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Cone-volume measure of general centered convex bodies

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ABSTRACT

We show that the cone-volume measure of a convex body with centroid at the origin satisfies the subspace concentration condition. This extends former results obtained in the discrete as well as in the symmetric case and implies, among others, a conjectured best possible inequality for the U-functional of a convex body.

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1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ having non-empty interiors, i.e., $K \in \mathcal{K}^n$ is a convex compact subset of the n-dimensional Euclidean space $\mathbb{R}^n$ with $\operatorname{int}(K) \neq \emptyset$. As usual, we denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^n \times \mathbb{R}^n$ with associated Euclidean norm $\| \cdot \|$, and $S^{n-1} \subset \mathbb{R}^n$ denotes the $(n-1)$-dimensional unit sphere, i.e., $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

For $K \in \mathcal{K}^n$ we write $S_K(\cdot)$ and $h_K(\cdot)$ to denote its surface area measure and support function, respectively, and ν_K to denote the Gauß map assigning the outer unit normal $\nu_K(x)$ to an $x \in \partial_* K$, where $\partial_* K$ consists of all points in the boundary ∂K of K having a unique outer normal vector. If the origin o lies in $K \in \mathcal{K}^n$, the *cone-volume measure of* K on S^{n-1} is given by

$$\mathrm{V}_K(\omega) = \int_{\omega} \frac{h_K(u)}{n} \, \mathrm{d}S_K(u) = \int_{\nu_K^{-1}(\omega)} \frac{\langle x, \nu_K(x) \rangle}{n} \, \mathrm{d}\mathcal{H}_{n-1}(x), \tag{1.1}$$

where $\omega \subseteq S^{n-1}$ is a Borel set and, in general, $\mathcal{H}_k(x)$ denotes the k-dimensional Hausdorff measure. Instead of $\mathcal{H}_n(\cdot)$, we also write $\mathrm{V}(\cdot)$ for the n-dimensional volume.

The name cone-volume measure stems from the fact that if K is a polytope with facets $F_1, \ldots, F_m$ and corresponding outer unit normals $u_1, \ldots, u_m$, then

$$\mathrm{V}_K(\omega) = \sum_{i=1}^{m} \mathrm{V}([o, F_i]) \delta_{u_i}(\omega). \tag{1.2}$$

Here δ_{u_i} is the Dirac delta measure on S^{n-1} concentrated at u_i, and for $x_1, \ldots, x_m \in \mathbb{R}^n$ and subsets $S_1, \ldots, S_l \subseteq \mathbb{R}^n$ we denote the convex hull of the set $\{x_1, \ldots, x_m, S_1, \ldots, S_l\}$ by $[x_1, \ldots, x_m, S_1, \ldots, S_l]$. With this notation $[o, F_i]$ is the cone with apex o and basis F_i.

In recent years, cone-volume measures have appeared and were studied in various contexts, see, e.g., F. Barthe, O. Guedon, S. Mendelson and A. Naor [6], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [10,11], M. Gromov and V.D. Milman [18], M. Ludwig [28], M. Ludwig and M. Reitzner [29], E. Lutwak, D. Yang and G. Zhang [32], A. Naor [34], A. Naor and D. Romik [35], G. Paouris and E. Werner [36], A. Stancu [42], G. Zhu [45,46], K.J. Böröczky and P. Hegedűs [8].

In particular, cone-volume measures are the subject of the *logarithmic Minkowski problem*, which is the particular interesting limiting case $p = 0$ of the general L_p-Minkowski problem – one of the central problems in convex geometric analysis. It is the task:

Find necessary and sufficient conditions for a finite Borel measure μ on S^{n-1} to be the cone-volume measure V_K of $K \in \mathcal{K}^n$ with o in its interior.

In the recent paper [11], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang solved the logarithmic Minkowski problem in the even case, i.e., they characterized the cone-volume measure of o-symmetric convex bodies $\{K \in \mathcal{K}^n : K = -K\}$. In order to state their result, we say that a finite Borel measure μ on S^{n-1} satisfies the *subspace concentration condition* if for any linear subspace $L \subseteq \mathbb{R}^n$

$$\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n}\,\mu(S^{n-1}), \tag{1.3}$$

and equality in (1.3) for some L implies the existence of a complementary linear subspace $\widetilde{L}$ such that

$$\mu(\widetilde{L} \cap S^{n-1}) = \frac{\dim \widetilde{L}}{n}\,\mu(S^{n-1}), \tag{1.4}$$

and hence $\operatorname{supp}\mu \subseteq L \cup \widetilde{L}$, i.e., the support of the measure "lives" in $L \cup \widetilde{L}$.

Via this condition, cone-volume measures of origin-symmetric convex bodies have been completely characterized by K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang.

Theorem I. *(See [11, Theorem 1.1].) A non-zero finite even Borel measure on S^{n-1} is the cone-volume measure of an origin-symmetric convex body if and only if it satisfies the subspace concentration condition.*

In the planar case, this result was proved earlier for discrete measures, i.e., for polygons, by A. Stancu [40,41]. For cone-volume measures of origin-symmetric polytopes (cf. (1.2)), the necessity of (1.3) was independently shown by M. Henk, A. Schürmann and J.M. Wills [24] and B. He, G. Leng and K. Li [22].

We recall that the centroid of a k-dimensional convex compact set $M \subset \mathbb{R}^n$ is defined as

$$\operatorname{cen}(M) = \mathcal{H}_k(M)^{-1} \int_M x \,\mathrm{d}\mathcal{H}_k(x),$$

and a convex body will be called *centered* if $\operatorname{cen}(K) = o$.

Centered bodies seem to be the right and natural class of convex bodies in order to extend Theorem I to general convex bodies. In fact, in [23] it was shown by M. Henk and E. Linke that the necessity part of Theorem I also holds for centered polytopes, i.e.,

Theorem II. *(See [23, Theorem 1.1].) Let $P \in \mathcal{K}^n$ be a centered polytope. Then its cone-volume measure V_P satisfies the subspace concentration condition.*

The proof of Theorem II relies heavily on the discrete structure of polytopes, in particular on the finite polytopal cell-decomposition of the projection of a polytope by its skeletons. Our main result extends Theorem II and thus the necessity part of Theorem I, to general convex bodies.

Theorem 1.1. *Let $K \in \mathcal{K}^n$ be centered. Then its cone-volume measure V_K satisfies the subspace concentration condition.*

While the subspace concentration condition is also sufficient to characterize cone-volume measures among even non-trivial Borel measures, the cone-volume measure of a centered convex body $K \in \mathcal{K}^n$ has to satisfy some extra properties. For example, in Theorem 4.1 we will prove that $\mathrm{V}_K(\Omega) \geq 1/(2n)$ for any open hemisphere $\Omega \subset S^{n-1}$ and we also provide a characterization of the equality case.

If K is not centered, then the subspace concentration condition may not hold any more. In fact, it was recently shown by G. Zhu [45] that for $u_1, \ldots, u_m \in S^{n-1}$ in general position, $m \geq n+1$, and arbitrary positive numbers $\gamma_1, \ldots, \gamma_m$ there always exists a (not necessarily centered) polytope $P \in \mathcal{K}^n$ with outer unit normals u_i and $\mathrm{V}_P(\{u_i\}) = \gamma_i$, $1 \leq i \leq m$. In other words, Zhu settled the logarithmic Minkowski problem for discrete measures whose support is in general position. In [9] this result was unified with the sufficiency part of the subspace concentration condition in the even discrete case by introducing the notation of *essential* subspaces. For a given finite Borel measure μ on S^{n-1} a subspace L, $1 \leq \dim L \leq n-1$, is called essential if $L \cap \operatorname{supp}\mu$ is not concentrated on any closed hemisphere of $L \cap \operatorname{supp}\mu$. K.J. Böröczky, P. Hegedűs and G. Zhu [9] proved that every finite discrete measure on S^{n-1} which satisfies the subspace concentration condition for all essential subspaces is the cone-volume measure of a polytope. In the case $n = 2$, this result was obtained before by A. Stancu [41].

In general, however, the centroid of such a polytope P is not the origin, and the characterization of cone-volume measures of general polytopes or convex bodies is still a challenging and important problem. We note that (1.4), i.e., the equality case of the subspace concentration condition is a kind of condition on the cone-volume measure which is independent of the choice of the origin (cf. Proposition 3.5 in Section 3).

In order to state a consequence of Theorem 1.1 we need the notation of an *isotropic measure*, going back to K.M. Ball's reformulation of the Brascamp–Lieb inequality in [2]. A Borel measure μ on S^{n-1} is called *isotropic* if

$$\mathrm{Id}_n = \int_{S^{n-1}} u \otimes u \,\mathrm{d}\mu(u),$$

where Id_n is the $n \times n$-identity matrix and $u \otimes u$ the standard tensor product, i.e., $u \otimes u = u\,u^\intercal$. Equating traces shows $\mu(S^{n-1}) = n$ for an isotropic measure. The subspace concentration condition holds for a Borel measure μ on S^{n-1} if and only if μ has an *isotropic normalized linear image*, i.e., there exists a $\Phi \in \mathrm{GL}(n)$ such that

$$\mathrm{Id}_n = \frac{n}{\mu(S^{n-1})} \int_{S^{n-1}} \frac{\Phi u}{\|\Phi u\|} \otimes \frac{\Phi u}{\|\Phi u\|} \,\mathrm{d}\mu(u). \tag{1.5}$$

The equivalence in this general form is due to K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [12], while the discrete case was established earlier by E.A. Carlen and

D. Cordero-Erausquin [14], and J. Bennett, A. Carbery, M. Christ and T. Tao [7] in their study of the Brascamp–Lieb inequality. Moreover, the case when strict inequality holds for all subspaces in (1.3) for a measure μ is due to B. Klartag [27]. Isotropic measures on S^{n-1} are also discussed, e.g. in F. Barthe [3,4], E. Lutwak, D. Yang and G. Zhang [31,33]. We note that isotropic measures on $\mathbb{R}^n$ play a central role in the KLS conjecture by R. Kannan, L. Lovász and M. Simonovits [25], see, e.g., F. Barthe and D. Cordero-Erausquin [5], S. Brazitikos, A. Giannopoulos, P. Valettas and B.-H. Vritsiou [13], O. Guedon and E. Milman [20], and B. Klartag [26].

From Theorem 1.1 and the equivalence (1.5) we immediately conclude

Corollary 1.2. *Every convex body $K \in \mathcal{K}^n$ has an affine image, whose cone-volume measure is isotropic.*

This, in particular, answers a question posed by E. Lutwak, D. Yang and G. Zhang [32].

Another consequence of Theorem 1.1 is related to the $\mathrm{SL}(n)$ invariant U-*functional* $\mathrm{U}(K)$ of a convex body $K \in \mathcal{K}^n$ containing the origin in its interior. It was introduced by E. Lutwak, D. Yang and G. Zhang [30] and it is defined as

$$\mathrm{U}(K) = \left(\int\limits_{u_1 \wedge \ldots \wedge u_n \neq 0} \mathrm{dV}_K(u_1) \cdots \mathrm{dV}_K(u_n) \right)^{\frac{1}{n}},$$

where the integral is over all subsets $(u_1, \ldots, u_n) \in S^{n-1} \times \cdots \times S^{n-1}$ such that the vectors $u_1, \ldots, u_n$ are linearly independent. The U-functional has proved very useful in obtaining strong inequalities for the volume of projection bodies (see, e.g., [30]). For information on projection bodies we refer to the books by Gardner [17] and Schneider [39], and for more information on the importance of centro-affine functionals we refer to C. Haberl and L. Parapatits [21,29] and the references within.

We readily have $\mathrm{U}(K) \leq \mathrm{V}(K)$, and equality holds if and only if $\mathrm{V}_K(L \cap S^{n-1}) = 0$ for any non-trivial subspace $L \subset \mathbb{R}^n$ according to K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [12]. In [12, Theorem 1.3] an optimal lower bound on the U-functional of a measure satisfying the subspace concentration condition is given. In combination with Theorem 1.1 we immediately get the best possible bound on $\mathrm{U}(K)$ in terms of $\mathrm{V}(K)$ which was conjectured in [12].

Corollary 1.3. *Let $K \in \mathcal{K}^n$ be centered. Then*

$$\mathrm{U}(K) \geq \frac{(n!)^{1/n}}{n} \mathrm{V}(K),$$

with equality if and only if K is a parallelepiped.

In particular, $\mathrm{U}(K) > (1/\mathrm{e})\mathrm{V}(K)$. For polytopes, Theorem 1.3 was shown in [23], where the special cases if K is an origin-symmetric polytope, or if $n = 2, 3$ were verified by B. He, G. Leng and K. Li [22], and G. Xiong [44], respectively.

The paper is organized as follows. In the next section we will collect some basic facts on log-concave functions and notations from convexity which will be used later on. The third section is devoted to the proof of Theorem 1.1, and in the last section we will show another characteristic property of cone-volume measures of convex bodies with centroid at the origin (Theorem 4.1).

2. Preliminaries

Good general references for the theory of convex bodies are provided by the books by Gardner [17], Gruber [19], Schneider [39] and Thompson [43].

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of a convex body $K \in \mathcal{K}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\}.$$

A boundary point $x \in \partial K$ is said to have a unit outer normal (vector) $u \in S^{n-1}$ provided $\langle x, u\rangle = h_K(u)$. $x \in \partial K$ is called singular if it has more than one unit outer normal, and $\partial_* K$ is the set of all non-singular boundary points. It is well known that the set of singular boundary points of a convex body has $\mathcal{H}_{n-1}$-measure equal to 0. For each Borel set $\omega \subseteq S^{n-1}$, the inverse spherical image of ω is the set of all points of ∂K which have an outer unit normal belonging to ω. Since the inverse spherical image of ω differs from $\nu_K^{-1}(\omega) \subseteq \partial_* K$ by a set of $\mathcal{H}_{n-1}$-measure equal to 0, we will often make no distinction between the two sets. We recall that ν_K denotes the Gauß map assigning the outer unit normal $\nu_K(x)$ to an $x \in \partial_* K$.

For $K \in \mathcal{K}^n$ the Borel measure S_K on S^{n-1} given by

$$S_K(\omega) = \mathcal{H}_{n-1}(\nu_K^{-1}(\omega))$$

is called the (Aleksandrov–Fenchel–Jessen) surface area measure. Observe that

$$\mathrm{V}(K) = \mathrm{V}_K(S^{n-1}) = \int_{S^{n-1}} \frac{h_K(u)}{n} \mathrm{d}\, S_K(u).$$

As usual, for two subsets $C, D \subseteq \mathbb{R}^n$ and non-negative reals $\nu, \rho \geq 0$ the Minkowski combination is defined by

$$\nu\, C + \rho\, D = \{\nu\, c + \rho\, d : c \in C,\ d \in D\}.$$

By the celebrated Brunn–Minkowski inequality we know that the n-th root of the volume of the Minkowski combination is a concave function. More precisely, for two convex compact sets $K_0, K_1 \in \mathcal{K}^n$ and for $\lambda \in [0, 1]$ we have

$$V((1-\lambda)\,K_0+\lambda\,K_1)^{1/n}\geq(1-\lambda)\,V(K_0)^{1/n}+\lambda\,V(K_1)^{1/n} \tag{2.1}$$

with equality for some $0<\lambda<1$ if and only if K_0 and K_1 lie in parallel hyperplanes or they are homothetic, i.e., there exist $t\in\mathbb{R}^n$ and $\rho\geq 0$ such that $K_1=t+\rho\,K_0$ (see also [16]).

Let $f:C\to\mathbb{R}_{>0}$ be a positive function on an open convex subset $C\subseteq\mathbb{R}^n$ with the property that there exists a $k\in\mathbb{N}$ such that $f^{1/k}$ is concave. Then by the (weighted) arithmetic–geometric mean inequality

$$\begin{aligned} f((1-\lambda)\,x+\lambda\,y) &= \left(f^{1/k}((1-\lambda)\,x+\lambda\,y)\right)^k \\ &\geq \left((1-\lambda)f^{1/k}(x)+\lambda f^{1/k}(y)\right)^k \\ &\geq f^{1-\lambda}(x)\cdot f^{\lambda}(y). \end{aligned}$$

This means that f belongs to the class of log-concave functions which by the positivity of f is equivalent to

$$\ln f((1-\lambda)\,x+\lambda\,y)\geq(1-\lambda)\ln f(x)+\lambda\ln f(y)$$

for $\lambda\in[0,1]$. Hence, for all $x,y\in C$ there exists a subgradient $g(y)\in\mathbb{R}^n$ such that (cf., e.g., [38])

$$\ln f(x)-\ln f(y)\leq\langle g(y),x-y\rangle. \tag{2.2}$$

If f is differentiable at y, the subgradient is the gradient of $\ln f$ at y, i.e., $g(y)=\nabla\ln f(y)=\frac{1}{f(y)}\nabla f(y)$.

For a subspace $L\subseteq\mathbb{R}^n$, let $L^\perp$ be its orthogonal complement, and for $X\subseteq\mathbb{R}^n$ we denote by $X|L$ its orthogonal projection onto L, i.e., the image of X under the linear map forgetting the part of X belonging to $L^\perp$.

Here, for $K\in\mathcal{K}^n$ and a d-dimensional subspace L, $1\leq d\leq n-1$, we are interested in the function measuring the volume of K intersected with planes parallel to $L^\perp$, i.e., in the function

$$f_{K,L}:L\to\mathbb{R}_{\geq 0}\text{ with }x\mapsto\mathcal{H}_k(K\cap(x+L^\perp)),$$

where $k=n-d$ is the dimension of $L^\perp$. By the Brunn–Minkowski inequality (2.1) and the remark above, $f_{K,L}$ is a log-concave function on $K|L$ which is positive in the relative interior of $K|L$ (cf. [1]). $f_{K,L}$ is also called the k-dimensional X-ray of K parallel to $L^\perp$ (cf. [17]). By well-known properties of (log-)concave functions we have (see, e.g., [38,39]).

Proposition 2.1.

i) *$f_{K,L}$ is continuous on* $\mathrm{int}\,(K)|L$*. Moreover, $f_{K,L}$ is Lipschitzian on any compact subset of* $(\mathrm{int}\,K)|L$*.*
ii) *$f_{K,L}$ is on* $\mathrm{int}\,(K)|L$ *almost everywhere differentiable, i.e., there exists a dense subset $D \subseteq \mathrm{int}\,(K)|L$, where $\nabla f_{K,L}$ exists.*

Now for $K \in \mathcal{K}^n$ with centroid at o, i.e., $\mathrm{cen}(K) = o$, we have by Fubini's theorem with respect to the decomposition $L \oplus L^\perp$

$$\begin{aligned} o &= \int_K x \, \mathrm{d}\mathcal{H}_n(x) \\ &= \int_{K|L} \left(\int_{(\hat{x}+L^\perp)\cap K} \tilde{x} \, \mathrm{d}\mathcal{H}_k(\tilde{x}) \right) \mathrm{d}\mathcal{H}_d(\hat{x}) \\ &= \int_{K|L} f_{K,L}(\hat{x}) \, \mathrm{cen}((\hat{x} + L^\perp) \cap K) \, \mathrm{d}\mathcal{H}_d(\hat{x}). \end{aligned}$$

Writing $\mathrm{cen}((\hat{x} + L^\perp) \cap K) = \hat{x} + \tilde{y}$ with $\tilde{y} \in L^\perp$ gives

$$\int_{K|L} f_{K,L}(\hat{x}) \, \hat{x} \, \mathrm{d}\mathcal{H}_d(\hat{x}) = 0. \tag{2.3}$$

3. Proof of Theorem 1.1

For the proof of Theorem 1.1 we will first establish some more properties of the function $f_{K,L}$, where we always assume that $L \subset \mathbb{R}^n$ is a d-dimensional linear subspace, $1 \leq d \leq n-1$, with k-dimensional orthogonal complement $L^\perp$. We recall that a function f is said to be upper semicontinuous on $K|L$ if whenever $x, y_m \in K|L$ for $m \in \mathbb{N}$ and y_m tends to x, then

$$f(x) \geq \limsup_{m\to\infty} f(y_m).$$

Although in general $f_{K,L}$ is not continuous in the points in the relative boundary of $K|L$, the purpose of the first lemma is to show that $f_{K,L}$ behaves "continuously" for sequences from the relative interior of $K|L$.

Lemma 3.1. *Let $K \in \mathcal{K}^n$.*

i) *$f_{K,L} : K|L \to \mathbb{R}_{\geq 0}$ is upper semicontinuous.*

ii) *Let* $o \in \operatorname{int} K$ *and* $x \in K|L$. *Then*

$$\lim_{m\to\infty} f_{K,L}(e^{\frac{-1}{m}}x) = f_{K,L}(x).$$

Proof. For i) let $x, y_m \in K|L$, $m \in \mathbb{N}$, be such that $\lim_{m\to\infty} y_m = x$. According to the Blaschke selection principle (cf., e.g., [39]), we may assume that the sequence of compact convex sets

$$C_m = [(y_m + L^\perp) \cap K] - y_m \subset L^\perp$$

converges to a compact convex set $C \subset L^\perp$ in the Hausdorff topology. Since the k-volume in $L^\perp$ is a continuous functional we have $\mathcal{H}_k(C) = \lim_{m\to\infty} f_{K,L}(y_m)$. However, $x + C \subset K$, and therefore $f_{K,L}(x) \geq \mathcal{H}_k(C)$.

The second property follows immediately from i), since in view of $o \in \operatorname{int} K$ and the concavity of $f_{K,L}^{1/k}$ we also know

$$f_{K,L}(e^{-\frac{1}{m}}x) \geq e^{-\frac{k}{m}} f_{K,L}(x). \qquad \square$$

In general, and in contrast to the polytopal case, the gradient $\nabla f_{K,L}$ might not be bounded, but, as the next lemma will show, the integral $\int_{K|L} \langle \nabla f_{K,L}(x), x\rangle \mathrm{d}x$ exists.

Lemma 3.2. *Let* $K \in \mathcal{K}^n$. *Then* $\int_{K|L} \langle \nabla f_{K,L}(x), x\rangle \mathrm{d}x$ *exists.*

Proof. For short we write $f = f_{K,L}$. It suffices to show that $\|\nabla f_{K,L}\|$ belongs to the class $L^1(K|L)$ of absolute integrable functions on $K|L$.

Due to Proposition 2.1 ii), $\nabla f(x)$ exists almost everywhere in $K|L$ and so we may write

$$\nabla f(x) = \nabla \left((f^{\frac{1}{k}})^k(x)\right) = k f^{\frac{k-1}{k}}(x) \nabla f^{\frac{1}{k}}(x) \tag{3.1}$$

for almost all $x \in K|L$.

By the Brunn–Minkowski theorem (2.1) the set

$$M = \{(x, f^{\frac{1}{k}}(x)) : x \in K|L\}$$

is part of the boundary of a $(d+1)$-dimensional compact convex set. Thus

$$\int_{K|L} \|\nabla f^{\frac{1}{k}}\| \, \mathrm{d}\mathcal{H}_d(x) \leq \int_{K|L} \sqrt{1 + \|\nabla f^{\frac{1}{k}}\|^2} \, \mathrm{d}\mathcal{H}_d(x) = \mathcal{H}_d(M) < \infty.$$

In view of (3.1) we obtain $\|\nabla f_{K,L}\| \in L^1(K|L)$. $\square$

The next identity has been proved in the special case of polytopes in [23]. In the discrete case, however, the proof depends heavily on the polynomial character of the function $f_{K,L}$. Here we need a different approach.

Lemma 3.3. *Let $K \in \mathcal{K}^n$ with $o \in \operatorname{int} K$. Then*

$$n\,\mathrm{V}_K(L \cap S^{n-1}) = d\,\mathrm{V}(K) + \int\limits_{K|L} \langle \nabla f_{K,L}(x), x\rangle \,\mathrm{d}\mathcal{H}_d(x).$$

Proof. We remark that due to Lemma 3.2 the identity is well-defined. Again we write $f = f_{K,L}$, and let $F : K|L \to L$ be the vector field given by

$$F(x) = f(x)x.$$

By Proposition 2.1 i), F is actually a Lipschitz vector field on any compact subset of $(\operatorname{int} K)|L$. Apparently, for $m \in \mathbb{N}$ the set

$$E_m = e^{-\frac{1}{m}} K|L \subset (\operatorname{int} K)|L$$

is a compact Lipschitz domain, whose (relative) boundary with respect to the linear space L will be denoted by $\overline{\partial}(E_m)$. Now Proposition 4.1.2 and Theorem 6.5.4 from Pfeffer [37] give the following Gauß–Green divergence theorem for Lipschitz vector fields on Lipschitz domains (which goes back to H. Federer [15])

$$\int\limits_{E_m} \operatorname{div} F(x)\,\mathrm{d}\mathcal{H}_d(x) = \int\limits_{\overline{\partial} E_m} \langle F(x), \nu_{E_m}(x)\rangle \,\mathrm{d}\mathcal{H}_{d-1}(x). \tag{3.2}$$

For $y \in \overline{\partial}(K|L)$ we certainly have $\nu_{K|L}(y) = \nu_{E_m}(e^{-\frac{1}{m}} y)$, and thus the right hand side of (3.2) becomes

$$\begin{aligned} &\int\limits_{\overline{\partial} E_m} \langle F(x), \nu_{E_m}(x)\rangle \,\mathrm{d}\mathcal{H}_{d-1}(x) \\ &\quad = e^{-\frac{(d-1)}{m}} \int\limits_{\overline{\partial}(K|L)} \langle F(e^{-\frac{1}{m}} y), \nu_{K|L}(y)\rangle \,\mathrm{d}\mathcal{H}_{d-1}(y) \\ &\quad = e^{-\frac{d}{m}} \int\limits_{\overline{\partial}(K|L)} f(e^{-\frac{1}{m}} y)\langle y, \nu_{K|L}(y)\rangle \,\mathrm{d}\mathcal{H}_{d-1}(y). \end{aligned}$$

Thus, Lemma 3.1 ii) and the Lebesgue dominated convergence theorem imply

$$\lim_{m\to\infty} \int\limits_{\partial E_m} \langle F(x), \nu_{E_m}(x)\rangle \, \mathrm{d}\mathcal{H}_{d-1}(x) = \int\limits_{\overline{\partial}(K|L)} f(y)\langle y, \nu_{K|L}(y)\rangle \, \mathrm{d}\mathcal{H}_{d-1}(y). \tag{3.3}$$

In order to evaluate the right hand side of (3.3) let $M = \partial K \cap (L^\perp + \overline{\partial}(K|L))$. Then the set of points of ∂K in M with a unique normal vector, i.e., $\partial_* K \cap M$ coincides with the set of points in $\nu_K^{-1}(L \cap S^{n-1})$. In addition, if $z \in M \cap \partial_* K$, then $\nu_{K|L}(z|L) = \nu_K(z)$. Hence, (3.3) and (1.1) lead to

$$\begin{aligned} \lim_{m\to\infty} \int\limits_{\overline{\partial} E_m} \langle F(x), \nu_{E_m}(x)\rangle \, \mathrm{d}\mathcal{H}_{d-1}(x) &= \int\limits_{\overline{\partial}(K|L)} f(y)\langle y, \nu_{K|L}(y)\rangle \, \mathrm{d}\mathcal{H}_{d-1}(y) \\ &= \int\limits_{\overline{\partial}(K|L)} \langle z, \nu_K(z)\rangle \, \mathrm{d}\mathcal{H}_{n-1}(z) = n\,\mathrm{V}_K(L \cap S^{n-1}). \end{aligned} \tag{3.4}$$

For the left hand side of (3.2) we observe that if $\nabla f(x)$ exists at $x \in \mathrm{int}\,(K)|L$, then

$$\mathrm{div} F(x) = d\, f(x) + \langle x, \nabla f(x)\rangle.$$

Thus, in view of Proposition 2.1 ii) we may write

$$\int\limits_{E_m} \mathrm{div} F(x)\, \mathrm{d}\mathcal{H}_d(x) = d \int\limits_{E_m} f(x)\, \mathrm{d}\mathcal{H}_d(x) + \int\limits_{E_m} \langle x, \nabla f(x)\rangle \, \mathrm{d}\mathcal{H}_d(x).$$

Since $\int_{K|L} f(x)\, \mathrm{d}\mathcal{H}_d(x) = \mathrm{V}(K)$ and due to Lemma 3.2 we deduce

$$\lim_{m\to\infty} \int\limits_{E_m} \mathrm{div} F(x)\, d\mathcal{H}_d(x) = dV(K) + \int\limits_{K|L} \langle x, \nabla f(x)\rangle \, d\mathcal{H}_d(x). \tag{3.5}$$

Combining (3.2), (3.4) and (3.5) completes the proof. □

If K is an o-symmetric convex body, the Brunn–Minkowski inequality (2.1) implies that the function $f_{K,L}(x)$ attains its maximum at the origin o. Hence, with (2.2) we find

$$\langle \nabla f_{K,L}(x), x\rangle \le f_{K,L}(x)\,(\ln f_{K,L}(x) - \ln f_{K,L}(o)) \le 0,$$

for every $x \in \mathrm{int}\,(K)|L$ where $\nabla f_{K,L}(x)$ exists. Although this is no longer true for centered convex bodies, the next lemma shows that it holds in the average. The proof of this lemma is essentially the one of [23, Lemma 2.2] where it is stated under the additional assumption that the integral $\int_{K|L}\langle\nabla f_{K,L}(x),x\rangle\,\mathrm{d}\mathcal{H}_d(x)$ exists. The existence is guaranteed here by Lemma 3.2. For completeness' sake we will give the short proof.

Lemma 3.4. *Let $K\in\mathcal{K}^n$ be centered. Then*

$$\int\limits_{K|L}\langle\nabla f_{K,L}(x),x\rangle\,\mathrm{d}\mathcal{H}_d(x)\le 0,$$

with equality if and only if $f_{K,L}$ is constant on $K|L$.

Proof. Again, let $f=f_{K,L}$ and let $g:\operatorname{int}(K)|L\to L$ be a subgradient of f. For $z\in(\operatorname{int}K)|L$, applying (2.2) to $y=o$ and $x=z$ first, and next to $y=z$ and $x=o$, we deduce that

$$\langle g(z),z\rangle\le\ln f(z)-\ln f(o)\le\langle g(o),z\rangle,\tag{3.6}$$

where g is a subgradient of f. In particular, if ∇f exists at $z\in(\operatorname{int}K)|L$, then $\langle\nabla f(z),z\rangle\le\langle g(o),zf(z)\rangle$. Together with the property $\operatorname{cen}(K)=o$ we get from (2.3)

$$\int\limits_{K|L}\langle\nabla f(z),z\rangle\,\mathrm{d}\mathcal{H}_d(z)\le\int\limits_{K|L}\langle g(o),zf(z)\rangle\,\mathrm{d}\mathcal{H}_d(z)=0.\tag{3.7}$$

Obviously, if f is constant the integral vanishes. Let us assume that equality holds in (3.7) and hence for almost all $z\in(\operatorname{int}K)|L$ in (3.6). In particular, we have $\ln f(z)-\ln f(o)=\langle g(o),z\rangle$, and in turn $f(z)=f(o)e^{\langle g(o),z\rangle}$ for almost all $z\in(\operatorname{int}K)|L$. Since f is continuous on $(\operatorname{int}K)|L$, Lemma 3.1 ii) yields that $f(z)=f(o)e^{\langle g(o),z\rangle}$ for all $z\in K|L$. However, we even know that $f^{\frac{1}{k}}$ is concave and thus, $g(o)=o$, i.e., f is constant. □

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let L be a d-dimensional linear subspace. By Lemma 3.3 and Lemma 3.4, we immediately obtain the subspace concentration inequality

$$\mathrm{V}_K(L\cap S^{n-1})=\frac{d}{n}V(K)+\frac{1}{n}\int\limits_{K|L}\langle\nabla f_{K,L}(x),x\rangle\,\mathrm{d}\mathcal{H}_d(x)\le\frac{d}{n}\mathrm{V}(K).$$

Let us assume that equality holds, and hence $f_{K,L}(x)=f_{K,L}(o)$ for $x\in K|L$ according to Lemma 3.4. For $x\in K|L$ let $C_x=K\cap(x+L^\perp)$. Since for any $x\in K|L$ there exists $\eta>0$ with $-\eta x\in K|L$ we have

$$\frac{\eta}{1+\eta}C_x+\frac{1}{1+\eta}C_{-\eta x}\subseteq C_o.$$

The Brunn–Minkowski inequality (2.1) implies

$$f_{K,L}(o)^{\frac{1}{n-d}} = \mathcal{H}_{n-d}(C_o)^{\frac{1}{n-d}} \geq \mathcal{H}_{n-d}\left(\frac{\eta}{1+\eta}\,C_x + \frac{1}{1+\eta}\,C_{-\eta x}\right)^{\frac{1}{n-d}}$$
$$\geq \frac{\eta}{1+\eta} f_{K,L}(x)^{\frac{1}{n-d}} + \frac{1}{1+\eta} f_{K,L}(-\eta x)^{\frac{1}{n-d}}.$$

Since $f_{K,L}(x) = f_{K,L}(-\eta x) = f_{K,L}(o)$ we have equality in the above inequality and by the equality characterization in the Brunn–Minkowski inequality we conclude that C_x is a translate of C_o, i.e., we have $C_x = [\mathrm{cen}(C_x) - \mathrm{cen}(C_o)] + C_o$.

Now let $v_0, v_1, \ldots, v_d \in K|L$ be affinely independent with $\sum_{i=0}^d v_i = o$. Then $\sum_{i=0}^d \frac{1}{d+1} C_{v_i} \subseteq C_o$, and thus $\sum_{i=0}^d \frac{1}{d+1} \mathrm{cen}(C_{v_i}) = \mathrm{cen}(C_o)$. In particular,

$$\mathrm{cen}(C_o) \in A = \mathrm{aff}\{\mathrm{cen}(C_{v_0}), \ldots, \mathrm{cen}(C_{v_d})\},$$

where aff{} denotes the affine hull. For every $x \in K|L$ there exists $\eta > 0$ such that $-\eta x \in [v_0, \ldots, v_d]$. Hence we may write $\lambda x + \sum_{i=0}^d \lambda_i v_i = o$ with $\lambda + \sum_{i=0}^d \lambda_i = 1$ and $\lambda, \lambda_i \geq 0$ for $i = 0, \ldots, d$. As above it follows $\lambda\,\mathrm{cen}(C_x) + \sum_{i=0}^d \lambda_i \mathrm{cen}(C_{v_i}) = \mathrm{cen}(C_o)$, and so $\mathrm{cen}(C_x) \in A$ as well.

Therefore, setting $\widetilde{L} = A - \mathrm{cen}(C_o)$, $M = (K \cap \widetilde{L})$ we get $K = M + C_o$ and M and C_o are contained in complementary subspaces. In particular, $\mathrm{supp}\,\mathrm{V}_K \subseteq L \cup \widetilde{L}^\perp$ and $L \cap \widetilde{L}^\perp = \{o\}$.

The proof of the reverse direction of the equality characterization (1.4) is given in the next proposition (Proposition 3.5), since it is a condition on the cone-volume measure which is independent of the location of the origin. □

The proof of Proposition 3.5 uses Minkowski's characterization theorem of the surface area measure of a convex body (cf., e.g., [39, Theorem 8.2.2]). It says that for dimensions $n \geq 2$ a finite Borel measure μ on S^{n-1} is the surface area measure S_K of an n-dimensional convex body K in $\mathbb{R}^n$ if and only if

$$\text{(a)} \quad \mu\left(\{v \in S^{n-1} : \langle u, v\rangle > 0\}\right) > 0 \text{ for any } u \in S^{n-1},$$
$$\text{(b)} \quad \int_{S^{n-1}} v\,\mathrm{d}\mu(v) = o. \tag{3.8}$$

In addition, if M is another n-dimensional convex body in $\mathbb{R}^n$ with $S_M = S_K$ then M is a translate of K.

In dimension one, i.e., $n = 1$, we have for any convex body (segment) $K \subset \mathbb{R}^1$ that $S_K(\{1\}) = S_K(\{-1\}) = 1$, and therefore both (a) and (b) still hold for $\mu = S_K$ on S^0. However, μ is a finite Borel measure on S^0 satisfying (a) and (b) if and only if there exists a $s > 0$ such that $\mu(\{1\}) = \mu(\{-1\}) = s$.

Proposition 3.5. *Let $K \in \mathcal{K}^n$ with $o \in \operatorname{int} K$ and let $\operatorname{supp} \mathrm{V}_K \subseteq L \cup \widetilde{L}$ for proper complementary linear subspaces $L, \widetilde{L} \subset \mathbb{R}^n$. Then*

$$\mathrm{V}_K(L \cap S^{n-1}) = \frac{\dim L}{n} \mathrm{V}_K(S^{n-1}) = \frac{\dim L}{n} \mathrm{V}(K).$$

Proof. In order to shorten the following let $L_1 = L$, $L_2 = \widetilde{L}$, $m_1 = \dim L_1$, $m_2 = \dim L_2 = n - m_1$, and hence $1 \leq m_1, m_2 \leq n-1$. The concentration of V_K onto $L_1 \cup L_2$ also implies the same for the surface area measure S_K, i.e., $\operatorname{supp} S_K \subseteq L_1 \cup L_2$ (cf. (1.1)), and let $S_{K,i}(\cdot)$ be the restrictions of $S_K(\cdot)$ onto $L_i \cap S^{n-1}$, $i = 1, 2$.

Since S_K satisfies (3.8), both $S_{K,i}$ also satisfy (3.8). Hence, as discussed above, there exist $\lambda_i > 0$ and m_i-dimensional convex bodies $C_i \subset L_i$ such that for any Borel set $\omega \subseteq L_i \cap S^{n-1}$ and $i = 1, 2$

$$S_{C_i}(\omega) = \lambda_i \cdot S_{K,i}(\omega) = \lambda_i \cdot S_K(\omega).$$

Let $\overline{M}_1 = L_2^\perp \cap (C_1 + L_1^\perp)$, $\overline{M}_2 = L_1^\perp \cap (C_2 + L_2^\perp)$ and $\overline{K} = \overline{M}_1 + \overline{M}_2$. Then, in particular, $\overline{K}|L_i = \overline{M}_i|L_i = C_i$, $i = 1, 2$, and for any Borel set $\omega \subseteq L_1 \cap S^{n-1}$ we have

$$S_{\overline{K}}(\omega) = \mathcal{H}_{m_2}(\overline{M}_2) \cdot S_{C_1}(\omega) = \lambda_1 \mathcal{H}_{m_2}(\overline{M}_2) \cdot S_K(\omega)$$

and, analogously, we obtain

$$S_{\overline{K}}(\omega) = \mathcal{H}_{m_1}(\overline{M}_1) \cdot S_{C_2}(\omega) = \lambda_2 \mathcal{H}_{m_1}(\overline{M}_1) \cdot S_K(\omega)$$

for any Borel set $\omega \subseteq L_2 \cap S^{n-1}$.

Therefore there exists $\alpha_i > 0$ such that $S_K = S_{\alpha_1 \overline{M}_1 + \alpha_2 \overline{M}_2}$, and from the uniqueness of the solution of the Minkowski problem up to translations we deduce that there exist translates M_i of $\alpha_i \overline{M}_i$ such that $K = M_1 + M_2$, and $M_1 \subset L_2^\perp$, $M_2 \subset L_1^\perp$. In particular, both M_i contain o in their relative interiors and with $M_1' = M_1|L_1$ (which is homothetic to C_1) we get

$$\mathrm{V}(K) = \mathcal{H}_{m_1}(M_1') \cdot \mathcal{H}_{m_2}(M_2) = \mathcal{H}_{m_2}(M_2) \cdot \int\limits_{\partial_* M_1'} \frac{\langle x, \nu_{M_1'}(x) \rangle}{m_1} \, \mathrm{d}\mathcal{H}_{m_1-1}(x). \tag{3.9}$$

However (1.1) yields that

$$\begin{aligned} \mathrm{V}_K(L_1 \cap S^{n-1}) &= \int\limits_{M_2 + \partial_* M_1'} \frac{\langle x, \nu_K(x) \rangle}{n} \, \mathrm{d}\mathcal{H}_{n-1}(x) \\ &= \mathcal{H}_{m_2}(M_2) \cdot \int\limits_{\partial_* M_1'} \frac{\langle x, \nu_{M_1'}(x) \rangle}{n} \, \mathrm{d}\mathcal{H}_{m_1-1}(x). \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) implies the proposition. □

4. Another property of the cone-volume measure of centered convex bodies

We start by recalling two basic notions/concepts about convex bodies. Firstly, $K \in \mathcal{K}^n$ is called a *cylinder* if it is of the form $[p, q] + C$ for $p, q \in \mathbb{R}^n$ and an $(n-1)$-dimensional convex compact set C. We call $p + C$ and $q + C$ the bases of the cylinder, and $[p, q]$ a generating segment.

Secondly, let $v \in S^{n-1}$ and let $M \in \mathcal{K}^n$ with support function $h_M(\cdot)$. For any t with $-h_M(-v) < t < h_M(v)$, we replace the section $M \cap (t\,v + v^\perp)$ with the $(n-1)$-ball of the same $\mathcal{H}_{n-1}$-measure, centered at $t\,v$ in $tv + v^\perp$. Here, $v^\perp$ is the abbreviation for the linear subspace orthogonal to v. The closure $\widetilde{M}$ of the union of these $(n-1)$-balls is called the *Schwarz rounding of* M with respect to the line $\mathbb{R}v$. It is a convex body by the Brunn–Minkowski theorem, and apparently we have $\mathrm{V}(\widetilde{M}) = \mathrm{V}(M)$. If $\widetilde{M}$ is a cylinder whose bases are orthogonal to v, then all sections of the form $M \cap (t\,v + v^\perp)$ are of the same $\mathcal{H}_{n-1}$-measure, and hence M is a cylinder, as well. For more on Schwarz rounding we refer to [19].

Theorem 4.1. *Let $K \in \mathcal{K}^n$ be centered. Then*

$$\mathrm{V}_K(\Omega) \geq \frac{1}{2n}\mathrm{V}(K),$$

for any open hemisphere $\Omega = \{u \in S^{n-1} : \langle u, v\rangle > 0\}$, $v \in S^{n-1}$. Equality holds if and only if K is a cylinder whose generating segment is parallel to the center v of Ω.

Proof. For simplification we assume $\mathrm{V}(K) = 1$. Let

$$\Omega = \{u \in S^{n-1} : \langle u, v\rangle > 0\}$$

be an open hemisphere with $v \in S^{n-1}$. The idea of the proof is to construct a cylinder Z with rotational symmetry around $\mathbb{R}v$ such that $V(Z) = 1$, and

$$\mathrm{V}_K(\Omega) \geq \mathrm{V}_{Z-\mathrm{cen}(Z)}(\Omega) = \frac{1}{2n}.$$

To construct Z, first we apply a linear transform Φ to ensure that the supporting hyperplane H at $\lambda v \in \partial \Phi K$ for suitable $\lambda > 0$ is orthogonal to v and $\mathrm{V}_K(\Omega) = \mathrm{V}_{\Phi K}(\Omega)$. Then we shake ΦK onto H to obtain K' such that H is still a supporting hyperplane and $K' \cap H = K'|H$. The next step is to use Schwarz rounding with respect to $\mathbb{R}v$ and finally we compare the resulting $\widetilde{K}$ to a suitable cylinder Z with rotational symmetry around $\mathbb{R}v$.

For any convex body $M \in \mathcal{K}^n$ with $o \in \mathrm{int}\, M$ and $x \in M|v^\perp$ let

$$\rho_M(x) = \max\{t \in \mathbb{R} : x + t\,v \in M\},$$

and let $\varphi_M(x) = x + \rho_M(x)\,v$. In particular the points of ∂M where all outer normals have an acute angle with v are of the form $\varphi_M(x)$ for $x \in \operatorname{int} M|v^\perp$. Therefore

$$\mathrm{V}_M(\Omega) = \mathrm{V}(\Xi_M) \quad \text{for } \Xi_M = \bigcup_{x\in M|v^\perp} [o, \varphi_M(x)].$$

For $x \in (\operatorname{int} M|v^\perp)\backslash\{o\}$ let $z = \theta^{-1}x \in \partial M|v^\perp$ for some $\theta \in (0,1)$. Since $[\varphi_M(z), o, \varphi_M(o)] \subseteq \Xi_M$ we have

$$x + \mathbb{R}v \text{ intersects } \Xi_M \text{ in a segment of length at least } (1-\theta)\|\varphi_M(o)\|. \tag{4.1}$$

Let $\lambda = \rho_K(o)$ and hence $\lambda v \in \partial K$, and let H_0 be a supporting hyperplane of K at λv. For a basis $v_1, \ldots, v_{d-1}$ of the linear subspace $H_0 - \lambda v$, we define the linear map Φ by $\Phi v = v$, and $\Phi v_i = v_i|v^\perp$ for $i = 1, \ldots, d-1$. In particular, we have $\operatorname{cen}(\Phi K) = o$, $\det \Phi = 1$, $(\Phi K)|v^\perp = K|v^\perp$, and $H = \lambda v + v^\perp$ is a supporting hyperplane of ΦK at $\lambda v \in \partial(\Phi K)$. In addition, $\mathrm{V}(\Phi K) = \mathrm{V}(K) = 1$, and $\Xi_{\Phi K} = \Phi(\Xi_K)$ implies that

$$\mathrm{V}_{\Phi K}(\Omega) = \mathrm{V}_K(\Omega).$$

Next we shake ΦK down to the supporting hyperplane H, i.e., for each $x \in (\Phi K)|v^\perp$, we translate the section $(x + \mathbb{R}v) \cap (\Phi K)$ by $(\lambda - \rho_{\Phi K}(x))v$. Hence one endpoint of the translated section lies in H. The resulting convex body is denoted by K' and we have

$$K'|v^\perp = (\Phi K)|v^\perp = C - \lambda\, v \quad \text{for } C = K' \cap H.$$

In addition $\mathrm{V}(K') = \mathrm{V}(\Phi K) = 1$, and $\Xi_{K'}$ is the cone $[o, C]$.

For $x \in (\operatorname{int} K'|v^\perp)\backslash\{o\}$, it follows by (4.1) that $x + \mathbb{R}v$ intersects $\Xi_{\Phi K}$ in a segment of length at least the length of $\Xi_{K'} \cap (x + \mathbb{R}v)$ and so

$$\mathrm{V}_K(\Omega) = \mathrm{V}_{\Phi K}(\Omega) = \mathrm{V}(\Xi_K) \geq \mathrm{V}(\Xi_{K'}) = \mathrm{V}_{K'}(\Omega). \tag{4.2}$$

Next we consider the position of the centroid $\operatorname{cen}(K')$ of K'. Due to the definition of the shaking process we know that

$$\begin{aligned} &\langle \operatorname{cen}(K'), u\rangle = \langle \operatorname{cen}(\Phi K), u\rangle = 0 \quad \text{for } u \in v^\perp, \text{ and} \\ &\langle \operatorname{cen}(K'), v\rangle \geq \langle \operatorname{cen}(\Phi K), v\rangle = 0 \quad \text{with equality if and only if } K' = \Phi K. \end{aligned}$$

We deduce

$$\operatorname{cen}(K') = \eta\, v \quad \text{where } \eta \geq 0, \text{ with } \eta = 0 \text{ if and only if } K' = \Phi K, \tag{4.3}$$

and in view of (4.2)

$$\mathrm{V}_K(\Omega) \geq \mathrm{V}_{K'-\operatorname{cen}(K')}(\Omega) \text{ with equality if and only if } K' = \Phi K. \tag{4.4}$$

Now let $\widetilde{K}$ be the Schwarz rounding of K' with respect to the line $\mathbb{R}v$. Then $\mathrm{V}(\widetilde{K}) = \mathrm{V}(K') = 1$, $\mathrm{V}_{K'}(\Omega) = \mathrm{V}_{\widetilde{K}}(\Omega)$ and by the rotational symmetry of $\widetilde{K}$ with respect to $\mathbb{R}\,v$, we get from (4.3)

$$\mathrm{cen}(\widetilde{K}) = \mathrm{cen}(K') = \eta\, v \quad \text{where } \eta \geq 0, \text{ with } \eta = 0 \text{ if and only if } K' = \Phi K \tag{4.5}$$

and so by (4.4)

$$\mathrm{V}_K(\Omega) \geq \mathrm{V}_{\widetilde{K} - \mathrm{cen}(\widetilde{K})}(\Omega) \text{ with equality if and only if } K' = \Phi K. \tag{4.6}$$

Finally we compare $\widetilde{K}$ to the cylinder Z over the $(n-1)$-ball $H \cap \widetilde{K}$, where $\mathrm{V}(Z) = \mathrm{V}(\widetilde{K}) = 1$ and Z and K lie on the same side of H. Observe that $V_{\widetilde{K}}(\Omega) = V_Z(\Omega)$ and by the rotational symmetry of Z we certainly have $\langle \mathrm{cen}(Z), u\rangle = 0$ for $u \in v^{\perp}$. On the other hand the rotational symmetry of $\widetilde{K}$ and $\widetilde{K}|v^{\perp} = (H \cap \widetilde{K}) - \lambda v$ yield that

$$\langle x, v\rangle > -h_Z(-v) > \langle y, v\rangle \quad \text{for all } x \in (\mathrm{int} Z)\backslash\widetilde{K} \text{ and } y \in \widetilde{K}\backslash Z.$$

Therefore,

$$\mathrm{cen}(Z) = \tau\, v \quad \text{where } \tau \geq \eta, \text{ with } \tau = \eta \text{ if and only if } Z = \widetilde{K}.$$

Together with (4.5) and (4.6) we conclude

$$\mathrm{V}_K(\Omega) \geq \mathrm{V}_{Z - \mathrm{cen}(Z)}(\Omega) = 1/(2n) \quad \text{with equality iff } K' = \Phi K \text{ and } Z = \widetilde{K},$$

i.e., with equality if and only if K is a cylinder whose generating segment is parallel to v. □

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References

[1] K.M. Ball, Logarithmically concave functions and sections of convex sets in $\mathbb{R}^n$, Studia Math. 88 (1988) 69–84.
[2] K.M. Ball, Volume ratios and a reverse isoperimetric inequality, J. Lond. Math. Soc. 44 (1991) 351–359.
[3] F. Barthe, On a reverse form of the Brascamp–Lieb inequality, Invent. Math. 134 (1998) 335–361.
[4] F. Barthe, A continuous version of the Brascamp–Lieb inequalities, in: Geometric Aspects of Functional Analysis, in: Lecture Notes in Math., vol. 1850, Springer, Berlin, 2004, pp. 53–63.
[5] F. Barthe, D. Cordero-Erausquin, Invariances in variance estimates, Proc. Lond. Math. Soc. 106 (2013) 33–64.

[6] F. Barthe, O. Guedon, S. Mendelson, A. Naor, A probabilistic approach to the geometry of the l_p^n-ball, Ann. Probab. 33 (2005) 480–513.
[7] J. Bennett, A. Carbery, M. Christ, T. Tao, The Brascamp–Lieb inequalities: finiteness, structure, and extremals, Geom. Funct. Anal. 17 (2007) 1343–1415.
[8] K.J. Böröczky, P. Hegedűs, The cone volume measure of antipodal points, Acta Math. Hungar. 146 (2) (August 2015) 449–465, http://dx.doi.org/10.1007/s10474-015-0511-z.
[9] K.J. Böröczky, P. Hegedűs, G. Zhu, On the discrete logarithmic Minkowski problem, Int. Math. Res. Not. IMRN (2015), http://dx.doi.org/10.1093/imrn/rnv189, in press, arXiv:1409.7907.
[10] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, The log-Brunn–Minkowski inequality, Adv. Math. 231 (2012) 1974–1997.
[11] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, The logarithmic Minkowski problem, J. Amer. Math. Soc. 26 (2013) 831–852.
[12] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, Affine images of isotropic measures, J. Differential Geom. 99 (3) (2015) 407–442.
[13] S. Brazitikos, A. Giannopoulos, P. Valettas, B.-H. Vritsiou, Geometry of Isotropic Convex Bodies, Math. Surveys Monogr., vol. 196, AMS, 2014.
[14] E. Carlen, D. Cordero-Erausquin, Subadditivity of the entropy and its relation to Brascamp–Lieb type inequalities, Geom. Funct. Anal. 19 (2009) 373–405.
[15] H. Federer, The Gauss–Green theorem, Trans. Amer. Math. Soc. 58 (1945) 44–76.
[16] R.J. Gardner, The Brunn–Minkowski inequality, Bull. Amer. Math. Soc. 39 (2002) 355–405.
[17] R.J. Gardner, Geometric Tomography, 2nd edition, Encyclopedia Math. Appl., vol. 58, Cambridge University Press, Cambridge, 2006.
[18] M. Gromov, V.D. Milman, Generalization of the spherical isoperimetric inequality for uniformly convex Banach spaces, Compos. Math. 62 (1987) 263–282.
[19] P.M. Gruber, Convex and Discrete Geometry, Grundlehren Math. Wiss., vol. 336, Springer, Berlin, 2007.
[20] O. Guedon, E. Milman, Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures, Geom. Funct. Anal. 21 (2011) 1043–1068.
[21] C. Haberl, L. Parapatits, The centro-affine Hadwiger theorem, J. Amer. Math. Soc. 27 (3) (2014) 685–705.
[22] B. He, G. Leng, K. Li, Projection problems for symmetric polytopes, Adv. Math. 207 (2006) 73–90.
[23] M. Henk, E. Linke, Cone-volume measures of polytopes, Adv. Math. 253 (2014) 50–62.
[24] M. Henk, A. Schürmann, J.M. Wills, Ehrhart polynomials and successive minima, Mathematika 52 (2005) 1–16.
[25] R. Kannan, L. Lovász, M. Simonovits, Isoperimetric problems for convex bodies and a localization lemma, Discrete Comput. Geom. 13 (1995) 541–559.
[26] B. Klartag, A Berry–Esseen type inequality for convex bodies with an unconditional basis, Probab. Theory Related Fields 145 (2009) 1–33.
[27] B. Klartag, On nearly radial marginals of high-dimensional probability measures, J. Eur. Math. Soc. (JEMS) 12 (2010) 723–754.
[28] M. Ludwig, General affine surface areas, Adv. Math. 224 (2010) 2346–2360.
[29] M. Ludwig, M. Reitzner, A classification of $SL(n)$ invariant valuations, Ann. of Math. 172 (2010) 1223–1271.
[30] E. Lutwak, D. Yang, G. Zhang, A new affine invariant for polytopes and Schneider's projection problem, Trans. Amer. Math. Soc. 353 (2001) 1767–1779.
[31] E. Lutwak, D. Yang, G. Zhang, Volume inequalities for subspaces of L_p, J. Differential Geom. 68 (2004) 159–184.
[32] E. Lutwak, D. Yang, G. Zhang, L^p John ellipsoids, Proc. Lond. Math. Soc. 90 (2005) 497–520.
[33] E. Lutwak, D. Yang, G. Zhang, Volume inequalities for isotropic measures, Amer. J. Math. 129 (2007) 1711–1723.
[34] A. Naor, The surface measure and cone measure on the sphere of l_p^n, Trans. Amer. Math. Soc. 359 (2007) 1045–1079.
[35] A. Naor, D. Romik, Projecting the surface measure of the sphere of l_p^n, Ann. Inst. Henri Poincaré Probab. Stat. 39 (2003) 241–261.
[36] G. Paouris, E. Werner, Relative entropy of cone measures and L_p centroid bodies, Proc. Lond. Math. Soc. 104 (2012) 253–286.
[37] W.F. Pfeffer, The Divergence Theorem and Sets of Finite Perimeter, CRC Press, Boca Raton, FL, 2012.
[38] R.T. Rockafellar, Convex Analysis, Princeton University Press, 1997.

[39] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Encyclopedia Math. Appl., vol. 44, Cambridge University Press, Cambridge, 1993, second expanded edition, 2014.
[40] A. Stancu, The discrete planar L_0-Minkowski problem, Adv. Math. 167 (2002) 160–174.
[41] A. Stancu, On the number of solutions to the discrete two-dimensional L_0-Minkowski problem, Adv. Math. 180 (2003) 290–323.
[42] A. Stancu, Centro-affine invariants for smooth convex bodies, Int. Math. Res. Not. IMRN (2012) 2289–2320.
[43] A.C. Thompson, Minkowski Geometry, Encyclopedia Math. Appl., vol. 63, Cambridge University Press, Cambridge, 1996.
[44] G. Xiong, Extremum problems for the cone-volume functional of convex polytopes, Adv. Math. 225 (2010) 3214–3228.
[45] G. Zhu, The logarithmic Minkowski problem for polytopes, Adv. Math. 262 (2014) 909–931.
[46] G. Zhu, The centro-affine Minkowski problem for polytopes, J. Differential Geom. 101 (1) (2015) 159–174.