \ldots; C_m, i_m)\alpha_1^{i_1} \cdot \ldots \cdot \alpha_m^{i_m}.$$

The mixed volume $V(C_1, i_1; \ldots; C_m, i_m)$ actually depends only on the C_j with $i_j > 0$, does not depend on the order how the pairs C_j, i_j are indexed, and we frequently ignore the pairs C_j, i_j with $i_j = 0$. We have $V(C_1, k) = \mathcal{H}_k(C_1)$, and $V(C_1, i_1; \ldots; C_m, i_m) > 0$ if each C_j is k-dimensional. It follows by the Alexandrov-Fenchel inequality that

$$(4.2) \quad V(C_1, i_1; \ldots; C_m, i_m)^k \geq \prod_{j=1}^m \mathcal{H}_k(C_j)^{i_j}.$$

An important special case of (4.2) is the classical Minkowski inequality, which says

$$(4.3) \quad V(C_1, 1; C_2, k-1)^k \geq \mathcal{H}_k(C_1)\mathcal{H}_k(C_2)^{k-1}.$$

Equality holds for k-dimensional C_1 and C_2 in the Minkowski inequality (4.3) if and only if C_1 and C_2 are homothetic. We remark that the equality conditions in the Alexandrov-Fenchel inequality (4.2) are not yet clarified in general.

Now the Alexandrov-Fenchel inequality (4.2), and actually already the Minkowski inequality (4.3) yields the classsical (general) Brunn-Minkowski theorem stating that if $C_1, \ldots, C_m$ are compact convex sets in $\mathbb{R}^k$, and $\alpha_1, \ldots, \alpha_m \geq 0$, then (cf. (2.1))

$$(4.4) \quad \mathcal{H}_k\left(\sum_{j=1}^m \alpha_j C_j\right)^{1/k} \geq \sum_{j=1}^m \alpha_i \mathcal{H}_k(C_i)^{1/k}.$$

Equality holds for k-dimensional $C_1, \ldots, C_m$ and positive $\alpha_1, \ldots, \alpha_m$ in the Brunn-Minkowski inequality (4.4) if and only if C_1 and C_j are homothetic for $j = 2, \ldots, m$.

We need the following stability version of the Minkowski inequality (4.3) due to A. Figalli, F. Maggi and A. Pratelli [16]. If C_1, C_2 are k-dimensional compact convex sets in $\mathbb{R}^k$, and

$$(4.5) \qquad \mathrm{V}(C_1, 1; C_2, k-1)^k \leq (1+\varepsilon)\mathcal{H}_k(C_1)\mathcal{H}_k(C_2)^{k-1}$$

for small $\varepsilon \geq 0$, then [16] proves that

$$(4.6) \qquad \delta_{\mathrm{vol}}(C_1, C_2) \leq \tilde{\gamma}_v \varepsilon^{1/2}$$

where the explicit $\tilde{\gamma}_v > 0$ depends only on the dimension k.

We remark that here we only work out the estimate with respect to the symmetric volume distance δ_{vol}, and then just use Corollary 3.2 for δ_{hom}. Actually, V.I. Diskant [13] proved that (4.5) implies

$$(4.7) \qquad \delta_{\mathrm{hom}}(C_1, C_2) \leq \tilde{\gamma}_h \varepsilon^{1/k}$$

for an unknown $\tilde{\gamma}_h > 0$ depending only on k. We note that (4.6) and Corollary 3.2 readily yields a version of (4.7) with exponent $\frac{1}{2k}$ instead of $\frac{1}{k}$.

Combining the stability versions (4.6) and (4.7) with Lemma 3.3 and Lemma 3.4 leads to the following stability version of the Brunn-Minkowski inequality.

Lemma 4.1. *For any $k \geq 1$, $m \geq 2$ and $\omega \in (0, 1]$, there exist positive $\varepsilon_0(k, m, \omega)$ and $\gamma(k, m, \omega)$ depending only on k, m and ω such that if k-dimensional compact convex sets $C_0, C_1, \ldots, C_m$ in $\mathbb{R}^k$, and $\alpha_1, \ldots, \alpha_m > 0$ satisfy that $\alpha_i/\alpha_j \geq \omega$ and $\mathcal{H}_k(C_i) = V$ for $i, j = 1, \ldots, m$, and*

$$\alpha_1 C_1 + \ldots + \alpha_m C_m \subset C_0 \quad \text{and} \quad \mathcal{H}_k(C_0) \leq e^\varepsilon (\alpha_1 + \ldots + \alpha_m)^k V$$

for some $\varepsilon \in (0, \varepsilon_0(k, m, \omega))$, then for $i = 1, \ldots, m$, we have

$$\delta_{\mathrm{vol}}(C_i, C_0) \;\leq\; \gamma(k, m, \omega)\varepsilon^{1/2},$$

$$\left\| c(C_0) - \sum_{i=1}^m \alpha_i c(C_i) \right\|_{C_0 - C_0} \;\leq\; (\alpha_1 + \ldots + \alpha_m)\gamma(k, m, \omega)\varepsilon^{1/2}.$$

Proof. First we assume that $C_0 = \alpha_1 C_1 + \ldots + \alpha_m C_m$. For $1 \leq i < j \leq m$, we apply the Alexandrov-Fenchel inequality (4.2) to each term in (4.1) except for $k\alpha_i\alpha_j^{k-1}\mathrm{V}(C_i, 1; C_j, k-1)$ and deduce that

$$\begin{aligned} e^\varepsilon(\alpha_1 + \ldots + \alpha_m)^k V \;\geq\; & k\alpha_i\alpha_j^{k-1}\mathrm{V}(C_i, 1; C_j, k-1) + \\ & \left[(\alpha_1 + \ldots + \alpha_m)^k - k\alpha_i\alpha_j^{k-1} \right] V. \end{aligned}$$

In other words,

$$k\alpha_i\alpha_j^{k-1}\mathrm{V}(C_i, 1; C_j, k-1) \leq k\alpha_i\alpha_j^{k-1}V + (e^\varepsilon - 1)(\alpha_1 + \ldots + \alpha_m)^k V.$$

Here $(\alpha_1 + \ldots + \alpha_m)^k \leq (\frac{m}{\omega})^k \alpha_i\alpha_j^{k-1}$, and hence

$$\mathrm{V}(C_i, 1; C_j, k-1) \leq \left(1 + \frac{2}{k}\left(\frac{m}{\omega}\right)^k \varepsilon \right) V.$$

Thus (4.6) yield

$$\delta_{\text{vol}}(C_i, C_j) \leq \bar{\gamma}(k, m, \omega)\varepsilon^{1/2} \tag{4.8}$$

for $\bar{\gamma}(k, m, \omega)$ depending only on k, m and ω. To compare to C_0, we may assume that $V = 1$, $\alpha_1 + \ldots + \alpha_m = 1$ and $c(C_i) = o$ for $i = 1, \ldots, m$. Let $M = C_1 \cap \ldots \cap C_m$.

It follows from (4.8) that

$$\mathcal{H}_k(C_i \backslash M) \leq m \cdot \bar{\gamma}(k, m, \omega)\varepsilon^{1/2}, \ i = 1, \ldots, m,$$

and hence $\mathcal{H}_k(M) \geq 1 - m \cdot \bar{\gamma}(k, m, \omega)\varepsilon^{1/2}$. Since $M \subset C_i$ for $i = 1, \ldots, m$ yields $M \subset C_0 = \sum_{i=1}^{m} \alpha_i C_i$, and $\mathcal{H}_k(C_0) \leq e^\varepsilon$, we deduce

$$\mathcal{H}_k(C_0 \Delta C_i) \leq 2\bar{\gamma}(k, m, \omega)\varepsilon^{1/2}, \ i = 1, \ldots, m.$$

Therefore Lemma 3.3 and Lemma 3.4 imply the required estimates for $\delta_{\text{vol}}(C_i, C_0)$ and $c(C_0)$ in the case $C_0 = \alpha_1 C_1 + \ldots + \alpha_m C_m$.

Finally, in the general case, let $C_0' = \alpha_1 C_1 + \ldots + \alpha_m C_m$, and hence $C_0' \subset C_0$. We may assume again that $V = 1$, $\alpha_1 + \ldots + \alpha_m = 1$ and $c(C_i) = o$ for $i = 1, \ldots, m$. The argument above and $C_0' \subset C_0$ yield that

$$\delta_{\text{vol}}(C_i, C_0') \leq \gamma^*(k, m, \omega)\varepsilon^{1/2},$$
$$\left\| c(C_0') \right\|_{C_0 - C_0} \leq \left\| c(C_0') \right\|_{C_0' - C_0'} \leq \gamma^*(k, m, \omega)\varepsilon^{1/2}$$

for $\gamma^*(k, m, \omega) > 0$ depending on k, m, ω. It follows from the Brunn-Minkowski inequality that $1 \leq \mathcal{H}_k(C_0') \leq \mathcal{H}_k(C_0) \leq e^\varepsilon$. Since $C_0' \subset C_0$, we conclude Lemma 4.1 by Lemma 3.4. $\square$

To prove the next Proposition 4.3, we need the following observation.

Lemma 4.2. *If M is a convex body in $\mathbb{R}^d$ such that $-M \subset \eta M$ for some $\eta \geq 1$, then there exists an d-simplex $T \subset M$ whose centroid is the origin such that $M \subset \eta d^{3/2}T$.*

Proof. We may assume that the John ellipsoid E of maximal volume contained in $M \cap (-M)$ is Euclidean ball, and let $T \subset E$ be an inscribed regular simplex. Then $\eta^{-1}M \subset M \cap (-M) \subset \sqrt{d}E \subset d^{3/2}T$. $\square$

For Proposition 4.3 we use the notation of the previous sections, i.e., $K \in \mathcal{K}^n$ is a centered convex body, $d, k \in \{1, \ldots, n-1\}$ with $d + k = n$, and L is a d-dimensional linear subspace. For $x \in K|L$, we set

$$f(x) = f_{K,L}(x) = \mathcal{H}_k(K \cap (x + L^\perp)).$$

Proposition 4.3. *There exist $t_0, \gamma > 0$ depending on n with the following properties. Let $t \in (0, t_0)$, let $M_* \subset K|L$ be a d-dimensional convex compact set, and let $K_* = K \cap (M_* + L^\perp)$. If $e^{-t} \leq f(x)/f(o) \leq e^t$ holds for any $x \in M_*$, then there exist a k-dimensional compact convex set $C \subset L^\perp$, and a complementary d-dimensional compact convex set M such that*

$$\delta_{\text{vol}}(K, C + M) \leq \gamma \max\left\{ \frac{V(K \backslash K_*)}{V(K)}, t^{1/2} \right\}.$$

Proof. Since $c(K) = o$ we have $-K \subset nK$ (cf. [48, p. 155]). Hence $-K|L \subset nK|L$ and we may choose, according to Lemma 4.2, $v_0, \ldots, v_d \in e^{-s}K|L$, for some $s > 0$, such that $v_0 + \ldots + v_d = o$, and

$$(4.9) \qquad e^{-s}K|L \subset n^{5/2}[v_0, \ldots, v_d].$$

For $x \in e^{-s}K|L$, let $K(x) = K \cap (x + L^\perp)$, and let

$$(4.10) \qquad \widetilde{K}(x) = \frac{f(o)^{1/k}}{f(x)^{1/k}} K(x), \quad \text{and hence } \mathcal{H}_k(\widetilde{K}(x)) = f(o).$$

We define

$$
\begin{aligned}
A &= \operatorname{aff}\{c(K(v_0)), \ldots, c(K(v_d))\}, \\
M &= \{y \in A : (y + L^\perp) \cap e^{-s}K \neq \emptyset\}, \\
C &= K(o) - c(K(o)).
\end{aligned}
$$

We compare K_* with $M + C$. To this end we consider the affine bijection $\varphi : L \to A$ defined by the correspondance $\{\varphi(x)\} = A \cap (x + L^\perp)$ for $x \in L$. In particular,

$$(4.11) \qquad \varphi(v_i) = c(K(v_i)),\ i = 0, \ldots, d\ \text{ and }\ \varphi(o) = \frac{1}{d+1}\sum_{i=0}^{d} c(K(v_i)).$$

Let $x \in e^{-s}K|L$. We have $\frac{-1}{2n^{5/2}}x \in \frac{1}{2}[v_0, \ldots, v_d]$ according to (4.9), thus

$$\frac{-1}{2n^{5/2}}\,x = \sum_{i=0}^{d} \alpha_i v_i \ \text{ where } \ \sum_{i=0}^{d}\alpha_i = 1 \text{ and } \alpha_i \geq \frac{1}{2(d+1)},\ i = 0, \ldots, d.$$

We define

$$
\begin{aligned}
\tilde{\beta} &= \frac{\beta f(x)^{1/k}}{f(o)^{1/k}} \ \text{ where } \ \beta = \frac{1}{1 + 2n^{5/2}}; \\
\tilde{\beta}_i &= \frac{\beta_i f(v_i)^{1/k}}{f(o)^{1/k}} \ \text{ where } \ \beta_i = \frac{\alpha_i 2n^{5/2}}{1 + 2n^{5/2}},\ i = 0, \ldots, d,
\end{aligned}
$$

and hence $\beta + \sum_{i=0}^{d}\beta_i = 1$ and $\beta x + \sum_{i=0}^{d}\beta_i v_i = o$. The condition on the function f yields that

$$e^{-t/k} \leq \tilde{\beta} + \tilde{\beta}_0 + \ldots + \tilde{\beta}_d \leq e^{t/k},$$

and the ratio of any two of $\tilde{\beta}, \tilde{\beta}_0, \ldots, \tilde{\beta}_d$ is at least $1/(4n^{5/2})$. In particular,

$$e^t(\tilde{\beta} + \tilde{\beta}_0 + \ldots + \tilde{\beta}_d)^k f(o) \geq \mathcal{H}_k(K(o)),$$

and the convexity of K implies (cf. (4.10))

$$\tilde{\beta}\widetilde{K}(x) + \sum_{i=0}^{d} \tilde{\beta}_i\widetilde{K}(v_i) = \beta K(x) + \sum_{i=0}^{d} \beta_i K(v_i) \subset K(o).$$

We deduce from Lemma 4.1, the stability version of the Brunn-Minkowski inequality, that there exists $\gamma^* > 0$ depending on n such that for $i = 0, \ldots, d$, we have

$$(4.12) \qquad \delta_{\mathrm{vol}}(K(v_i), K(o)), \ \delta_{\mathrm{vol}}(K(x), K(o)) \le \gamma^* t^{1/2},$$

$$(4.13) \qquad \left\| c(K(o)) - \beta c(K(x)) - \sum_{i=1}^{d} \beta_i c(K(v_i)) \right\|_{K(o) - K(o)} \le \gamma^* t^{1/2}.$$

First we asssume that $x = o$. In this case, (4.11) and (4.13) yield

$$(4.14) \qquad \|c(K(o)) - \varphi(o)\|_{K(o) - K(o)} \le \gamma^* t^{1/2}.$$

Next let $x \in e^{-s}K|L$ be arbitrary. We have $\beta \varphi(x) + \sum_{i=0}^{d} \beta_i \varphi(v_i) = \varphi(o)$ because φ is affine. We recall that $C = K(o) - c(K(o))$. Let

$$w = c(K(o)) - \beta c(K(x)) - \sum_{i=1}^{d} \beta_i c(K(v_i)).$$

Since $\beta \varphi(x) = \varphi(o) - \sum_{i=0}^{d} \beta_i \varphi(v_i)$, we have

$$\begin{aligned}
\left\| c(K(x)) - \varphi(x) \right\|_{C-C} &= \frac{\|\beta c(K(x)) - \beta \varphi(x)\|_{C-C}}{\beta} \\
&\le \frac{\|\beta c(K(x)) + w - \beta \varphi(x)\|_{C-C}}{\beta} + \frac{\| - w\|_{C-C}}{\beta} \\
&= \frac{\|c(K(o)) - \varphi(o) - \sum_{i=0}^{d} \beta_i (c(K(v_i)) - \varphi(v_i))\|_{C-C}}{\beta} \\
&\quad + \frac{\|c(K(o)) - \beta c(K(x)) - \sum_{i=1}^{d} \beta_i c(K(v_i))\|_{C-C}}{\beta}.
\end{aligned}$$

As $\varphi(v_i) = c(K(v_i))$ according to (4.11), it follows by (4.13) and (4.14) that

$$(4.15) \qquad \|c(K(x)) - \varphi(x)\|_{C-C} \le \frac{2\gamma^*}{\beta} t^{1/2} < 6n^{5/2} \gamma^* t^{1/2}.$$

For $x \in e^{-s}K|L$, by (4.15), (4.12) and (4.10),

$$\begin{aligned}
\mathcal{H}_k\big((C + \varphi(x))\Delta K(x)\big) &\le \mathcal{H}_k\big((C + \varphi(x))\Delta(C + c(K(x)))\big) \\
&\quad + \mathcal{H}_k\big((C + c(K(x)))\Delta(\widetilde{K}(x) - c(\widetilde{K}(x)) + c(K(x)))\big) \\
&\quad + \mathcal{H}_k\big((\widetilde{K}(x) - c(\widetilde{K}(x)) + c(K(x)))\Delta K(x)\big) \\
&< 9n^{5/2} \gamma^* t^{1/2} \mathcal{H}_k(C).
\end{aligned}$$

Hence, by Fubini's theorem we get

$$\mathrm{V}(K_* \Delta(M + C)) < 9n^{5/2} \gamma^* t^{1/2} \mathrm{V}(M + C).$$

This and Lemma 3.4 yield the required estimate for δ_{vol}. $\qquad \square$

5. Some more properties of $f_{K,L}(x)$

Here we establish some more properties of the log-concave function (cf. (2.3))

$$f_{K,L} : L \to \mathbb{R}_{\geq 0} \text{ with } x \mapsto \mathcal{H}_k(K \cap (x + L^\perp)),$$

and use the notation introduced in Section 2, i.e., $K \in \mathcal{K}^n$ is an n-dimensional centered convex body, L is a d-dimensional subspace L, $1 \leq d \leq n - 1$, and we set $k = n - d$. Since we will keep K and L fixed, we just write $f(x)$ instead of $f_{K,L}(x)$. As in Section 2 let $g(x)$ be the subgradient of $f(x)$, and we recall that $g(x) = \nabla f(x)/f(x)$ almost everywhere on $\mathrm{int}\,(K)|L$.

For $\eta \geq 0$, we set

$$
\begin{aligned}
M_\eta &= \{x \in K|L : \ln f(x) - \ln f(o) \geq \langle g(o), x \rangle - \eta\}, \\
K_\eta &= K \cap (M_\eta + L^\perp).
\end{aligned}
$$

Since $\ln f$ is concave, both M_η and K_η are compact and convex.

Lemma 5.1. *Let $\eta \geq 0$. Then*

$$\int_{K|L} \langle \nabla f(x), x \rangle d\mathcal{H}_d(x) \leq -\eta \mathrm{V}(K \backslash K_\eta).$$

Proof. Let $x \in (\mathrm{int}\,K)|L$ and $\eta \geq 0$, and let us assume $\ln f(x) - \ln f(o) \leq \langle g(o), x \rangle - \eta$. Then by (2.2) we have $\langle g(x), x \rangle \leq \langle g(o), x \rangle - \eta$. Hence if ∇f exists at $x \in (\mathrm{int}\,K)|L$, then

$$
\begin{aligned}
\langle \nabla f(x), x \rangle &\leq 0 \text{ provided that } x \in M_\eta, \\
\langle \nabla f(x), x \rangle &\leq \langle g(o), f(x)x \rangle - f(x)\eta \text{ provided that } x \notin M_\eta.
\end{aligned}
$$

We conclude the lemma by (2.4) and the fact that $\mathrm{V}(K \backslash K_\eta) = \int_{(K|L) \backslash M_\eta} f(x)\, dx$. $\square$

Lemma 5.2. *Let $\eta \in [0, 1]$. If $\mathrm{V}(K \backslash K_\eta) \leq \mathrm{V}(K)/(2^n e)$, then*

$$e^{-\tau} \leq \frac{f(x)}{f(o)} \leq e^\tau \text{ for } \tau = 7n^{3/2}\eta^{1/2} \text{ and } x \in M_\eta.$$

Proof. By Lemma 3.1 we have $\frac{1}{2}K \subset K_\eta$, and $f(x) \geq f(o)e^{\langle g(o), x \rangle - \eta}$ for $x \in K_\eta$. We claim that for $\pm y \in K_\eta$

$$(5.1) \qquad\qquad\qquad |\langle g(o), y \rangle| \leq 3\sqrt{k\eta}.$$

The concavity of $f^{1/k}$ yields that

$$
\begin{aligned}
f(o)^{1/k} &\geq \frac{f(y)^{1/k} + f(-y)^{1/k}}{2} \geq f(o)^{1/k}e^{-\eta/k}\frac{e^{\langle g(o), y \rangle/k} + e^{\langle g(o), -y \rangle/k}}{2} \\
&\geq f(o)^{1/k}e^{-\eta/k}\left(1 + \left(\frac{\langle g(o), y \rangle}{2k}\right)^2\right).
\end{aligned}
$$

Since $e^t < 1 + 2t$ for $t \in [0, 1]$, we conclude (5.1).

It follows from $\frac{1}{2}K \subset K_\eta$ and $-K \subset nK$ (since $c(K) = o$) that $\frac{1}{2}(K|L) \subset M_\eta$ and $-(K|L) \subset n(K|L)$. In particular, if $x \in M_\eta$ is arbitrary, then $\pm y \in$

M_η for $y = \frac{1}{2n}x$. We deduce from (5.1) that $|\langle g(o), x \rangle| = 2n|\langle g(o), y \rangle| \leq 6n\sqrt{k}\eta$. Therefore, the lemma follows from $f(o)e^{\langle g(o),x \rangle - \eta} \leq f(x) \leq f(o)e^{\langle g(o),x \rangle}$. $\qquad\square$

6. Proofs of Theorem 1.1 and Theorem 1.2

For the proofs of the two stability theorems 1.1 and 1.2, let $K \in \mathcal{K}^n$ be centered, and let

$$\mathrm{V}_K(L \cap S^{n-1}) > \frac{d - \varepsilon}{n} \mathrm{V}(K)$$

for a proper linear subspace L with $\dim L = d$ and some $\varepsilon \in (0, (2^n e)^{-5})$. As before, for $x \in K|L$ let

$$f(x) = \mathcal{H}_k(K \cap (x + L^\perp)).$$

According to Lemma 2.2, the condition on $\mathrm{V}_K(L \cap S^{n-1})$ is equivalent to

(6.1) $$\int_{K|L} \langle \nabla f(x), x \rangle d\mathcal{H}_d(x) > -\varepsilon \mathrm{V}(K).$$

Proof of Theorem 1.1. We set $\eta = \varepsilon^{4/5}$, and use the notation of Lemma 5.1. It follows from (6.1) and Lemma 5.1 that

$$\mathrm{V}(K \backslash K_\eta) < \varepsilon^{1/5}\mathrm{V}(K) < \mathrm{V}(K)/(2^n e),$$

and from Lemma 5.2 that

$$e^{-t} \leq \frac{f(x)}{f(o)} \leq e^t \ \text{ for } t = 7n^{3/2}\varepsilon^{2/5} \text{ and } x \in M_\eta.$$

We assume that ε is small enough in order to apply Proposition 4.3 with $M_* = M_\eta$ and $t = 7n^{3/2}\varepsilon^{2/5}$. We deduce the existence of an $(n - d)$-dimensional compact convex set $C \subset L^\perp$, and complementary d-dimensional compact convex set M such that

$$\delta_{\mathrm{vol}}(K, C + M) \leq \gamma_v \varepsilon^{1/5}.$$

By Corollary 3.2 implies that

$$\delta_{\mathrm{hom}}(K, C + M) \leq \gamma_h \varepsilon^{1/(5n)},$$

completing the proof of Theorem 1.1. $\qquad\square$

Proof of Theorem 1.2. We may assume $K|L = [-a, b]$ where $0 < a \leq b$. Since $c(K) = o$ implies $-K \subset nK$ according to B. Grünbaum (cf.[24], [48, p. 155]) we have $b \leq na$.

We set $\eta = \varepsilon^{2/3}$, and use again the notation of Lemma 5.1. We deduce from (6.1) and Lemma 5.1 that

(6.2) $$\mathrm{V}(K \backslash K_\eta) < \varepsilon^{1/3}\mathrm{V}(K) < \mathrm{V}(K)/(2^n e),$$

and from Lemma 5.2 that

(6.3) $$e^{-t} \leq \frac{f(x)}{f(o)} \leq e^t \ \text{ for } t = 7n^{3/2}\varepsilon^{1/3} \text{ and } x \in M_\eta.$$

It follows from Lemma 3.1 and (6.2) that $\frac{1}{2}[-a, b] \subset M_\eta$, therefore the concavity of $\ln f$ and (6.3) yield that

$$(6.4) \qquad\qquad f(x) \le e^{2t} f(o) \ \text{ for } x \in [-a, b].$$

Let $M_\eta = [-a_\eta, b_\eta]$ for $a_\eta, b_\eta > 0$. Since $K \backslash K_\eta$ contains two cones, one with base $K(-a_\eta)$ and height $a - a_\eta$, and the other with base $K(b_\eta)$ and height $b - b_\eta$, we get by (6.3), (6.2) and (6.4) that

$$\frac{a - a_\eta + b - b_\eta}{n} e^{-t} f(o) \ \le \ \frac{a - a_\eta + b - b_\eta}{n}(f(-a_\eta) + f(b_\eta))$$

$$\le \ \mathrm{V}(K \backslash K_\eta) < \varepsilon^{\frac{1}{3}} \mathrm{V}(K) \le \varepsilon^{\frac{1}{3}} e^{2t} f(o)(a + b).$$

In particular,

$$\mathcal{H}_1(M_\eta) = a_\eta + b_\eta > (1 - 2n\varepsilon^{\frac{1}{3}})(a + b).$$

Here and below $\gamma_1, \gamma_2, \ldots$ denote positive constants depending on n. We deduce by (6.3) that if ε is small enough, then

$$af(-a) + bf(b) \ = \ n\mathrm{V}_K(L \cap S^{n-1}) > (1 - \varepsilon)\mathrm{V}(K) > (1 - \varepsilon)\mathcal{H}_1(M_\eta)e^{-t}f(o)$$

$$> \ (1 - \gamma_1\varepsilon^{\frac{1}{3}})(a + b)f(o).$$

Since $b \ge a$ and $\frac{a}{a+b} \ge \frac{1}{n+1}$ by $b \le na$, (6.4) implies that if ε is small enough, then

$$f(-a), f(b) \ge (1 - \gamma_2\varepsilon^{\frac{1}{3}})f(o).$$

As $\ln f$ is concave, we have

$$f(x) \ge (1 - \gamma_2\varepsilon^{\frac{1}{3}})f(o) \ \text{ for } x \in [-a, b].$$

However, $\frac{a}{a+b} C(b) + \frac{b}{a+b} C(-a) \subset C(o)$, where $C(x) = K \cap (x + L^\perp)$. Thus, Lemma 4.1 yields that

$$(6.5) \qquad \delta_{\mathrm{vol}}(C(o), C(-a)) \le \gamma_3\varepsilon^{\frac{1}{6}} \ \text{ and } \ \delta_{\mathrm{vol}}(C(o), C(b)) \le \gamma_3\varepsilon^{\frac{1}{6}}.$$

Hence,

$$\widetilde{C} = (C(-a) - \tilde{x}) \cap (C(b) - \tilde{y}) \ \text{ for } \tilde{x} = c(C(-a)) \text{ and } \tilde{y} = c(C(b)).$$

From (6.4) and (6.5) we get

$$[\tilde{x}, \tilde{y}] + \widetilde{C} \subset K \ \text{ and } \ \mathrm{V}(K) \le (1 + \gamma_4\varepsilon^{\frac{1}{6}})\mathrm{V}([\tilde{x}, \tilde{y}] + \widetilde{C}).$$

Using Lemma 3.4, we replace $\widetilde{C}$ by a suitably smaller homothetic copy C such that $c(C) = o$, and obtain that there exist $x \in \tilde{x} + \widetilde{C}$ and $y \in \tilde{y} + \widetilde{C}$ satisfying $o \in [x, y]$, $e^{-s}\|x\| \le \|y\| \le e^s\|x\|$ for $s = \gamma_5\varepsilon^{\frac{1}{6}}$, and

$$[x, y] + C \subset K \ \text{ and } \ \mathrm{V}(K) \le (1 + \gamma_6\varepsilon^{\frac{1}{6}})\mathrm{V}([x, y] + C).$$

Finally, if $z \in [-a, b]$, then $-z/n \in [-a, b]$ and $\frac{1}{n+1} C(z) + \frac{n}{n+1} C(-z/n) \subset C(o)$. Therefore, Lemma 3.1, Lemma 4.1 and the estimates above imply

$$K \subset [x, y] + (1 + \gamma_5\varepsilon^{\frac{1}{6n}})C.$$

Which completes the proof of Theorem 1.2. $\qquad\qquad\qquad\qquad\qquad\square$

7. STABILITY OF THE U-FUNCTIONAL $U(K)$

Let $m \in \{1, \ldots, n\}$. In this section, a finite sequence $u_1, \ldots, u_m$ always denote points of S^{n-1}, and by $\lin\{X\}$ we denote the linear hull of a set X. As in [30], we define $\sigma_m(K) > 0$ by

$$\sigma_m(K)^m = \int_{u_1 \wedge \ldots \wedge u_m \neq 0} 1 \, dV_K(u_1) \cdots dV_K(u_m).$$

In particular, $\sigma_1(K) = V(K)$, $\sigma_n(K) = U(K)$, and for $m < n$, we have

(7.1)
$$\sigma_{m+1}(K)^{m+1} =$$
$$\int_{u_1 \wedge \ldots \wedge u_m \neq 0} \left(V(K) - V_K(S^{n-1} \cap \lin\{u_1, \ldots, u_m\}) \right) dV_K(u_1) \cdots dV_K(u_m).$$

As $V_K(S^{n-1} \cap \lin\{u_1, \ldots, u_m\}) \leq \frac{m}{n} V(K)$ for linearly independent $u_1, \ldots, u_m$ according to Theorem II, we deduce that

(7.2)
$$\sigma_{m+1}(K)^{m+1} \geq \left(1 - \frac{m}{n}\right) V(K) \sigma_m(K)^m.$$

Therefore the inequality of Theorem III follows from

$$U(K)^n \geq \frac{1}{n} V(K) \sigma_{n-1}(K)^{n-1} \geq \ldots \geq \frac{(n-1)!}{n^{n-1}} V(K)^{n-1} \sigma_1 = \frac{n!}{n^n} V(K)^n.$$

Now we assume that

$$U(K) \leq (1+\varepsilon) \frac{(n!)^{1/n}}{n} V(K),$$

where $\varepsilon > 0$ is small enough to satify all estimates below. In particular, $\varepsilon < \frac{1}{4n^3} \tilde{\varepsilon}_0$, where $\tilde{\varepsilon}_0$ comes from Theorem 1.2. Applying (7.1) for $m = 1$, (7.2) for $m \geq 2$, and using $(1+\varepsilon)^n \frac{n-1}{n} < \frac{n-1}{n} + 2n\varepsilon$ gives

(7.3)
$$\int_{S^{n-1}} (V(K) - V_K(S^{n-1} \cap \lin\{u\})) \, dV_K(u) \leq \left(\frac{n-1}{n} + 2n\varepsilon\right) V(K)^2.$$

For any $X \subset S^{n-1}$, there exists $u \in X$ maximizing $V_K(S^{n-1} \cap \lin\{u\})$ because different 1-dimensional subspaces have disjoint intersections with S^{n-1}. We consider linearly independent $v_1, \ldots, v_n \in S^{n-1}$ such that v_1 maximizes $V_K(S^{n-1} \cap \lin\{u\})$ for $u \in S^{n-1}$, and v_i maximizes $V_K(S^{n-1} \cap \lin\{u\})$ for all $u \in S^{n-1} \setminus \lin\{v_1, \ldots, v_{i-1}\}$ if $i = 2, \ldots, n$. Let $L = \lin\{v_1, \ldots, v_{n-1}\}$, and let $V_K(S^{n-1} \cap \lin\{v_n\}) = (\frac{1}{n} - t)V(K)$, and hence $t \in [0, \frac{1}{n}]$ (cf. (1.3)). Thus, we have

(7.4) $\quad V_K(S^{n-1} \cap \lin\{v_i\}) \geq (\frac{1}{n} - t)V(K)$ for $i = 1, \ldots, n$,

(7.5) $\quad V_K(S^{n-1} \cap \lin\{u\}) \leq (\frac{1}{n} - t)V(K)$ for $u \in S^{n-1} \setminus L$.

We deduce from (7.3), (7.5) and $V_K(S^{n-1} \cap \lin\{u\}) \leq \frac{1}{n} V(K)$ for $u \in S^{n-1} \cap L$ that

$$\left(\frac{n-1}{n} + t\right) V(K) V_K(S^{n-1} \setminus L) + \frac{n-1}{n} V(K) V_K(S^{n-1} \cap L) \leq \left(\frac{n-1}{n} + 2n\varepsilon\right) V(K)^2.$$

Since $V_K(S^{n-1}\backslash L) \geq \frac{1}{n}V(K)$ according to Theorem II, we conclude that $t \leq 2n^2\varepsilon$. In particular, $V_K(S^{n-1} \cap \operatorname{lin}\{v_i\}) \geq (\frac{1}{n} - 2n^2\varepsilon)V(K)$ for $i = 1,\ldots,n$ by (7.4).

From Theorem 1.2 we find for $i = 1,\ldots,n$, that there exist an $(n-1)$-dimensional compact convex set $C_i \subset v_i^\perp$ with $c(C_i) = o$, and $x_i, y_i \in \partial K$ such that $y_i = -e^{s_i}x$, where $|s_i| < n\tilde{\gamma}_v\varepsilon^{\frac{1}{6}}$, and for $i = 1,\ldots,n$, we have

$$(7.6) \qquad\qquad [x_i, y_i] + C_i \subset K,$$

$$(7.7) \qquad\qquad V(K\backslash([x_i, y_i] + C_i)) \leq n\tilde{\gamma}_v\varepsilon^{\frac{1}{6}}V(K),$$

$$(7.8) \qquad\qquad\qquad K \subset [x_i, y_i] + (1 + 2\tilde{\gamma}_h\varepsilon^{\frac{1}{6n}})C_i.$$

Observe that v_i is an exterior normal at x_i, $i = 1,\ldots,n$. After a linear transformation of K, we may also assume that $v_1,\ldots,v_n$ form and orthonormal system, and $\langle v_i, x_i - y_i\rangle = 2$. In particular,

$$(7.9) \qquad e^{-\tau} < \langle v_i, x_i\rangle, \langle -v_i, y_i\rangle < e^\tau, \ \tau = n\tilde{\gamma}_v\varepsilon^{\frac{1}{6}}.$$

In what follows, we write $\gamma_1, \gamma_2, \ldots$ for positive constants depending on n only. It follows from combining (7.6), (7.7) and (7.9) that

$$(7.10) \quad 1 - \gamma_1\varepsilon^{\frac{1}{6}} < \mathcal{H}_{n-1}(C_i)/\mathcal{H}_{n-1}(C_j) < 1 + \gamma_1\varepsilon^{\frac{1}{6}} \ \text{ for } i,j \in \{1,\ldots,n\}.$$

For any $i \neq j \in \{1,\ldots,n\}$, we write

$$\begin{aligned} w_i(v_j) &= h_{C_i}(v_j) + h_{C_i}(-v_j), \\ a_i(v_j) &= \max\left\{\mathcal{H}_{n-2}(C_i \cap (tv_j + v_j^\perp)) : -h_{C_i}(-v_j) \leq t \leq h_{C_i}(v_j)\right\}, \end{aligned}$$

and recall that $h_{C_i}(x)$ denotes the support function. Hence, $w_i(v_j)$ is the width of C_i in the direction of v_j. Observe that C_i contains a bipyramid whose basis has volume $a_i(v_j)$ and of height $w_i(v_j)$ which gives the lower bound in (7.11). For the upper bound we integrate along $\mathbb{R}v_j$ to get

$$(7.11) \quad \frac{1}{n-1}w_i(v_j)a_i(v_j) \leq \mathcal{H}_{n-1}(C_i) \leq w_i(v_j)a_i(v_j) \ \text{ for } i \neq j \in \{1,\ldots,n\}.$$

Let $p \neq q \in \{1,\ldots,n\}$, and let $t_1, t_0 \in \mathbb{R}$ be defined by the properties that t_1x_p and t_0x_p lie in the supporting hyperplanes to $C_q + x_q$ with exterior normals v_p and $-v_p$, respectively. In particular, we can choose $t_1 > t_0$ and $t_* \in \mathbb{R}$ such that

$$\begin{aligned} \langle v_p, t_1x_p\rangle &= h_{C_q+x_q}(v_p) \\ \langle -v_p, t_0x_p\rangle &= h_{C_q+x_q}(-v_p) \\ \mathcal{H}_{n-2}(C_q \cap (t_*x_p + v_p^\perp)) &= a_q(v_p). \end{aligned}$$

It follows from (7.8) and (7.9) that

$$(7.12) \qquad\qquad t_1 - t_0 > w_q(v_p)/2,$$

$$C_q \cap (t_*x_p + v_p^\perp) \subset t_*x_p + (1 + 2\tilde{\gamma}_h\varepsilon^{\frac{1}{6n}})C_p.$$

Therefore, since $C_q \subset v_q^\perp$, we get $a_p(v_q) \geq (1 + 2\tilde{\gamma}_h \varepsilon^{\frac{1}{6n}})^{-(n-2)} a_q(v_p)$, and hence interchanging the role of p and q leads to

$$1 - \gamma_2 \varepsilon^{\frac{1}{6n}} < a_q(v_p)/a_q(v_p) < 1 + \gamma_2 \varepsilon^{\frac{1}{6n}}.$$

We deduce from (7.10) and (7.11) that

(7.13) $$\frac{1}{2n} < \frac{w_p(v_q)}{w_q(v_p)} < 2n.$$

Let $m \in \{0, 1\}$. According to (7.6) and $t_m x_p \in [x_p, y_p]$, we have $t_m x_p + C_p \subset K$, and hence

$$\langle t_m x_p, v_q \rangle + h_{C_p}(v_q) \leq h_K(v_q) = \langle x_q, v_q \rangle.$$

On the other hand, the definition of t_m shows that $C_q + x_q$ intesects $t_m x_p + v_p^\perp$ in some z, and hence z is contained in $t_m x_p + (1 + 2\tilde{\gamma}_h \varepsilon^{\frac{1}{6n}}) C_p$ by (7.8), which in turn yields that

$$\langle t_m x_p, v_q \rangle + (1 + 2\tilde{\gamma}_h \varepsilon^{\frac{1}{6n}}) h_{C_p}(v_q) \geq \langle z, v_q \rangle = \langle x_q, v_q \rangle.$$

We conclude that

(7.14) $$h_{C_p}(v_q) \leq \langle x_q - t_m x_p, v_q \rangle \leq (1 + 2\tilde{\gamma}_h \varepsilon^{\frac{1}{6n}}) h_{C_p}(v_q) \quad \text{for } m = 0, 1,$$

and hence

$$|\langle (t_1 - t_0) x_p, v_q \rangle| \leq 2\tilde{\gamma}_h \varepsilon^{\frac{1}{6n}} h_{C_p}(v_q) < 2\tilde{\gamma}_h \varepsilon^{\frac{1}{6n}} w_p(v_q).$$

Applying (7.12), (7.13), and the analoguous argument to y_q implies that

(7.15) $$|\langle x_p, v_q \rangle|, |\langle y_p, v_q \rangle| \leq \gamma_3 \varepsilon^{\frac{1}{6n}}.$$

Let P be the parallepiped

$$P = \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leq \langle x_i, v_i \rangle, \ \langle x, -v_i \rangle \leq \langle y_i, -v_i \rangle, \ i = 1, \dots, n\},$$

and hence each facet of P contains one of $x_i + C_i$, $y_i + C_i$, $i = 1, \dots, n$. We claim that

(7.16) $$\frac{1}{4n} P \subset K.$$

We suppose that (7.16) does not hold and seek a contradiction. Possibly reversing the orientation of some of the v_i, we may asssume that

(7.17) $$z = \frac{1}{4n} \sum_{i=1}^{n} \langle x_i, v_i \rangle v_i \notin K.$$

In particular, $\|z\| \leq \frac{1}{2\sqrt{n}}$ by (7.9), and there exists $u \in S^{n-1}$ such that

(7.18) $$\langle u, z \rangle > \langle u, x \rangle \quad \text{for } x \in K.$$

There exists v_p such that $|\langle u, v_p \rangle| \geq 1/\sqrt{n}$, and hence (7.9) and (7.15) yield that $\langle u, x_p \rangle \geq \frac{1}{\sqrt{n}} - \gamma_4 \varepsilon^{\frac{1}{6n}}$ if $\langle u, v_p \rangle \geq 1/\sqrt{n}$, and $\langle u, y_p \rangle \geq \frac{1}{\sqrt{n}} - \gamma_4 \varepsilon^{\frac{1}{6n}}$ if $\langle u, v_p \rangle \leq -1/\sqrt{n}$. However $\langle u, z \rangle \leq \|z\| \leq \frac{1}{2\sqrt{n}}$, which contradicting (7.18). Therefore we conclude (7.16).

For $i = 1, \ldots, n$, let

$$\Xi_{2i-1} = [o, x_i + C_i] \quad \text{and} \quad \Xi_{2i} = [o, y_i + C_i].$$

Since the basis of the cones $\Xi_1, \ldots, \Xi_{2n}$ lie in different facets of P, the interiors of $\Xi_1, \ldots, \Xi_{2n}$ are pairwise disjoint. By (7.7) and (7.9) we know $V(\Xi_j) \geq (\frac{1}{2n} - \gamma_5 \varepsilon^{\frac{1}{6}}) V(K)$, and so we get

$$V(\Xi) > (1 - 2n\gamma_5 \varepsilon^{\frac{1}{6}}) V(K) \quad \text{for } \Xi = \bigcup_{j=1}^{2n} \Xi_j \subset K.$$

We conclude from (7.16) that

$$V(P \backslash K) \leq V(P \backslash \Xi) = (4n)^n V\left(\tfrac{1}{4n} P \backslash \Xi\right) \leq (4n)^n V(K \backslash \Xi) \leq \gamma_6 \varepsilon^{\frac{1}{6}} V(K),$$

and hence Lemma 3.1 (ii) implies that $(1 - \gamma_8 \varepsilon^{\frac{1}{6n}}) P \subset K$, completing the proof of Theorem 1.3.

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