About the centroid body and the ellipsoid of inertia

T. Bisztriczky and K. Böröczky*, Jr.

In honor of Helge Tverberg

1 The centroid body

Recall that the support function of a compact convex set K is defined as $h_K(u) = \max_{x\in K}\{\langle u, x\rangle\}$. The support function h_K is positive homogeneous and convex, and any function with these properties is the support function of some compact convex set (see the illuminating paper of M. Berger [2], or the classic [5] by T. Bonnesen & W. Fenchel).

Let C be a convex body in $\mathbb{R}^n$; namely, C is compact convex and the interior is non–empty. Then there exists some o–symmetric convex body ΓC, the so called *centroid body*, whose support function is

$$h_{\Gamma C}(u) = \frac{1}{V(C)} \int_C |\langle u, x\rangle|\, dx.$$

The name originates from the fact that if C is o–symmetric then there exists a nice description of ΓC: For any $u \in S^{n-1}$, denote by $\gamma(u)$ the centroid of the convex set $\{x \in C : \langle u, x\rangle \geq 0\}$. Then $\gamma(u)$ just parameterizes the boundary of ΓC, and u is actually the exterior unit normal at $\gamma(u)$.

Centroid bodies were introduced by C.M. Petty [13], but in some form they already appeared in the works of C. Dupin (*cf.* [8]) and W. Blaschke (*cf.* [3]). For all the basic properties of centroid bodies mentioned in this section, consult the paper [12] of V.D. Milman & A. Pajor, or the survey article [11] by E. Lutwak, or Chapter 9 of the book [9] by R.J. Gardner.

If φ is linear then the centroid body of $\varphi(C)$ is $\varphi(\Gamma C)$. If the origin $o \in C$ then a characteristic property is that $V(\Gamma C)$ is proportional to the average volume of simplices in C such that o is one of the vertices. To put this into a more precise form,

*Supported by OTKA, Hungary, and NSERC, Canada. The paper was written during a visit to University College London

we use the notation $[x_1, \ldots, x_m]$ to denote the convex hull of the points $x_1, \ldots, x_m$. Then regardless of whether or not $o \in C$, we have

$$V(\Gamma C) = \frac{2^n}{V(C)^n} \int_C \cdots \int_C V\left([0, x_1, \ldots, x_n]\right) \, dx_1 \ldots dx_n. \tag{1}$$

Denote by κ_n the volume of the Euclidean unit ball B in $\mathbb{R}^n$. The Busemann–Petty projection inequality states that (*cf.* C.M. Petty [13])

$$V(\Gamma C) \geq \left(\frac{2\kappa_{n-1}}{(n+1)\kappa_n}\right)^n \cdot V(C), \tag{2}$$

where equality holds if and only if C is an ellipsoid. The equivalent statement for the average of the volume of the simplices (*cf.* H. Busemann [7]) is called the Busemann random simplex inequality.

Our first goal is to provide a converse of (2) in the planar case. We start with o–symmetric domains because most of the applications are concerned with them.

THEOREM 1.1 *Assume that C is an o–symmetric convex body in $\mathbb{R}^2$. Then the area $A(\Gamma C) \leq \frac{5}{27} \cdot A(C)$, and equality holds if and only if C is a parallelogram.*

Note that if E is an ellipse and P is a parallelogram with the same area then (see (2) and Lemma 3.1)

$$\frac{A(\Gamma P)}{A(\Gamma E)} = \frac{5\pi^2}{48} = 1.0280\ldots$$

Therefore the area of the centroid domain is almost completely determined by the area of C. This points to one formulation of the so called *slicing problem*: On one hand, (2) and the Stirling formula yield that

$$V(\Gamma C)^{\frac{1}{n}} > \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{n}} \cdot V(C)^{\frac{1}{n}}.$$

Now the slicing problem asks whether there exists an absolute constant c such that

$$V(\Gamma C)^{\frac{1}{n}} < c \cdot \frac{1}{\sqrt{n}} \cdot V(C)^{\frac{1}{n}}. \tag{3}$$

An early formulation baptized the problem (due to J.D. Vaaler, *ca.* 1980); namely, is there an absolute constant c' such that if C is o–symmetric then

$$V(C)^{\frac{n-1}{n}} \leq c' \cdot \max_{\theta \in S^{n-1}} \left(C \cap \theta^{\perp}\right)?$$

The equivalence of the two formulations of the slicing problem is presented in V. Milman & A. Pajor [12], Section 5. A third formulation is given in Section 2, and see V.D. Milman & A. Pajor [12], Section 5, or R.J. Gardner [9], Notes to Section 9, for thorough discussion.

CONJECTURE 1.2 *Given the volume of an o–symmetric convex body in $\mathbb{R}^n$, $n \geq 3$, the volume of the centroid body is maximized by the parallelotopes.*

Most probably, the parallelotopes are the only extremal bodies. If the conjecture holds then it yields the existence of the absolute constant c for the slicing problem (see Lemma 3.1).

Next, let us turn to a conjecture of E. Lutwak (personal communication):

$$\begin{array}{c}\textit{If } C \textit{ is an o–symmetric convex body in } \mathbb{R}^n \textit{ then the points} \\ \left|\Gamma(C \cap \theta^\perp)\right| \cdot \theta \textit{ where } \theta \in S^{n-1}, \textit{ describe the boundary of a convex body.}\end{array} \qquad (4)$$

Here $|\cdot|$ stands for the $(n-1)$–dimensional Lebesgue measure. Now the Busemann intersection inequality (*cf.* H. Busemann [6]) says that if we replace the $(n-1)$–measure of the centroid body of the section with the $(n-1)$–measure of the section then the resulting surface is convex. Therefore the conjecture holds in $\mathbb{R}^2$, and (2) and Theorem 1.1 yield that in $\mathbb{R}^3$, the surface of E. Lutwak is convex up to a constant 1.0280.

On the other hand, we prove that the conjecture (4) fails to hold in $\mathbb{R}^n$ for infinitely many n (see Lemma 7.2). We would like to point out the following interesting phenomenon: Let C and C' be o–symmetric convex bodies in $\mathbb{R}^n$ such that $C \subset C'$. Then for certain n, it may happen that $V(\Gamma C') < V(\Gamma C)$ (see (19)). On the hand, $A(\Gamma C') \geq A(\Gamma C)$ in $\mathbb{R}^2$ (see Lemma 7.3). It would be interesting to know whether the conjecture (4) holds in $\mathbb{R}^3$.

Finally, we discuss convex bodies, which may not be o–symmetric. Given the the volume of a convex body C in $\mathbb{R}^n$, it is meaningless to ask for the maximum of $V(\Gamma C)$ because moving C to infinity increases $V(\Gamma C)$ beyond any bound. Therefore we assume that $o \in C$. This condition is also natural from the point of view that (1) has a geometric meaning in this case.

We consider again only the planar version.

THEOREM 1.3 *Let C be a convex body in $\mathbb{R}^2$. If $o \in C$ then $A(\Gamma C) \leq \frac{16}{27} \cdot A(C)$, and equality holds if and only if C is a triangle with o as a vertex.*

We conjecture that the analogous statement holds also in higher dimensions.

2 The ellipsoid of inertia

The main reference to this section is the paper V. Milman & A. Pajor [12], and the basic statements are summarized in R. Gardner [9], Notes to Section 9.

Let C be a convex body in $\mathbb{R}^n$. Then the function

$$\mathcal{B}_C(u,v) = \frac{1}{V(C)} \cdot \int_C \langle u,x\rangle \cdot \langle v,x\rangle \, dx$$

is symmetric, bilinear and $\mathcal{B}_C(u,u) > 0$ for $u \neq 0$. Therefore there exists an o–symmetric ellipsoid $\Gamma_2 C$ whose support function is

$$h_{\Gamma_2 C}(u) = \sqrt{\frac{1}{V(C)} \cdot \int_C \langle u,x\rangle^2 \, dx}.$$

This ellipsoid is usually called the *ellipsoid of inertia*, but certain associated homothetic ellipsoids are known as Fenchel ellipsoid (*cf.* C.M. Petty [13]), or Legendre ellipsoid (*cf.* W. Blaschke [4] or V. Milman & A. Pajor [12], Section 1.1). We were not able to find out, who initiated the investigation of this notion, which definitely traces back at least to the 19th century physics. The relation to physics is that the so called Legendre ellipsoid has the same second moment of inertia as C with respect to any hyperplane through the origin (*cf.* C.M. Petty [13]). Note that the ellipsoid of inertia is also invariant under linear transformations (see E. Lutwak [10] for a detailed proof).

According to a classical observation (going back to the 19th century, but can be found say in W. Blaschke [4], or in C.M. Petty [13])),

$$V(\Gamma_2 C) = \kappa_n \cdot \sqrt{n!} \cdot \sqrt{\frac{1}{V(C)^n} \int_C \cdots \int_C V\left([0,x_1,\ldots,x_n]^2\right) \, dx_1 \ldots dx_n}. \qquad (5)$$

It is also a classical fact that given the volume of C, the volume of $\Gamma_2 C$ is minimized by the ellipsoids (*cf.* W. Blaschke [4]).

Now assume that C is o–symmetric. This restriction is not essential for various considerations below, but we would like to emphasize the relation to the slicing problem. The Hölder inequality yields right away that $h_{\Gamma C}(u) \leq h_{\Gamma_2 C}(u)$, and hence

$$\Gamma C \subset \Gamma_2 C. \qquad (6)$$

On the other hand, there exists an absolute constant $c_0 > 1$ such that

$$\Gamma_2 C \subset c_0 \cdot \Gamma C \qquad (7)$$

(see V. Milman & A. Pajor [12], Section 1.4). Now the method of proving Theorem 1.1 also yields

THEOREM 2.1 *Assume that C is an o–symmetric convex body in $\mathbb{R}^2$. Then the area $A(\Gamma_2 C) \leq \frac{\pi}{12} \cdot A(C)$, and equality holds if and only if C is a parallelogram.*

We believe

CONJECTURE 2.2 *Given the volume of an o–symmetric convex body in $\mathbb{R}^n$, $n \geq 3$, the volume of the ellipsoid of inertia is maximized by the parallelotopes.*

Most probably, the parallelotopes are again the only extremal bodies. If the conjecture holds then it solves the the slicing problem (see Lemma 3.1).

For the sake of completeness, we recall yet an other formulation of the slicing problem due to J. Bourgain *ca.* 1982 (see also V. Milman & A. Pajor [12], Section 5): Assume that C is in isotropic position; namely, $V(C) = 1$ and $\Gamma_2 C$ is a ball. Then the slicing problem asks for an absolute constant c_1 such that if θ is a unit vector then

$$\int_C \langle \theta, x \rangle^2 \, dx < c_1.$$

Here the left hand side is independent of θ because of the isotropic position.

In case of possibly not o–symmetric planar convex bodies, the proof of Theorem 1.3 can be easily adopted to the ellipse of inertia:

THEOREM 2.3 *Let C be a convex body in $\mathbb{R}^2$. If $o \in C$ then $A(\Gamma_2 C) \leq \frac{\pi}{2\sqrt{3}} \cdot A(C)$, and equality holds if and only if C is a triangle with o as a vertex.*

We conjecture that the analogous statement holds also in higher dimensions.

3 The bodies associated to parallelotopes

Let W^n be the unit cube $[-\frac{1}{2}, \frac{1}{2}]^n$ in $\mathbb{R}^n$. The symmetries of W^n yield that $\Gamma_2 W^n$ is a ball, and it follows that the radius is $\frac{1}{\sqrt{12}}$. We deduce by (6) that

$$V(\Gamma W^n) \leq V(\Gamma_2 W^n) = \left(\frac{1}{\sqrt{12}} \right)^n \cdot \kappa_n \cdot V(W^n). \tag{8}$$

Next, we consider the planar case, and even determine the centroid body. For $u = (\sin\alpha, \cos\alpha)$, $\alpha \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, denote the set of points $x \in W^2$ satisfying $\langle u, x \rangle \geq 0$ by $W(\alpha)$, and the centroid of $W(\alpha)$ by $\gamma(\alpha)$. Since

$$\gamma(\alpha) = 2 \cdot \int_{W(\alpha)} x \, dx,$$

some elementary calculations yield that

$$\gamma'(\alpha) = \frac{1}{6} \cdot \left(\frac{1}{\cos^2\alpha}, -\tan\alpha \cdot \frac{1}{\cos^2\alpha} \right).$$

Therefore the part of the boundary of ΓW^2 parameterized by γ is the graph of the function

$$f(t) = \tfrac{1}{4} - 3t^2, \qquad t \in [-\tfrac{1}{6}, \tfrac{1}{6}].$$

The actual boundary of ΓW^2 consists of four arcs, and each of them is congruent with γ.

Using this representation in the planar case, and (8) and the Stirling formula in the higher dimensional case, we obtain

LEMMA 3.1 *Let P be an o–symmetric parallelotope in $\mathbb{R}^n$.*

(i) *If $n = 2$ then $A(\Gamma P) = \frac{5}{27} \cdot A(P)$ and $A(\Gamma_2 P) = \frac{\pi}{12} \cdot A(P)$.*

(ii) *If $n \geq 3$ then $V(\Gamma P)^{\frac{1}{n}} \leq V(\Gamma_2 P)^{\frac{1}{n}} < \sqrt{\frac{e\pi}{12}} \cdot \frac{1}{\sqrt{n}} \cdot V(P)^{\frac{1}{n}}$.*

4 Shaking in the plane

The proof of the Theorems uses "desymmetrization" of a planar convex body C. Let l be a line in $\mathbb{R}^2$, and choose a half plane l^+ bounded by l. For any line $\tilde{l}$, perpendicular to l and intersecting C, translate the intersection along $\tilde{l}$ so that the segment lands in l^+ and one endpoint lies in l. The union of the translated segments is a planar convex body C', which satisfies $A(C') = A(C)$. Usually it is obvious which half plane determined by l we need, and we simply speak about shaking with respect to l. This process, named shaking ("Schüttelung") was developed by W. Blaschke. One may think about the Blaschke shaking as the dual of the Steiner symmetrization, when each segment is translated so that the midpoint lands in l.

Now let $\tilde{l}$ be a line containing o. We say that the planar convex body C is *more symmetric than C' in the direction of $\tilde{l}$* if the conditions 1. and 2. below hold for any pair of lines l_1 and l_2 parallel to $\tilde{l}$:

1. $l_j \cap C$ and $l_j \cap C'$ have the same length, $j = 1, 2$;
2. Assume that $l_j \cap C$ is a segment and $l_j \neq \tilde{l}$, $j = 1, 2$, and denote the midpoint of $l_j \cap C$ and $l_j \cap C'$ by m_j and m'_j, respectively, $j = 1, 2$. If $\operatorname{lin} m_1$ and $\operatorname{lin} m'_1$ intersects l_2 in p and p', respectively, then $d(p, m_2) \leq d(p', m'_2)$.

The next observation is used in several related arguments, and a proof is included for the sake of completeness.

PROPOSITION 4.1 *Let $\tilde{l}$ be a line containing o, and let C_1, C_2 and C'_1, C'_2 be planar convex bodies such that C_1 and C_2 are more symmetric than C'_1 and C'_2, respectively, in the direction of $\tilde{l}$. Then*

$$\int_{C_1} \int_{C_2} A([o, x, y])\, dxdy \leq \int_{C'_1} \int_{C'_2} A([o, x, y])\, dxdy,$$

and equality holds if and only if $d(p, m_2) = d(p', m_2')$ holds for any pair l_1 and l_2 in the condition 2. above.

Proof: The proposition follows from the following claim: Assume that σ_1 and σ_2 are parallel and not collinear segments such that their midpoints are contained in the line l through o. If σ_1 is moved parallel to σ_2 and away from l then

$$\int_{\sigma_1}\int_{\sigma_2} A([o,x,y])\,dxdy \qquad \text{strictly increases.} \tag{9}$$

Since the problem is invariant under affine transformations which do not change the direction of σ_1, we may assume that l is actually the perpendicular bisector of σ_2. As σ_1 moves, the change of the integral is caused by the endpoints of σ_1. Therefore it is enough to verify that if $y - z$ is parallel to σ_2 and z is closer to l than y then

$$\int_{\sigma_2} A([o,x,y])\,dx > \int_{\sigma_2} A([o,x,z])\,dx. \tag{10}$$

Assume that w moves away from l along a line l_0 perpendicular to l, and l_0 avoids o and σ_2. Set up a coordinate system such that l is the first axis, and consider the points $x = (a,b)$ and $x' = (a,-b)$ of σ_2, and $w = (s,t)$, which satisfy

$$2 \cdot A([o,x,w]) + 2 \cdot A([o,x',w]) = |at - sb| + |at + sb| = \begin{cases} 2|sb| & \text{if } |at| \le |sb| \\ 2|at| & \text{if } |at| \ge |sb|. \end{cases}$$

The condition on w yields that $|t|$ increases as w is moving, and hence

$$A([o,x,w]) + A([o,x',w]) \qquad \text{increases.} \tag{11}$$

We deduce the weak form of (10), allowing the equality sign. On the other hand, if x is so close to l that $|at| > |sb|$ then $A([o,x,w]) + A([o,x',w])$ strictly increases, which in turn finally yields (10). □

Similarly, we have

PROPOSITION 4.2 *We use the notation and conditions of Proposition 4.1. We have*

$$\int_{C_1}\int_{C_2} A([o,x,y])^2\,dxdy \le \int_{C_1'}\int_{C_2'} A([o,x,y])^2\,dxdy,$$

and equality holds if and only if $d(p, m_2) = d(p', m_2')$ for any pair l_1 and l_2.

Proof: We use the same idea and notation as for the proof of Proposition 4.1. In particular, it is sufficient to prove that if w moves away from l parallel to $\mathrm{aff}\{x, x'\}$ then

$$A([o,x,w])^2 + A([o,x',w])^2 \quad \text{strictly increases.} \tag{12}$$

Since $|t|$ increases, and

$$4 \cdot A([o,x,w])^2 + 4 \cdot A([o,x',w])^2 = (at - sb)^2 + (at + sb)^2 = 2 \cdot a^2t^2 + 2 \cdot s^2b^2,$$

we conclude (12). □

Observe that Proposition 4.1 and Proposition 4.2 combined with the Steiner symmetrization yield (2) and its analogue for the ellipsoid of inertia in the planar case. Actually, the higher dimensional cases can be also handled using the Steiner symmetrization.

5 The extremal property of the parallelogram

We present the proof only for Theorem 1.1, the proof of Theorem 2.1 is completely analogous (one uses Proposition 4.2 instead of Proposition 4.1).

Let C be an o–symmetric convex body in $\mathbb{R}^2$. If E is the ellipse with maximal area contained in C (the so–called Löwner ellipse) then (see K. Ball [1] or R. Gardner [9]) $C \subset \sqrt{2}\,E$. Transforming E into the unit circular disc B, we may assume that

$$B \subset C \subset \sqrt{2}B.$$

Therefore the Blaschke selection theorem yields the existence of an o–symmetric planar convex body K which maximizes $A(\Gamma C)/A(C)$ among all o–symmetric planar convex bodies. We prove that K is a parallelogram.

Let $z \in \partial K$, and assume that the line l through z is tangent to one of the arcs of ∂K meeting at z. Denote the half plane determined by $\mathrm{lin}z$ and containing this arc by z^+, and the tangent line to the arc $\partial K \cap z^+$ at $-z$ by $\tilde{l}$.

We claim that either

$$\begin{array}{c} \textit{the line through } w \textit{ parallel to } z \textit{ supports } K \textit{ where } [z,w] = l \cap K \cap z^+, \\ \textit{or the line through } \tilde{w} \textit{ parallel to } z \textit{ supports } K \textit{ where } [-z,\tilde{w}] = \tilde{l} \cap K \cap z^+. \end{array} \tag{13}$$

Set $K^+ = K \cap z^+$. Then $A([o,x,y]) = A([o,x,-y])$ yields that

$$\int_K \int_K A([0,x,y])\,dxdy = 4 \cdot \int_{K^+} \int_{K^+} A([0,x,y])\,dxdy,$$

and hence we concentrate on K^+.

We may assume that l is perpendicular to lin$z$. Let $[x, \tilde{x}]$ and $[y, \tilde{y_0}]$ be sections of K^+ by lines parallel with z such that x and y are the endpoints closer to l, and x is closer to lin$z$ than y ($x \neq z$). Since K is convex, $[y, \tilde{y_0}]$ lies between the parallel lines aff$\{x, z\}$ and aff$\{\tilde{x}, -z\}$.

Shake down K^+ to l. The role of l and $\tilde{l}$ is actually symmetric because o is the midpoint of $[z, -z]$. We deduce by Proposition 4.1 that $A(\Gamma K)$ is not decreased by the shaking.

Assume that $l \cap K \cap z^+ = [z, w]$, and $\tilde{l} \cap K \cap z^+ = [-z, \tilde{w}]$ (possibly say $w = z$). The only case when $A(\Gamma K)$ is not increased by shaking is if for any pair x and y as above, either y lies in aff$\{x, z\}$ or $\tilde{y}$ lies in aff$\{\tilde{x}, -z\}$. Therefore there exists no section $[y, \tilde{y}] \in K^+$ whose projection into l (into $\tilde{l}$) lands outside of $[z, w]$ (outside of $[-z, \tilde{w}]$), which in turn yields the claim (13).

We deduce by (13) that there exists a segment $[v, w]$ in ∂K. We may assume that this segment is maximal, and denote by u its midpoint. Now if z is an interior point of $[u, w]$ then (13) yields the line through $-v$ parallel with z, which is a supporting line of K. It follows by continuity that aff$\{-v, w\}$ is also a supporting line of K, and hence K is a parallelogram.

6 The extremal property of the triangle

We present the proof only for Theorem 1.3, the proof of Theorem 2.3 is completely analogous.

Let C be any planar convex body containing o. If E is the o–symmetric ellipse such that $x + E$ is the ellipse with maximal area contained in C then $C \subset x + 2E$ (see K. Ball [1] or R.J. Gardner [9]). We may assume that $E = B$ by affine invariance. Therefore the Blaschke selection theorem yields again the existence of a K which maximizes $A(\Gamma C)/A(C)$ and $o \in K$.

First we consider a special case. Recall that the positive hull pos $\{x, y\}$ of two vectors x and y is in general the cone they enclose; more precisely, the set of linear combinations $\lambda\, x + \mu\, y$ with $\lambda, \mu \geq 0$.

PROPOSITION 6.1 *If C is a planar convex body and $o \in \partial C$ then $A(\Gamma C)/A(C)$ is maximized by the triangles with o as a vertex.*

Proof: We may assume that $A(\Gamma C)/A(C)$ is maximal under the condition $o \in \partial C$. Suppose that either C is not a triangle or o is not a vertex.

Let $[o, u_1]$ and $[o, u_2]$ be the two (possibly degenerate) maximal segments in ∂C meeting at o. If $u_1 = u_2$ ($= o$) then define l' as some supporting line at o. If $u_1 \neq u_2$ then define l' as the line through o parallel to $u_2 - u_1$, which is again a supporting line. Then there exists a line $\tilde{l}$ intersecting $C \backslash [0, u_1, u_2]$ and parallel to l'. We deduce that $[x, o] \cap \text{int} C \neq \emptyset$ holds for any $x \in \tilde{l} \cap C$.

Shake C down to the line l through o and orthogonal to $\tilde{l}$ (use either half plane determined by l). If $[x,x']$, $[y,y']$ are sections of C parallel to l' and $[x,x']$ is closer to o than $[y,y']$ then $[y,y'] \subset \mathrm{pos}\{x,x'\}$. We deduce by Proposition 4.1 that $A(\Gamma C') \geq A(\Gamma C)$ for the resulting C'.

Now if $[y,y'] = \tilde{l} \cap C$ and $[x,x']$ is a section of C closer to o than $[y,y']$ then $[y,y'] \subset \mathrm{int}\,\mathrm{pos}\{x,x'\}$. Therefore $A(\Gamma C') > A(\Gamma C)$, which contradicts the maximality of $A(\Gamma C)$.

□

We need the following technical but useful statement:

PROPOSITION 6.2 *If C is a planar convex body and $o \in C$ then there exist points $x_1, x_2 \in \partial C$ and supporting lines l_1, l_2 at x_1, x_2, respectively, such that l_1 and l_2 are parallel, and $o \in [x_1, x_2]$.*

Proof: We may assume by continuity that C is smooth, strictly convex and $o \in \mathrm{int}\, C$. Orient ∂C. For any $x \in \partial C$, define $\alpha(x)$ as the angle of the half line xo and the tangent half line at x of the arc of ∂K on the positive side of x. In addition, x' denotes the point on ∂C with $o \in [x,x']$.

Let $x_0 \in \partial C$. If $\alpha(x_0) = \alpha(x_0')$ then we are done. Otherwise, move x from x_0 to x_0' along ∂C in the positive direction. Then $\alpha(x) - \alpha(x')$ is continuous, and changes its sign at some x_1. Then choosing $x_2 = x_1'$, the supporting lines at x_1 and x_2 are parallel.

□

Now let K be the planar convex body maximizing $A(\Gamma C)/A(C)$ under the condition that $o \in C$. We have already discussed the case when o is contained in the boundary of K, so suppose that $o \in \mathrm{int}\, K$.

Let the points x_1, x_2 and the supporting lines l_1, l_2 be given as in Proposition 6.2. Assume that o is not farther from l_2 than from l_1. The line $\mathrm{aff}\,\{x_1, x_2\}$ divides K into two halves K^+ and K^-. Assume that l_1 is orthogonal to $z = x_1 - x_2$, and shake K down to the line $l = l_1 - x_2$ so that the segment $[x_1, x_2]$ moves to $[z, 0]$. Denote by $\widetilde{K}^+$ ($\widetilde{K}^-$) the image of K^+ (K^-).

Choose the points $w_+ \in l$ ($w_- \in l$) on the side of $\mathrm{lin}\, y$ containing K^+ (K^-) so that $A([o,z,w_+]) = A(K^+)$ ($A([o,z,w_-]) = A(K^-)$). Now we deduce by Proposition 6.1 that

$$\int_{[o,z,w_+]} \int_{[o,z,w_+]} A([o,x,y])\, dxdy > \int_{K^+} \int_{K^+} A([o,x,y])\, dxdy, \tag{14}$$

and the analogous statement holds for $[o,z,w_-]$ and K^-.

Since K^+ and K^- lie between l_1 and l_2, and o is not farther from l_2 than from l_1, we deduce by Proposition 4.1 that

$$\int_{\widetilde{K}^+} \int_{\widetilde{K}^-} A([o,x,y])\, dxdy > \int_{K^+} \int_{K^-} A([o,x,y])\, dxdy. \tag{15}$$

Define u_+ and u_- so that $\widetilde{K}^+ \cap [0, w_+] = [o, u_+]$ and $\widetilde{K}^- \cap [0, w_-] = [o, u_-]$. Observe that if $x \in \widetilde{K}^-$ then the line passing through u_+ and parallel to x separates $[0, w_+, z] \backslash \widetilde{K}^+$ from $\widetilde{K}^+ \backslash [0, w_+, z]$. On the other hand, the areas of $[0, w_+, z] \backslash \widetilde{K}^+$ and $\widetilde{K}^+ \backslash [0, w_+, z]$ are the same. It follows that

$$\int_{[0,w_+,z]} \int_{[0,w_-,z]} A([o, x, y])\, dxdy \geq \int_{\widetilde{K}^+} \int_{\widetilde{K}^-} A([o, x, y])\, dxdy.$$

Combining the inequality with (14) and (15) yields that $A(\Gamma[0, w_+, w_-]) > A(\Gamma K)$. This contradicts the maximality of $A(\Gamma K)$, and hence K is a triangle, and o is a vertex.

Finally, we deduce by formula (1) that $A(\Gamma K) = \frac{16}{27} \cdot A(K)$.

7 About the monotonicity of $V(\Gamma C)$

First we construct counterexamples to E. Lutwak's conjecture (4) in certain $\mathbb{R}^n$. Our argument is based on

PROPOSITION 7.1 *Assume that the conjecture (4) holds in $\mathbb{R}^{n+1}$, $n \geq 2$, and let $C \subset C'$ be o–symmetric convex bodies in $\mathbb{R}^n$ such that $L \cap C = L \cap C'$ for a linear $(n-1)$–plane L in $\mathbb{R}^n$. Then*

$$V(\Gamma C') \geq V(\Gamma C).$$

Proof: Let u_1 and u_2 be orthogonal unit vectors in $\mathbb{R}^{n+1}$, and for $i = 1, 2$, choose an isomorphism φ_i between $\mathbb{R}^n$ and $u_i^\perp$ such that φ_1 and φ_2 coincide on L. In addition, define the convex body K as the convex hull of $\varphi_1(C)$ and $\varphi_2(C)$ in $\mathbb{R}^{n+1}$.

Now the unit vector $u = \frac{1}{\sqrt{2}} \cdot (u_1 + u_2)$ and the linear map $\varphi = \frac{1}{2} \cdot (\varphi_1 + \varphi_2)$ satisfy that $\varphi(C) = u^\perp \cap K$, and the point $V(\Gamma(u^\perp \cap K)) \cdot u$ lies on the segment between $V(\Gamma C) \cdot u_1$ and $V(\Gamma C) \cdot u_2$. Set

$$K' = \mathrm{conv}(\varphi(C') \cup K).$$

Then the conjecture (4) applied to K' yields that $V(\Gamma(u^\perp \cap K')) \cdot u$ lies beyond the segment connecting $V(\Gamma C) \cdot u_1$ and $V(\Gamma C) \cdot u_2$, and hence $V(\Gamma C') \geq V(\Gamma C)$. □

In order to abbreviate the integral formulae appearing in the random simplex representation for an o–symmetric convex body C in $\mathbb{R}^n$, set

$$I(C) = \int_C \cdots \int_C V([0, x_1, \ldots, x_n])\, dx_1 \ldots dx_n,$$
$$S(C, z) = \int_C \cdots \int_C V([0, y_1, \ldots, y_{n-1}, z])\, dy_1 \ldots dy_{n-1}.$$

Then (1) can be written in the form

$$I(C) = 2^{-n} \cdot V(C)^n \cdot V(\Gamma C).$$

If z_0 is the center of one facet of the unit cube $W^n = [-\frac{1}{2}, \frac{1}{2}]^n$, then

$$S(W^n, z_0) = \frac{1}{2n} \cdot I(W^{n-1}). \tag{16}$$

On the other hand, (2) and the Stirling formula yield that

$$I(W^n) > n^{-\frac{1}{2}\cdot n + o(n)}.$$

We deduce by $n! = n^{n+o(n)}$ that there exists infinitely many n such that $I(W^n) > \frac{8}{n} \cdot I(W^{n-1})$, and hence

$$S(W^n, z_0) < \frac{1}{16} \cdot I(W^n). \tag{17}$$

Let n satisfy (17). There exists an o–symmetric, strictly convex body C in $\mathbb{R}^n$ such that z_0 lies on its boundary, and

$$S(C, z_0) < \frac{1}{8} \cdot \frac{I(C)}{V(C)}. \tag{18}$$

For $\varepsilon > 0$, define

$$C_\varepsilon = \mathrm{conv}\{\pm(1+\varepsilon)z_0, C\}.$$

Assume that $\varepsilon > 0$ tends to zero. Then $C_\varepsilon \backslash C$ consists of two components (one at z_0 and one at $-z_0$), and the diameter of both of them tends to zero. We deduce that

$$S(C, z) < \frac{1}{4} \cdot \frac{I(C)}{V(C)}$$

holds for any $z \in C_\varepsilon \backslash C$. In addition, $V(C_\varepsilon \backslash C)$ tends to zero, and hence

$$\begin{aligned} I(C_\varepsilon) - I(C) &= n \cdot \int_{C_\varepsilon \backslash C} S(C, z)\, dz + O\left(V(C_\varepsilon \backslash C)^2\right) \\ &< 2 \cdot n \cdot \int_{C_\varepsilon \backslash C} S(C, z)\, dz. \end{aligned}$$

Combining the last two estimates shows that

$$\frac{I(C_\varepsilon) - I(C)}{V(C_\varepsilon) - V(C)} < n \cdot \frac{I(C)}{V(C)}.$$

Therefore if $\varepsilon > 0$ is small then

$$\frac{I(C_\varepsilon) - I(C)}{I(C)} < \frac{n \cdot (V(C_\varepsilon) - V(C))}{V(C)} < \frac{V(C_\varepsilon)^n - V(C)^n}{V(C)^n},$$

which in turn yields that

$$V(\Gamma C_\varepsilon) < V(\Gamma C) \qquad \text{while} \qquad C \subset C_\varepsilon. \tag{19}$$

We conclude by Proposition 7.1,

LEMMA 7.2 *There exist infinitely many n such that the conjecture (4) fails in $\mathbb{R}^n$.*

Finally, we discuss the conjecture (4) in $\mathbb{R}^3$, which is in turn related to the monotonicity of the area of the centroid body in the plane. We prove

LEMMA 7.3 *Let C and K be o–symmetric convex bodies in $\mathbb{R}^2$. If $C \subset K$ then $A(\Gamma C) \leq A(\Gamma K)$, and equality holds if and only if $C = K$.*

Before proving Lemma 7.3, let us consider an o–symmetric convex body M in $\mathbb{R}^2$, and a $z \in \partial M$. Assume that l is a tangent line at z to M, and denote by P the parallelotope with area $A(M)$ whose two sides are contained in l and $-l$, and the other two sides are parallel to z. Then

$$S(M, z) \geq S(P, z) = \frac{1}{16} \cdot A(M)^2. \tag{20}$$

Now let C and K be o–symmetric convex bodies in $\mathbb{R}^2$ such that C is strictly contained in K. Define $K_t = (1-t)\,C + t\,K$ for $0 \leq t \leq 1$. Lemma 7.3 follows by proving that $A(\Gamma K_t)$ is strictly increasing. Since K_t is a linear family, it is sufficient to prove that

$$A(\Gamma K_t) - A(\Gamma K_0) > 0 \tag{21}$$

holds for small $t > 0$.

Note that (21) can be written in the form

$$\frac{I(K_t) - I(K_0)}{I(K_0)} > \frac{A(K_t)^2 - A(K_0)^2}{A(K_0)^2},$$

where $A(K_t)$ tends to $A(K_0)$. Thus if $t > 0$ is small then (21) is the consequence of the inequality

$$\frac{I(K_t) - I(K_0)}{A(K_t) - A(K_0)} > 2.5 \cdot \frac{I(K_0)}{A(K_0)}. \tag{22}$$

We give a lower bound for the variation of $I(K_t)$ by considering the two cases when one of the x_i's, $i = 1, 2$, in the definition of $I(K_t)$ lies in the difference set, and the other variable is chosen from K_0. Therefore

$$I(K_t) - I(K_0) > 2 \cdot \int_{K_t \setminus K_0} S(K_0, z)\, dz,$$

while $I(K_0) \leq \frac{5}{4 \cdot 27} \cdot A(K_0)^3$ holds by Theorem 1.1. We deduce by (20) that

$$\begin{aligned} \frac{I(K_t) - I(K_0)}{A(K_t) - A(K_0)} &> 2 \cdot \frac{\int_{K_t \setminus K_0} S(K_0, z)\, dz}{A(K_t \setminus K_0)} \\ &\geq \frac{1}{8} \cdot A(K_0)^2 \quad > \quad 2.5 \cdot \frac{I(K_0)}{A(K_0)}. \end{aligned}$$

Therefore the inequality (22) follows for small t, which in turn yields Lemma 7.3. □

We conjecture that (4) holds at least in $\mathbb{R}^3$.

Acknowledgment: We wish to thank Endre Makai and the referee for their many useful comments.

References

[1] K. Ball: *An elementary introduction to modern convex geometry.* In: Flavors of convex geometry, ed. S. Levy, Cambridge University Press, Cambridge, 1997.

[2] M. Berger: *Convexity.* Amer. Math. Monthly, **97** (1990), 650–678.

[3] W. Blaschke: *Über affine Geometrie IX: Verschiedene Bemerkungen un Aufgaben.* Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.–Phys. Kl., **69** (1917), 412–420.

[4] W. Blaschke: *Über affine Geometrie XI: eine minimum Aufgabe für Legendres trägheits Ellipsod.* Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.–Phys. Kl., **70** (1918), 72–75.

[5] T. Bonnesen & W. Fenchel: *Theorie der konvexen Körper.* Springer, Berlin, 1934. English translation: *Theory of convex bodies.* BCS Associates, Moscow, Idaho, U.S.A., 1987.

[6] H. Busemann: *A theorem on convex bodies of Brunn–Minkowski type.* Proc. Nat. Akad. Sci. U.S.A. **35** (1949), 27–31.

[7] H. Busemann: *Volume in terms of concurrent cross–sections.* Pacific J. Math. **3** (1953), 1–12.

[8] C. Dupin: *Application de Géometrie et de Méchanique à la Marine, aux Points et Chaussées.* Bachelier, Paris, 1822.

[9] R.J. Gardner: *Geometric Tomography.* Cambridge University Press, Cambridge, 1995.

[10] E. Lutwak: *On a conjectured projection inequality of Petty.* Contemp. Math., **113** (1990), 171–182.

[11] E. Lutwak: *Selected affine isoperimetric inequalities.* In: Handbook of convex geometry A, 151–176. North-Holland, Amsterdam, 1993.

[12] V.D. Milman & A. Pajor: *Isotropic position and inertie ellipsoids and zonoids of the unit ball of a normed n–dimensional space.* Geometric Aspects of Functional Analysis, ed. J. Lindenstrauss and V.D. Milman, Lec. Notes Math. 1376 Springer, (1989), 64–104.

[13] C.M. Petty: *Centroid surfaces.* Pacific. J. Math., **11** (1961), 1535–1547.

T. Bisztriczky
Dept. of Mathematics
University of Calgary
Calgary, Alberta
Canada, T2N 1N4
tbisztri@math.ucalgary.ca

K. Böröczky, Jr.
Rényi Institute of Mathematics
Budapest Pf. 127
1364 Hungary
carlos@math–inst.hu