Isoperimetric inequalities, Brunn-Minkowski theory and Minkowski-type Monge-Ampère equations on the sphere

Károly J. Böröczky, Alessio Figalli, João P.G. Ramos

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For Abel and Csilla

Preface

The main topic of the book is how Geometric Isoperimetric-type inequalities intertwine with functional inequalities such as the Prékopa-Leindler inequality, the Sobolev inequality, Poincaré inequality, on the one hand, and also with the uniqueness of the solution of certain Minkowski-type Monge-Ampère equations on the sphere, on the other hand. The classical set up, due to Minkowski around 1900, is that the characterization of the equality case in the Brunn-Minkowski inequality for convex bodies (cf. Theorem 1.12.3) yields the uniqueness result about the Minkowski problem (cf. Section 9.2), and the similar pattern is followed in the case of Firey's L_p-Brunn-Minkowski inequality and Lutwak's L_p-Minkowski problem for $p > 1$ (cf. Section 9.4), or in the case of capacity and torsional rigidity (cf. Section 9.5.7). On the other hand, the pattern is reversed in the case of the L_p-Brunn-Minkowski and L_p-Minkowski conjectures when $p < 1$ is close to 1. In this case, first the uniqueness of the solution of the even L_p-Minkowski problem had been recently established using spectral gap estimates à la Hilbert and the continuity method from PDE, and this uniqueness result in turn yields the L_p-Brunn-Minkowski inequality (cf. Section 9.4). In some sense, the first eight chapters provide the necessary background for Chapter 9 - and this fact explains the length of this book.

It is a mission of this work to clarify the relation between two frequently but sometimes carelessly used notions of surface area of bodies in $\mathbb{R}^n$: on the one hand, the $(n-1)$-dimensional Hausdorff measure of the boundary, and on the other hand, the outer Minkowski content based on the parallel domain (see Chapter 4 and Section 5.4). The advantage of the first approach is that it fits nicely the Coarea formula, while the second approach suits the use of measure concentration and the Brunn-Minkowski inequality.

We introduce various tools useful in related problems. For example, we discuss proofs of the Brunn-Minkowski inequality in $\mathbb{R}^n$ (a generalisation of the Isoperimetric Inequality) using symmetrization (à la Steiner) in Section 1.10, parametrization of "height" (à la Brunn and Minkowski) in Section 1.12, combinatorial ideas (à la Hadwiger and Ohmann) in Section 3.2, optimal transport (à la Knothe and Gromov) as a special case of the Prékopa-Leindler inequality in Section 3.4, and spectral theory (à la Hilbert and Aleksandrov) in Section 8.5.2. A close-to-optimal stability version of the Brunn-Minkowski inequality for convex bodies in terms of volume difference is also presented in Section 8.6.

The book has various readings. It discusses:

- a basic introduction into convexity in Chapters 1 and 2, that is appended by the more advanced theory of associated ellipsoids and the Blaschke-Santaló inequality in Chapter 6;

- a survey into the theory of sets of finite perimeter and functions of bounded variation in $\mathbb{R}^n$ in Chapter 5;
- the Isoperimetric and Anisotropic Isoperimetric inequality from various angles, including the case of convex bodies in Section 2.4 together with a stability version in Section 8.6, the case of bodies with Lipschitz boundary in the Euclidean, Hyperbolic, and Spherical spaces, or with respect to a log-concave density in Chapter 4, and the most general case of sets of finite perimeter in $\mathbb{R}^n$ in Chapter 5;
- functional versions of geometric inequalities, including the Prékopa-Leindler inequality in Section 3.4 and the anisotropic Sobolev inequality in Sections 4.2, 4.3 and 5.3 as versions of the Brunn-Minkowski inequality, the log-Sobolev inequality in Section 4.6.1 derived from the Gaussian Isoperimetric Inequality, or the functional Santaló inequality and its reverse forms as functional versions of the Blaschke-Santaló inequality and its reverse forms in Chapter 6;
- the classical theory of mixed volumes and the corresponding Minkowski and Aleksandrov-Fenchel inequalities (stemming from the Brunn-Minkowski inequality) in Chapter 7 (in terms of polytopes) and in Chapter 8 (in terms of smooth convex bodies);
- the interplay between Brunn-Minkowski type inequalities in $\mathbb{R}^n$ and Monge-Ampère equations on S^{n-1} within Lutwak's L_p-Minkowski Theory in Sections 7.6, 8.7 and 8.8, and in Chapter 9.

We have made an effort to ensure that the chapters can be read independently, quoting the relevant results from the other chapters where necessary. The diagram below explains the direct interrelations among chapters.

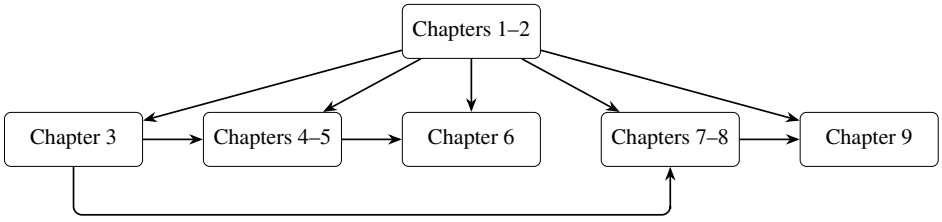

Figure 1. Chapter flow.

As we explained above, Chapters 1 and 2 contain the fundamental material about convex sets in $\mathbb{R}^n$, but these chapters can be skipped if somebody is already familiar with the basics.

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Contents

Preface vii
0.1 Basic notions and notation 1

1 Preliminaries in convex geometry in $\mathbb{R}^n$ 5
1.1 Affine hull, Convex hull 5
1.2 Closed convex sets in $\mathbb{R}^n$ 7
1.3 Additional properties of convex cones 11
1.4 Polyhedra, polytopes 14
1.5 Regular boundary points, Differentiability of convex functions 18
1.6 Support function of a compact convex set 21
1.7 Hausdorff distance of compact convex sets 25
1.8 Simple polytopes and Strongly Isomorphic polytopes 27
1.9 The polar (dual) of a convex body 30
1.10 Steiner symmetrization and Schwarz symmetrization 32
1.11 Centroid 37
1.12 Equality in the Brunn-Minkowski and the Isodiametric inequalities . . 42
1.13 Comments to Chapter 1 45
1.A Supplement: Epigraph and Legendre transform 46
1.B Supplement: Recession cone and support function of closed convex sets 51
1.C Supplement: Iterated Steiner symmetrizations 54
1.D Supplement: The Hausdorff Measure for subsets of $\mathbb{R}^n$ 56

2 Surface area, surface area measure and cone volume measure for convex bodies in $\mathbb{R}^n$ 61
2.1 Some integral formulas involving exterior unit normals 61
2.2 Parametrizing the boundary via the radial function 63
2.3 Surface area, Cauchy formula 65
2.4 Isoperimetric inequality and Anisotropic Isoperimetric Inequality . . . 69
2.5 Surface area measure 71
2.6 Cone Volume measure 76
2.7 Well approximating polytopes in terms of the Hausdorff distance and the Banach-Mazur distance 82
2.8 Comments to Section 2 89
2.A Supplement: Divergence Theorem for convex bodies 91
2.B The Projection Body and the Isoperimetric Inequality 91

3 The Brunn-Minkowski and the Prékopa-Leindler inequalities in the measurable case 93
3.1 The Brunn-Minkowski inequality for measurable sets 93
3.2 The Hadwiger-Ohman proof 95
3.3 Green-Tao via Brunn-Minkowski 97
3.4 The Prékopa-Leindler inequality 98
3.5 Proof of Prékopa-Leinder via optimal transport 101
3.6 Coordinate-wise product of unconditional convex bodies 105
3.7 Hausdorff metric on the space of compact subsets 106
3.8 Brunn-Minkowski via Steiner symmetrization 108
3.9 Comments to Chapter 3 111
3.A Supplement: Brunn-Minkowski and Prékopa-Leindler 115
3.B Supplement: Equality in Brunn-Minkowski via Hadwiger-Ohmann . . 116

4 The Isoperimetric inequality in the case of Lipschitz boundary 119
4.1 The Isoperimetric inequality in the Euclidean space 119
4.2 Sobolev inequality 122
4.3 The Anisotropic Isoperimetric inequality 125
4.4 Wulff shape and minimizing surface energy 128
4.5 Isodiametric and Isoperimetric inequalities in H^n and S^n 131
4.6 The Gaussian Isoperimetric and the log-Sobolev inequalities 135
4.7 The Cheeger constant, a Poincaré inequality and the KLS conjecture for log-concave densities 143
4.8 Comments to Chapter 4 153
4.A Supplement: The Spherical Space and the Hyperpolic Space 158
4.B Supplement: Equality in the Isodiametric Inequality in H^n and S^n . . . 163
4.C Supplement: Isoperimetric inequality for k-gons in H^2, S^2 or $\mathbb{R}^2$ 165
4.D Supplement: Concentration of measure via Isoperimetric type inequalities 169

5 The Isoperimetric Inequality for sets of Finite Perimeter in $\mathbb{R}^n$ and the Sobolev inequality for BV functions 175
5.1 Sets of Finite Perimeter and BV functions 176
5.2 Characterization of isoperimetric sets 185
5.3 The anisotropic Sobolev inequality for BV functions 190
5.4 Sets of finite weighted perimeter 192
5.5 Comments to Chapter 5 199

6 Associated ellipsoids, Blaschke-Santaló inequality and the Reverse Isoperimetric Inequality 201
6.1 John and Löwner ellipsoid 201

6.2 Some characterizations of ellipsoids . . . 204
6.3 Reverse isoperimetric inequality . . . 205
6.4 Ellipsoid of inertia, Isotropic position of a convex body . . . 210
6.5 The Blaschke-Santaló inequality for the polar body . . . 220
6.6 Reverse Blaschke-Santaló inequality and Mahler's conjecture . . . 223
6.7 Functional Santaló inequality and reverse form . . . 225
6.8 Volume approximation by polytopes . . . 228
6.9 The M-ellipsoid, the Covering number and the Reverse Brunn-Minkowski inequality . . . 233
6.10 Comments to Chapter 6 . . . 235
6.A Supplement: The John condition on the inscribed maximal volume ellipsoid . . . 240
6.B Supplement: The rank one Geometric Brascamp-Lieb inequality . . . 243
6.C Supplement: Minimal surface area position - isotropic surface area measure . . . 246
6.D Supplement: Maximal area of inscribed polygons and the extremality of ellipses . . . 248
6.E Supplement: The Blaschke-Santaló inequality for general convex bodies 250
6.F Supplement: From the Affine Isoperimetric to the Blaschke-Santaló inequality . . . 257

7 Steiner's formula and Mixed Volumes . . . 259
7.1 Steiner formula, Intrinsic volumes and Mean projections . . . 259
7.2 Isoperimetric inequality for Mean projections . . . 262
7.3 Mixed volumes . . . 264
7.4 The Minkowski inequality and the Aleksandrov-Fenchel inequality . . . 269
7.5 Aleksandrov's lemma for Wulff shapes . . . 273
7.6 The L_p-Brunn-Minkowski inequality for $p \geqslant 1$ . . . 275
7.7 Isoperimetric type problems for polytopes . . . 281
7.8 Comments to Section 7 . . . 286
7.A Supplement: Proof of the Aleksandrov-Fenchel inequality based on polytopes . . . 289
7.B Supplement: The Simplex Mean Width conjecture and some related results . . . 297

8 Convex bodies and Gaussian curvature . . . 301
8.1 Second order differentiability of the boundary . . . 301
8.2 Surface area measure and the curvature function . . . 308
8.3 Mixed volumes and smooth convex bodies . . . 310
8.4 Minkowski inequality, the Surface area measure and the other Area measures . . . 315

8.5 The Hilbert-Aleksandrov operator and the Aleksandrov-Fenchel inequality . . . 318
8.6 Stability of the (Anistropic) Isoperimetric and the Brunn-Minkowski inequalities . . . 323
8.7 The Logarithmic Brunn-Minkowski and the Logarithmic Minkowski conjectures . . . 333
8.8 The L_p-Brunn-Minkowski and the L_p-Minkowski conjectures and inequalities when $p \in (0, 1)$ . . . 347
8.9 L_p-Affine Surface Areas for $p > 0$ . . . 357
8.10 Comments to Chapter 8 . . . 368
8.A Supplement: Aleksandrov's Mixed Discriminant Inequality . . . 372
8.B Supplement: The upper semi-continuity of the L_p-Affine Surface Area 375
8.C Supplement: Affine Isoperimetric Inequality via Steiner symmetrization . . . 379
8.D Supplement: Valuations on convex bodies . . . 381

9 The Minkowski Problem, the L_p-Minkowski problem, and the L_p-Brunn-Minkowski inequality/conjecture . . . 389
9.1 Monge-Ampère equations in $\mathbb{R}^d$ . . . 389
9.2 The Minkowski Problem . . . 391
9.3 The L_p-Minkowski problem . . . 396
9.4 Uniqueness in the L_p-Minkowski problem . . . 406
9.5 Comments to Chapter 9 . . . 410
9.A Supplement: L_p surface area measures: Weak continuity and diameter bounds . . . 429
9.B Supplement: The even L_p-Minkowski problem for $p \in [0, 1)$ . . . 434
9.C Supplement: About the L_p-Minkowski problem in general for $p \geqslant 0$. 439

10 Appendix: Background from Analysis and Algebra . . . 453
10.1 Weak convergence, Regular measures and the Lebesgue measure . . . 453
10.2 Hölder's, Jensen's and Minkowski's inequalities and the L^p function spaces . . . 455
10.3 Lebesgue Integral, Convolution, Fourier transform . . . 458
10.4 Hausdorff measure and Lipschitz functions . . . 460
10.5 The Stone-Weierstrass theorem . . . 464
10.6 Differentiability of Convex Functions . . . 464
10.7 Self-adjoint Elliptic Linear Operators on the sphere S^{n-1} . . . 465
10.8 Matrices: The Hilbert-Schmidt norm, Hadamard's inequality and the Perron-Frobenius Theorem . . . 468
10.9 Log-concave functions . . . 470

References 479

Index 517

0.1 Basic notions and notation

$C_c(X)$:
The space of continuous real functions of compact support (zero outside of a compact set) for a topological space X. In addition, $C_c^k(\mathbb{R}^n)$, $k = 1, 2, \ldots$, denotes the space of real C^k functions on $\mathbb{R}^n$ with compact support, allowing $k = \infty$.

Minkowski sum:
$X + Y = \{x + y : x \in X \ \& \ y \in Y\}$ for $X, Y \subset \mathbb{R}^n$.
$\alpha X + \beta Y = \{\alpha x + \beta y : x \in X \ \& \ y \in Y\}$ for $\alpha, \beta \in \mathbb{R}$.

Origin in $\mathbb{R}^n$:
$o = o_{\mathbb{R}^n}$.

Scalar product and ℓ_2 norm in $\mathbb{R}^n$:
$\langle \cdot, \cdot \rangle$ and $\|x\| = \sqrt{\langle x, x \rangle}$.

Orthogonal complement in $\mathbb{R}^n$:
$L^\perp = \{x \in \mathbb{R}^n : \langle x, y \rangle = 0 \ \forall y \in L\}$ for a linear subspace $L \subset \mathbb{R}^n$.
$u^\perp = \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\}$ for a vector $u \in \mathbb{R}^n \backslash \{o\}$.

Orthogonal projection in $\mathbb{R}^n$:
$X|L = \Pi_L(X) = L \cap (X + L^\perp)$ for $X \subset \mathbb{R}^n$ and a linear subspace $L \subset \mathbb{R}^n$.

Topology in $\mathbb{R}^n$:
For $X \subset \mathbb{R}^n$, let $\operatorname{int} X$, ∂X, and $\operatorname{cl} X$ denote the interior, the boundary, and the closure, respectively, with respect to the vector space topology of $\mathbb{R}^n$.

Relative topology with respect to an affine subspace in $\mathbb{R}^n$:
$\operatorname{relint} C$ and $\operatorname{relbd} C$ denote, respectively, the relative interior and relative boundary of a closed convex set $C \subset \mathbb{R}^n$ with respect to the affine subspace $\operatorname{aff} C$ generated by C (cf. Lemma 1.1.9).

Lebesgue measure, integration in $\mathbb{R}^n$:
$|X|$ is the Lebesgue measure for a measurable $X \subset \mathbb{R}^n$.
$\int_X f = \int_X f(x)\, dx$ denotes the Lebesgue integral for a measurable $f : X \to \mathbb{R}$ (if the integral exists).

Unit (Euclidean or ℓ_2) ball in $\mathbb{R}^n$:
$B^n = \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}$ and $\omega_n = |B^n|$.

Hausdorff measure:
$\mathcal{H}^d$ for $d \geqslant 0$, normalized in a way such that $\mathcal{H}^d(X) = |X|$ for Lebesgue measurable $X \subset \mathbb{R}^d$ and integer $d \geqslant 1$ (see Sections 1.D and 10.4).

Unit sphere S^{n-1} in $\mathbb{R}^n$, and integration on S^{n-1}:
$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.
$\int_X f(x)\, dx = \int_X f\, d\mathcal{H}^{n-1}$ for $\mathcal{H}^{n-1}$ measurable set $X \subset S^{n-1}$ and function $f : X \to \mathbb{R}$ (if the integral exists).

ε-net:
Given $X \subset \mathbb{R}^n$ and $\varepsilon > 0$, an ε-net $\Xi \subset X$ is a discrete set such that, for any $x \in X$, there exists $y \in \Xi$ with $\|x - y\| \leqslant \varepsilon$.

Convex set $X \subset \mathbb{R}^n$:
$X \subset \mathbb{R}^n$ is convex if $(1-t)x + ty \in X$ for $x, y \in X$ and $t \in [0,1]$.

Normal cone:
If $X \subset \mathbb{R}^n$ is closed convex and $y \in \partial X$, then the normal cone at y is

$$N_X(y) = \{u \in \mathbb{R}^n : \langle u, x - y\rangle \leqslant 0 \ \forall x \in X\}.$$

Closest point map:
If $X \subset \mathbb{R}^n$ convex closed and $z \in \mathbb{R}^n$, then the unique point $y \in X$ closest to z given by Lemma 1.2.2 is $\Pi_X(z) = y$. If $L \subset \mathbb{R}^n$ is a linear subspace, then $\Pi_L(z) = z|L$ is the orthogonal projection of z to L.

Hausdorff distance of compact sets:
If $X, Y \subset \mathbb{R}^n$ are non-empty compact sets, then

$$\delta_H(X,Y) = \min\{\varrho \geqslant 0 : X \subset Y + \varrho\, B^n \text{ and } Y \subset X + \varrho\, B^n\}$$

where δ_H is a metric on the space of (non-empty) compact subsets of $\mathbb{R}^n$.

Space of compact convex sets:
$\mathcal{K}^n$ is the space of compact convex subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric δ_H.

Support function of a compact convex set $K \subset \mathbb{R}^n$:
$h_K(u) = \max\{\langle x, u\rangle : x \in K\}$ for $u \in \mathbb{R}^n$ (see Section 1.6).
When speaking about compact convex sets, we always assume that the sets are non-empty.

Face of a $K \in \mathcal{K}^n$:
$F_K(u) = \{x \in K : \langle x, u\rangle = h_K(u)\}$ for $u \in \mathbb{R}^n \backslash \{o\}$.

Convex body:
$K \subset \mathbb{R}^n$ is convex body if it is compact, convex, and $\operatorname{int} K \neq \varnothing$.

- $\mathcal{K}^n_o$ is the family of convex bodies $K \subset \mathbb{R}^n$ such that $o \in K$.
- $\mathcal{K}^n_{(o)}$ is the family of convex bodies $K \subset \mathbb{R}^n$ such that $o \in \operatorname{int} K$.

Radial function of a $K \in \mathcal{K}^n_o$:
For $u \in S^{n-1}$, $\varrho_K(u) = \max\{t \geqslant 0 : tu \in K\}$ is the radial function (see Section 1.9 if $o \in \operatorname{int} K$, and Section 2.6 in general).

Polar (dual) body K^* for a $K \in \mathcal{K}^n_{(o)}$:
$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \ \forall\ y \in K\}$ (see Section 1.9).

Norm $\|x\|_K$ with respect to a convex body $K \in \mathcal{K}^n_{(o)}$:
$\|x\|_K = \min\{t \geqslant 0 : x \in tK\}$ for $x \in \mathbb{R}^n$ (cf. Sections 1.6 and 1.9) where $\|x\|_K = h_{K*}(x)$ for $x \in \mathbb{R}^n$, and $\|u\|_K = 1/\varrho_K(u)$ for $u \in S^{n-1}$. If $K = -K$, then $\|\cdot\|_K$ is an actual norm.

Regular boundary points, $\partial' X$, exterior unit normal $\nu_X(x)$:
For a closed set $X \subset \mathbb{R}^n$ with non-empty interior and locally Lipschitz boundary, $x \in \partial X$ is a regular boundary point (equivalently, $x \in \partial' X$) if ∂X is differentiable at x and; therefore; there exists a unique exterior unit normal $\nu_X(x)$ at x (see Section 1.5 if X is convex, and Chapter 4 if ∂X is locally Lipschitz).

Surface area measure S_K for a compact convex $K \subset \mathbb{R}^n$:
The surface area measure S_K is a Borel measure on S^{n-1}. If K is a convex body and $\omega \subset S^{n-1}$ is Borel, then $S_K(\omega) = \mathcal{H}^{n-1}\left(\nu_K^{-1}(\omega)\right)$ (see Section 2.5).

Cone volume measure V_K for a $K \in \mathcal{K}^n_o$:
The cone volume measure V_K is a Borel measure on S^{n-1} with $dV_K = \frac{1}{n} h_K \, dS_K$ (see Section 2.6).

L_p-surface area measure $S_{K,p}$ for a $K \in \mathcal{K}^n_o$:
The L_p-surface area measure $S_{K,p}$ is a Borel measure on S^{n-1} with $dS_{K,p} = h_K^{1-p} \, dS_K$ (see Section 9.3).

Derivative of a locally Lipschitz (for example, convex) function on an open $\Omega \subset \mathbb{R}^n$:
$D\varphi(x)$ is the $n \times m$ matrix derivative of a locally Lipschitz function $\varphi : \Omega \to \mathbb{R}^m$ for $x \in \Omega$ wherever the derivative exists (so, for $\mathcal{H}^n$ a.e. $x \in \Omega$).

Hessian of a C^2 or a convex function on an open $\Omega \subset \mathbb{R}^n$:
$D^2\varphi(x)$ is the Hessian of the function $\varphi : \Omega \to \mathbb{R}^m$ for $x \in \Omega$ where either φ is C^2 on Ω, or φ is locally convex and $D^2\varphi(x)$ exists at x (it exists at $\mathcal{H}^n$ a.e. $x \in \Omega$, see Section 10.6).

Identity matrix:
I_d is the $d \times d$ identity matrix.

Spherical gradient and Hessian:
For a C^2 function $h : S^{n-1} \to \mathbb{R}$, let $\tilde{h}(tu) = t \cdot h(u)$ and $\bar{h}(tu) = h(u)$ for $t > 0$ and $u \in S^{n-1}$; then (see Definition 8.1.6)

$$\nabla h(u) = D\bar{h}(u)|_{u^\perp} = D\tilde{h}(u)|_{u^\perp}, \qquad \nabla^2 h(u) = D^2\bar{h}(u)|_{u^\perp},$$
$$\widetilde{D}^2 h(u) = \widetilde{D}^2\tilde{h}(u) = D^2\tilde{h}(u)|_{u^\perp} = \nabla^2 h(u) + h(u) I_{n-1}.$$

Subdifferential of a convex function:
$\partial\varphi(z) = \{u \in \mathbb{R}^n : \varphi(x) - \varphi(z) \geqslant \langle u, x - z\rangle \ \forall x \in \Omega\}$ for a convex function $\varphi : \Omega \to \mathbb{R}$ and $z \in \Omega$ where $\Omega \subset \mathbb{R}^n$ is open and convex. In particular, $\partial\varphi(z)$ is nonempty, convex and compact.

Log-concave functions and measures:
For a convex set $C \subset \mathbb{R}^n$, a function $f : C \to [0, \infty)$ is log-concave if $f((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} f(y)^{\lambda}$ holds for $x, y \in C$ and $\lambda \in (0,1)$; or in other words, $f = e^{-\varphi}$ for a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$. If $\{f > 0\}$ is not contained in a hyperplane, then $d\mu = f\, d\mathcal{H}^n$ is a log-concave measure (cf. Section 10.9).

Transpose of a matrix:
For an $n \times m$ matrix Φ, its transpose is the $m \times n$ matrix Φ^t.

Tensor product as a matrix:
$u \otimes v = uv^t$ for $u, v \in \mathbb{R}^n$.

Cardinality of a finite set X:
$\#X$

Chapter 1

Preliminaries in convex geometry in $\mathbb{R}^n$

In this chapter, we introduce the basic notions in convexity - bounded and unbounded closed convex sets, supporting hyperplanes, closest point map, convex polytopes and polyhedra, centroid, polarity, almost everywhere differentiability of the boundary of a convex body, convex functions, and encoding a compact convex set with the help of the convex and 1-homogeneous support function. Via Steiner symmetrization, we also prove two fundamental geometric inequalities, namely, the Brunn-Minkowski inequality and the Isodiametric inequality (the latter also for non-convex sets).

1.1 Affine hull, Convex hull

In this section we introduce the basic notions of affine geometry.

Definition 1.1.1 (Affine subspace). $A \subset \mathbb{R}^n$ is an affine subspace of dimension d if there exists $w \in A$ such that $A - w$ is a d-dimensional linear subspace.

If $A \subset \mathbb{R}^n$ is a d-dimensional affine subspace, then $A - x$ is a d-dimensional linear subspace for any $x \in A$, where points, lines and hyperplanes are the 0-dimensional, 1-dimensional and $(n-1)$-dimensional affine subspaces, respectively. In addition, a non-empty intersection of affine subspaces is an affine subspace.

Definition 1.1.2 (Affine hull). For $X \subset \mathbb{R}^n$, aff X is the smallest affine subspace containing X.

We observe that if $x_1, \ldots, x_k \in \mathbb{R}^n$, then

$$\text{aff}\,\{x_1, \ldots, x_k\} = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_k \in \mathbb{R} \right\}.$$

Definition 1.1.3. We say $x_0, \ldots, x_d \in \mathbb{R}^n$ are affinely independent if $\sum_{i=0}^{d} \lambda_i x_i = o$ and $\sum_{i=0}^{d} \lambda_i = 0$ for $\lambda_0, \ldots, \lambda_d \in \mathbb{R}$ imply that each $\lambda_i = 0$.

Remark. When $d \geqslant 1$, the above definition is equivalent to saying that the vectors $x_1 - x_0, \ldots, x_d - x_0$ are independent. In this case, aff $\{x_0, \ldots, x_d\}$ is of dimension d, and for any $z \in$ aff $\{x_0, \ldots, x_d\}$ there exist unique $\lambda_0, \ldots, \lambda_d \in \mathbb{R}$ with $z = \sum_{i=1}^{d} \lambda_i x_i$ and $\sum_{i=1}^{d} \lambda_i = 1$.

Definition 1.1.4 (Convex sets). We say that $X \subset \mathbb{R}^n$ is convex if $(1-\lambda)x + \lambda y \in X$ for any $x, y \in X$ and $\lambda \in [0,1]$.

As the intersection of convex sets is convex, the following concept is well-defined.

Definition 1.1.5 (Convex hull). For $X \subset \mathbb{R}^n$ the smallest convex set containing X is its convex hull $\operatorname{conv} X$; namely,

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \geqslant 1,\ \forall\, x_i \in X,\ \forall\, \lambda_i \geqslant 0 \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}. \tag{1.1}$$

Proposition 1.1.6 (Carathédory). *If $X \subset \mathbb{R}^n$ and $x \in \operatorname{conv} X$, then there exist an integer $1 \leqslant k \leqslant n+1$ and $x_1, \ldots, x_k \in X$ such that $x \in \operatorname{conv}\{x_1, \ldots, x_k\}$.*

Proof. For $x \in \operatorname{conv} X$, let $k \geqslant 1$ be minimal with the property that there exist $x_1, \ldots, x_k \in X$ and $\lambda_1, \ldots, \lambda_k \geqslant 0$ such that $x = \sum_{i=1}^{k} \lambda_i x_i$ and $\sum_{i=1}^{k} \lambda_i = 1$. We claim that

$$k \leqslant n+1. \tag{1.2}$$

We suppose that $k \geqslant n+2$, and seek a contradiction. The minimality of k yields that $\lambda_1, \ldots, \lambda_k > 0$. Since $k \geqslant n+2$, there exist $\mu_1, \ldots, \mu_k \in \mathbb{R}$ not all zero with $\sum_{i=1}^{k} \mu_i x_i = o$ and $\sum_{i=1}^{k} \mu_i = 0$, and hence there exists some index $i \in \{1, \ldots, k\}$ with $\mu_i > 0$. Let

$$t = \min \left\{ \frac{\lambda_i}{\mu_i} : \mu_i > 0 \right\} > 0,$$

and let $j \in \{1, \ldots, k\}$ such that $\mu_j > 0$ and $t = \frac{\lambda_j}{\mu_j}$. Then, for $i = 1, \ldots, k$, it follows that $\lambda_i' = \lambda_i - t\mu_i \geqslant 0$, $\sum_{i=1}^{k} \lambda_i' = 1$, and $x = \sum_{i=1}^{k} \lambda_i' x_i$. Now $\lambda_j' = 0$ contradicts the minimality of k, and hence proves the claim (1.2), and in turn Proposition 1.1.6. ∎

For compact $X \subset \mathbb{R}^n$, its diameter is defined as

$$\operatorname{diam} X = \max\{\|y - z\| : y, z \in X\}.$$

Proposition 1.1.7. *If $X \subset \mathbb{R}^n$ is compact, then* $\operatorname{conv} X$ *is compact, and* $\operatorname{diam} \operatorname{conv} X = \operatorname{diam} X$.

Remark. If $X \subset \mathbb{R}^n$ is closed, then $\operatorname{conv} X$ may not be closed; for example, for the closed set $X = \left\{ \left(t, \frac{\pm 1}{t}\right) : t > 0 \right\} \subset \mathbb{R}^2$, we have $\operatorname{conv} X = \{(t, s) : t > 0 \ \&\ s \in \mathbb{R}\}$.

Proof. $\widetilde{X} = \operatorname{conv} X$ is compact as a consequence of Proposition 1.1.6. Now, let $D = \operatorname{diam} X$, and let $y, z \in \widetilde{X}$. We have $y \in X + DB^n$ for $x \in X$ because $\widetilde{X}$ is the intersection of all convex sets containing X, and hence $X \subset y + DB^n$, which in turn yields $\widetilde{X} \subset y + D\,B^n$. Therefore, $\|y - z\| \leqslant D$. ∎

The following statement is equivalent to Carathédory's Proposition 1.1.6.

Proposition 1.1.8 (Radon). *If $x_1, \ldots, x_k \in \mathbb{R}^n$ with $k \geqslant n+2$, then there exists $I \subset \{1, \ldots, k\}$ with $1 \leqslant \#I < k$ such that* $\operatorname{conv}\{x_i\}_{i \in I} \cap \operatorname{conv}\{x_i\}_{i \notin I} \neq \varnothing$.

Proof. For each i, let $y_i = (x_i, 1) \in \mathbb{R}^{n+1}$. Since $k \geqslant n+2$, there exist $\alpha_1, \ldots, \alpha_k$ not all zero with $\sum_{i=1}^k \alpha_i y_i = o_{\mathbb{R}^{n+1}}$; or equivalently, $\sum_{i=1}^k \alpha_i x_i = o_{\mathbb{R}^n}$ and $\sum_{i=1}^k \alpha_i = 0$. It follows that $1 \leqslant \#I < k$ for $I = \{i \in \{1, \ldots, k\} : \alpha_i \geqslant 0\}$. In addition, we have

$$A = \sum_{i \in I} \alpha_i = \sum_{i \notin I} (-\alpha_i) > 0$$

and $z \in \operatorname{conv}\{x_i\}_{i \in I} \cap \operatorname{conv}\{x_i\}_{i \notin I}$ for $z = \sum_{i \in I} \frac{\alpha_i}{A} x_i = \sum_{i \notin I} \frac{-\alpha_i}{A} x_i$. ∎

Lemma 1.1.9. *If $X \subset \mathbb{R}^n$ is a (non-empty) closed convex set and $d = \dim A$ for $A = \operatorname{aff} X$, then its relative interior* $\operatorname{relint} X$ *(with respect to the topology of A) is non-empty and convex and X is its closure.*

Remark. We write $d = \dim X$ where $|X| > 0$ if and only if $d = n$.

Proof. We may assume that $A = \mathbb{R}^n$, hence there exist affinely independent $x_0, \ldots, x_n \in X$. For $i = 0, \ldots, n$, there exists (unique) $u_i \in S^{n-1}$ such that $\langle u_i, x_i \rangle > \langle u_i, x_j \rangle$ for $j \neq i$, and $\langle u_i, x_k \rangle = \langle u_i, x_j \rangle$ for $k, j \neq i$. Choosing $s > 0$ such that $\langle u_i, x_i - x_j \rangle > s$ for all $j \neq i$, and setting $r = \frac{s}{n+1}$, we have $z_0 + r\,B^n \subset \operatorname{conv}\{x_0, \ldots, x_n\} \subset X$ for $z_0 = \frac{1}{n+1}(x_0 + \ldots + x_n)$. In particular, the convex set $\operatorname{int} X$ is non-empty.

Finally, for any $y \in \partial X$ and $\lambda \in (0, 1]$, we have that $(1-\lambda)y + \lambda z_0 + \lambda r B^n \subset X$; therefore, $X = \operatorname{cl} \operatorname{int} X$. ∎

1.2 Closed convex sets in $\mathbb{R}^n$

While one of the main objectives of this book is to understand properties of compact convex sets in $\mathbb{R}^n$, we will need many properties of possibly unbounded closed convex sets (see Section 1.B for further properties).

Lemma 1.2.1 (Unbounded closed convex sets). *Let $X \subset \mathbb{R}^n$ be a convex closed set. Then X is unbounded if and only if there exists $u \in S^{n-1}$ such that $x + [0, \infty)u \subset X$ for any $x \in X$; this, in turn, is equivalent to saying that X contains a half-line.*

Proof. The only non-trivial implication is the fact that if X is unbounded, then there exists $u \in S^{n-1}$ such that $x + [0, \infty)u \subset X$ for any $x \in X$.

Since X is unbounded, there exists a sequence $\{x_k\} \subset X$ with $\|x_k\| \to \infty$. Fix $y \in X$, and hence $\|x_k - y\| \to \infty$. Up to a taking a subsequence, we may assume that $u_k = \frac{x_k - y}{\|x_k - y\|} \in S^{n-1}$ converges to a $u \in S^{n-1}$. Since

$$\|x_k - y\| - \|x - y\| \leqslant \|x_k - x\| \leqslant \|x_k - y\| + \|x - y\|$$

for any $x \in X$, this also implies that $\lim_{k \to \infty} \frac{x_k - x}{\|x_k - x\|} = u$. For any $t > 0$, it follows from the convexity of X that $x + tu_k \in X$ whenever $\|x_k - x\| > t$, which in turn yields that $x + tu \in X$ as X is closed. ∎

Lemma 1.2.2. *Let $X \subset \mathbb{R}^n$ be convex closed, and let $z \notin X$. Then there exists a unique closest point $y \in X$ to z, and $X \subset H^+ = \{x \in \mathbb{R}^n : \langle x, z-y\rangle \leqslant \langle y, z-y\rangle\}$.*

Proof. The existence of a closest point $y \in X$ to z follows from the fact that as $\lim_{x\to\infty} \|x - z\|^2 = \infty$, the continuous function $x \mapsto \|x-z\|^2$ has a minimum on the closed set X.

Now, given $x \in X$, it follows that $y + t(x-y) = (1-t)y + tx \in X$ for $t \in [0,1]$ as X is convex; therefore, $\|z - y - t(x-y)\|^2 \geqslant \|z-y\|^2$ for $t \in (0,1)$, which inequality in turn is equivalent to saying that

$$\langle z-y, z-y\rangle - 2t\langle z-y, x-y\rangle + t^2\langle x-y, x-y\rangle \geqslant \langle z-y, z-y\rangle.$$

Letting $t > 0$ tend to zero, we deduce that $\langle z-y, x-y\rangle \leqslant 0$, and hence $x \in H^+$. Finally, if $x \in X$ with $x \neq y$, then $\langle z-y, x-y\rangle \leqslant 0$ yields that $\|x-z\|^2 = \|(x-y)+(y-z)\|^2 > \|y-z\|^2$, showing the uniqueness of y as a closest point. ∎

The rest of the section collects various consequences of Lemma 1.2.2.

Lemma 1.2.3 (Exterior normal). *Let $X \subset \mathbb{R}^n$ be convex closed and $y \in \partial X$. Then there exists $u \in \mathbb{R}^n \backslash \{o\}$ such that $\langle x, u\rangle \leqslant \langle y, u\rangle$ for $x \in X$.*

Remark. Any such u is called an exterior normal at y, and $y + u^\perp$ is called a supporting hyperplane, which then exists at any boundary point.

Proof. Let $z_k \in \mathbb{R}^n \backslash X$ a sequence tending to y, and let y_k be the closest point of X to z_k (cf. Lemma 1.2.2). As $\|y_k - z_k\| \leqslant \|y - z_k\|$, also the sequence y_k tends to y. Since S^{n-1} is compact, we may assume that $u_k = \frac{z_k - y_k}{\|z_k - y_k\|}$ tends to $u \in S^{n-1}$. Using Lemma 1.2.2, we deduce that if $x \in X$ then $\langle u, x-y\rangle = \lim_{k\to\infty}\langle u_k, x - y_k\rangle \leqslant 0$. ∎

Corollary 1.2.4. *An $X \subset \mathbb{R}^n$ with $X \neq \mathbb{R}^n$ is a closed convex set if and only if X is the intersection of closed half-spaces; namely, the supporting half-spaces.*

The boundary of a closed convex set $X \subset \mathbb{R}^n$ is structured into faces.

Definition 1.2.5 (Face). For a convex closed $X \subset \mathbb{R}^n$ and $u \in \mathbb{R}^n \backslash \{o\}$, let us assume that there exists a $y \in \partial X$ where u is an exterior normal. Then $F_X(u) = \{x \in X : \langle x, u\rangle = \langle y, u\rangle\}$ is the face of X with exterior normal u.

Remarks.

- It follows from Lemma 1.2.3 that $F_X(u) \subset \partial X$.
- An equivalent definition of a face of X is that it is the intersection of X with a supporting hyperplane.
- What is called face here, following Grünbaum [352], is frequently called *exposed face* in the literature (see, for example, Rockafellar [633]).

Lemma 1.2.6 (Separation of a closed convex set and a compact set). *Let $X \subset \mathbb{R}^n$ be convex closed, and $K \subset \mathbb{R}^n$ compact convex with $X \cap K = \varnothing$. Then there exist $v \in S^{n-1}$ and $\alpha \in \mathbb{R}$ such that $\langle v, x\rangle < \alpha$ for $x \in X$ and $\langle v, y\rangle > \alpha$ for $y \in K$.*

Proof. Since K is compact and X is closed, there exist $x_0 \in X$ and $y_0 \in K$ such that $\|x_0 - y_0\|$ is the minimum of the distances of points of X and K. Therefore, the hyperplane normal to $x_0 - y_0$ and passing through $\frac{x_0+y_0}{2}$ strictly separates X and K (thanks to Lemma 1.2.2). ∎

Lemma 1.2.7 (Separation of closed convex sets). *If $X, Y \subset \mathbb{R}^n$ are convex closed, $\operatorname{int} Y \neq \varnothing$, and $X \cap \operatorname{int} Y = \varnothing$, then there exist $v \in S^{n-1}$ and $\alpha \in \mathbb{R}$ such that $\langle v, x\rangle \leqslant \alpha$ for $x \in X$ and $\langle v, y\rangle \geqslant \alpha$ for $y \in Y$.*

Proof. We may assume that $o \in \operatorname{int} Y$, and chose a $x_0 \in X$. For $k > \|x_0\|$, the disjoint compact sets $Y_k = kB^n \cap (1 - \frac{1}{k})Y \subset \operatorname{int} Y$ and $X_k = kB^n \cap X$ can be separated by a hyperplane H_k by Lemma 1.2.6, and hence H_k intersects $\operatorname{conv}\{o, x_0\}$ in a point z_k. We consider a subsequence $\{H_{k'}\} \subset \{H_k\}$ such that $z_{k'}$ tends to a $z \in \operatorname{conv}\{o, x_0\}$, and some exterior unit vector $u_{k'}$ of $H_{k'}$ tends to a $v \in S^{n-1}$, and then choose $\alpha = \langle v, z\rangle$. ∎

Lemma 1.2.3 ensures that the normal cone below is not empty.

Definition 1.2.8 (Cone and Normal Cone).

(a) $C \subset \mathbb{R}^n$ is a convex cone if $\alpha x + \beta y \in C$ holds for any $x, y \in C$ and $\alpha, \beta \geqslant 0$. We say that C is non-trivial if $C \neq \{o\}$ and $C \neq \mathbb{R}^n$.

(b) If $X \subset \mathbb{R}^n$ is closed convex and $y \in \partial X$, then the normal cone at y is

$$N_X(y) = \{u \in \mathbb{R}^n : \langle u, x - y\rangle \leqslant 0 \ \forall x \in X\} \tag{1.3}$$

which is either a non-trivial closed convex cone, or $N_X(y) = \mathbb{R}^n$ in the case $X = \{y\}$.

Remarks.

- $v \in N_X(y)\backslash\{o\}$ for $y \in \partial X$ if and only if v is an exterior normal at y.
- If $C \subset \mathbb{R}^n$ is a closed convex cone with $C \neq \mathbb{R}^n$, then $\mathcal{U} = S^{n-1} \cap N_C(o)$ satisfies $C = \{x \in \mathbb{R}^n : \langle u, x\rangle \leqslant 0 \ \forall u \in \mathcal{U}\}$.

  Corollary 1.2.4 yields the following representation of closed convex cones.

Corollary 1.2.9. *A $C \subset \mathbb{R}^n$ with $C \neq \mathbb{R}^n$ is a closed convex cone if and only if X is the intersection of closed half-spaces containing the origin o on their boundary.*

Lemma 1.2.10. *If $X \subset \mathbb{R}^n$ is closed and convex, $y_m, y \in \partial X$ with $y_m \to y$ and $u_m \in N_X(y_m) \cap S^{n-1}$, then $u \in N_X(y)$ for any accumulation point u of the sequence $\{u_m\}$.*

Proof. We may assume that $\lim_{m\to\infty} u_m = u$. For $x \in X$, we have $\langle u_m, x - y_m\rangle \leqslant 0$ for any m; therefore, $\langle u, x - y\rangle \leqslant 0$ also in the limit. ∎

Next, we show that there always exists an exterior unit normal vector such that the opposite vector points inward.

Lemma 1.2.11. *Let $X \subset \mathbb{R}^n$ be closed, convex, with $\operatorname{int} X \neq \varnothing$, and let $x \in \partial X$. Then there exist $u \in \operatorname{relint} N_X(x)$ and $t > 0$ such that $x - tu \in \operatorname{int} X$.*

Proof. We may assume that $x = o$.

First we claim that there exist $w \in S^{n-1} \cap N_X(x)$ and $t > 0$ such that

$$x - tw \in \operatorname{int} X. \tag{1.4}$$

To prove the claim, let $C = \operatorname{cl}\{[0,\infty)z : z \in \operatorname{int} X\}$, that is a closed convex cone with $\operatorname{int} C \neq \varnothing$. Then the claim (1.4) is equivalent to proving that $N_X(o) \cap \operatorname{int}(-C) \neq \varnothing$.

By contradiction, we suppose that $N_X(o) \cap \operatorname{int}(-C) = \varnothing$. Then Lemma 1.2.7 provides a $v \in S^{n-1}$ such that $\langle v, y\rangle \geqslant 0$ for $y \in -C$ and $\langle v, z\rangle \leqslant 0$ for $z \in N_X(o)$. Hence, the first property of v yields that $v \in N_X(o)$, which combined with the second property of v implies that $1 = \langle v, v\rangle \leqslant 0$. This contradiction proves the claim (1.4).

Finally, as $x - tw \in \operatorname{int} X$, wiggling w a bit if needed results in a $u \in \operatorname{relint} N_X(x)$ such that $x - tu \in \operatorname{int} X$. ∎

Lemma 1.2.2 makes it possible to consider the closest point map.

Definition 1.2.12 (Closest point map). If $X \subset \mathbb{R}^n$ convex closed and $z \in \mathbb{R}^n$, then the unique point $y \in X$ closest to z given by Lemma 1.2.2 is $\Pi_X(z) = y$.

Remarks:

(a) For $z \in \mathbb{R}^n \backslash X$, we have $\Pi_X(z) = y \in \partial X$ if and only if $z \in y + N_X(y)$;

(b) For a linear subspace $L \subset \mathbb{R}^n$, Π_L is the orthogonal projection.

The closest point map is actually a contraction, as we show it now.

Lemma 1.2.13. *If $X \subset \mathbb{R}^n$ is convex and closed and $z_1, z_2 \in \mathbb{R}^n$, then*

$$\|\Pi_X(z_1) - \Pi_X(z_2)\| \leqslant \|z_1 - z_2\|.$$

Proof. We set $y_i = \Pi_X(z_i)$ and $u_i = z_i - y_i$. Then $\langle u_i, y_i - x\rangle \geqslant 0$ for $x \in X$ by Lemma 1.2.2 for $i = 1, 2$. Hence, the substitution $z_i = y_i + u_i$ implies

$$\|z_1 - z_2\|^2 = \|y_1 - y_2\|^2 + \|u_1 - u_2\|^2 + 2\langle u_1, y_1 - y_2\rangle + 2\langle u_2, y_2 - y_1\rangle \geqslant \|y_1 - y_2\|^2.$$

∎

1.3 Additional properties of convex cones

As the normal cones at a boundary point of a closed convex set is a closed convex cone, we need to better understand such structures, which is the main aim of this section. We recall that $C \subset \mathbb{R}^n$ is a convex cone if $\alpha x + \beta y \in C$ for any $x, y \in C$ and $\alpha, \beta \geqslant 0$, and C is non-trivial if $C \neq \{o\}, \mathbb{R}^n$.

Definition 1.3.1 (Dual cone). For a closed convex cone $C \subset \mathbb{R}^n$, its dual cone is

$$C^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leqslant 0 \ \forall\, y \in C\}.$$

Remarks. In particular, $C^* = N_C(o)$. Since $C^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \ \forall\, y \in C\}$ as C is a cone, our notation is consistent with the notation for the polar of a compact convex set in Definition 1.9.1.

Lemma 1.3.2. *If $C \subset \mathbb{R}^n$ is a closed convex cone, then C^* is a closed convex cone, and $C^{**} = C$.*

Proof. The definition immediately yields that C^* is a closed convex cone and $C \subset C^{**}$.

If $z \notin C$, then Lemma 1.2.6 provides $v \in S^{n-1}$ and $\alpha \in \mathbb{R}$ such that

$$\langle v, z\rangle > \alpha \quad \text{and} \quad \langle v, x\rangle < \alpha \ \forall\, x \in C.$$

As C is a cone, choosing $x = o$ we deduce that $\alpha > 0$. Also, replacing x with tx and letting $t \to \infty$, we get $\langle v, x\rangle \leqslant 0$ for all $x \in C$. It follows that $v \in C^*$ and $\langle v, z\rangle > 0$, therefore $z \notin C^{**}$. ∎

Definition 1.3.3 (Pointed convex cones). A closed non-trivial convex cone $C \subset \mathbb{R}^n$ is pointed if it contains no line.

Remark. Pointed convex cones are also frequently called *strongly convex cones*.

As the convex hull of two linear subspaces is their sum, we have the following statement.

Lemma 1.3.4. *Any closed convex cone $C \subset \mathbb{R}^n$ of dimension at least 1 that is not a linear subspace can be written as $L + C'$ where L is a linear subspace and $C' \subset L^\perp$ is a pointed convex cone.*

Remark. If $X \subset \mathbb{R}^n$ is a closed convex set with $\operatorname{int} X \neq \varnothing$, then the normal cone $N_X(x)$ is pointed convex at any $x \in \partial X$.

Now we establish various ways to characterize pointed convex cones.

Lemma 1.3.5 (Pointed convex cones). *For a closed convex cone $C \subset \mathbb{R}^n$, the following statements are equivalent:*

(i) $C \cap (-C) = \{o\}$;

(ii) C contains no line (and hence it is a pointed convex cone);

(iii) $\dim C^* = n$*;*

(iv) $C \neq \mathbb{R}^n$ and for any $z \notin C$ there exists $w \in \mathbb{R}^n$ such that $\langle w, z\rangle > 0$ and

$$\langle w, x\rangle < 0 \text{ for all } x \in C\backslash\{o\}.$$

Remark. It follows from (iv) that a closed convex cone C is pointed convex if and only if there exists a supporting hyperplane H such that $H \cap C = \{o\}$. Because of this property, o is sometimes called an apex of the pointed convex cone.

Proof. (i) $\Longrightarrow$ (ii): If $\ell \subset C$ is a line, then Lemma 1.2.1 implies that $\ell' \subset C$ for the parallel linear 1-subspace by Lemma 1.2.1. Therefore, $\ell' \subset C \cap (-C)$.

(ii) $\Longrightarrow$ (iii): If $C^* \subset u^\perp$ for $u \in S^{n-1}$, then $\mathbb{R}u \subset C^{**} = C$.

(iii) $\Longrightarrow$ (iv): As $z \notin C$, Lemma 1.2.6 implies that there exists $w_0 \in \mathbb{R}^n$ such that

$$\langle w_0, z\rangle > 0 \quad \text{and } \langle w_0, x\rangle \leqslant 0 \ \forall\, x \in C.$$

Hence $w_0 \in C^*$. By $\operatorname{int} C^* \neq \varnothing$ and continuity, there exists a $w \in \operatorname{int} C^*$ such that $\langle w, z\rangle > 0$. Now, for $x \in C\backslash\{o\}$, it follows that $w + \varepsilon x \in C^*$ for some small $\varepsilon > 0$. Thus $\langle x, w + \varepsilon x\rangle \leqslant 0$, which implies that $\langle x, w\rangle < 0$.

(iv) $\Longrightarrow$ (i): Since there exists $w \in \mathbb{R}^n$ such that $\langle w, x\rangle < 0$ for all $x \in C\backslash\{o\}$, we deduce that $C \cap (-C) = \{o\}$.

■

As $\dim C^* = n$ for any pointed closed convex cone $C \subset \mathbb{R}^n$ by Lemma 1.3.5, the following property is a useful consequence of Lemma 1.2.11 and Lemma 1.3.4.

Lemma 1.3.6. *For any closed convex cone $C \subset \mathbb{R}^n$ that is not a linear subspace, there exists $w \in \operatorname{relint} C$ such that $-w \in \operatorname{relint} C^*$.*

Finally, we consider a natural method to construct convex cones.

Definition 1.3.7 (Positive hull). If $X \subset \mathbb{R}^n$, then its positive hull is

$$\operatorname{pos} X = \left\{ \sum_{i=1}^{k} \lambda_i x_i : k \geqslant 1,\ \forall\, x_i \in X \text{ and } \forall\, \lambda_i \geqslant 0 \right\},$$

which is a convex cone (that may not be closed even if X is compact, cf. Example 1.3.10).

Since any finite set of vectors in $\mathbb{R}^n$ of cardinality at least $n + 1$ is dependent, we have the following analogue of Carathédory's Proposition 1.1.6.

Lemma 1.3.8. *If $X \subset \mathbb{R}^n$ and $z \in \operatorname{pos} X$, then there exist $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \geqslant 0$ such that $z = \sum_{i=1}^{n} \lambda_i x_i$.*

Lemma 1.3.9. *If $X \subset \mathbb{R}^n$ is compact and $o \notin \operatorname{conv} X$, then the convex cone* $\operatorname{pos} X$ *is a closed set.*

Remark. As X is compact, $o \notin \operatorname{conv} X$ is equivalent with the property that there exist $\alpha > 0$ and $v \in \mathbb{R}^n$ such that $\langle v, x\rangle \geqslant \alpha$ for $x \in X$.

Proof. Let $\alpha > 0$ and $v \in \mathbb{R}^n$ such that $\langle v, x\rangle \geqslant \alpha$ for $x \in X$. We consider $z = \lim_{k\to\infty} z_k$ for $z_k \in \operatorname{pos} X$, and hence $z_k = \sum_{i=1}^n \lambda_{ki} x_{ki}$ for $\lambda_{ki} \geqslant 0$ and $x_{ki} \in X$ by Lemma 1.3.8. There exists a $\gamma > 0$ such that $\langle v, z_k\rangle \leqslant \gamma$ for each k, and we may assume that $\lim_{k\to\infty} x_{ki} = y_i \in X$ for $i = 1, \ldots, n$ by the compactness of X. The first property and $\langle v, x_{ki}\rangle \geqslant \alpha$ imply that each $\lambda_{ki} \leqslant \gamma/\alpha$, which in turn yields $z \in \operatorname{pos}\{y_1, \ldots, y_n\} \subset \operatorname{pos} X$. ∎

We note that if $X \subset \mathbb{R}^n$ is compact and $o \notin X$, then $\operatorname{pos} X$ may not be closed.

Example 1.3.10. If $X = \{(x, y, z) \in \mathbb{R}^3 : |x| = 1 \text{ and } y^2 + (z-1)^2 = 1\}$, then X is compact and $o \notin X$, but $\operatorname{pos} X$ is the union of the open half-space $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ and the x axis; therefore, it is not closed.

Lemma 1.3.11. *If $X \subset \mathbb{R}^n$ is finite, then* $\operatorname{pos} X$ *is a closed convex cone.*

Proof. Since $\operatorname{pos} X$ is a convex cone, the only issue is whether it is closed. We proceed by induction on $n \geqslant 1$ where the case $n = 1$ trivially holds as a positive hull in $\mathbb{R}$ is either a point, or a half-line, or $\mathbb{R}$.

In the case $n \geqslant 2$, let $P = \operatorname{conv} X$. If $o \notin P$, then Lemma 1.3.9 verifies Lemma 1.3.11. If $o \in \operatorname{int} P$, then $\operatorname{pos} X = \mathbb{R}^n$, and hence closed.

Therefore, the only case left open is when $n \geqslant 2$ and $o \in \partial P$, and let $u \in S^{n-1}$ be an exterior normal at o to P. For $X_0 = X \cap u^\perp$, $C_0 = \operatorname{pos} X_0$ is a closed convex cone by induction, thus we may assume that $X_+ = X \backslash u^\perp$ is non-empty. Here $\langle -u, x\rangle > 0$ for $x \in X_+$, and hence the finiteness of X_+ yields that $o \notin \operatorname{conv} X_+$, and there exists a $\beta > 0$ such that $\langle x, -u\rangle \geqslant \beta\|x\|$ for $x \in X_+$. In particular, $C_+ = \operatorname{pos} X_+$ is a closed convex cone by Lemma 1.3.9, and

$$\operatorname{pos} X = C_0 + C_+. \tag{1.5}$$

It also follows by the triangle inequality for $\|\cdot\|$ that $\langle y, -u\rangle \geqslant \beta\|y\|$ for $y \in C_+$.

Finally, to prove that $\operatorname{pos} X$ is closed, let $z_k \in \operatorname{pos} X$ tend to a $z \in \mathbb{R}^n$; in particular, $\langle -u, z_k\rangle \leqslant \gamma$ for some constant $\gamma > 0$. According to (1.5), we have $z_k = w_{k0} + w_{k+}$ for $w_{k0} \in C_0 \subset u^\perp$ and $w_{k+} \in C_+$ where

$$\gamma \geqslant \langle -u, z_k\rangle = \langle -u, w_{k+}\rangle \geqslant \beta\|w_{k+}\|.$$

Therefore, we may assume that w_{k+} tends to w_+, and hence $w_{k0} = z_k - w_{k+}$ tends to some w_0. We have $w_+ \in C_+$ and $w_0 \in C_0$ as these cones are closed sets, and hence $z \in \operatorname{pos} X$ by (1.5). ∎

We note that for two closed convex cones, their sum, which is always a convex cone, may not be closed.

Example 1.3.12. For the closed convex cones $C = \{(-t, -t, 0) : t \geqslant 0\}$ and $\widetilde{C} = \{(x, y, z) \in \mathbb{R}^3 : x \geqslant 0 \ \& \ y^2 + z^2 \leqslant x^2\}$, we have that $p = (0, 0, 1) \notin C + \widetilde{C}$, but $p = \lim_{t\to\infty}(q_t + \tilde{q}_t)$ where $q_t = (-t, -t, 0) \in C$ and

$$\tilde{q}_t = \left(\sqrt{t^2 + \left(1 + \frac{1}{t}\right)^2},\ t, 1 + \frac{1}{t}\right) \in \widetilde{C}.$$

1.4 Polyhedra, polytopes

According to Corollary 1.2.4, closed convex subsets of $\mathbb{R}^n$ different from $\mathbb{R}^n$ are the intersections of closed half-spaces. In this section, we consider polyhedra that are the intersections of finitely many closed halfspaces.

Definition 1.4.1 (Polyhedron). A polyhedron $P \subset \mathbb{R}^n$ is the non-empty intersection of finitely many closed half-spaces in $\mathbb{R}^n$, allowing $P = \mathbb{R}^n$ (empty family of closed half-spaces). A 0-dimensional and a 1-dimensional face are called vertex and edge, respectively. If $\dim P = n$ and $P \neq \mathbb{R}^n$, then the $(n-1)$-dimensional faces are called facets.

The following properties (including Lemma 1.4.2) are immediate from the definition:

- Any linear subspace is a polyhedron, and
- non-empty intersection of a finite family of polyhedra is a polyhedron.

Lemma 1.4.2. *If $P = H_1^+ \cap \ldots \cap H_k^+$ is a polyhedron for closed half-spaces $H_i^+ \subset \mathbb{R}^n$, and $L \subset \mathbb{R}^n$ is a proper linear subspace, then the orthogonal projection $\Pi_L P$ is a polyhedron. In the case $\Pi_L P \neq L$, P is the intersection of all $H_i^+ \cap L$ such that H_i^+ contains a translate of $L^\perp$.*

Next we consider the "face lattice" of a polyhedron P; namely, the partially ordered set whose elements are the faces of P, and the ordering is by inclusion.

Proposition 1.4.3. *Let $P \subset \mathbb{R}^n$ be a polyhedron with $P \neq \mathbb{R}^n$.*

(a) P has somes faces, any face of P is a polyhedron, and the number of faces is finite.

(b) Any face of a face of P is a face of P.

(c) If $\dim P = n$, then

(c1) ∂P is the union of the facets;

(c2) any face F of P of dimension at most $n-2$ is the face of a facet, and is the intersection of the facets containing it;

(c3) any $(n-2)$-dimensional face is contained in exactly two facets.

Proof. We prove Proposition 1.4.3 by induction on n where the cases $n = 1, 2$ trivially hold. We may assume that $n \geqslant 3$, $o \in \operatorname{int} P$, and $P = \cap_{i=1}^{k} H_i^+$ where $H_1^+, \dots, H_k^+$ is a minimal family of closed half-spaces whose intersection is P. Let $H_i = \partial H_i^+$ and $F_i = H_i \cap P$ for $i = 1, \dots, k$.

For the first statement of (a), $P \neq \mathbb{R}^n$ yields that P has a boundary point z_0, that is contained in a supporting hyperplane H_0, and hence $H_0 \cap P$ is a face. For the second statement of (a), if $F = H \cap P$ is a face for a supporting hyperplane H of P, then $F = H \cap P = \cap_{i=1}^{k} (H \cap H_i^+)$, thus F is a polyhedron.

Turning to (c1), for any $z \notin P$, there exists an H_i^+ such that $z \notin H_i^+$, and hence the segment $\operatorname{conv}\{o, z\}$ intersects H_i. It follows that $\partial P = \cup_{i=1}^{k} F_i$. The minimality of $H_1^+, \dots, H_k^+$ yields that for any H_j^+, there exists a $z_j \in \cap_{i \neq j}^{k} H_i^+$ with $z_j \notin P$. Then $\operatorname{conv}\{o, z_j\}$ intersects F_j in a w_j that is the center of a $(n-1)$-ball contained in F_j as $o \in \operatorname{int} P$, thus each F_j is a facet, namely, $\dim F_j = n - 1$.

For (c2), let $F = H \cap P$ be a face for a supporting hyperplane H, and let $z \in \operatorname{relint} F$. Here $z \in F_j$ for a facet F_j by (c1), and hence $F \subset H_j$ by $z \in \operatorname{relint} F$ and $F \subset H_j^+$, which in turn yields that $F = H \cap P \cap H_j = H \cap F_j$, thus F is a face of F_j.

To continue with the second half of (c2), we observe that $F_j = \bigcap_{i \neq j} (H_j \cap H_i^+)$, and let $I_j \subset \{1, \dots, n\} \backslash \{j\}$ be minimal such that $F_j = \bigcap_{i \in I_j} (H_j \cap H_i^+)$, and hence $G_{ji} = H_i \cap F_j$, $i \in I_j$, are the $(n-2)$ faces of F_j by induction. Here $G_{ji} = P \cap H_i \cap F_j = F_i \cap F_j$, and hence the induction hypothesis implies that

$$F = \bigcap_{F \subset G_{ji}} G_{ji} = \bigcap_{F \subset F_j \cap F_i} F_i \cap F_j = \bigcap_{F \subset F_i} F_i.$$

This completes the proof of (c2), and turn the proof of (a) by induction.

For (c3), let F be an $(n-2)$-dimensional face of P, and let $L \subset \mathbb{R}^n$ be the two-dimensional linear subspace orthogonal to $\operatorname{aff} F$, and hence $\Pi_L P$ is a polyhedron according to Lemma 1.4.2. The vertex $\Pi_L F$ of $\Pi_L P$ is contained in two edges f_1 and f_2 of Π_L, thus F is contained in exactly two facets of P by Lemma 1.4.2 whose affine hulls are $\operatorname{aff}(f_1 + F)$ and $\operatorname{aff}(f_2 + F)$.

For (b), let G be a face of F for a face F of P. According to (c2), there exists a facet F_i such that F is either a face of F_i, or $F = F_i$, and hence G is a face of F_i by induction. In particular, $A \cap F_i = G$ for an $(n-2)$-dimensional affine subspace $A \subset H_i$. If $L \subset \mathbb{R}^n$ is a two-dimensional linear subspace orthogonal to A, then $v = \Pi_L G$ is an endpoint of the edge $\Pi_L F_i$ of the polyhedron $\Pi_L P$. It follows that v is a vertex of $\Pi_L P$ by induction, and hence $v = \ell \cap \Pi_L P$ for a line $\ell \subset L$; therefore, $G = P \cap (\ell + A)$. ∎

Definition 1.4.4 (Polytopes). A polytope is a bounded polyhedron.

In other words, a polytope is a bounded intersection of finitely many closed half-spaces. Now we show that this property is equivalent with the dual definition; namely, the convex hull of finitely many points.

Proposition 1.4.5. *A set $P \subset \mathbb{R}^n$ is a polytope if and only if it is the convex hull of finitely many points.*

Remark. The polytope is actually the convex hull of its vertices.

Proof. We use induction on $n \geqslant 1$ where the case $n = 1$ is trivial.

To show that a polytope $P \subset \mathbb{R}^n$ is the convex hull of its vertices, we may assume that $\dim P = n$. For any $x \in P$, a line ℓ through x intersects ∂P in $y_1 \in F_1$ and $y_2 \in F_2$ for some facets F_1, F_2 of P by Proposition 1.4.3 (c1) (possibly $F_1 = F_2$), and hence $x \in \operatorname{conv}\{y_1, y_2\}$. Now y_i is the convex hull of the finitely many vertices of F_i by induction and Proposition 1.4.3 (a), which are vertices of P, as well, by Proposition 1.4.3 (c2).

On the other hand, let us verify that $P = \operatorname{conv} X$ is a polytope where $X = \{x_1, \dots, x_k\} \subset \mathbb{R}^n$. We may assume that $\dim P = n$ and $o \in \operatorname{int} P$. Readily, P is bounded. For any supporting hyperplane $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = \alpha\}$ for $u \in S^{n-1}$ and $\alpha \in \mathbb{R}$ where $\langle x_i, u\rangle \geqslant \alpha$ for $x_i \in X$, any $z \in H \cap P$ is of the form $z = \sum_{i=1}^k \lambda_i x_i$ where $\sum_{i=1}^k \lambda_i = 1$ and each $\lambda_i \geqslant 0$. Since $\langle z, u\rangle = \alpha$ and each $\langle x_i, u\rangle \geqslant \alpha$, we deduce that $\lambda_i = 0$ if $x_i \notin H$; or in other words, the face $H \cap P$ is the convex hull of $H \cap X$. It follows that P has only finitely many faces.

Let $F_1, \dots, F_m$ denote the $(n-1)$-dimensional faces of P, and H_i^+ denote the "supporting" half-space containing P with $P \cap \partial H_i^+ = F_i$. In addition, let Ξ be the union of $\operatorname{lin} F$ over all faces F of P with $\dim F \leqslant n-2$, and hence $|\Xi| = 0$. For any $z \in P \backslash \Xi$, the half-line $\operatorname{pos} z$ intersects ∂P in a point y, which is contained in an $(n-1)$-dimensional face of P, and hence $P = \bigcap_{i=1}^m H_i^+$. ∎

These two dual representations of a polytope (namely, bounded intersection of finitely many half-spaces or convex hull of finitely many points) are used in different setups. For example, given two polytopes $P, Q \subset \mathbb{R}^n$,

$$P \cap Q \text{ is a polytope provided } P \cap Q \neq \varnothing \tag{1.6}$$

because of the representation as a bounded intersection of finitely many half-spaces, and the Minkowski sum

$$P + Q \text{ is a polytope} \tag{1.7}$$

because of the representation as a convex hull of finitely many points.

Let us consider some basic examples of polytopes that will be used in this book. For the definition and main properties of the Hausdorff measure $\mathcal{H}^d$, see Section 1.D.

Example 1.4.6 (Polytopes). Let $e_1, \dots, e_n$ form an orthonormal basis of $\mathbb{R}^n$.

Cube, Parallelepiped: A cube is congruent to $[-a,a]^n$ for $a>0$, with $2n$ facets (each is an $(n-1)$-cube) and 2^n vertices. In particular, $[-1,1]^n = B_\infty^n$ is the unit ball of the l_∞-norm.
A parallelepiped is of the form $x+\Phi[-1,1]^n$ for $\Phi \in \mathrm{GL}(n)$ and $x \in \mathbb{R}^n$.

Cross-polytope: $\mathrm{conv}\{\pm e_1,\dots,\pm e_n\}$ is the unit ball of the l_1-norm.

Simplex: Convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$.

Regular simplex: Simplex whose edges have the same length, for example, $\mathrm{conv}\{e_1,\dots,e_n\}$ is the regular $(n-1)$-simplex of edge length $\sqrt{2}$.
If $v_1,\dots,v_{n+1} \in S^{n-1}$ satisfy $\langle v_i, v_j\rangle = \frac{-1}{n}$ for $i \neq j$, then $\mathrm{conv}\{v_1,\dots,v_{n+1}\}$ is the regular simplex inscribed into B^n, and $\{x \in \mathbb{R}^n : \langle x, v_i\rangle \leqslant 1,\ i=1,\dots,n+1\}$ is the regular simplex circumscribed around B^n

Pyramid or "bounded Cone": Congruent to $C = \mathrm{conv}\{o,F\}$ where F is an $(n-1)$-dimensional polytope with $o \notin \mathrm{aff}\, F$ with exterior unit normal u, and hence $|C| = \frac{1}{n}\, h \cdot \mathcal{H}^{n-1}(F)$ where h is the distance of $\mathrm{aff}\, F$ from o.

Definition 1.4.7 (Polyhedral Cone). If $C=\mathbb{R}^n$ or $C = \bigcap_{i=1}^m H_i^+$ where each H_i^+ is a half-space of $\mathbb{R}^n$ with $o \in \partial H_i^+$, then C is called a polyhedral cone. Writing $d = \dim C$, the "normalized angle" of C is

$$\beta(C) = \int_C e^{-\pi\|x\|^2}\, d\mathcal{H}^d(x) = \frac{\mathcal{H}^d(C\cap B^n)}{\omega_d} = \frac{\mathcal{H}^{d-1}(C\cap S^{n-1})}{d\omega_d}, \qquad (1.8)$$

and hence $\beta(\mathbb{R}^n)=1$.

The dual representation of a polyhedral cone is the positive hull of finitely many vectors. We observe that if $L \subset \mathbb{R}^n$ is a linear subspace of dimension $d \geqslant 1$, then $L = \mathrm{pos}\{\pm x_1,\dots,\pm x_d\}$ for any basis $x_1,\dots,x_d$ of L.

Proposition 1.4.8. *A $C \subset \mathbb{R}^n$ is a polyhedral cone if and only if $C = \mathrm{pos}\{x_1,\dots,x_k\}$ for $x_1,\dots,x_k \in \mathbb{R}^n$.*

Proof. Both in the cases if C is a polyhedral cone or if it is the positive hull of a finite set, C is a closed convex cone (cf. Lemma 1.3.11), thus Lemma 1.3.4 yields that $C = L + C'$ where L is a linear subspace and $C' \subset L^\perp$ is a pointed convex cone. We may assume that $d = \dim C' \geqslant 1$, and let $A \subset L^\perp$ be a $(d-1)$-dimensional linear subspace such that $A \cap C = \{o\}$ (cf. Lemma 1.3.5). It follows that if $z \in \mathrm{relint}\, C'$, then $P' = (A+z)\cap C'$ is bounded, thus it is a compact convex set satisfying that $\mathrm{pos}\, P' = C'$.

Now, if C is a polyhedral cone, then P is a polyhedron, and hence a polytope; that is, the convex hull of its vertices $v_1,\dots,v_m$ by Proposition 1.4.5. Therefore, C is the positive hull of $v_1,\dots,v_m$ and a finite set of vectors whose positive hull is L.

Next, we assume that $C = \mathrm{pos}\{x_1,\dots,x_k\}$. We may assume that for some m with $1 \leqslant m \leqslant k$, $x_i \in L$ if and only if $i > m$. It follows that $C' = \mathrm{pos}\{x_1',\dots,x_k'\}$ for $x_i' =$

$\Pi_{L^\perp} x_i$. Let $y_i = t_i x_i' \in A + z$ for suitable $t_i > 0$. It follows that $P = \operatorname{conv}\{y_1, \dots, y_m\}$, and hence P is a polyhedron by Proposition 1.4.5, which in turn yields that $C' = \bigcap_{i=1}^p H_i'^+$ where each $H_i'^+$ is a closed half-space of $L^\perp$ and o lies on the relative boundary of $H_i'^+$. Therefore $C = \bigcap_{i=1}^p H_i^+$ where $H_i^+ = H_i'^+ + L$. ∎

Next we show that the dual of a polyhedral cone is also a polyhedral cone.

Lemma 1.4.9. *For $u_1, \dots, u_k \in \mathbb{R}^n \backslash \{o\}$, $C = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant 0,\ i = 1, \dots, k\}$ satisfies $C^* = \operatorname{pos}\{u_1, \dots, u_k\}$.*

Proof. We have $\operatorname{pos}\{u_1, \dots, u_k\} \subset C^*$ by the definition of a dual cone. If z does not belong to $\operatorname{pos}\{u_1, \dots, u_k\}$, then Lemma 1.2.6 yields the existence of a $v \in \mathbb{R}^n$ such that $\langle v, z \rangle > 0$ and $\langle v, u_i \rangle \leqslant 0$ for $i = 1, \dots, k$. It follows that $v \in C$, and hence $z \notin C^*$. ∎

It follows from the definition of the normal cone (cf. (1.3)) and the dual cone that for any compact convex set $K \subset \mathbb{R}^n$ and $z \in \partial K$,

$$N_K(z) = C^* \text{ for the closed convex cone } C = \operatorname{cl}\operatorname{pos}(K - z). \tag{1.9}$$

For a polytope $P \subset \mathbb{R}^n$, we deduce (iii) of Lemma 1.4.10 below from (1.9), Proposition 1.4.3 and Lemma 1.4.9, next (iii) and (1.9) yield (i) (cf. also (1.8) for (1.10)), and in turn (i) implies (1.11). If $\dim P = n$, then we set $N_P(P) = \{o\}$.

Lemma 1.4.10 (Normal cone at a face). *Let $P \subset \mathbb{R}^n$ be a polytope, $P \neq \mathbb{R}^n$.*

(i) If F is a face of P, or $\dim P < n$ and $F = P$, then we can set $N_P(F) = N_P(z)$ for any $z \in \operatorname{relint} F$, and $y \in (\operatorname{relint} F) + N_P(F)$ if and only if $\Pi_P(y) \in \operatorname{relint} F$. In addition, if $\dim F = d$, then

$$\left|(N_P(F) \cap \varrho B^n) + \operatorname{relint} F\right| = \varrho^{n-d} \omega_{n-d} \cdot \beta\left(N_P(F)\right) \mathcal{H}^d(F). \tag{1.10}$$

(ii) If $\varrho > 0$, then using $\sqcup$ for disjoint union, we have

$$P + \varrho B^n = \left((N_P(P) \cap \varrho B^n) + \operatorname{relint} P\right) \sqcup \bigsqcup_{F \text{ face}} \left((N_P(F) \cap \varrho B^n) + \operatorname{relint} F\right). \tag{1.11}$$

(iii) If $\dim P = n$, $F_1, \dots, F_k$ are the facets of P and u_i is the exterior unit normal to F_i, then for any face F of P and $z \in \operatorname{relint} F$, we have

$$N_P(z) = \operatorname{pos}\{u_i : F \subset F_i\} = N_P(F).$$

1.5 Regular boundary points of closed convex sets and convex functions

At a boundary point of a closed convex set $X \subset \mathbb{R}^n$, there might be various supporting hyperplanes; or in other words, various exterior unit normals (think about the vertices

of a polytope). However, as we will soon see, there is a unique supporting hyperplane at a typical boundary point provided $\operatorname{int} X \neq \varnothing$ (like at the points in the relative interior of a facet of a polytope).

Definition 1.5.1 (Regular boundary points). For a closed convex set $X \subset \mathbb{R}^n$, $X \neq \mathbb{R}^n$, with $\operatorname{int} X \neq \varnothing$, we say that $x \in \partial X$ is a regular boundary point if there exists a unique supporting hyperplane at x; or equivalently, there exists a unique exterior unit normal $\nu_X(x) \in N_X(x) \cap S^{n-1}$. Let $\partial' X \subset \partial X$ be the set of regular boundary points.

Actually $\mathcal{H}^{n-1}$ a.e. boundary points are regular, as the next result shows.

Theorem 1.5.2. *For a closed convex set $X \subset \mathbb{R}^n$ with $\operatorname{int} X \neq \varnothing$, $\partial X \backslash \partial' X$ is σ-compact with σ-finite $\mathcal{H}^{n-2}$-measure, thus $\partial' X$ is Borel and $\mathcal{H}^{n-1}(\partial X \backslash \partial' X) = 0$.*

Proof. We may assume that X is bounded (compact) and $o \in X$.

Let $u_1, \dots, u_n$ be an orthonormal basis of $\mathbb{R}^n$, and let $u_{i+n} = -u_i$ for $i = 1, \dots, n$. For an integer $a > \operatorname{diam} X + 2$, let $W = [-a, a]^n$, and let F_i be the facet of W with exterior unit normal u_i, $i = 1, \dots, 2n$. For $k \geqslant 2$, let $\Gamma_k \subset \partial W$ be the subset such that

$$\Gamma_k \cap F_i = \left\{ x \in F_i : \exists\, u_j \in u_i^{\perp} \text{ such that } \langle x, u_j \rangle \in \tfrac{1}{kn}\mathbb{Z} \right\} \quad \text{for } i = 1, \dots, 2n.$$

It follows that $\mathcal{H}^{n-2}(\Gamma_k) < \infty$, $\operatorname{relbd} F_i \subset \Gamma_k$ for $i = 1, \dots, 2n$, and if $s \subset F_i$ for a segment of length at least $\frac{1}{k}$, then $s \cap \Gamma_k \neq \varnothing$.

Let $X_k \subset \partial X$ be the set of points of $x \in \partial X$ such that there exist $u, v \in S^{n-1} \cap N_X(x)$ with $\angle(u, v) \geqslant \frac{1}{k}$. Then X_k is compact (cf. Lemma 1.2.10), and as any point of ∂W is of distance at least 2 from X, the properties of Γ_k yield that $N_X(x) \cap \Gamma_k \neq \varnothing$ for $x \in X_k$. In particular, $X_k \subset \Pi_X(\Gamma_k)$, and as Π_X is a contraction by Lemma 1.2.13, it follows that $\mathcal{H}^{n-2}(X_k) < \infty$ (cf. Corollary 1.D.10). We conclude Theorem 1.5.2 from $\partial X \backslash \partial' X = \bigcup_k X_k$. ∎

Lemma 1.2.10 yields that ν_X is continuous on regular boundary points of X.

Lemma 1.5.3. *If $X \subset \mathbb{R}^n$ is a closed convex set with $\operatorname{int} X \neq \varnothing$, then $\nu_X : \partial' X \to S^{n-1}$ is continuous.*

Next, we turn to convex functions. In particular, for a convex and open $\Omega \subset \mathbb{R}^d$, a function $\varphi : \Omega \to \mathbb{R}$ is convex if

$$\varphi((1-\lambda)x + \lambda y) \leqslant (1-\lambda)\varphi(x) + \lambda\, \varphi(y)$$

for $x, y \in \Omega$ and $\lambda \in [0, 1]$. It is known that a convex function φ on a convex open set is continuous. For this and other basic properties of convex functions used below, see the references in Section 10.6.

Definition 1.5.4 (Subdifferential of a convex function). If $\varphi : \Omega \to \mathbb{R}$ convex for a convex and open $\Omega \subset \mathbb{R}^d$ and $x \in \Omega$, then

$$\partial\varphi(x) = \{u \in \mathbb{R}^d : \varphi(y) \geqslant \varphi(x) + \langle u, y - x\rangle \text{ for any } y \in \Omega\}.$$

Remarks.

- $\partial\varphi(x) \neq \varnothing$ and is compact for any $x \in \Omega$.
- $\partial\varphi(x)$ is one point if and only if $D\varphi(x)$ exists, and we have $\partial\varphi(x) = \{D\varphi(x)\}$ in this case.
- For the epigraph $X = \{(x,t) \in \mathbb{R}^{d+1} : x \in \Omega \text{ and } t \geqslant \varphi(x)\} \subset \mathbb{R}^{d+1}$ of φ, it is convex, its boundary contains the graph of φ, and if $x \in \Omega$ and $y = (x, \varphi(x)) \in \partial X$, then
$$u \in \partial\varphi(x) \text{ if and only if } (u, -1) \in N_X(y) \subset \mathbb{R}^{d+1}.$$
Here $X \subset \mathbb{R}^{d+1}$ may not be a closed subset if $\Omega \neq \mathbb{R}^d$, but if $x + rB^d \subset \Omega$ for $r > 0$, then the epigraph of φ above $x + rB^d$ is a closed convex set of $\mathbb{R}^{d+1}$ by the continuity of φ; therefore, $N_X(y)$ can be defined with respect to this closed convex set and is independent of the small $r > 0$.

The remark above shows the intimate relation between derivative (subdifferential) of a convex function and the outer normals to the epigraph. In particular, Theorem 1.5.2 yields a stronger form Rademacher's theorem about the a.e. differentiability of convex functions (cf. Lemma 10.4.7).

Corollary 1.5.5. *For a convex function $\varphi : \Omega \to \mathbb{R}$ on an open convex $\Omega \subset \mathbb{R}^d$, φ is differentiable at every $x \in \Omega$ but at a σ-compact subset of Ω with σ-finite $\mathcal{H}^{d-1}$-measure, and hence φ is $\mathcal{H}^d$ a.e. differentiable.*

Proof. Write Ω as the countable union of the sets of the form $z_j + r_j B^n \subset \Omega$, $j \in \mathbb{N}$, where $z_j \in \Omega$ and $r_j > 0$, and apply Theorem 1.5.2 to the epigraphs $X_j = \{(x,t) \in \mathbb{R}^{d+1} : x \in z_j + r_j B^n \text{ and } t \geqslant \varphi(x)\}$. ∎

According to Section 10.6, a convex function on an open convex set Ω is actually locally Lipschitz, namely, Lipschitz on any compact subset of Ω. Therefore, the Remarks after Definition 1.5.4 on the subdifferential yield the following correspondence between regular boundary points and differentiability of convex functions.

Lemma 1.5.6. *For closed convex $X \subset \mathbb{R}^n$ with $\operatorname{int} X \neq \varnothing$ and $X \neq \mathbb{R}^n$, ∂X is a Lipschitz manifold (cf. Remark-10.4.8), and if a relatively open $U \subset \partial X$ is the graph of a convex function φ on a convex relatively open $\Omega \subset v^\perp$ for $v \in S^{n-1}$, then $y = x + \varphi(x)v \in U$ is a regular point for $x \in \Omega$ if and only if $D\varphi(x)$ exists.*

Lemma 1.2.10 yields that an everywhere differentiability convex function is C^1.

Proposition 1.5.7 (Differentiable and convex is C^1).

(i) If $\varphi : \Omega \to \mathbb{R}$ is convex and differentiable for open convex $\Omega \subset \mathbb{R}^d$, then φ is C^1.

(ii) If every $x \in \partial X$ is a regular boundary point for a closed convex $X \subset \mathbb{R}^n$ with $\operatorname{int} X \neq \emptyset$*, then ∂X is a C^1 manifold.*

1.6 Support function of a compact convex set

This section shows how the support function encodes a compact convex set in a rather natural manner, and builds a fundamental bridge between Geometry and Analysis. We note that for possibly unbounded closed convex sets, their faces have been already defined in Definition 1.2.5, and their support function is discussed in Section 1.B.

Definition 1.6.1 (Support function, Faces). Let $K \subset \mathbb{R}^n$ be compact and convex.

- The function $h_K(u) = \max\{\langle x, u\rangle : x \in K\}$ of $u \in \mathbb{R}^n$ is the *support function* of K.
- For $u \in \mathbb{R}^n \backslash \{o\}$, $F_K(u) = \{x \in K : \langle x, u\rangle = h_K(u)\}$ is the *face* of K with exterior normal u.

Remark. The name "support function" reflects the fact that for $u \in \mathbb{R}^n \backslash \{o\}$ and a compact convex $K \subset \mathbb{R}^n$, the hyperplane $\{x \in \mathbb{R}^n : \langle x, u\rangle = h_K(u)\}$ is the supporting hyperplane of K with exterior normal u. In particular, if $o \in K$ and $u \in S^{n-1}$, then $h_K(u)$ is the distance of the origin o from this supporting hyperplane.

The following properties directly follow from the definition and the fact that any point not contained in a compact convex set $K \subset \mathbb{R}^n$ is cut off by a supporting hyperplane of K (cf. Lemma 1.2.2). We recall that for a proper linear subspace $L \subset \mathbb{R}^n$, the orthogonal projection of K into L is denoted by $K|L$.

Lemma 1.6.2. *Let $K, C \subset \mathbb{R}^n$ be compact convex and $u, v \in \mathbb{R}^n$.*

(i) $K = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_K(u)\ \forall u \in \mathbb{R}^n\} = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_K(u)\ \forall u \in S^{n-1}\}$.

(ii) h_K is convex and one-homogeneous; namely, $h_K(\gamma u) = \gamma h_K(u)$ for $\gamma \geqslant 0$, and for any $\alpha, \beta \geqslant 0$, we have

$$h_K(\alpha u + \beta v) \leqslant \alpha h_K(u) + \beta h_K(v).$$

(iii) For $x \in \partial K$, $u \in N_K(x)$ if and only if $h_K(u) = \langle u, x\rangle$.

(iv) $h_{\Phi K}(u) = h_K(\Phi^t u)$ for $\Phi \in \mathrm{GL}(n)$.

(v) $h_{\alpha K + \beta C} = \alpha\, h_K + \beta\, h_C$ and $F_{\alpha K + \beta C}(u) = \alpha\, F_K(u) + \beta\, F_C(u)$ for $\alpha, \beta \geqslant 0$; in particular, $h_{K-z}(u) = h_K(u) - \langle z, u\rangle$ for $z, u \in \mathbb{R}^n$.

(vi) $h_K(u) = h_{K|L}(u)$ for a proper linear subspace $L \subset \mathbb{R}^n$ and $u \in L$.

(vii) If $K \subset C$, then $h_K \leqslant h_C$.

A basic property of the faces of a *convex body* $K \subset \mathbb{R}^n$ (K is a compact convex set with $\operatorname{int} K \neq \varnothing$) is that as the unique exterior unit normal $\nu_K(x)$ exists at $\mathcal{H}^{n-1}$ a.e. $x \in \partial K$ (cf. Theorem 1.5.2), if $\omega \subset S^{n-1}$ is a Borel set, then $\nu_K^{-1}(\omega) = \{x \in \partial' K : \nu_K(x) \in \omega\}$ satisfies

$$\mathcal{H}^{n-1}\left(\nu_K^{-1}(\omega)\right) = \mathcal{H}^{n-1}\left(\bigcup_{u\in\omega} F(K,u)\right). \tag{1.12}$$

Let us consider some fundamental examples of the support function.

Example 1.6.3. In the following examples, we always have $u \in \mathbb{R}^n$.

- *Polytopes:* $h_P(u) = \max\{\langle u, v_i\rangle : i = 1,\dots,m\}$ for a polytope $P = \operatorname{conv}\{v_1,\dots,v_m\}$; for example, if $v_1,\dots,v_m$ are the vertices of P.
- *Euclidean ball:* $h_{rB^n}(u) = r \cdot \|u\|$ for $r > 0$.
- *Unit ball of l_p norm:* Let $1 \leqslant p \leqslant \infty$.
  $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leqslant 1\}$ if $p \in [1,\infty)$ where $\|(x_1,\dots,x_n)\|_p = \sqrt[p]{|x_1|^p + \dots + |x_n|^p}$ if $p \geqslant 1$ and $\|(x_1,\dots,x_n)\|_\infty = \max\{|x_1|,\dots,|x_n|\}$ (cf. Definition 10.2.6), and hence $B_2^n = B^n$, $B_\infty^n = [-1,1]^n$ and $B_1^n = \operatorname{conv}\{\pm e_1,\dots,\pm e_n\}$ is a regular cross-polytope for the orthonormal basis $e_1,\dots,e_n$ of $\mathbb{R}^n$.
  For $q \geqslant 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ (where $q = \infty$ if $p = 1$), we claim that

$$h_{B_p^n}(u) = \|u\|_q. \tag{1.13}$$

  On the one hand, Hölder inequality (10.4) yields that $\langle x, u\rangle \leqslant \|x\|_p \|u\|_q \leqslant \|u\|_q$ for $x \in B_p^n$, and hence $h_{B_p^n}(u) \leqslant \|u\|_q$. On the other hand, there exists $x \in \partial B_p^n$ with $\langle x, u\rangle = \|x\|_p \|u\|_q = \|u\|_q$, which in turn yields (1.13).
- *Zonotopes, Zonoids:* Zonotopes are Minkowski sums of finitely many segments, and zonoids are limits of zonotopes (limits are discussed in the next Section 1.7). Zonoids are always centrally symmetric, and an o-symmetric convex body $K \subset \mathbb{R}^n$ is a zonoid if and only if there exists a finite even Borel measure μ on S^{n-1} not concentrated on any great subsphere such that

$$h_K(x) = \int_{S^{n-1}} |\langle x, v\rangle| \, d\mu(v).$$

  Here K is a zonotope if and only if μ is discrete.
  Any planar centrally symmetric convex body is a zonoid, but this property fails in higher dimensions. A typical example for zonoids is the projection body, see Section 2.B. For "classical" properties of zonoids, see Schneider, Weil [668].

Extreme points form the smallest subset a convex compact set $K \subset \mathbb{R}^n$ such that for any $u \in \mathbb{R}^n$, the maximum of the linear function $x \mapsto \langle u, x\rangle$ over $x \in K$; namely, the value of the support function h_K at u, is realized at an extreme point.

Definition 1.6.4 (Extreme point). If $K \subset \mathbb{R}^n$ is convex, compact, then an $x \in K$ is called an extreme point if $x = (1-\lambda)y + \lambda z$ for $y, z \in K$ and $\lambda \in (0,1)$ implies $y = z = x$ (and hence $x \in \partial K$). The set of extreme points is $\operatorname{ext} K \subset \operatorname{relbd} K$.

Lemma 1.6.5. *Let $K \subset \mathbb{R}^n$ be convex, compact.*

(i) If $F \subset K$ is a face, then $\operatorname{ext} F \subset \operatorname{ext} K$.

(ii) The set $\operatorname{ext} K$ *is non-empty, and is the smallest set such that* $K = \operatorname{conv} \operatorname{ext} K$.

Proof. For (i), there exists a half-space $H^+ \subset \mathbb{R}^n$ such that $K \subset H^+$ and $F = \partial H^+ \cap K$. If $x \in \operatorname{ext} F$ and $x = (1-\lambda)y + \lambda z$ for $y, z \in K$ and $\lambda \in (0,1)$, then $y, z \in \partial H^+$, and hence $y = z = x$.

For (ii), if $K = \operatorname{conv} X$, then $\operatorname{ext} K \subset X$, as $\operatorname{conv} X$ is the family of all convex linear combinations of the points of X.

Finally, we show that $K = \operatorname{conv} \operatorname{ext} K$ by induction on $\dim K$ where the statement readily holds if $\dim K = 0$. If $\dim K \geqslant 1$ and x is a point of the relative boundary of K, then $x \in F$ for a face $F \subset K$; therefore, $x \in \operatorname{conv} \operatorname{ext} F \subset \operatorname{conv} \operatorname{ext} K$ by induction and (i). ∎

Next, we show how the support function encodes the facial structure of a compact convex set as subdifferentials (cf. Definition 1.5.4 for the notion of subdifferential). A compact convex set $K \subset \mathbb{R}^n$ is called a *strictly convex body* if ∂K contains no segment.

Lemma 1.6.6. *If $K \subset \mathbb{R}^n$ is compact convex and $u \in \mathbb{R}^n \backslash \{o\}$, then $\partial h_K(u) = F_K(u)$. In particular, if $K \subset \mathbb{R}^n$ is a strictly convex body, and $u \in \mathbb{R}^n \backslash \{o\}$ is an exterior normal at an $x \in \partial K$, then $Dh_K(u) = x$.*

Proof. If $z \in F_K(u)$, then $\langle z, u\rangle = h_K(u)$, and hence $h_K(v) - h_K(u) \geqslant \langle z, v\rangle - \langle z, u\rangle = \langle z, v-u\rangle$ for any $v \in \mathbb{R}^n$. Therefore, $z \in \partial h_K(u)$.

On the other hand, if $z \in \partial h_K(u)$, then for any $t \in (0,1)$, we have

- $t \cdot h_K(u) = h_K((1+t)u) - h_K(u) \geqslant \langle z, (1+t)u - u\rangle = t\langle z, u\rangle$ and
- $-t \cdot h_K(u) = h_K((1-t)u) - h_K(u) \geqslant \langle z, (1-t)u - u\rangle = -t\langle z, u\rangle$,

thus $\langle z, u\rangle = h_K(u)$. In addition, for any $v \in \mathbb{R}^n$, $z \in \partial h_K(u)$ yields that

$$h_K(v) = h_K(u) + (h_K(v) - h_K(u)) \geqslant \langle z, u\rangle + \langle z, v-u\rangle = \langle z, v\rangle,$$

and hence $z \in K$, which in turn yields $z \in F_K(u)$ by $\langle z, u\rangle = h_K(u)$. ∎

We note that whenever $o \in \operatorname{int} K$, then we can consider it as the unit ball of a norm type function, which function is in some sense "dual" to the notion of support function (cf. Section 1.9).

Definition 1.6.7. For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, if $x \in \mathbb{R}^n$, then its associated "norm" is

$$\|x\|_K = \min\{t \geqslant 0 : x \in t\,K\}.$$

Remarks.

- The function $x \mapsto \|x\|_K$ is convex and $\|\lambda x\|_K = \lambda \|x\|_K$ for $\lambda \geqslant 0$.
- $\|\cdot\|_K$ is an actual norm if and only if $K = -K$ (K is o-symmetric).

The support function h_K is typically not differentiable at o for a compact convex set K (unless K is a point), and so we consider the differentiability of h_K on $\mathbb{R}^n \backslash \{o\}$.

Lemma 1.6.8. *Let $K \subset \mathbb{R}^n$ be compact and convex. Then h_K is C^1 on $\mathbb{R}^n \backslash \{o\}$ if and only if K is a strictly convex body, that is in turn equivalent to saying that $\|u+v\|_{K-z} > \|u\|_{K-z} + \|v\|_{K-z}$ for any $z \in \operatorname{int} K$ and independent $u, v \in \mathbb{R}^n \backslash \{o\}$.*

Proof. As h_K is convex, h_K is C^1 on $\mathbb{R}^n \backslash \{o\}$ if and only if $\partial h_K(u) = F_K(u)$ is a unique point for any $u \in \mathbb{R}^n \backslash \{o\}$ (cf. Lemma 1.6.6), that is equivalent to saying that ∂K contains no segment. This is in turn equivalent to saying that $\|\cdot\|_{K-z}$ is as "strictly convex" as possible for any $z \in \operatorname{int} K$. ∎

Next, we prove that support functions are characterized by the properties of being convex and one-homogeneous. A more general version of Proposition 1.6.9 for any closed convex set is provided in Proposition 1.B.1 using elementary properties of the Legendre transform.

Proposition 1.6.9. *Let $h : \mathbb{R}^n \to \mathbb{R}$. There exists a compact convex set $K \subset \mathbb{R}^n$ satisfying $h = h_K$ if and only if h is convex and $h(\gamma u) = \gamma h(u)$ holds for $u \in \mathbb{R}^n$ and $\gamma \geqslant 0$. In addition, the compact convex set $K \subset \mathbb{R}^n$ satisfying $h = h_K$ is unique.*

Proof. If $h = h_K$ for a compact convex set $K \subset \mathbb{R}^n$, then h_K is convex and one-homogeneous by Lemma 1.6.2 (ii), and the uniqueness of K follows from Lemma 1.6.2 (i).

Therefore, let h be convex and one-homogeneous. If $z \in \partial h(u)$ for $u \in \mathbb{R}^n \backslash o$, then any $v \in \mathbb{R}^n$ satisfies

$$h(v) - h(u) \geqslant \langle z, v-u \rangle. \tag{1.14}$$

Applying this with $v = 2u$, we deduce that $h(u) = h(2u) - h(u) \geqslant \langle z, u \rangle$. On the other hand, using $v = \frac{1}{2}u$ in (1.14), it follows that $\frac{-1}{2} h(u) = h(\frac{1}{2}u) - h(u) \geqslant \langle z, \frac{-1}{2}u \rangle$, and hence $h(u) \leqslant \langle z, u \rangle$. In particular, if $z \in \partial h(u)$ for $u \in \mathbb{R}^n \backslash o$, then

$$\langle z, u \rangle = h(u) \text{ and } h(v) \geqslant \langle z, v \rangle \text{ for } v \in \mathbb{R}^n. \tag{1.15}$$

We deduce that $z \in K = \{x \in \mathbb{R}^n : \langle x, w \rangle \leqslant h(w)\ \forall w \in \mathbb{R}^n\}$, and hence $\partial h(u) \subset K$ for any $u \in \mathbb{R}^n$.

Therefore, K is non-empty, and its definition shows that $h_K(u) \leqslant h(u)$ for any $u \in \mathbb{R}^n$. In addition, if $u \in \mathbb{R}^n$ and $z \in \partial h(u) \subset K$, then $h_K(u) \geqslant \langle z, u \rangle = h(u)$ (cf. (1.14)), proving that $h_K(u) = h(u)$ for any $u \in \mathbb{R}^n$. ∎

Next we describe the normal cones of the Minkowski sums of polytopes (see Proposition 1.4.3 for the face structure of a polytope).

Lemma 1.6.10 (Normal cones at a Minkowski sum). *If $P, Q \subset \mathbb{R}^n$ are polytopes, then the family of normal cones of $P+Q$ is the family of all $N_P(F) \cap N_Q(G)$ with $N_P(F) \cap N_Q(G) \neq \{o\}$ where F (G) is a face of P (Q), or P is lower dimensional and $F = P$ (or Q is lower dimensional and $G = Q$).*

Proof. If F (G) is a face of P (Q), or P is lower dimensional and $F = P$ (or Q is lower dimensional and $G = Q$), and $N_P(F) \cap N_Q(G) \neq \{o\}$, then $F = F_P(u)$ and $G = F_Q(u)$ for some $u \in N_P(F) \cap N_Q(G) \cap S^{n-1}$, and hence $F + G = F_{P+Q}(u)$ is a face of $P+Q$ with $N_{P+Q}(F+G) = N_P(F) \cap N_Q(G)$.

On the other hand, for any face M of $P+Q$, we have $M = F_{P+Q}(u)$ for a $u \in S^{n-1}$; therefore, $M = F + G$ and $N_{P+Q}M = N_P(F) \cap N_Q(G)$ for $F = F_P(u)$ and $G = F_Q(u)$. ∎

1.7 Hausdorff distance of compact convex sets

A useful property of the space $\mathcal{K}^n$ of compact convex subsets of $\mathbb{R}^n$ is that it can be equipped with a natural topology. The easiest way to show this is to provide an explicit metric, which turns $\mathcal{K}^n$ into a metric space. Next, we introduce the most commonly used metric.

Definition 1.7.1 (Hausdorff distance). The Hausdorff distance between non-empty compact convex sets $K, C \subset \mathbb{R}^n$ is

$$\delta_H(K,C) = \min\{\varrho \geqslant 0 : K \subset C + \varrho B^n \text{ and } C \subset K + \varrho B^n\}.$$

The Hausdorff distance $\delta_H(K,C)$ is a metric on the space $\mathcal{K}^n$ of compact convex subsets of $\mathbb{R}^n$ (see Section 3.7 for the version on the space of compact subsets). We always consider the space of compact convex sets with the topology induced by δ_H (where we always assume that the compact convex sets are non-empty), and write $K_m \to K$ or $\lim_{m\to\infty} K_m = K$ to signal that $\lim_{m\to\infty} \delta_H(K_m, K) = 0$.

The Hausdorff distance of convex compact sets $K, C \subset \mathbb{R}^n$ can be expressed in terms of the restrictions of the support functions to S^{n-1}. Abusing the notation, we write $\|h_K - h_C\|_\infty$ to denote the L_∞ norm of the restriction of $h_K - h_C$ to S^{n-1}.

Lemma 1.7.2. *For convex compact sets $K, C \subset \mathbb{R}^n$, we have*

$$\|h_K - h_C\|_\infty = \max\left\{|h_K(u) - h_C(u)| : u \in S^{n-1}\right\} = \delta_H(K,C). \tag{1.16}$$

Proof. Let $\varrho = \delta_H(K,C)$. Using the linearity and monotonicity of the support function (cf. Lemma 1.6.2), $C \subset K + \varrho B^n$ yields that $h_K(u) - h_C(u) \geqslant -\varrho$. We deduce via exchanging the role of C and K that $\|h_K - h_C\|_\infty \leqslant \varrho$ for $u \in S^{n-1}$.

For the reverse inequality, possibly after exchanging the role of C and K, we may assume that there exists an $x_0 \in \partial C$ with $x_0 \in \partial(K + \varrho B^n)$ since $\delta_H(K, C)$ is defined as a minimum. For the exterior normal $u_0 \in S^{n-1}$ at x_0 to $K + \varrho B^n$ (and hence to $C \subset K + \varrho B^n$, as well), we have $h_{K+\varrho B^n}(u_0) = h_C(u_0)$, thus $|h_K(u_0) - h_C(u_0)| = \varrho$. This proves that $\|h_K - h_C\|_\infty \geqslant \varrho$, and in turn yields $\|h_K - h_C\|_\infty = \varrho$. ∎

Now we show that the support function has a Lipschitz constant that depends only on the radius of a centered ball containing the compact convex set.

Lemma 1.7.3. *If $R > 0$ and $K \subset R\,B^n$ is a compact convex set, then*

$$|h_K(u) - h_K(v)| \leqslant R\|u - v\| \quad \textit{for } u, v \in \mathbb{R}^n.$$

Proof. If $y, z \in K$ satisfy $\langle y, u\rangle = h_K(u)$ and $\langle z, v\rangle = h_K(v)$, then $\langle z, v\rangle \geqslant \langle y, v\rangle$, and

$$h_K(u) - h_K(v) = \langle y, u\rangle - \langle z, v\rangle = \langle y, u - v\rangle + \langle y - z, v\rangle \leqslant \langle y, u - v\rangle \leqslant R\|u - v\|.$$

A similar argument shows that $h_K(v) - h_K(u) \leqslant R\|v - u\|$. ∎

The following statement is actually due to Hausdorff in the more general context of compact subsets of $\mathbb{R}^n$ (cf. Section 3.7); however, we go by the name Blaschke Selection Theorem as most textbooks do.

Theorem 1.7.4 (Blaschke Selection Theorem). *Any bounded sequence $\{K_m\}$ of compact convex sets in $\mathbb{R}^n$ has a subsequence that tends to a compact convex set.*

Proof. Assume that each $K_m \subset RB^n$ for $R > 0$, set $\tilde{h}_{K_m} = h_{K_m}|_{S^{n-1}}$, and choose a countable dense set $X \subset S^{n-1}$. As $\tilde{h}_{K_m} \leqslant R$, there exists a subsequence $\{\tilde{h}_{K_{m'}}\}$ such that $\tilde{h}_{K_{m'}}(x)$ is convergent for any $x \in X$. It follows from Lemma 1.7.3 that $\{\tilde{h}_{K_{m'}}\}$ is a Cauchy sequence with respect to $\|\cdot\|_\infty$, and hence $\tilde{h}_{K_{m'}}$ tends to a continuous $\tilde{h}$ on S^{n-1} uniformly.

We define $h(tu) = t\tilde{h}(u)$ for $u \in S^{n-1}$ and $t \geqslant 0$. As $h_{K_{m'}}$ tends pointwise to h on $\mathbb{R}^n$, we deduce that h is convex and one-homogeneous; therefore, $h = h_C$ for a compact convex set C. In turn, $K_{m'}$ tends to C by (1.16). ∎

Lemma 1.7.5. *The volume (Lebesgue measure) is continuous as a function of compact convex subsets in $\mathbb{R}^n$.*

Remark. As we will see in Section 3.7, the Hausdorff distance can be naturally extended to compact subsets of $\mathbb{R}^n$; however, the Lebesgue measure is not continuous as a function of compact subsets.

Proof. Let $K_m \to K$ for compact convex sets $K, K_m \subset \mathbb{R}^n$. We may assume that $o \in \operatorname{relint} K$. If $\dim K = n$, then $rB^n \subset K$ for some $r > 0$, and hence $\varrho B^n \subset \frac{\varrho}{r} K$ for any $\varrho > 0$. If $\delta_H(K_m, K) < \varrho$ for $\varrho \in (0, r)$, then $(1 - \frac{\varrho}{r})\, K \subset K_m \subset (1 + \frac{\varrho}{r})\, K$; therefore, $|K_m| \to |K|$.

On the other hand, if $\dim K < n$, then $K \subset u^{\perp}$ for some $u \in S^{n-1}$. For an $R > 0$ such that $K \subset RB^n$, if $\delta_H(K_m, K) < \varrho$ for $\varrho \in (0,1)$, then $K_m \subset (R+1)(B^n \cap u^{\perp}) + [-\varrho, \varrho]u$, and hence $|K_m| \leqslant (R+1)^{n-1}\omega_{n-1} \cdot 2\varrho$. In particular, $|K_m| \to 0 = |K|$. ∎

Any compact convex set $K \subset \mathbb{R}^n$ can be arbitrarily well approximated by polytopes (see (1.17) and (1.18) below and Section 2.7 for more precise estimates), and by convex bodies of C^{∞}_{+} boundary (see Theorem 8.1.10) in terms of the Hausdorff distance. One simple tool to handle polytopal approximation is the notion of an ε-net. For $\varepsilon > 0$ and $X \subset \mathbb{R}^n$, a finite set $\Xi \subset X$ is called an ε-net if for any $x \in X$ there exists $z \in \Xi$ with $\|x - z\| < \varepsilon$. Now if $\varepsilon > 0$ and $K \subset \mathbb{R}^n$ is a compact convex set, then

$$\delta_H(K, P) \leqslant \varepsilon \text{ for any } \varepsilon\text{-net } \Xi \subset K \text{ and } P = \operatorname{conv} \Xi. \tag{1.17}$$

Sometimes we need approximations by full-dimensional polytopes containing the compact convex set $K \subset \mathbb{R}^n$. If $\varepsilon > 0$, P is a polytope with $\delta_H(K, P) \leqslant \frac{\varepsilon}{4n}$, then

$$\delta_H(K, Q) < \varepsilon \text{ and } K \subset \operatorname{int} Q \text{ for the polytope } Q = P + \left[-\tfrac{\varepsilon}{2n}, \tfrac{\varepsilon}{2n}\right]^n \tag{1.18}$$

where both the P in (1.17) and the Q in (1.18) can be assumed o-symmetric if K is o-symmetric.

1.8 Simple polytopes and Strongly Isomorphic polytopes

The main goal of this section is to introduce the families of strongly isomorphic simple polytopes. We will need the following special case of Lemma 1.6.10:

Lemma 1.8.1. *If $P, Q \subset \mathbb{R}^n$ are n-polytopes, then the family of normal cones of $P + Q$ is the family of all $N_P(F) \cap N_Q(G)$ where F and G are faces of P and Q, respectively, such that $N_P(F) \cap N_Q(G) \neq \{o\}$.*

According to Proposition 1.4.3, if $P \subset \mathbb{R}^n$ is an n-dimensional polytope, then any vertex v is the intersection the facets containing v; therefore, v is contained in at least n facets. It follows by induction on the dimension n that v is also contained in at least n edges. It also follows that exactly n facets meet at v if and only if exactly n edges meet at v, and in this case, any $(n-1)$ of these edges are edges of one of the facets containing v, as well.

Definition 1.8.2 (Simple polytopes). We say that a d-dimensional polytope $P \subset \mathbb{R}^n$, $1 \leqslant d \leqslant n$, is simple if exactly d edges meet at any vertex of P.

Remark. The considerations above and Lemma 1.4.10 show the following properties:

- Any at least two-dimensional face of a simple polytope is simple.

- For an n-dimensional polytope $P \subset \mathbb{R}^n$, $n \geqslant 2$, P is simple if and only if the normal cone at any vertex is simplicial; namely, it is the positive hull of n independent vectors.

We note that Lemma 1.4.10 yields the property that if $u_1, \dots, u_k$ are the facet exterior unit normas of an n-polytope Q with corresponding facets $F_1, \dots, F_k$, and F is a face of Q, then

$$N_Q(F) = \operatorname{pos}\{u_i : F \subset F_i\}. \tag{1.19}$$

Next, we show that any n-polytope P can be approximated by simple polytopes that have the same facet unit exterior normals as P.

Lemma 1.8.3. *If $P \subset \mathbb{R}^n$ is an n-polytope and $\varepsilon > 0$, then there exists a simple n-polytope P' such that $\delta_H(P, P') < \varepsilon$ for the Hausdorff distance, the families of unit exterior normals to the facets of P and P' coincide, and the normal cone at any vertex of P' is contained in the normal cone of some vertex of P.*

Proof. Let $u_1, \dots, u_k$ be the facet exterior unit normals of P, and for $\tau = (\tau_1, \dots, \tau_k) \in \mathbb{R}^k$, let $P_\tau = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant \tau_i \ i = 1, \dots, k\}$. Thus there exists a $\varepsilon_0 > 0$ such that if $\delta_H(P, P_\tau) < \varepsilon_0$, then P_τ is an n-polytope and $u_1, \dots, u_k$ are the exterior normals of the facets of P_τ.

If $\delta_H(P, P_\tau) < \varepsilon_0$ and P is not simple, then there exist a subset $I \subset \{1, \dots, k\}$ of cardinality $n+1$ and a vertex v of P_τ such that $\langle v, u_i \rangle = \tau_i$ for $i \in I$; therefore, the determinant of the $(n+1) \times (n+1)$ matrix made up from the vectors (u_i, τ_i), $i \in I$, is 0. This shows that except for the $\tau \in \mathbb{R}^k$ lying in the union of certain $\binom{k}{n+1}$ proper linear subspaces of $\mathbb{R}^k$, any $\tau \in \mathbb{R}^k$ such that $\delta_H(P, P_\tau) < \varepsilon_0$ gives us a simple polytope P_τ.

Finally, indirectly, we suppose that there exist $\tau^{(m)} = (\tau_1^{(m)}, \dots, \tau_k^{(m)}) \in \mathbb{R}^k$ such that $\lim_{m \to \infty} \tau_i^{(m)} = h_P(u_i)$ for $i = 1, \dots, k$, and each $P_{\tau^{(m)}}$ has a vertex $v^{(m)}$ where the normal cone is not contained in the in the normal cone of any vertex of P. As $P_{\tau^{(m)}}$ tends to P, we may assume that $v^{(m)}$ tends to an $x_0 \in \partial P$, and hence $x_0 \in \operatorname{relint} F$ for a face F of P. According to (1.19), $N_P(x_0) = N_P(F)$ is the positive hull of a subset of $\mathcal{U} \subset \{u_1, \dots, u_k\}$. It also follows from (1.19) (with $Q = P_{\tau^{(m)}}$) and $N_{P_{\tau^{(m)}}}(v^{(m)}) \not\subset N_P(F)$ that we may assume that there exists a $u_j \notin \mathcal{U}$ such that $u_j \notin N_P(F)$ but $u_j \in N_{P_{\tau^{(m)}}}(v^{(m)})$ for each m. However, the last property yields $u_j \in N_P(x_0) = N_P(F)$, which is a contradiction, completing the proof of Lemma 1.8.3. ∎

Strongly isomorphic have similar face structures, which is very useful in certain calculations (see for example the proof the Aleksandrov Fenchel inequality in Section 7.A).

Definition 1.8.4 (Strongly Isomorphic Polytopes). The polytopes $P_1, \dots, P_k \subset \mathbb{R}^n$ are strongly isomorphic if for any $1 \leqslant i < j \leqslant n$, P_i and P_j have exactly the same set of normal cones; or equivalently, $\dim F_{P_i}(u) = \dim F_{P_j}(u)$ for any $u \in S^{n-1}$.

It follows that if the polytopes $P_1, \dots, P_k \subset \mathbb{R}^n$ are strongly isomorphic, then they have the same dimension, and if the common dimension is less than n, then their affine hulls are parallel.

Proposition 1.8.5. *For compact convex sets $C_1, \dots, C_k \subset \mathbb{R}^n$ and $\varepsilon > 0$, there exist strongly isomorphic simple n-polytopes $P_1, \dots, P_k \subset \mathbb{R}^n$ such that $\delta_H(C_i, P_i) < \varepsilon$ for $i = 1, \dots, k$.*

Proof. Choose $R > 0$ such that $C_1, \dots, C_k \subset \operatorname{int} RB^n$. According to (1.18), there exist n polytopes $Q_1, \dots, Q_k \subset RB^n$ such that $\delta_H(C_i, Q_i) < \varepsilon/3$ for $i = 1, \dots, k$. It follows from Lemma 1.8.1 that $Q, \widetilde{Q}_1, \dots, \widetilde{Q}_k$ are strongly isomorphic for $Q = Q_1 + \ldots + Q_k \subset \operatorname{int}(kRB^n)$ and

$$\widetilde{Q}_i = Q_i + \frac{\varepsilon}{3kR} \sum_{j \neq i} Q_j,$$

and hence $\delta_H(Q_i, \widetilde{Q}_i) < \varepsilon/3$ and $\delta_H(C_i, \widetilde{Q}_i) < 2\varepsilon/3$ for $i = 1, \dots, k$.

Finally, Lemma 1.8.3 provides a simple n-polytope $Q' \subset kRB^n$ such that the families of unit exterior normals to the facets of Q and Q' coincide, and the normal cone at any vertex of Q' is contained in the normal cone of some vertex of Q. Readily,

$$P_i = \widetilde{Q}_i + \frac{\varepsilon}{3kR} \cdot Q'$$

satisfies that $\delta_H(C_i, P_i) < \varepsilon$ for $i = 1, \dots, k$. As Q is strongly isomorphic to $\widetilde{Q}_i$, $i = 1, \dots, k$, we deduce from that the normal cone at any vertex of Q' is contained in the normal cone of some vertex of $\widetilde{Q}_i$, and hence Lemma 1.6.10 for the Minkowski sum of polytopes yields that P_i is strongly isomorphic to the simple polytope Q'. ∎

Simple polytopes have the advantage that local deformations keeping the exterior unit normals of the facets do not change the face structure (see Lemma 1.8.6). For independent $u_1, \dots, u_n \in \mathbb{R}^n$, let $u_1^*, \dots, u_n^* \in \mathbb{R}^n$ be the dual basis; namely, $\langle u_i^*, u_j \rangle = 0$ if $i \neq j$, and $\langle u_i^*, u_i \rangle = 1$, and let $R = \max_{i=1}^n \|u_i^*\|$. It follows that if $p, q \in \mathbb{R}^n$ satisfy that $\langle p, u_i \rangle = t_i$ and $\langle q, u_i \rangle = s_i$ for $i = 1, \dots, n$ (and hence $p = \sum_{i=1}^n t_i u_i^*$ and $q = \sum_{i=1}^n s_i u_i^*$), then

$$\|p - q\| \leqslant nR \max_{i=,\dots,n} |t_i - s_i|. \tag{1.20}$$

Since any vertex of an n-dimensional simple polytope is contained in n facets, (1.20) yields the folowing:

Lemma 1.8.6. *For a simple n-polytope $P \subset \mathbb{R}^n$, there exists $\varepsilon_0 > 0$, such that if $u_1, \dots, u_n$ are the unit exterior normals of the facets of P, and $|t_i - h_P(u_i)| < \varepsilon_0$ for $i = 1, \dots, k$, then $P' = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant t_i\}$ is a simple n-polytope strongly isomorphic to P.*

1.9 The polar (dual) of a convex body

We recall (cf. Section 1.6) that if $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then the corresponding "norm" is

$$\|x\|_K = \min\{t \geqslant 0 : x \in t\,K\}.$$

In this section, we consider the "dual norm" via the polar K^* of K.

Definition 1.9.1 (Polar (dual) of a convex body). If $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then its polar is

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \ \forall\ y \in K\},$$

which is a convex body with $o \in \operatorname{int} K^*$, and is o-symmetric if K is o-symmetric.

In particular, if K is o-symmetric, then $\|\cdot\|_{K^*}$ is the norm of the space dual to the one with norm $\|\cdot\|_K$. In general, the following statement summarizes the fundamental properties of polarity of convex bodies.

Proposition 1.9.2. *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int} K$.*

(i) $z \in \partial K$ and $y \in \mathbb{R}^n$ is an exterior normal at z to K with $\langle y, z\rangle = 1$ if and only if $y \in \partial K^*$ and z is an exterior normal at y to K^* with $\langle y, z\rangle = 1$;

(ii) $(K^*)^* = K$;

(iii) $h_{K^*}(u) = \|u\|_K$ and $h_K(u) = \|u\|_{K^*}$ for $u \in \mathbb{R}^n$;

(iv) $(\Phi\,K)^* = \Phi^{-t}K^*$ for $\Phi \in \mathrm{GL}(n)$.

Remark. $h_K(y) = h_{K^*}(z) = 1$ in (i).

Proof. If $z \in \partial K$ and $y \in \mathbb{R}^n$ is an exterior normal at z to K with $\langle y, z\rangle = 1$ in (i), then $\langle y, x\rangle \leqslant \langle y, z\rangle = 1$ for $x \in K$, and hence $y \in K^*$, and $\langle w, z\rangle \leqslant 1 = \langle y, z\rangle$ for $w \in K^*$, which in turn yields that z is exterior normal at $y \in \partial K^*$, proving half of (i).

For (ii), readily $K \subset (K^*)^*$. If $p \notin K$, then $p = \lambda z$ for $\lambda > 1$ and $z \in \partial K$, and hence $\langle p, y\rangle = \lambda > 1$ where y is an exterior normal at z to K with $\langle y, z\rangle = 1$. Since $y \in K^*$, we have $p \notin (K^*)^*$. Having (ii), interchanging the role of K and K^* verifies the other implication in (i).

For (iii), $h_K(y) = 1$ is equivalent to $\|y\|_{K^*} = 1$ by (i).

For (iv), $\langle \Phi^{-t}x, \Phi y\rangle = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$. ∎

We deduce the following examples for polars from Example 1.6.3 for support functions and from Proposition 1.9.2.

Example 1.9.3.

l_p-balls: $(B_p^n)^* = B_q^n$ if $\frac{1}{p} + \frac{1}{q} = 1$ for $p, q \in [1, \infty]$.

Polytopes: If $P = \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leqslant 1,\ i = 1, \dots, k\}$ is bounded for given $v_1, \dots, v_k \in \mathbb{R}^n \backslash \{o\}$, then $o \in \operatorname{int} P$ and $P^* = \operatorname{conv}\{v_1, \dots, v_k\}$.

Duality for convex bodies and convex cones: Let $\mathbb{R}^n = e^\perp$ for an $e \in S^n \subset \mathbb{R}^{n+1}$. If $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then the closed pointed convex cone $C = \operatorname{pos}(K + e) \subset \mathbb{R}^{n+1}$ (cf. Section 1.3) satisfies that $C^* = \operatorname{pos}(K^* - e)$.

Remark 1.9.4. Example 1.9.3 yields that if $P \subset \mathbb{R}^n$ is a polytope with $o \in \operatorname{int} P$ and k facets, then P^* is a polytope with k vertices. Moreover, if the facets of P touch B^n ("P is circumscribed around B^n"), then the vertices of P^* lie on S^{n-1} ("P^* is inscribed into B^n").

Next, we consider the notion of radial function that is in some sense dual to the notion of support function (cf. (1.21)).

Definition 1.9.5 (Radial function). For a convex body $K \subset \mathbb{R}^n$ with $o \in K$, if $u \in S^{n-1}$, then $\varrho_K(u) = \max\{t \geqslant 0 : tu \in K\}$.

Remark. In particular, $\varrho_K(u) \cdot u \in \partial K$ for $u \in S^{n-1}$.

Proposition 1.9.2 yields that if $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ and $u \in S^{n-1}$, then

$$\varrho_K(u) = \frac{1}{\|u\|_K} = \frac{1}{h_{K^*}(u)}. \tag{1.21}$$

We observe that the function $u \mapsto \varrho_K(u) \cdot u = (\|u\|_K)^{-1} \cdot u$ parametrizes ∂K of $u \in S^{n-1}$ (see Section 2.2).

Polarity (cf. Proposition 1.9.2), the parametrization of the boundary by the radial function and the characterization of the differentiablity of the support function by Lemma 1.6.8 yields the following:

Lemma 1.9.6. *Let $K \subset \mathbb{R}^n$ be a convex body. Then ∂K is a C^1 manifold if and only if $\| \cdot \|_{K-z}$ is C^1 on S^{n-1} for $z \in \operatorname{int} K$, that is equivalent to saying that $h_K(u + v) < h_K(u) + h_K(v)$ for independent $u, v \in \mathbb{R}^n$.*

The next statement follows from Proposition 1.9.2 via Lemma 1.6.8 and Lemma 1.9.6.

Corollary 1.9.7. *Given convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, its boundary ∂K is a C^1 manifold if and only if K^* is strictly convex.*

We note that the polar body satisfies various fundamental inequalities. The probably most widely used one is the Blaschke-Santaló inequality and its reverse form (see Sections 6.5 and 6.6), which states, for example, in the case of an o-symmetric convex body $K \subset \mathbb{R}^n$ that

$$2^{-n}|B^n|^2 < |K| \cdot |K^*| \leqslant |B^n|^2$$

where equality holds in the upper bound (the Blaschke-Santaló inequality) if and only if K is an ellipsoid. For related inequalities, see Section 2.B about the so-called projection body, and Section 6.9 about the Reverse Brunn-Minkowski inequality. Polarity of convex bodies also plays a prominent role, for example, in the study of L_p affine surface areas (cf. Section 8.9), and of $\mathrm{SL}(n)$ intertwining valuations (cf. Theorem 8.D.7).

1.10 Steiner and Schwarz symmetrizations of convex bodies in $\mathbb{R}^n$, and the Isodiametric and Brunn-Minkowski inequalities

In this section, we discuss two classical symmetrization methods, the Steiner symmetrization (the "mother of all symmetrizations") and a variant of it, the Schwarz symmetrization. The main reason to use these tools is, in general, that when proving an inequality, the symmetrization reduces the problem to more symmetric objects. We demonstrate this method by proving the Isodiametric and the Brunn-Minkowski inequalities *via* Steiner symmetrization. We recall that $X|L$ is the orthogonal projection of an $X \subset \mathbb{R}^n$ into a proper linear subspace $L \subset \mathbb{R}^n$.

Definition 1.10.1 (Steiner symmetrization). Let $K \subset \mathbb{R}^n$ convex body, and let $u \in S^{n-1}$. The followings are equivalent definitions of the Steiner symmetrial $\Theta_{u^\perp} K$.

(a) $\Theta_{u^\perp} K = \bigcup \left\{ x + [-q,q]u : x \in K|u^\perp \text{ and } \mathcal{H}^1\Big(K \cap (x + \mathbb{R}u)\Big) = 2q \right\}$.

(b) $\Theta_{u^\perp} K = \left\{ x + \dfrac{t-s}{2}\, u : x \in K|u^\perp \text{ and } x + tu, x + su \in K \text{ for } s,t \in \mathbb{R} \right\}$.

(c) If $f, g : K|u^\perp \to \mathbb{R}$ are concave functions such that

$$K = \left\{ x + t\,u : x \in K|u^\perp \text{ and } -g(x) \leqslant t \leqslant f(x) \right\},$$

then

$$\Theta_{u^\perp} K = \left\{ x + t\,u : x \in K|u^\perp \text{ and } -\frac{f(x)+g(x)}{2} \leqslant t \leqslant \frac{f(x)+g(x)}{2} \right\}.$$

Definition 1.10.2 (Inradius and circumradius). For a convex body $K \subset \mathbb{R}^n$, the inradius $r(K)$ is the maximal radius of any ball contained in K, and the circumradius $R(K)$ is the minimal radius of any ball containing K.

Remark. There exists a unique circumscribed ball of radius $R(K)$ containing K as the intersection of two distinct balls of radius $R(K)$ is contained in a ball with smaller radius. On the other hand, there might exist several balls of radius $r(K)$ contained in K, like in the case of a rectangle. If K is o-symmetric, then $r(K)B^n \subset K \subset R(K)B^n$.

Proposition 1.10.3. *Let $K, C \subset \mathbb{R}^n$ be convex bodies, and let $u \in S^{n-1}$.*

(i) $\Theta_{u^\perp} K$ is a convex body symmetric through $u^\perp$;

(ii) $\Theta_{u^\perp}(\lambda K) = \lambda \cdot \Theta_{u^\perp} K$ *if* $\lambda \in \mathbb{R}\backslash\{0\}$;

(iii) $r(\Theta_{u^\perp} K) \geqslant r(K)$ *and* $R(\Theta_{u^\perp} K) \leqslant R(K)$;

(iv) $|\Theta_{u^\perp} K| = |K|$;

(v) $\operatorname{diam} \Theta_{u^\perp} K \leqslant \operatorname{diam} K$;

(vi) $|\alpha \Theta_{u^\perp} K + \beta \Theta_{u^\perp} C| \leqslant |\alpha K + \beta C|$ *for* $\alpha, \beta > 0$;

(vii) $\Theta_{u^\perp} K = (w|u^\perp) + rB^n$ *assuming that* $K = w + rB^n$ *for* $r > 0$ *and* $w \in \mathbb{R}^n$.

Proof. The statements (i), (ii) and (vii) follow by definition (c), while (iii) follows by (i), (vii) and containment, and (iv) follows from Fubini's theorem and definition (a).

For (v), we use definition (b). Let $\operatorname{diam} \Theta_{u^\perp} K = \|z_1 - z_2\|$ where $z_i = x_i + \frac{1}{2}(t_i - s_i), x_i \in K|u^\perp, y_i = x_i + t_i u \in K$ and $w_i = x_i + s_i u \in K$ for $t_i, s_i \in \mathbb{R}$ and $i = 1, 2$. Since $z_1 - z_2 = \frac{1}{2}(y_1 - y_2 + w_2 - w_1)$, the triangle inequality yields that either $\|z_1 - z_2\| \leqslant \|y_1 - y_2\|$ or $\|z_1 - z_2\| \leqslant \|w_2 - w_1\|$ where $\|y_1 - y_2\|, \|w_2 - w_1\| \leqslant \operatorname{diam} K$.

For (vi), it is sufficient to prove that $\alpha \Theta_{u^\perp} K + \beta \Theta_{u^\perp} C \subset \Theta_{u^\perp}(\alpha K + \beta C)$ according to (iv), which in turn follows from definition (b) of the Steiner symmetrization. ∎

We say that a convex body $K \subset \mathbb{R}^n$ is unconditional with respect to an orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$ if it is symmetric through $u_1^\perp, \ldots, u_n^\perp$; or in other words, if $\sum_{i=1}^n t_i u_i \in K$ for $t_1, \ldots, t_n \in \mathbb{R}$ implies that $\sum_{i=1}^n \varepsilon_i t_i u_i \in K$ where $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$.

Lemma 1.10.4. *If* $u_1, \ldots, u_n$ *form an orthonormal basis of* $\mathbb{R}^n$ *and* $K \subset \mathbb{R}^n$ *is a convex body, then* $\Theta_{u_n^\perp} \circ \ldots \circ \Theta_{u_1^\perp} K$ *is unconditional, and hence it is o-symmetric.*

Proof. We observe that if $C \subset \mathbb{R}^n$ convex body and $j = 2, \ldots, n$, and C is symmetric through $u_i^\perp$ for $i = 1, \ldots, j-1$, then $\Theta_{u_j^\perp} C$ is also symmetric through $u_i^\perp$ for $i = 1, \ldots, j-1$. ∎

We now demonstrate how Lemma 1.10.4 leads to the Isodiametric Inequality.

Theorem 1.10.5 (Isodiametric Inequality for convex bodies). *If* $K \subset \mathbb{R}^n$ *is a convex body with* $|K| = |rB^n|$ *for* $r > 0$, *then* $\operatorname{diam} K \geqslant 2r$; *or in other works,* $\operatorname{diam} K \geqslant 2\omega_n^{-1/n} |K|^{1/n}$.

Proof. Let $u_1, \ldots, u_n$ form an orthonormal basis of $\mathbb{R}^n$, and hence $K' = \Theta_{u_n^\perp} \circ \ldots \circ \Theta_{u_1^\perp} K$ is an o-symmetric convex body by Lemma 1.10.4 with $\operatorname{diam} K' \leqslant \operatorname{diam} K$ by Proposition 1.10.3 (v). Since $|K'| = |rB^n|$ by Proposition 1.10.3 (iv), it follows that $K' \backslash \operatorname{int}(rB^n) \neq \varnothing$, which in turn yields that $\operatorname{diam} K' \geqslant 2r$. ∎

For a bounded measurable set $X \subset \mathbb{R}^n$, its diameter is $\operatorname{diam} X = \sup\{\|x - y\| : x, y \in X\}$. Since $\operatorname{diam} X = \operatorname{diam} \operatorname{conv} \operatorname{cl} X$ by Proposition 1.1.7, Theorem 1.10.5 directly yields its extension to measurable sets.

Theorem 1.10.6 (Isodiametric Inequality). *If $X \subset \mathbb{R}^n$ is bounded and measurable, then* $\operatorname{diam} X \geqslant 2\omega_n^{-1/n}|X|^{1/n}$.

The Isodiametric Inequality indicates what the right normalization of the Hausdorff measure is in order to conincide with Lebesgue measure (see Section 1.D).

Typically, Steiner symmetrization is useful to prove inequalities where balls are extremizers because iterated Steiner symmetrizations can lead to a ball.

Theorem 1.10.7 (Iterated Steiner symmetrizations). *If $K \subset \mathbb{R}^n$ is a convex body with $|K| = |rB^n|$ for $r > 0$, and $\varepsilon \in (0, \frac{1}{2})$, then there exist finitely many Steiner symmetrizations starting with K producing a convex body K' with*

$$(1-\varepsilon)rB^n \subset K' \subset (1+\varepsilon)rB^n. \tag{1.22}$$

Or equivalently, there exists a sequence $\{K_m\}$ of convex bodies tending to rB^n where $K_0 = K$ and $K_{m+1} = \Theta_{u_m^\perp} K_m$ for some $u_m \in S^{n-1}$.

Remark. Klartag [464] proves that $cn^4|\log\varepsilon|^2$ Steiner symmetrizations are enough to produce a K' as in (1.22) where $c > 1$ is an absolute constant.

Proof. According to Lemma 1.10.4, we may assume that K is o-symmetric and $K \neq rB^n$. Writing $\mathcal{F}_K$ to denote the family of convex bodies resulting from finitely many iterated Steiner symmetrizations starting from K, the Blaschke Selection Theorem 1.7.4, the continuity of the volume (cf. Lemma 1.7.5) and Proposition 1.10.3 (iv) yield that Theorem 1.10.7 is equivalent proving that

$$\Xi = \sup\{|C \cap rB^n| : C \in \mathcal{F}_K\} = |rB^n|. \tag{1.23}$$

The argument is indirect, so we suppose that $\Xi < |rB^n|$, and seek a contradiction. For $k \geqslant 2$, let $C_k \in \mathcal{F}_K$ such that $|C_k \cap rB^n| > \Xi - \frac{1}{k}$. As $C_k \subset R(K)B^n$, we may assume that $C_k \to C_0$ for an origin symmetric convex body C_0 by the Blaschke Selection Theorem 1.7.4 where $|C_0| = |rB^n|$ by the continuity of volume (cf. Lemma 1.7.5). As $|C_0 \cap rB^n| = \Xi < |rB^n|$, there exist balls $z_1 + \varrho B^n \subset (\operatorname{int} rB^n)\backslash C_0$ and $z_2 + \varrho B^n \subset (\operatorname{int} C_0)\backslash(rB^n)$ for some $\varrho > 0$. Now, there exists a k large enough such that $\frac{1}{k} < |\varrho B^n|$, $z_1 + \varrho B^n \subset (rB^n)\backslash C_k$ and $z_2 + \varrho B^n \subset C_k\backslash(rB^n)$; therefore, if $u = \frac{z_1 - z_2}{\|z_1 - z_2\|}$ and $\|x - x_0\| < \varrho$ for $x \in u^\perp$ and $x_0 = z_i|u^\perp$, then the line $\ell_x = x + \mathbb{R}u$ satisfies

$$\begin{aligned}\mathcal{H}^1\left(\ell_x \cap rB^n \cap \Theta_{u^\perp} C_k\right) &= \min\left\{\mathcal{H}^1\left(\ell_x \cap rB^n\right), \mathcal{H}^1\left(\ell_x \cap C_k\right)\right\} \\ &\geqslant \mathcal{H}^1\left(\ell_x \cap rB^n \cap C_k\right) + \mathcal{H}^1\left(\ell_x \cap \left(z_2 + \varrho B^n\right)\right),\end{aligned}$$

and hence

$$|rB^n \cap \Theta_{u^\perp} C_k| \geqslant |C_k \cap rB^n| + |\varrho B^n| > \Xi,$$

which is a contradiction verifying (1.23). ∎

Remark 1.10.8. According to Theorem 1.C.3, we can choose a sequence of Steiner symmetrizations whose results tend to a ball in a way such that the hyperplanes are chosen independently of the convex body. Let $v_1, \dots, v_n \in S^{n-1}$ be independent such that $\angle(v_i, v_j)/\pi$ is irrational for $i \neq j$, and let $u_{kn+i} = v_i$ for $k \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. Now if $K \subset \mathbb{R}^n$ is an o-symmetric convex body with $|K| = |rB^n|$ for $r > 0$, then K_m tends to rB^n where $K_0 = K$ and $K_{m+1} = \Theta_{u_m^\perp} K_m$.

As an example for how to use Steiner symmetrizations to prove geometric inequalities, we provide the proof of the Brunn-Minkowski inequality.

Theorem 1.10.9 (Brunn-Minkowski inequality). *If $K, C \subset \mathbb{R}^n$ are compact convex, and $\alpha, \beta \geqslant 0$, then*

$$|\alpha K + \beta C|^{\frac{1}{n}} \geqslant \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}}. \tag{1.24}$$

Proof. We may assume that $\alpha = \beta = 1$, and $|K| = |rB^n|$ and $|C| = |RB^n|$ for $r, R > 0$. We apply simultaneous iterated Steiner symmetrization to K and C, which does not increase the volume of the Minkowski sum by Proposition 1.10.3 (vi). We may assume that K and C are unconditional, and hence o-symmetric according to Lemma 1.10.4.

According to Proposition 1.10.3 (iii), Steiner symmetrization does not increase the Hausdorff distance from rB^n or RB^n. Therefore, we apply iterated Steiner symmetrizations simultaneously to K and C in stages, and at each stage, we apply finitely many Steiner symmetrizations based on Theorem 1.10.7 to make sure that either the image of K, or the image of C (always just one of them), gets closer to the respective ball. Eventually, we construct a sequence $\{\Theta_{u_m^\perp}\}$ of Steiner symmetrizations such that writing $K_0 = K$, $C_0 = C$, $K_{m+1} = \Theta_{u_m^\perp} K_m$ and $C_{m+1} = \Theta_{u_m^\perp} C_m$, the sequences $\{K_m\}$ and $\{C_m\}$ tend to rB^n and RB^n, respectively. It follows from Proposition 1.10.3 (vi) and the continuity of volume Lemma 1.7.5 that

$$|K + C|^{\frac{1}{n}} \geqslant \lim_{m \to \infty} |K_m + C_m|^{\frac{1}{n}} = |rB^n + RB^n|^{\frac{1}{n}} = |rB^n|^{\frac{1}{n}} + |RB^n|^{\frac{1}{n}} = |K|^{\frac{1}{n}} + |C|^{\frac{1}{n}}.$$

■

Using the AM-GM inequality, and the fact that $\frac{1}{2}(\alpha K + \beta C) + \frac{1}{2}(\gamma K + \delta C) = \frac{\alpha+\gamma}{2} K + \frac{\beta+\delta}{2} C$ holds for convex $K, C \subset \mathbb{R}^n$ and $\alpha, \beta, \gamma, \delta \geqslant 0$, the Brunn-Minkowski inequality (1.24) has the following frequently used consequences (cf. Lemma 1.12.2).

Corollary 1.10.10. *Let $K, C \subset \mathbb{R}^n$ be compact convex.*

(i) $|(1-\lambda) K + \lambda C| \geqslant |K|^{1-\lambda} |C|^{\lambda}$ *for* $\lambda \in (0, 1)$;

(ii) $t \mapsto |C + tK|^{\frac{1}{n}}$ *and* $t \mapsto |(1-t)C + tK|^{\frac{1}{n}}$ *are concave functions of* $t \in [0, 1]$.

It is not hard to see (using triangulations similarly as in Remark 1.10.11 below) that the process of Steiner symmetrization produces a new polytope from a polytope; however, the resulting polytope has typically more vertices. Lemma 1.10.12 exhibits

some special cases when the number of vertices does not grow during Steiner symmetrization.

Remark 1.10.11. Given a polytope $P \subset \mathbb{R}^n$ and some facets $F_1, \ldots, F_q$ of P, a triangulation $\mathcal{F}$ of $\{F_1, \ldots, F_q\}$ is a finite family $\mathcal{F}$ of $(n-1)$-simplices whose union is $F_1 \cup \ldots \cup F_q$, each vertex of simplex in $\mathcal{F}$ is a vertex of some F_i, and the nonempty intersection of any two simplices in $\mathcal{F}$ is a common face. $\mathcal{F}$ can be constructed by constructing a triangulation $\mathcal{F}_m$ of the family of m-dimensional polytopes that are faces of some F_i into m-dimensional simplices by induction on $m = 0, \ldots, n-1$ in a way such that if F is a face of some F_i, and $G \in \mathcal{F}_m$ intersects F in a k-dimensional set, $0 \leqslant k \leqslant m$, then $F \cap G \in \mathcal{F}_k$.

Lemma 1.10.12. *For an n-dimensional polytope $P \subset \mathbb{R}^n$, assume that there exists $u \in S^{n-1}$ such that if v is a vertex of P, then either $(v + \mathbb{R}u) \cap P = \{v\}$, or $(v + \mathbb{R}u) \cap P$ is a segment whose other endpoint is also a vertex of P. Then the Steiner symmetrical $\Theta_{u^\perp} P$ is a polytope with the same number of vertices as P.*

Proof. We use the notations as in Definition 1.10.1 (c) of the Steiner symmetrization, only with $K = P$. Let $\mathcal{F}$ be a triangulation of the faces of P lying in the graph of f, and let $\mathcal{F}'$ be the family of the projections of the elements of $\mathcal{F}$ into $u^\perp$. Now elements of $\mathcal{F}'$ partition $P|u^\perp$, and the condition in Lemma 1.10.12 yields that for any $S \in \mathcal{F}'$ with vertices $x_1, \ldots, x_n$, the points $x_i + f(x_i)u$ and $x_i + g(x_i)u$ are vertices of P for $i = 1, \ldots, n$, and both f and g are linear on S. In turn, we deduce that $\frac{1}{2}(f+g)$ is linear on S for any $S \in \mathcal{F}'$; therefore, any line ℓ parallel to u contains the same number of vertices of P and of $\Theta_{u^\perp} P$. ∎

A close relative of the Steiner symmetrization is the Schwarz symmetrization. We recall that $\omega_n = |B^n|$.

Definition 1.10.13 (Schwarz symmetrization). If $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then the Schwarz symmetrial is

$$\Theta_{\mathbb{R}u} K = \bigcup \Big\{ tu + (u^\perp \cap r_t B^n) : t \in \mathbb{R},\ (tu + u^\perp) \cap K \neq \varnothing,\ r_t \geqslant 0, \\ \mathcal{H}^{n-1}\Big(K \cap (tu + u^\perp)\Big) = r_t^{n-1} \omega_{n-1} \Big\}.$$

Lemma 1.10.14. *For a convex body $K \subset \mathbb{R}^n$ and $u \in S^{n-1}$, there exists a sequence $\{\Theta_{v_m^\perp}\}$ of Steiner symmetrizations for $v_m \in u^\perp \cap S^{n-1}$ such that if $K_0 = K$ and $K_{m+1} = \Theta_{u_m^\perp} K_m$, then $\{K_m\}$ tends to $\Theta_{\mathbb{R}u} K$.*

Proof. Let $K|(\mathbb{R}u) = [a, b]u$ for $a < b$. According to Lemma 1.10.4, we may assume that K is unconditional with respect to a basis involving u, and hence $K \cap (tu + u^\perp)$ is symmetric through tu for $t \in (a, b)$.

For $k \geqslant 2$ and $i = 1, \ldots, 2^k - 1$, consider the hyperplanes $H_{k,i} = u^\perp + (a + i \cdot \frac{b-a}{2^k})u$, and apply finitely many Steiner symmetrizations based on Theorem 1.10.7 using vectors from $u^\perp$ to ensure that the ratio of the relative circumradius and inradius of $H_{k,i} \cap K$ becomes less than $1 + 2^{-k}$ for each $i = 1, \ldots, 2^k - 1$, and then let k tend to infinity to complete the proof of Lemma 1.10.14.

Alternatively, one may apply Theorem 1.C.3 instead of Theorem 1.10.7 in the following way. We deduce from Theorem 1.C.3 (cf. Remark 1.10.8) that choosing independent $v_1, \ldots, v_{n-1} \in S^{n-1} \cap u^\perp$ such that $\angle(v_i, v_j)/\pi$ is irrational for $i \neq j$, the iterated Steiner symmetrizations through $v_1^\perp, \ldots, v_{n_1}^\perp$ lead to a convex body K' such that $K' \cap (tu + u^\perp)$ is an $(n-1)$-ball centered at tu for $t \in (a, b)$, and hence $K' = \Theta_{\mathbb{R}u} K$. ∎

Combining Proposition 1.10.3 and Lemma 1.10.14 leads to the following properties of the Schwarz symmetrization.

Corollary 1.10.15. *$K, C \subset \mathbb{R}^n$ convex body, $u \in S^{n-1}$.*

(i) $\Theta_{\mathbb{R}u} K$ is a convex body that has axial rotational symmetry through $\mathbb{R}u$;

(ii) $\Theta_{\mathbb{R}u}(rB^n) = rB^n$ for $r > 0$;

(iii) $\Theta_{\mathbb{R}u}(\lambda K) = \lambda \Theta_{\mathbb{R}u} K$ if $\lambda \in \mathbb{R}\backslash\{0\}$;

(iv) $r(\Theta_{\mathbb{R}u} K) \geqslant r(K)$ and $R(\Theta_{\mathbb{R}u} K) \leqslant R(K)$;

(v) $|\Theta_{\mathbb{R}u} K| = |K|$;

(vi) $\operatorname{diam} \Theta_{\mathbb{R}u} K \leqslant \operatorname{diam} K$;

(vii) $|\alpha \Theta_{\mathbb{R}u} K + \beta \Theta_{\mathbb{R}u} C| \leqslant |\alpha K + \beta C|$ for $\alpha, \beta > 0$.

Similarly argument to the one leading to Theorem 1.10.7 yields

Theorem 1.10.16 (Iterated Schwarz symmetrizations). *If $K \subset \mathbb{R}^n$ is a convex body with $|K| = |rB^n|$ for $r > 0$, then there exists sequence $\{K_m\}$ of convex bodies tending to rB^n where $K_0 = K$ and $K_{m+1} = \Theta_{\mathbb{R}u_m} K_m$ for some $u_m \in S^{n-1}$.*

1.11 Centroid

In physics, in many circumstances, we can replace a body by a single point; namely, by the body's centroid having assigned the same mass. In mathematics, centroids prove to be the most natural "centers" a convex body, even if there are some other useful candidates. In this section, we prove the elementary properties of the centroid that are sufficient for most of the applications, and point to references for some more advanced properties.

Definition 1.11.1 (Centroid). For a convex body $K \subset \mathbb{R}^n$, its centroid is

$$\sigma_K = \frac{1}{|K|} \int_K x \, dx,$$

and hence if $u \in \mathbb{R}^n$, then

$$\langle u, \sigma_K \rangle = \frac{1}{|K|} \int_K \langle u, x \rangle \, dx = \frac{1}{|K|} \int_{\mathbb{R}} t \cdot \mathcal{H}^{n-1}\Big(K \cap (u^\perp + tu)\Big) \, dt. \tag{1.25}$$

The convex body K is called *centered* if $\sigma_K = o$.

Lemma 1.11.2 (Affine equivariance of the centroid). *If $K \subset \mathbb{R}^n$ is a convex body, then $\sigma_{\Phi K} = \Phi \sigma_K$ where $\Phi x = Ax + b$ for $A \in \mathrm{GL}(n)$ and $b \in \mathbb{R}^n$.*

Examples

- If $K \subset \mathbb{R}^n$ is an o-symmetric convex body, then $\sigma_K = o$ by affine equivariance.
- If $C = \mathrm{conv}\{o, F\}$ where $F \subset u^\perp + hu$ is an $(n-1)$-dimensional compact convex set for $u \in S^{n-1}$ and $h > 0$, then for $t \in [0, h]$, we have $\mathcal{H}^{n-1}(C \cap (u^\perp + tu)) = (t/h)^{n-1} \mathcal{H}^{n-1}(F)$, and hence (1.25) yields

$$|C| = \tfrac{1}{n} \, h \cdot \mathcal{H}^{n-1}(F), \tag{1.26}$$

$$\langle u, \sigma_C \rangle = \tfrac{n}{n+1} \, h, \tag{1.27}$$

$$\big|\{x \in C : \langle u, x \rangle \leqslant \langle u, \sigma_C \rangle\}\big| = \big(\tfrac{n}{n+1}\big)^n \, |C|. \tag{1.28}$$

The centroid is also equivariant under Steiner and Schwarz symmetrizations.

Lemma 1.11.3. *If $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then*

$$\sigma_{\Theta_{u^\perp} K} = \sigma_K | u^\perp \quad \text{and} \quad \sigma_{\Theta_{\mathbb{R}u} K} = \sigma_K | \mathbb{R}u$$

Proof. Affine equivariance (cf. Lemma 1.11.2) yields that $\sigma_{\Theta_{u^\perp} K} \in u^\perp$ and $\sigma_{\Theta_{\mathbb{R}u} K} \in \mathbb{R}u$. On the other hand, (1.25) and Definition 1.10.1 (a) of the Steiner symmetrization imply $\langle v, \sigma_{\Theta_{u^\perp} K} \rangle = \langle v, \sigma_K \rangle$ for $v \in u^\perp$, and (1.25) also yields $\langle u, \sigma_{\Theta_{\mathbb{R}u} K} \rangle = \langle u, \sigma_K \rangle$. ∎

Now we show that the centroid is sitting in the "middle" of a convex body where the first property was probably already known to Minkowski, and the second property was proved by Grünbaum [351].

Lemma 1.11.4. *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) If $\sigma_K = o$, then $-K \subset nK$;

(ii) If H^+ is a closed half-space with $\sigma_K \in H^+$, then $|H^+ \cap K| \geqslant (\frac{n}{n+1})^n |K| > \frac{1}{e} |K|$.

Remark. Simplices are optimal for (i) and (ii) (cf. (1.27) and (1.28)).

Proof. Now, (i) is equivalent to saying that if $u \in S^{n-1}$, then $h_{K-\sigma_K}(-u) \leqslant n h_{K-\sigma_K}(u)$; or in other words, using that $h_{K-\sigma_K}(-u) = h_K(-u) + \langle \sigma_K, u\rangle$ and $h_{K-\sigma_K}(u) = h_K(u) - \langle \sigma_K, u\rangle$, (i) is equivalent to the claim that for any convex body $K \subset \mathbb{R}^n$ and $u \in S^{n-1}$, we have

$$\text{if } h_K(-u) = 0, \text{ then } \langle \sigma_K, u\rangle \leqslant \tfrac{n}{n+1}\, h_K(u). \tag{1.29}$$

Set $h = h_K(u) > 0$, and define $K_t = K \cap (tu + u^\perp)$ and $f(t) = \mathcal{H}^{n-1}(K_t)$ for $t \in [0, h]$. Since $\frac{1}{2}(K_t + K_s) \subset K_{\frac{1}{2}(t+s)}$ follows from convexity, the Brunn-Minkowski inequality (1.24) yields that $f^{\frac{1}{n-1}}$ is concave. Choose $a > 0$ such that $\int_0^h (at)^{n-1}\, dt = \int_0^h f(t)\, dt = |K|$. As $f^{\frac{1}{n-1}}$ is concave, we have $ah \geqslant f(h)^{\frac{1}{n-1}}$, and there exists some $t_0 \in (0, h)$ such that $(at)^{n-1} \leqslant f(t)$ if $t \in [0, t_0]$, and $(at)^{n-1} \geqslant f(t)$ if $t \in [t_0, h]$. We deduce that

$$\begin{aligned} |K|\langle \sigma_K, u\rangle - \frac{a^{n-1}h^{n+1}}{n+1} &= \int_K \langle u, x\rangle\, dx - \frac{a^{n-1}h^{n+1}}{n+1} = \int_0^h t\left(f(t) - (at)^{n-1}\right) \\ &= \int_0^{t_0} t\left(f(t) - (at)^{n-1}\right) - \int_{t_0}^h t\left((at)^{n-1} - f(t)\right) \\ &\leqslant t_0 \int_0^{t_0} \left(f(t) - (at)^{n-1}\right) - t_0 \int_{t_0}^h \left((at)^{n-1} - f(t)\right) = 0. \end{aligned}$$

Since $|K| = a^{n-1}h^n/n$, we conclude (1.29), and in turn (i).

For (ii), we may assume that $H^+ = \{x \in \mathbb{R}^n : \langle t, u\rangle \leqslant \langle \sigma_K, u\rangle\}$. We consider $K_t^+ = \{x \in K : \langle x, u\rangle \leqslant t\}$ and $F(t) = |K_t^+| = \int_0^t f$ for $t \in [0, h]$, and hence $F' = f$. Since $\frac{1}{2}(K_t^+ + K_s^+) \subset K^+_{\frac{1}{2}(t+s)}$ follows from convexity, the Brunn-Minkowski inequality (1.24) yields that $F^{\frac{1}{n}}$ is concave. Therefore, the Jensen inequality Lemma 10.2.4 applied to the probability measure $|K|^{-1} f(t)\, dt$ on $[0, h]$ and the substitution $s = F(t)$ yield

$$\begin{aligned} |H^+ \cap K|^{\frac{1}{n}} &= F\left(\langle \sigma_K, u\rangle\right)^{\frac{1}{n}} = F^{\frac{1}{n}}\left(\int_0^h t \cdot |K|^{-1} f(t)\, dt\right) \geqslant \int_0^h F(t)^{\frac{1}{n}} \cdot |K|^{-1} f(t)\, dt \\ &= |K|^{-1} \int_0^h F(t)^{\frac{1}{n}} \cdot F'(t)\, dt = |K|^{-1} \int_0^{|K|} s^{\frac{1}{n}}\, ds = \frac{n}{n+1} |K|^{\frac{1}{n}}, \end{aligned}$$

proving (ii). ∎

The following statement is proved in Kannan, Lovász, Simonovits [454] where the factor n is optimal, for example, for simplices.

Lemma 1.11.5 (KLS ellipsoid). *If $K \subset \mathbb{R}^n$ is a convex body, then there exists an o-symmetric ellipsoid E with $\sigma_K + E \subset K \subset \sigma_K + nE$.*

Remark. See Proposition 6.4.9 for a more precise statement. In particular, E is a ball if the ellipsoid of inertia of K is a ball; namely, if $\int_{K-\sigma_K}\langle u,x\rangle^2\,dx$ is the same value for any $u\in S^{n-1}$.

The proof of Lemma 1.11.6 uses polar coordinates; namely, the formula that if $f\in L^1(\mathbb{R}^n,\mathcal{H}^n)$, then (cf. (10.13))

$$\int_{\mathbb{R}^n} f=\int_{S^{n-1}}\int_0^\infty f(ru)r^{n-1}\,dr\,d\mathcal{H}^{n-1}(u). \tag{1.30}$$

Lemma 1.11.6. *If $K\subset\mathbb{R}^n$ is a convex body with $o\in\operatorname{int}K$, then*

(i) $|K|=\dfrac{1}{n}\displaystyle\int_{S^{n-1}}\varrho_K(u)^n\,du$;

(ii) $\sigma_K=o$ *if and only if* $\displaystyle\int_{S^{n-1}}u\cdot\varrho_K(u)^{n+1}\,du=o$.

Proof. For (i), take $f=\mathbf{1}_K$ in (1.30).

For (ii), $\sigma_K=o$ if and only if $\int_{\mathbb{R}^n}\langle v,x\rangle\cdot\mathbf{1}_K(x)\,dx=0$ for any $v\in\mathbb{R}^n$, which is equivalent to (ii) by taking $f(x)=\langle v,x\rangle\cdot\mathbf{1}_K(x)$ in (1.30). ∎

The following statements are proved by Milman, Pajor [583].

Theorem 1.11.7 (Milman, Pajor). *If $K,C\subset\mathbb{R}^n$ are centered convex bodies, then*

$$|K-C|\cdot|K\cap C|\geqslant|K|\cdot|C|\ \text{and}\ |K\cap(-K)|\geqslant 2^{-n}|K|. \tag{1.31}$$

Remarks.

- Here $|K-C|\cdot|K\cap C|\leqslant\binom{2n}{n}|K|\cdot|C|<4^n|K|\cdot|C|$ by (1.32) below.
- The first inequality in (1.31) readily yields the second by taking $C=-K$.
- Combining Theorem 4.1 in Campos, van Hintum, Morris, Tiba [178] and the solution of Ball's and Bourgain's Slicing conjecture by Klartag, Lehec [470] (cf. Theorem 6.4.12), we deduce the existence of an absolute constant $c>0$ such that for any centered convex body $K\subset\mathbb{R}^n$, we have

  $$|K\cap(-K)|\geqslant 2^{-n+cn}|K|.$$

  On the other hand, if $K\subset\mathbb{R}^n$ is a centered simplex, then $|K\cap(-K)|<\left(\frac{2}{e}\right)^n\sqrt{n}|K|$ according to Fáry, Rédei [258] (see the formula after (70) in [258]).

  While do not provide the proof of Theorem 1.11.7, we verify the following variant.

Theorem 1.11.8 (Rogers-Shephard). *If $K, C \subset \mathbb{R}^n$ re convex bodies, then*

$$|K|\,|C| \leqslant |K - C| \cdot \max_{x \in \mathbb{R}^n} |K \cap (x + C)| \leqslant \tbinom{2n}{n} |K|\,|C| < 4^n |K|\,|C|; \tag{1.32}$$

$$|K - K| \leqslant \tbinom{2n}{n} |K| < 4^n |K|, \tag{1.33}$$

$$|\widetilde{K} \cap (-\widetilde{K})| \geqslant 2^{-n} |K| \ \textit{for some } \widetilde{K} = K - w,\ w \in K. \tag{1.34}$$

Remarks.

- $w = \sigma_K$ works in (1.34) by (1.31).
- $|K - K| = \binom{2n}{n} |K|$ if K is a simplex in (1.33).
- If K, C are o-symmetric, then $|K \cap C| = \max_{x \in \mathbb{R}^n} |K \cap (x + C)|$ by the Brunn-Minkowski inequality (1.24).

Proof. Taking convolution, $\mathbf{1}_K * \mathbf{1}_{-C}(x) = \int_{\mathbb{R}^n} \mathbf{1}_K(y) \cdot \mathbf{1}_{-C}(x - y)\, dy = |K \cap (x + C)|$ follows for $x \in \mathbb{R}^n$ where $K \cap (x + C) \neq \varnothing$ if and only $x \in D = K - C$; therefore,

$$|K| \cdot |C| = \int_{\mathbb{R}^n} \mathbf{1}_K * \mathbf{1}_{-C} = \int_D |K \cap (x + C)|\, dx. \tag{1.35}$$

Now (1.35) yields the first inequality in (1.32).

For the second inequality in (1.32), we assume that $|K \cap C| = \max_{x \in \mathbb{R}^n} |K \cap (x + C)|$, and hence $o \in \operatorname{int} D$. For $x \in D$, there exists $z \in \partial D$ with $x = \|x\|_D \cdot z$ where $z = a - b$ for $a \in K, b \in L$, thus $x = \|x\|_D (a - b)$. We deduce the containment relation

$$\|x\|_D \cdot a + (1 - \|x\|_D)(K \cap C) \subset K \cap (x + C)$$

implying that $|K \cap (x + C)| \geqslant (1 - \|x\|_D)^n |K \cap C|$, and in turn we conclude from (1.35) and from $\frac{d}{dt} \int_{tD} (1 - \|x\|_D)^n\, dx = |D|(1 - t)^n t^{n-1}$ for $t \in (0, 1)$ that

$$|K| \cdot |C| \geqslant |K \cap C| \cdot |D| \int_0^1 (1 - t)^n t^{n-1} dt = \tbinom{2n}{n}^{-1} |K \cap C| \cdot |K - C|.$$

Here $\binom{2n}{n} < 4^n$ follows by the Stirling formula (10.2), and finally (1.32) implies (1.33) and (1.34). ∎

If $K \subset \mathbb{R}^n$ is convex body, then the Brunn-Minkowski inequality (1.24) and the Rogers-Shephard inequality (1.33) yield that

$$2^n |K| \leqslant |K - K| \leqslant \tbinom{2n}{n} |K| < 4^n |K|. \tag{1.36}$$

1.12 Equality in the Brunn-Minkowski inequality for convex bodies and in the Isodiametric inequality

We note that we have already proved the Brunn-Minkowski inequality

$$|\alpha K + \beta C|^{\frac{1}{n}} \geqslant \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}} \tag{1.37}$$

for convex bodies $K, C \subset \mathbb{R}^n$ and $\alpha, \beta \geqslant 0$ via Steiner symmetrization in Theorem 1.10.9. In this section, we provide Brunn's original ideas in [164] at the end of the 19th centrury that also leads to the characterization of the case of equality (see Theorem 1.12.3). While the Brunn-Minkowski inequality is one of the most widely used estimates in convexity, Theorem 1.11.7 shows that it is far from optimal if, for example, $|K| = |C|$ and $|K \cap C|$ is small for centered convex bodies $K, C \subset \mathbb{R}^n$.

Before proving the Brunn-Minkowski inequality, we state some useful equivalent forms of it.

Definition 1.12.1 (Homothetic convex bodies). Convex bodies $K, C \subset \mathbb{R}^n$ are homothetic if there exist $\gamma > 0$ and $x \in \mathbb{R}^n$ such that $K = \gamma C + x$ (it is readily a symmetric relation).

Lemma 1.12.2 (Equivalent forms of Brunn-Minkowski). *The following are equivalent assuming that they hold for any convex bodies $K, C \subset \mathbb{R}^n$.*

(i) $|\alpha K + \beta C|^{\frac{1}{n}} \geqslant \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}}$ *for* $\alpha, \beta > 0$;

(ii) $|K + C|^{\frac{1}{n}} \geqslant |K|^{\frac{1}{n}} + |C|^{\frac{1}{n}}$;

(iii) $|(1-\lambda) K + \lambda C|^{\frac{1}{n}} \geqslant (1-\lambda)|K|^{\frac{1}{n}} + \lambda |C|^{\frac{1}{n}}$ *for* $\lambda \in (0,1)$;

(iv) $|(1-\lambda) K + \lambda C| \geqslant |K|^{1-\lambda} |C|^{\lambda}$ *for* $\lambda \in (0,1)$;

(v) $f(t) = |C + tK|^{\frac{1}{n}}$ *and* $g(t) = |(1-t)C + tK|^{\frac{1}{n}}$ *are concave for* $t \in [0,1]$.

Equality holds in (i), (ii) or (iii) if and only if K and C are homothetic, equality holds in (iv) if and only if K and C are translates, and f or g in (v) is linear if and only if K and C are homothetic.

Proof. (i), (ii) and (iii) are equivalent as $|\alpha K| = \alpha^n |K|$ for $\alpha > 0$, and (iii) yields (iv) by the AM-GM inequality.

To show that (iv) implies (i), set $\alpha_0 = |X|^{\frac{1}{n}}, \beta_0 = |Y|^{\frac{1}{n}}, X_0 = X/\alpha_0$ and $Y_0 = Y/\beta_0$, and hence $|X_0| = |Y_0| = 1$. Setting $\lambda = \frac{\beta\beta_0}{\alpha\alpha_0 + \beta\beta_0}$, (iv) yields $|(1-\lambda) X_0 + \lambda Y_0|_* \geqslant 1$, thus $|\alpha X + \beta Y|_*^{\frac{1}{n}} = (\alpha\alpha_0 + \beta\beta_0) \left| \frac{\alpha\alpha_0}{\alpha\alpha_0 + \beta\beta_0} X + \frac{\beta\beta_0}{\alpha\alpha_0 + \beta\beta_0} Y \right|_*^{\frac{1}{n}} \geqslant \alpha\alpha_0 + \beta\beta_0$.

(v) yields either (ii) (if f is concave) or (iii) (if g is concave).

To show that (i) implies (v), we observe that the convexity of K and C yields that $\frac{1}{2}(C + sK) + \frac{1}{2}(C + rK) = C + tK$ and

$$\frac{1}{2}((1-s)C + sK) + \frac{1}{2}((1-r)C + rK) = (1-t)C + tK$$

hold for $s, r \geqslant 0$ and $t = \frac{1}{2}(s + r)$.

Concerning equality, the conditions for (i), (ii), (iii) and (iv) are readily equivalent.

In (v), f is linear on $[0, 1]$ if and only if $|C + \frac{1}{2}K|^{\frac{1}{n}} = \frac{1}{2}|C|^{\frac{1}{n}} + \frac{1}{2}|C + K|^{\frac{1}{n}}$, which is in turn equivalent with $C + K = \gamma C + z$ for $\gamma > 0$, $z \in \mathbb{R}^n$. In other words, $h_C + h_{K-z} = \gamma h_C$, and since there exists some $u \in S^{n-1}$ with $h_C(u) > 0$ and $h_{K-z}(u) > 0$, we deduce that $\gamma > 1$ and $K = (\gamma - 1)C + z$.

In addition, g is linear on $[0, 1]$ if and only if $|\frac{1}{2}C + \frac{1}{2}K|^{\frac{1}{n}} = \frac{1}{2}|C|^{\frac{1}{n}} + \frac{1}{2}|K|^{\frac{1}{n}}$, that is equivalent to saying that K, C are homothetic. ∎

Theorem 1.12.3 (Brunn-Minkowski with equality). *If $K_1, K_2 \subset \mathbb{R}^n$ are convex bodies and $\alpha_1, \alpha_2 > 0$, then $|\alpha_1 K_1 + \alpha_2 K_2|^{\frac{1}{n}} \geqslant \alpha_1 |K_1|^{\frac{1}{n}} + \alpha_2 |K_2|^{\frac{1}{n}}$, and equality holds if and only if K_1 and K_2 are homothetic.*

Proof. The argument is by induction on $n \geqslant 1$ where the case $n = 1$ trivial.

Let $n \geqslant 2$, and assume that $\alpha_1 + \alpha_2 = 1$, moreover $|K_i| = 1$ and $\sigma_{K_i} = \int_{K_i} x\, dx = o$ for $i = 1, 2$. Therefore writing $K = \alpha_1 K_1 + \alpha_2 K_2$, the Brunn-Minkowski inequality with characterization of equality is equivalent to saying that

$$|K| = |\alpha_1 K_1 + \alpha_2 K_2| \geqslant 1, \tag{1.38}$$

with equality if and only if $K_1 = K_2$. To simplify notation, the $(n-1)$-dimensional Hausdorff measure is denoted by $\mathcal{H} = \mathcal{H}^{n-1}$ in this proof.

As a first step, for a fixed $u \in S^{n-1}$, we prove (1.38) using the sections of K_1 and K_2 orthogonal to u, and verify the claim that equality in (1.38) yields that $h_{K_1}(u) = h_{K_2}(u)$.

If $-h_{K_i}(-u) < r_i \leqslant h_{K_i}(u)$ for $i = 1, 2$ and $-h_K(-u) < s < h_K(u)$, then let

$$\begin{aligned} V_i(r_i) &= |\{x \in K_i : \langle x, u\rangle \leqslant r_i\}|; \\ \widetilde{C}_i(r_i) &= K_i \cap (r_i u + u^\perp); \\ C(s) &= K \cap (su + u^\perp). \end{aligned}$$

In particular, $\int_{-h_{K_i}(-u)}^{r_i} \mathcal{H}(\widetilde{C}_i(t))\, dt = V_i(r_i)$, thus $\frac{\partial}{\partial t} V_i(t) = \mathcal{H}(\widetilde{C}_i(t))$ and $V_i(h_{K_i}(u)) = 1$, and in addition, we have $\alpha_1 \widetilde{C}_1(r_1) + \alpha_2 \widetilde{C}_2(r_2) \subset C(\alpha_1 r_1 + \alpha_2 r_2)$.

The key idea of the argument is that the level sets $\widetilde{C}_i(r_i)$ of K_i are parametrized by the volume $v \in (0, 1)$ below the level set. Thus for $v \in (0, 1)$, we define $t_i(v) \in \mathbb{R}$ by $V_i(t_i(v)) = v$, and set $C_i(v) = \widetilde{C}_i(t_i(v))$, and hence

$$\frac{\partial}{\partial v} t_i(v) = \frac{1}{\mathcal{H}(C_i(v))}. \tag{1.39}$$

Using substitution $t = \alpha_1 t_1(v) + \alpha_2 t_2(v)$, the notation $A_i(v) = \mathcal{H}(C_i(v))$ and the fact that the Brunn-Minkowski inequality (1.37) is known in dimension $n - 1$ by induction,

we deduce that

$$\begin{aligned}|K| &= \int_{-\alpha_1 h_{K_1}(-u)-\alpha_2 h_{K_2}(-u)}^{\alpha_1 h_{K_1}(u)+\alpha_2 h_{K_2}(u)} \mathcal{H}(\widetilde{C}(t))\,dt \\ &= \int_0^1 \mathcal{H}(C(\alpha_1 t_1(v)+\alpha_2 t_2(v)))\left(\frac{\alpha_1}{A_1(v)}+\frac{\alpha_2}{A_2(v)}\right)dv \\ &\geqslant \int_0^1 \left(\alpha_1 A_1(v)^{\frac{1}{n-1}}+\alpha_2 A_2(v)^{\frac{1}{n-1}}\right)^{n-1}\left(\frac{\alpha_1}{A_1(v)}+\frac{\alpha_2}{A_2(v)}\right)dv.\end{aligned}$$

Since $\left(\alpha_1 A_1^{\frac{1}{n-1}}+\alpha_2 A_2^{\frac{1}{n-1}}\right)^{n-1} \geqslant A_1^{\alpha_1}A_2^{\alpha_2} \geqslant \left(\frac{\alpha_1}{A_1}+\frac{\alpha_2}{A_2}\right)^{-1}$ for $A_1, A_2 > 0$ by the AM-GM inequality where equality yields $A_1 = A_2$, we conclude that

$$|\alpha_1 K_1 + \alpha_2 K_2| \geqslant \int_0^1 1\,dv = 1,$$

proving the Brunn-Minkowski inequality (1.38).

We claim that equality in Brunn-Minkowski yields that

$$h_{K_2}(u) = h_{K_1}(u) \quad \text{for every} \quad u \in S^{n-1}. \tag{1.40}$$

Applying the argument above for a fixed $u \in S^{n-1}$, equality in Brunn-Minkowski in (1.38) implies that $A_1(v) = A_2(v)$ for $v \in (0,1)$, and hence (1.39) yields the existence of a constant $c \in \mathbb{R}$ with $t_2(v) = t_1(v) + c$ for every $v \in (0,1)$. Using $\sigma_{K_1} = \sigma_{K_2} = o$ and $\mathcal{H}(\widetilde{C}_2(t_i(v))) = A_i(v)$, it follows from (1.25) that

$$\begin{aligned}0 &= \int_{K_2}\langle x,u\rangle\,dx = \int_{-h_{K_2}(-u)}^{h_{K_2}(u)} t\cdot\mathcal{H}(\widetilde{C}_2(t))\,dt = \int_0^1 t_2(v)\cdot A_2(v)\cdot\frac{1}{A_2(v)}\,dv \\ &= \int_0^1 t_1(v)+c\,dv = c+\int_0^1 t_1(v)\,dv = c+\int_{K_1}\langle x,u\rangle\,dx = c.\end{aligned}$$

We deduce that $t_2(v) = t_1(v)$ for $v \in (0,1)$, and hence the claim (1.40) follows from $h_{K_2}(u) = \lim_{v\to 1^-} t_2(v) = \lim_{v\to 1^-} t_1(v) = h_{K_1}(u)$.

In turn, the claim (1.40) yields that $K_1 = K_2$. ∎

Finally, we show how the Brunn-Minkowski inequality with equality yields the characterization of the equality case in the Isodiametric Inequality Theorem 1.10.5.

Theorem 1.12.4 (Isodiametric Inequality for convex bodies with equality). *If $K \subset \mathbb{R}^n$ is a convex body, then* $\operatorname{diam} K \geqslant 2\omega_n^{-1/n}|K|^{1/n}$, *with equality if and only if K is a ball.*

Proof. Let $|K| = |rB^n|$ for $r > 0$, and let us consider the o-symmetric convex body $K' = \frac{1}{2}(K-K)$. As $\operatorname{diam} C = \max_{u\in S^{n-1}} h_C(u) + h_C(-u)$ for a convex body $C \subset \mathbb{R}^n$,

we deduce that $\operatorname{diam} K' = \operatorname{diam} K$. It follows from the Brunn-Minkowski inequality Theorem 1.12.3 that $|K'| \geqslant |K|$. Since $K' \backslash \operatorname{int}(rB^n) \neq \varnothing$, we deduce that $\operatorname{diam} K = \operatorname{diam} K' \geqslant 2r$.

If $\operatorname{diam} K = 2r$, then $K' = rB^n$, and hence $|K| = |\frac{1}{2}K + \frac{1}{2}(-K)|$. It follows from the equality case of Brunn-Minkowski inequality Theorem 1.12.3 that K' is a translate of K; therefore, K is a ball. ∎

Let us discuss the rather simple extension of the equality case of the Isodiametric Inequality to any bounded measurable set $X \subset \mathbb{R}^n$ where the diameter of X is

$$\operatorname{diam} X = \sup\{\|x - y\| : x, y \in X\}.$$

Theorem 1.12.5 (Isodiametric Inequality in $\mathbb{R}^n$ with equality). *If $X \subset \mathbb{R}^n$ is bounded, measurable and* $\operatorname{diam} X > 0$, *then* $\operatorname{diam} X \geqslant 2\omega_n^{-1/n}|X|^{1/n}$, *with equality if and only if there exists a Euclidean ball $B \supset X$ with $|B \backslash X| = 0$.*

Proof. We consider the following equivalent form: If $\operatorname{diam} X = 2r > 0$, then $|X| \leqslant |rB^n|$. It follows from the definition that $\operatorname{diam} \operatorname{cl} X = \operatorname{diam} X$, and $\widetilde{X} = \operatorname{conv} \operatorname{cl} X$ is a compact convex set with $\operatorname{diam} \widetilde{X} = \operatorname{diam} X$ by Proposition 1.1.7. We deduce from Theorem 1.12.4 that $|X| \leqslant |\widetilde{X}| \leqslant |rB^n|$.

If $|X| = |rB^n|$, then $|\widetilde{X}| = |rB^n|$, and hence Theorem 1.12.4 yields that $\widetilde{X}$ is a ball of radius r. As $|X| = |\widetilde{X}|$, we conclude that $|\widetilde{X} \backslash X| = 0$. ∎

1.13 Comments to Chapter 1

Concerning additional properties of the closed and compact convex sets and the support function, see the survey Colesanti [207], and the monographs Bonnesen, Fenchel [107], Gruber [349], Hug, Weil [433] and Schneider [661], Chapters 1-3. For properties of the polar (dual) body, see also Artstein-Avidan, Giannopoulos, Milman [38] and Schneider [661], Section 1.6. For more advanced properties of normal cones, polytopes and polyhedral sets, see Barvinok [73], Grünbaum [352] and Ziegler [739].

Families of strongly isomorphic polytopes have been already used by Aleksandrov [7, 11] in one of his proofs of the Aleksandrov-Fenchel inequality (see Section 7.A and Schneider [661], Section 2.4 and Section 7.3). The main idea is that mixed volumes of strongly isomorphic polytopes can be interperted as a multilinear forms of the values of the support functions at the common exterior unit vectors of the facets (see Theorem 7.A.7). This idea has been generalized into his polytope algebra by McMullen [568, 570]. Putterman [629] used strongly isomorphic polytopes in his argument about the Logarithmic (L_0) Brunn-Minkowski conjecture for $p \in [0, 1)$ origin symmeric convex bodies (see Section 8.7). Strongly isomorphic polytopes can represent ample divisors on projective toric varieties (see Cox, Little, Schenck [222], Ewald

[252], Fulton [308] and Oda [606]), or certain properties of compact hyperbolic manifolds (see Fillastre [283]),.

For in depth studies on Steiner symmetrization and Schwarz symmetrization, see Bianchi, Gardner, Gronchi [93, 94] (see also Bianchi, Gronchi [95]). The papers [93, 94] provide a broader class of n hyperplanes for the iterated Steiner Symmetrization than Theorem 1.C.3; namely, $v_1, \dots, v_n \in S^{n-1}$ are independent in a way such that $\langle v_i, v_j \rangle \neq 0$ for $i \neq j$, and $\angle(v_1, v_2) = \alpha\pi$ for an irrational $\alpha \in (0, 1)$.

The Brunn-Minkowski inequality for convex bodies was proved by Brunn [164] in dimensions $n = 2, 3$ in 1889, and by Minkowski in any dimensions a couple years later (see Section 1.12 for their argument). It was Minkowski's work [591] where the importance of the inequality was recognized, and it has found its place within a whole, now called Brunn-Minkowski, theory as reviewed by the thorough monograph Schneider [661]. Various natural strengthened versions of the Brunn-Minkowski inequality for convex bodies are known or conjectured, like the stability version Theorem 8.6.4 essentially due to Figalli, Maggi, Pratelli [278], and the Logarithmic Brunn-Minkowski conjecture for o-symmetric convex bodies (see Section 8.7). Dar's conjectured strengthening of the Brunn-Minkowski inequality states in [230] that if K and C are convex bodies in $\mathbb{R}^n$, and $\Theta = \max_{x \in \mathbb{R}^n} V(K \cap (x + C))$, then

$$|K + C|^{\frac{1}{n}} \geqslant \Theta^{\frac{1}{n}} + \left(\frac{|K| \cdot |C|}{\Theta} \right)^{\frac{1}{n}}. \tag{1.41}$$

Dar's conjecture is only known to hold in the plane (see Xi, Leng [723]), and in some very specific cases in higher dimension (see Dar [230]). For extensions of the Brunn-Minkowski inequality to non-convex sets or to functions - that is, the Prékopa-Leindler inequality - see Chapter 3.

The Isodiametric Inequality in $\mathbb{R}^n$ was known to Steiner, as it readily follows from his symmetrization method. Optimal stability version is provided by Maggi, Ponsiglione, Pratelli [558].

1.A Supplement: Epigraph and Legendre transform (convex conjugate) of convex functions

This section is based on Section A.2 in Artstein-Avidan, Giannopoulos, V. Milman [38]. We say that a function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is non-trivial if there exists $x \in \mathbb{R}^n$ with $\varphi(x) < \infty$. In addition, a non-trivial function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is convex, if whenever $\varphi(x), \varphi(y) < \infty$ and $\lambda \in (0, 1)$, then

$$(1 - \lambda)\varphi(x) + \lambda\, \varphi(y) \geqslant \varphi\Big((1 - \lambda)x + \lambda\, y\Big).$$

We observe that if $\varphi : \Omega \to \mathbb{R}$ is a convex function for convex $\Omega \subset \mathbb{R}^n$ in the classical sense, then it can be extended to a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ by defining $\varphi(x) = \infty$ for $x \in \mathbb{R}^n \backslash \Omega$.

On the other hand, if $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is a non-trivial convex function, then its domain

$$\operatorname{dom} \varphi = \{x \in \mathbb{R}^n : \varphi(x) < \infty\}$$

is a non-empty convex subset of $\mathbb{R}^n$.

Definition 1.A.1 (Epigraph of a convex function). For a non-trivial convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$, its epigraph $\operatorname{epi} \varphi \subset \mathbb{R}^{n+1}$ is the unbounded convex set lying above the graph of φ; namely,

$$\operatorname{epi} \varphi = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{dom} \varphi \text{ and } t \geqslant \varphi(x)\}.$$

Remarks.

- The epigraph is a convex subset of $\mathbb{R}^{n+1}$, and the graph of φ over $\operatorname{dom} \varphi$ is a subset of the boundary of the epigraph.
- If $\ell : \mathbb{R}^n \to \mathbb{R}$ is a linear function; namely, $\ell(x) = \langle a, x \rangle + b$ for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, then $\operatorname{epi} \ell$ is a closed half-space of $\mathbb{R}^{n+1}$.

We say that a non-trivial convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is *lower semi-continuous* if whenever $\lim_{k \to \infty} x_k = x$ for $x_k \in \operatorname{dom} \varphi$ and $x \in \mathbb{R}^n$, then

$$\liminf_{k \to \infty} \varphi(x_k) \geqslant \varphi(x).$$

Since φ is continuous on the relative interior of $\operatorname{dom} \varphi$ (see Section 10.6), lower semi-continuity is determined by the behaviour of φ near the relative boundary of $\operatorname{dom} \varphi$.

The definitions of epigraph and lower semi-continuity directly yield the following.

Lemma 1.A.2. *A non-trivial convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is lower semi-continuous if and only if its epigraph is closed.*

Because of Lemma 1.A.2, lower semi-continuous convex functions are frequently called "closed". The following lemma is the analogue of the statement that any non-trivial closed convex subset of $\mathbb{R}^n$ is the intersection of closed half-spaces (cf. Corollary 1.2.4), and the intimate relation between convex functions and their epigraphs (see the Remarks after Definition 1.5.4).

Lemma 1.A.3. *A non-trivial convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is lower semi-continuous if and only if for any $x \in \mathbb{R}^n$, we have*

$$\varphi(x) = \sup\{\ell(x) : \ell \text{ is a linear function and } \ell(z) \leqslant \varphi(z) \text{ for } z \in \mathbb{R}^n\}. \tag{1.42}$$

Proof. If (1.42) holds, then epi φ is closed as it is the intersection of closed half spaces (the epigraphs of certain linear functions); therefore, φ is lower semi-continuous by Lemma 1.A.2.

On the other hand, let us assume that φ is lower semi-continuous, and hence epi $\varphi \subset \mathbb{R}^{n+1}$ is a closed convex set, and we write $(x,t) \in \mathbb{R}^{n+1}$ to denote the points of $\mathbb{R}^{n+1}$ for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, and set $\Omega = \operatorname{dom} \varphi$.

If $x \in \operatorname{relint} \Omega$, then there exists a supporting hyperplane $H \subset \mathbb{R}^{n+1}$ to epi $\varphi \subset \mathbb{R}^{n+1}$ at $(x, \varphi(x))$ such that H is the graph of a linear function ℓ_x on $\mathbb{R}^n$; therefore, we may even write maximum in (1.42) in this case. Here if $\operatorname{int} \Omega \neq \varnothing$, then we can actually choose any supporting hyperplane $H \subset \mathbb{R}^{n+1}$ to epi $\varphi \subset \mathbb{R}^{n+1}$ at $(x, \varphi(x))$.

Next if $x \in \operatorname{relbd} \Omega$, then $\varphi(x) = \sup_{y \in \operatorname{relint} \Omega} \ell_y(x)$ as epi $\varphi \subset \mathbb{R}^{n+1}$ is the closure of its relative interior.

Finally, for $x \notin \operatorname{cl} \Omega$, we need to verify that for any $t > 0$ there exists a linear function ℓ on $\mathbb{R}^n$ such that $\ell(z) \leqslant \varphi(z)$ for $z \in \operatorname{dom} \varphi$ and $\ell(x) > t$. Let $p \in \operatorname{int} \Omega$, and we may translate Ω and x simultaneously in a way such that $\operatorname{conv}\{x, p\}$ intersects $\partial\Omega$ in o. Since φ is convex on the segment $\operatorname{conv}\{p, o\} \subset \mathbb{R}^n$, there exists an $m > 0$ such that $\varphi(\lambda p) \geqslant -m$ for $\lambda \in [0,1]$. Therefore, choosing large s satisfying

$$\frac{s+t}{m+t} > \frac{\|x\| + \|p\|}{\|x\|},$$

the segment $\sigma \subset \mathbb{R}^{n+1}$ connecting $(p, -s)$ and (x,t) lies below epi φ, thus it is disjoint from epi φ. It follows that σ can be separated by a hyperplane $H \subset \mathbb{R}^{n+1}$ from epi φ according to Lemma 1.2.6. This H is the graph of a linear function ℓ on $\mathbb{R}^n$ because H separates $(p, -s)$ from epi φ; moreover, ℓ satisfies that $\ell(x) > t$ and $\ell(z) \leqslant \varphi(z)$ for $z \in \operatorname{dom} \varphi$, completing the proof of (1.42). ∎

Lemma 1.A.3 inspires the notion of the convex closure of a function.

Definition 1.A.4 (The closure of a convex function). For a non-trivial convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$, its convex closure $\overline{\operatorname{conv}}\, \varphi$ is the lower semi-continuous convex function satisfying that for any $x \in \mathbb{R}^n$, we have

$$\overline{\operatorname{conv}}\, \varphi(x) = \sup\{\ell(x) : \ \ell \text{ is a linear function and } \ell(z) \leqslant \varphi(z) \text{ for } z \in \mathbb{R}^n\}. \quad (1.43)$$

In particular, the epigraph of $\overline{\operatorname{conv}}\, \varphi$ is the closure of epi φ in $\mathbb{R}^{n+1}$.

Moreover, $\overline{\operatorname{conv}}\, \varphi = \varphi$ if φ is also lower semi-continuous.

Remark. $\overline{\operatorname{conv}}\, \varphi(x) = \varphi(x)$ for $x \in \operatorname{relint} \operatorname{dom} \varphi$ for any non-trivial convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ where the relative interiors of $\overline{\operatorname{conv}}\, \varphi$ and φ coincide. As the epigraph of $\overline{\operatorname{conv}}\, \varphi$ is a closed convex subset of $\mathbb{R}^{n+1}$, for any $x \in \operatorname{relbd} \operatorname{dom} \varphi$, there exists a sequence $x_k \in \operatorname{relint} \operatorname{dom} \varphi$ such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} \varphi(x_k) = \overline{\operatorname{conv}}\, \varphi(x)$.

Actually, for any non-trivial function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ such that there exists a linear function ℓ satisfying that $\ell(z) \leqslant \varphi(z)$ for $z \in \mathbb{R}^n$, its convex closure $\overline{\operatorname{conv}}\,\varphi$ can be defined *via* (1.43), and the convex closure is a lower semi-continuous convex function whose epigraph is the closure of the convex hull of the epigraph $\{(x,t) \in \mathbb{R}^{n+1} : \varphi(x) \leqslant t < \infty\}$ of φ.

We recall the notion of subdifferential at $x \in \Omega$ for a convex function $\varphi : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is open and convex (cf. Definition 10.6.3); namely,

$$\partial\varphi(x) = \{a \in \mathbb{R}^n : \varphi(z) \geqslant \langle a, z - x\rangle + \varphi(x)\ \forall z \in \Omega\}, \tag{1.44}$$

where $\partial\varphi(x)$ is nonempty, convex and compact.

Now let $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ be a non-trivial convex function. If $x \in \operatorname{dom}\varphi$, then we say that an $a \in \mathbb{R}^n$ is a *subgradient* at x if

$$\varphi(z) \geqslant \langle a, z - x\rangle + \varphi(x) \quad \text{for all } z \in \operatorname{dom}\varphi, \tag{1.45}$$

and hence such a subgradient a always exists if x lies in the relative interior of $\operatorname{dom}\varphi$ by (1.44). On the other hand, no subgradient may exist if x lies at the relative boundary of $\operatorname{dom}\varphi$; for example, consider the case $n = 1$, $\operatorname{dom}\varphi = [-1, 1]$, $x = 1$ and $\varphi(t) = -\sqrt{1 - t^2}$ for $t \in [-1, 1]$.

We observe that if $\Omega = \operatorname{int}\operatorname{dom}\varphi \neq \varnothing$ for a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$, then for any $x \in \Omega$, the subdifferential $\partial\varphi(x)$ consists of all subgradients at x, and φ is differentiable at x if and only if there exists a unique subgradient a at x; that is,

$$a = D\varphi(x). \tag{1.46}$$

Definition 1.A.5 (Legendre transform, or convex conjugate). For a non-trivial function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$, its Legendre transform (also called Fenchel conjugate or convex conjugate) is the function $\mathcal{L}(\varphi) : \mathbb{R}^n \to (-\infty, \infty]$ satisfying

$$\mathcal{L}\varphi(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - \varphi(y)\}.$$

Remark. If there exists $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $\varphi(x) \geqslant \langle a, x\rangle - b$ (for example, if φ is convex, cf. (1.45)), then $\mathcal{L}\varphi(a) \leqslant b$, and hence the Legendre transform $\mathcal{L}\varphi$ is a non-trivial, convex and lower semi-continuous function as its epigraph in $\mathbb{R}^{n+1}$ - being the intersection of closed half spaces - is a closed convex set (cf. Lemma 1.A.2).

We note that the notation φ^* is also frequently used for the Legendre transform. We prefer to use $\mathcal{L}\varphi$ because f° is a notation used for the polar of a non-negative function (see Definition 6.7.1). The Legendre transform is typically applied to convex functions. We note that if $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is a non-trivial and convex function, then the Remark after Definition 1.A.4 yields that

$$\mathcal{L}(\overline{\operatorname{conv}}\,\varphi) = \mathcal{L}\varphi. \tag{1.47}$$

Remark 1.A.6. Let $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ be non-trivial and convex with $\Omega = \operatorname{dom} \varphi$.

(i) An equivalent definition of the Legendre transform $\mathcal{L}\varphi(x)$ for $x \in \mathbb{R}^n$ is that it is the smallest number (possibly ∞) such that

$$\mathcal{L}\varphi(x) + \varphi(y) \geqslant \langle x, y \rangle \quad \text{for all } y \in \Omega. \tag{1.48}$$

(ii) In particular, $a \in \partial\varphi(x)$ (is a subgradient) at an $x \in \Omega$ if and only if

$$\mathcal{L}\varphi(a) + \varphi(x) = \langle a, x \rangle. \tag{1.49}$$

According to (1.44), there exists a subgradient $a \in \mathbb{R}^n$ at any $x \in \operatorname{relint} \Omega$.

Lemma 1.A.7. *If $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is a non-trivial convex function, then*

$$\mathcal{L}\mathcal{L}\varphi = \overline{\operatorname{conv}}\, \varphi.$$

In particular, if φ is also lower semi-continuous, then $\mathcal{L}\mathcal{L}\varphi = \varphi$.

Proof. It follows from (1.43) and (1.48) that $\mathcal{L}\varphi(x) + \overline{\operatorname{conv}}\, \varphi(y) \geqslant \langle x, y \rangle$ for $x, y \in \mathbb{R}^n$, and hence (1.48) applied $\mathcal{L}\varphi$ and (1.43) yield that $\overline{\operatorname{conv}}\, \varphi(x)(x) \geqslant \mathcal{L}\mathcal{L}\varphi(x)$ for any $x \in \mathbb{R}^n$. Indirectly, we suppose that $\overline{\operatorname{conv}}\, \varphi(x) > \mathcal{L}\mathcal{L}\varphi(x)$ for some $x \in \mathbb{R}^n$, and seek a contradiction.

Now (1.43) yields the existence of a linear function $\ell(z) = \langle a, z \rangle - b$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ such that $\ell(z) \leqslant \overline{\operatorname{conv}}\, \varphi(z)$ for $z \in \Omega$ and $\mathcal{L}\mathcal{L}\varphi(x) < \langle a, x \rangle - b$. We deduce $\mathcal{L}\varphi(a) \leqslant b$ from $\langle a, x \rangle \leqslant b + \varphi(x)$ and (1.48), therefore,

$$\mathcal{L}\mathcal{L}\varphi(x) < \langle a, x \rangle - b \leqslant \langle a, x \rangle - \mathcal{L}\varphi(a).$$

Thus $\mathcal{L}\mathcal{L}\varphi(x) + \mathcal{L}\varphi(a) < \langle a, x \rangle$, contradicting (1.48), and proving Lemma 1.A.7. ∎

While any non-trivial, convex and lower semi-continuous function $\varphi : \mathbb{R} \to (-\infty, \infty]$ is continuous on its domain, this property may fail if φ is defined on $\mathbb{R}^n$ for $n \geqslant 2$. For example, let us fix a $p \in S^{n-1}$. Let $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ be the convex function such that $\operatorname{dom} \varphi = B^n$, $\varphi(p) = 0$, and if $z \in S^{n-1} \backslash \{p\}$, then $\varphi(z) = 1$ and the restriction of φ to the segment $\operatorname{conv}\{p, z\}$ is a linear function.

In this book, the Legendre transform is used, for example, to represent the support function of a closed convex set (cf. (1.53)), or to define the polar of a log-concave function in the functional Santaló inequality and in its reverse (cf. Definition 6.7.1). Originally, the Legendre transform was a tool in differential equations applied to convex differentiable functions $\varphi : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is convex and open. According to (1.47), we may assume that φ is the restriction of a lower semi-continuous function $\tilde{\varphi} : \mathbb{R}^n \to [-\infty, \infty]$ to $\Omega = \operatorname{int} \operatorname{dom} \tilde{\varphi}$.

Remark 1.A.8. Let $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ be a non-trivial convex function such that the interior Ω of $\operatorname{dom} \varphi$ is non-empty.

- If φ is C^1 and strictly convex on Ω, then (1.49) yields that the gradient map $x \mapsto D\varphi(x)$ is a homeomorphism between Ω and the interior of dom $\mathcal{L}\varphi$, and satisfies

$$\begin{aligned} D\mathcal{L}\varphi(a) &= x \qquad \text{for } x \in \Omega \text{ and } a = D\varphi(x), \text{ and} \\ \mathcal{L}\varphi(a) + \varphi(x) &= \langle x, a \rangle. \end{aligned} \tag{1.50}$$

- If φ is twice differentiable on Ω and the Hessian $D^2\varphi(x)$ is positive definite for any $x \in \Omega$, then (1.50) yields that for $x \in \Omega$, $a = D\varphi(x)$ and $A = D^2\varphi(x)$, we have

$$D^2\mathcal{L}\varphi(a) = A^{-1}. \tag{1.51}$$

- We recall that φ is twice differentiable at a.e $x \in \Omega$ in the Aleksandrov sense (cf. Theorem 10.6.2). Now, if $A = D^2\varphi(x)$ exists and positive definite at some $x \in \Omega$, then Gorni [332] proves that $\mathcal{L}\varphi$ is twice differentiable at $a = D\varphi(x)$, which is contained in the interior of dom$\mathcal{L}\varphi$, and (1.51) holds at x and a.

1.B Supplement: Recession cone and support function of closed convex sets

A typical application of Legendre transform is concerning the support function h_K of a (non-empty) closed convex set $K \subset \mathbb{R}^n$; namely, $h_K : \mathbb{R}^n \to (-\infty, \infty]$ is defined by

$$h_K(x) = \sup_{y \in K} \langle x, y \rangle. \tag{1.52}$$

Readily, h_K is convex and one-homogeneous ($h_K(\lambda x) = \lambda h_K(x)$ for $\lambda \geqslant 0$ and $x \in \mathbb{R}^n$). We observe that the "logarithmic characteristic function $\mathbf{1}_K^\infty = -\log \mathbf{1}_K$" defined by

$$\mathbf{1}_K^\infty(x) = \begin{cases} 0 & \text{if } x \in K; \\ \infty & \text{if } x \in \mathbb{R}^n \backslash K \end{cases}$$

is a convex lower semi-continuous function that satisfies

$$\mathcal{L}\mathbf{1}_K^\infty = h_K. \tag{1.53}$$

Proposition 1.B.1.

(i) *If $K \subset \mathbb{R}^n$ is a non-empty closed convex set, then h_K is a non-trivial homogeneous, convex and lower semi-continuous function.*

(ii) *If $h : \mathbb{R}^n \to (-\infty, \infty]$ is a non-trivial homogeneous, convex and lower semi-continuous function, then there exists a unique non-empty closed convex set $K \subset \mathbb{R}^n$ such that $h = h_K$.*

Proof. (i) directly follows from the properties of the Legendre transform and (1.53).

For (ii), we observe that for any $\lambda > 0$ and $x \in \mathbb{R}^n$, the homogeneity of h and the use of the substitution $z = \lambda y$ yield that

$$\lambda \cdot \mathcal{L}h(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, \lambda y\rangle - h(\lambda y)\} = \sup_{z \in \mathbb{R}^n} \{\langle x, z\rangle - h(z)\} = \mathcal{L}h(x).$$

It follows that $\mathcal{L}h(x) \in \{0, \infty\}$ for any $x \in \mathbb{R}^n$. Since $\mathcal{L}h$ is a non-trivial convex and lower semi-continuous function, we conclude that $\mathcal{L}h = \mathbf{1}_K^\infty$ for the closed convex set $K = \operatorname{dom} \mathcal{L}h$, and hence Lemma 1.A.7 and (1.53) yield that $h = \mathcal{L}\mathbf{1}_K^\infty = h_K$. ∎

We note that the support function of a compact convex set is discussed in more detail in Section 1.6.

To understand the structure of an unbounded closed convex set, a useful tool is the so-called *recession cone* (see (1.54) and Schneider [666]). We recall that according to Lemma 1.2.1, a closed convex set $K \subset \mathbb{R}^n$ is unbounded if and only if there exists $u \in \mathbb{R}^n \backslash \{o\}$ and $x_0 \in \mathbb{R}^n$ such that $x_0 + [0, \infty)u \subset K$ (or in other words, K contains a half-line), which in turn equivalent to the property that $x + [0, \infty)u \subset K$ for any $x \in K$. We also recall that $C \subset \mathbb{R}^n$ is a convex cone if and only if $\alpha x + \beta y \in C$ for any $x, y \in C$ and $\alpha, \beta \geqslant 0$ (see Section 1.3 for the properties of closed convex cones that we need in this section).

If $K \subset \mathbb{R}^n$ with $K \neq \mathbb{R}^n$ is an unbounded closed convex set, then the set

$$\operatorname{res} K = \{u \in \mathbb{R}^n : x + [0, \infty)u \subset K \text{ for any } x \in K\} \tag{1.54}$$

is readily a closed convex cone, which is called the *recession cone* of K and satisfies $\operatorname{res} K \neq \mathbb{R}^n$. It follows that $\operatorname{res} K$ is pointed (line free, see Lemma 1.3.5) if and only if K is line free, and

$$\operatorname{res}(K_1 + K_2) = \operatorname{res} K_1 + \operatorname{res} K_2$$

for unbounded closed convex sets $K_1, K_2 \subset \mathbb{R}^n$.

Example 1.B.2 (Recession cone of unbounded polyhedra).
If $K = \{x \in \mathbb{R}^n : \langle u_i, x\rangle \leqslant \alpha_i,\ i = 1, \dots, \ell\}$ is an unbounded polyhedron for $u_i \in \mathbb{R}^n \backslash \{o\}$ and $\alpha_i \in \mathbb{R}$, $i = 1, \dots, \ell$, then $\operatorname{res} K = \{x \in \mathbb{R}^n : \langle u_i, x\rangle \leqslant 0,\ i = 1, \dots, \ell\}$ is a polyhedral cone.

According to Lemma 1.3.5, a closed convex cone $C \subset \mathbb{R}^n$ is pointed (line free) if and only if $\operatorname{int} C^* \neq \varnothing$.

Proposition 1.B.3. *Let $K \subset \mathbb{R}^n$, $K \neq \mathbb{R}^n$, be an unbounded closed convex set with $C = \operatorname{res} K$.*

(i) $\operatorname{cl} \operatorname{dom} h_K = C^*$.

(ii) *If $u \in \operatorname{relint} C^*$, then there exists an $x \in \partial K$ such that $h_K(u) = \langle x, u\rangle$, and hence u is an exterior normal at x.*

(iii) *If C is pointed and $u \in \operatorname{int} C^*$, then*

$$\{z \in K : h_K(u) = \langle z, u\rangle\} = \partial h_K(u) \tag{1.55}$$

is a compact face of K with exterior normal u; moreover, if $\Theta \subset \operatorname{int} C^*$ is compact, then its inverse Gaussian image

$$\bigcup\{\partial h_K(u) : u \in \Theta\} = \{z \in \partial K : \exists u \in \Theta \text{ that is an exterior normal at } z\}$$

is also compact.

Proof. We may assume that $o \in K$.

First we prove (ii). Let $L = \operatorname{lin} C^*$, and hence $\dim L \geqslant 1$ and $C = \widetilde{C} + L^\perp$ where $\widetilde{C} = C \cap L$ is the polar of C^* within L. Now $u + \varrho\,(B^n \cap L) \subset C^*$ for some $\varrho > 0$, and hence if $v \in \widetilde{C} \cap S^{n-1}$, then $u + \varrho v \in C^*$ and $\langle u + \varrho v, v\rangle \leqslant 0$ yield that $\langle u, v\rangle \leqslant -\varrho$. We deduce that if $w \in \widetilde{C}$, then

$$\langle u, w\rangle \leqslant -\varrho\,\|w\|. \tag{1.56}$$

As K is a closed convex set with $C = \operatorname{res} K$ and $o \in K$, if $x_m \in K$ and $\lim_{m\to\infty} \|x_m\| = \infty$, then any accumulation point of the sequence $x_k/\|x_m\| \in S^{n-1}$ lies in C. Therefore, (1.56) implies the existence of some $M > 0$ such that if $x \in K \cap L$ and $\|x\| \geqslant M$, then

$$\langle u, x\rangle \leqslant -\frac{\varrho}{2} \cdot \|x\|. \tag{1.57}$$

Now, let $z_m \in K$ be a sequence such that $h_K(u) = \lim_{m\to\infty}\langle z_m, u\rangle$. As $z_m + L^\perp \subset K$ for each z_m, we may assume that $z_m \in L \cap K$. Since $h_K(u) \geqslant \langle u, o\rangle = 0$, if m is large, then $\langle z_m, u\rangle > -\frac{\varrho M}{2}$; therefore, (1.57) yields that the sequence $\{z_m\}$ is bounded. Thus we may assume that $\lim_{m\to\infty} z_m = x \in K$, proving (ii).

It follows from (ii) that if $u \in \operatorname{relint} C^*$, then $h_K(u) < \infty$. On the other hand, if $u \in \mathbb{R}^n \backslash C^*$, then there exists $v \in C$ such that $\langle u, v\rangle > 0$. As $o \in K$, we have $tv \in K$ for $t > 0$, and hence $h_K(u) \geqslant \sup_{t>0}\langle u, tv\rangle = \infty$, proving (i).

For (iii), if C is pointed and $\Theta \subset \operatorname{int} C^*$ is compact, then there exists $\varrho > 0$ such that $\Theta + \varrho\, B^n \subset C^*$. The argument for (ii) yields that there exists $M > 0$ such that if $u \in \Theta$ and $x \in K$ with $\|x\| \geqslant M$, then $\langle u, x\rangle \leqslant -\frac{\varrho}{2} \cdot \|x\|$. As $o \in K$, if $u \in \Theta$ is an exterior normal at an $x \in \partial K$, then we deduce first that $\|x\| \leqslant M$, and in turn the compactness of the inverse Gaussian image $\{z \in \partial K : \exists u \in \Theta$ exterior normal at $z\}$.

Finally, (1.55) can be proved similarly to Lemma 1.6.6. ∎

Combining Proposition 1.B.1 and Proposition 1.B.3, we deduce the following.

Corollary 1.B.4. *If $C \subset \mathbb{R}^n$ is a closed convex pointed cone and $h : C^* \to (-\infty, \infty]$ is a convex, lower semi-continuous and one-homogeneous function that is finite on* $\operatorname{int} C^*$*, then there exists a unique unbounded closed convex set $K \subset \mathbb{R}^n$ such that $C = \operatorname{res} K$ and $h = h_K$ on C^*.*

1.C Supplement: Iterated Steiner symmetrizations with respect to n fixed hyperplanes

The main goal of this section is to prove Theorem 1.C.3 about iterated Steiner symmetrizations. First, we consider Lemma 1.C.1 and Lemma 1.C.2, which are needed in the proof of Theorem 1.C.3.

Lemma 1.C.1. *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in K$, and let $u \in S^{n-1}$.*

(i) If $\varrho > 0$, then $|\varrho B^n \cap \Theta_{u^\perp} K| \geqslant |\varrho B^n \cap K|$;

(ii) If K is not symmetric through $u^\perp$, then there exists $\varrho > 0$ such that $|\varrho B^n \cap \Theta_{u^\perp} K| > |\varrho B^n \cap K|$.

Proof. (i) follows from the fact that $(x + \mathbb{R}u) \cap rB^n$ is symmetric through $u^\perp$ for any $x \in K|u^\perp$.

If K is not symmetric through $u^\perp$, then possibly after changing u into $-u$, there exists $x \in (\mathrm{int}K)|u^\perp$ such that $x + tu, x + su \in \partial K$ where $t > |s|$ and $s < 0$. Let ϱ satisfy $\|x + su\| < \varrho < \|x + tu\|$. Now, there exists some $\eta > 0$ such that if $\|z - x\| < \eta$ holds for $z \in u^\perp$, then $\mathcal{H}^1\big((z + \mathbb{R}u) \cap \varrho B^n\big) < \mathcal{H}^1\big((z + \mathbb{R}u) \cap K\big)$. In turn, we conclude (ii). ∎

Lemma 1.C.2. *If $v_1, \ldots, v_n \in S^{n-1}$, $n \geqslant 2$, are independent such that $\angle(v_i, v_j) = \alpha_{ij}\pi$ for $i \neq j$ and irrational $\alpha_{ij} \in (0, 1)$, and a non-empty o-symmetric $X \subset S^{n-1}$ is compact and symmetric through the hyperplane $v_i^\perp$ for $i = 1, \ldots, n$, then $X = S^{n-1}$.*

Proof. The proof is by induction on $n \geqslant 2$. If $n = 2$, then the composition of the reflections through $v_1^\perp$ and $v_2^\perp$ is a rotation of angle $\alpha_{12}2\pi$; therefore, it is sufficient to prove that numbers of the form $\{m\alpha_{12}\}$ for $m \in \mathbb{N}$ are dense in $(0, 1)$ where $\{t\} = t - \lfloor t \rfloor$ is the fractional part of a $t \in \mathbb{R}$. In turn, this follows from Diophantine approximation; namely, that there exists arbitrary large integer q such that $|\alpha_{12} - \frac{p}{q}| \leqslant \frac{1}{q^2}$ for some integer p.

Now let $n \geqslant 3$, and assume that Lemma 1.C.2 is known in $\mathbb{R}^{n-1}$. For $i = 1, \ldots, n$, let $w_i \in S^{n-1}$ be such that $\langle w_i, v_j \rangle = 0$ for $j \neq i$, and hence $w_1, \ldots, w_n$ re independent.

We suppose that there exists $z \in S^{n-1} \backslash X$, and seek a contradiction. Let $x_0 \in X$ such that $\mu = \max\{\langle x, z \rangle : x \in X\} = \langle x_0, z \rangle$, and hence $\mu \in [0, 1)$ as X is o-symmetric. Choose a $w_i \notin \mathrm{lin}\{z, x_0\}$, which exists by $n \geqslant 3$. The set $S = \{y \in S^{n-1} : \langle y, w_i \rangle = \langle x_0, w_i \rangle\}$ is an $(n-2)$-dimensional sphere that is symmetric through each $v_j^\perp$ with $j \neq i$. As $x_0 \in X \cap S$, the induction hypothesis yields $S \subset X$, and hence $S \subset H^+$ for the half-space $H^+ = \{y \in \mathbb{R}^n : \langle y, z \rangle \leqslant \mu\}$.

Now, the tangent affine $(n-2)$-space at x_0 to S in aff S is $x_0 + L$ for the linear $(n-2)$-space $L = x_0^\perp \cap w_i^\perp$ by the definition of w_i and S. Thus $S \subset H^+$ and $x_0 \in \partial H^+$ imply that $x_0 + L \subset H^+$, which in turn yields $L \subset z^\perp$. However, this is a contradiction as x_0, w_i, z are independent, proving Lemma 1.C.2. ∎

Remark (Existence of the basis in Lemma 1.C.2). The family of bases (as a vector space) $v_1, \dots, v_n \in S^{n-1}$ of $\mathbb{R}^n$ up to isometry can be parametrized by the family C of symmetric positive definite so-called Gram matrices $A = [a_{ij}]$ with $a_{ii} = 1$ on the diagonal where the correspondence is provided by $a_{ij} = \langle v_j, v_j \rangle$ where we consider $C \subset \mathbb{R}^N$ for $N = \frac{n(n-1)}{2}$ taking a_{ij}, $i < j$ as coordinates. Since any symmetric matrix $A = [a_{ij}]$ with $a_{ii} = 1$ and $|a_{ij}| < \frac{1}{n-1}$ for $i < j$ lies in C, it follows that C is a bounded open convex subset of $\mathbb{R}^N$. If $A = [a_{ij}] \in C$ does not satisfy the condition in Theorem 1.C.2, then there exists $i < j$ such that a_{ij} is an algebraic number; therefore, $\mathcal{H}^N$ a.e. matrices $A = [a_{ij}] \in C$ represents a basis of $\mathbb{R}^n$ satisfying the conditions in Lemma 1.C.2.

The following elegant way to handle iterated Steiner symmetrizations is due to Klain [462].

Theorem 1.C.3 (Iterated Steiner symmetrizations). *Let $v_1, \dots, v_n \in S^{n-1}$ be independent such that $\angle(v_i, v_j) = \alpha_{ij}\pi$ for $i \neq j$ and irrational $\alpha_{ij} \in (0, 1)$, and let $u_{kn+i} = v_i$ for $k \in \mathbb{N}$ and $i \in \{1, \dots, n\}$.*

If $K \subset \mathbb{R}^n$ is an o-symmetric convex body with $|K| = |rB^n|$ for $r > 0$, then the iterated Steiner symmetrial K_m tends to rB^n where $K_0 = K$ and $K_{m+1} = \Theta_{u_m^\perp} K_m$.

Proof. According to the Blaschke Selection Theorem 1.7.4 and $K_m \subset R(K)B^n$, it is equivalent to prove that if a subsequence $\{K_{m_j}\}$ of $\{K_m\}$ tends to an o-symmetric convex body C, then $C = rB^n$. We may assume the existence of a $\xi \in \{1, \dots, n\}$ such that n divides $m_j - \xi$ for any m_j. We shift the indices of $v_1, \dots, v_n$ to obtain $\{w_1, \dots, w_n\} = \{v_1, \dots, v_n\}$ where $w_1 = v_\xi$, and in general, $w_{i+\xi-1} = v_j$ if n divides $i + \xi - 1 - j$.

We claim that C is symmetric through each $v_i^\perp$, $i = 1, \dots, n$; or equivalently, C is symmetric through each $w_i^\perp$. We suppose that the claim does not hold, and seek a contradiction. Let $\ell \in \{1, \dots, n\}$ be the smallest index such that C is not symmetric through $w_\ell^\perp$. According to Lemma 1.C.1, there exist some $\varrho, \eta > 0$ such that $\left|\varrho B^n \cap \Theta_{w_\ell^\perp} C\right| > |\varrho B^n \cap C| + 3\eta$. Choose $\lambda > 1$ such that $(\lambda^n - 1)|C| < \eta$, and hence $(1 - \lambda^{-n})|C| < \eta$, and choose a $J > 1$ such that $\lambda^{-1} C \subset K_{m_j} \subset \lambda C$ if $j \geqslant J$. It follows from Proposition 1.10.3 (ii) and from $\Theta_{w_i^\perp} C = C$ for $i < \ell$ that

$$\begin{aligned}
\left|\varrho B^n \cap \Theta_{w_\ell^\perp} \dots \Theta_{w_1^\perp} K_{m_J}\right| &\geqslant \left|\varrho B^n \cap \Theta_{w_\ell^\perp} \dots \Theta_{w_1^\perp} \lambda^{-1} C\right| \\
&= \left|\varrho B^n \cap \lambda^{-1} \Theta_{w_\ell^\perp} C\right| \\
&\geqslant \left|\varrho\varrho B^n \cap \Theta_{w_\ell^\perp} C\right| - \left(\left|\Theta_{w_\ell^\perp} C\right| - \left|\lambda^{-1} \Theta_{w_\ell^\perp} C\right|\right) \\
&\geqslant \left|rB^n \cap \Theta_{w_\ell^\perp} C\right| - \eta \geqslant |\varrho B^n \cap C| + 2\eta.
\end{aligned}$$

As $K_{m_{J+1}}$ is obtained by further Steiner symmetrizations, Lemma 1.C.1 implies that

$$\left|\varrho B^n \cap K_{m_{J+1}}\right| \geqslant |\varrho B^n \cap C| + 2\eta. \tag{1.58}$$

On the other hand, $K_{m_{J+1}} \subset \lambda C$ yields that

$$\left|\varrho B^n \cap K_{m_{J+1}}\right| \leqslant |\varrho B^n \cap \lambda C| \leqslant |\varrho B^n \cap C| + |\lambda C| - |C| \leqslant |\varrho B^n \cap C| + \eta,$$

contradicting (1.58), and in turn verifying the claim that C is symmetric through each $v_i^{\perp}$.

Finally, for $R = \max\{\|x\| : x \in C\}$, applying Lemma 1.C.2 to the set $Z = C \cap RS^{n-1}$ yields $RS^{n-1} \subset C$, and hence $C = RB^n = rB^n$. ∎

Remark. Klain [462] actually proves a much stronger statement, which yields, for example, the following: If $v_1, \ldots, v_\ell \in S^{n-1}$, and $K \subset \mathbb{R}^n$ is a convex compact set, then the iterated Steiner symmetrial K_m tends to a compact convex set C that is symmetric through $v_i^{\perp}$ for $i = 1, \ldots, \ell$ where $K_0 = K$, $K_{m+1} = \Theta_{u_m^{\perp}} K_m$ and $u_{k\ell+i} = v_i$ for $k \in \mathbb{N}$ and $i \in \{1, \ldots, \ell\}$.

1.D Supplement: The Hausdorff Measure for subsets of $\mathbb{R}^n$

In this section, we introduce the Hausdorff measure for subsets of $\mathbb{R}^n$ and establish its basic properties (see Section 10.4, Falconer [254] and Federer [260] for more advanced properties). Given $s \geqslant 0$, let $\omega_s := \frac{\pi^{s/2}}{\Gamma(1+s/2)}$ where $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t}\, dt$ denotes the Γ function for $\lambda > 0$, and hence $\omega_n = |B^n|$.

Definition 1.D.1 (Hausdorff measure on $\mathbb{R}^n$). Given $s \geqslant 0$ and $\delta > 0$, for any $E \subset \mathbb{R}^n$, we define

$$\mathcal{H}^s_\delta(E) = \inf\left\{\omega_s \sum \left(\frac{\operatorname{diam} F_i}{2}\right)^s : E \subset \bigcup_{i=1}^{\infty} F_i \ \&\ \forall \operatorname{diam} F_i < \delta\right\}.$$

The *s-dimensional Hausdorff measure* $\mathcal{H}^s$ is defined by $\mathcal{H}^s(E) := \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E)$.

Remark 1.D.2. The following properties hold:

- $\mathcal{H}^s_\delta$ and $\mathcal{H}^s$ are outer measures.
- If E is a segment, then $\mathcal{H}^1(E) = \mathcal{H}^1_\delta(E)$ is the length of E for every $\delta > 0$.
- If E is a curve (image of a Lipschitz function $\varphi : [0, 1] \to \mathbb{R}^n$), then $\mathcal{H}^1_\delta(E) = \operatorname{diam}(E)$ whenever $\delta \geqslant \operatorname{diam}(E)$. On the other hand, $\mathcal{H}^1_\delta(E) \to \mathcal{H}^1(E)$ as $\delta \to 0^+$ where $\mathcal{H}^1(E)$ is the "length of the curve E" if φ is injective.
- If E is countable, then $\mathcal{H}^s(E) = 0$ for every $s > 0$.

- If E is finite, then $\mathcal{H}^0(E) = \#E$.
- $\mathcal{H}^s(E) = \mathcal{H}^s(x+E) = \mathcal{H}^s(\Phi E)$ for every $x \in \mathbb{R}^n$ and every $\Phi \in O(n)$.
- $\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$ for $\lambda > 0$.

Proposition 1.D.3. *$\mathcal{H}^s$ is a Borel measure on $\mathbb{R}^n$ for $s \geqslant 0$.*

Remark 1.D.4. $\mathcal{H}^0$ is the counting measure.

The normalization constant appearing in the definition of the Hausdorff measure is explained by the fact that $\mathcal{H}^n$ agrees with the Lebesgue measure $\mathcal{L}^n$ for measurable subsets of $\mathbb{R}^n$:

Theorem 1.D.5. *$\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$ for $n \geqslant 1$.*

Proof. In order to prove

$$\mathcal{L}^n \leqslant \mathcal{H}^n, \tag{1.59}$$

it is sufficient to verify that if $E \subset \mathbb{R}^n$ is Borel measurable and $\varepsilon, \delta > 0$, then $\mathcal{L}^n(E) \leqslant \mathcal{H}^n_\delta(E) + \varepsilon$. We choose a countable covering $\mathcal{F}$ of E such that $\operatorname{diam} F < \delta$ for $F \in \mathcal{F}$ and $\sum_{F\in\mathcal{F}} \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n < \mathcal{H}^n_\delta(E) + \varepsilon$. Since $\mathcal{L}^n(F) \leqslant \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n$ for any $F \in \mathcal{F}$ according to the Isodiametric Inequality Theorem 1.10.6, we conclude (1.59) by the σ-subadditivity of the Lebesgue measure.

In the reverse direction, first we verify that

$$\mathcal{H}^n \leqslant \alpha_n \mathcal{L}^n \quad \text{for } \alpha_n = \omega_n n^{n/2} 2^{1-n} \tag{1.60}$$

by proving that if $\delta > 0$ and $E \subset \mathbb{R}^n$ is compact, then $\mathcal{H}^n_\delta(E) \leqslant \alpha_n \mathcal{L}^n(E)$. Let us consider a tiling of $\mathbb{R}^n$ by translates of the cube $[0,a]^n$ for an $a > 0$ such that $\operatorname{diam}[0,a]^n = a\sqrt{n} < \delta$ and $\mathcal{L}^n(E + a\sqrt{n}\, B^n) \leqslant 2\mathcal{L}^n(E)$, and let $\mathcal{F}$ be the family of tiles intersecting E. It follows that $\#\mathcal{F} \leqslant 2\mathcal{L}^n(E)/a^n$, and hence $\mathcal{H}^n_\delta(E) \leqslant \#\mathcal{F} \cdot \omega_n \left(\frac{a\sqrt{n}}{2}\right)^n \leqslant \alpha_n \mathcal{L}^n(E)$.

Finally, we prove

$$\mathcal{H}^n \leqslant \mathcal{L}^n, \tag{1.61}$$

which is equivalent to verify that $\mathcal{H}^n_\delta(E) \leqslant \mathcal{L}^n(E) + \varepsilon$ holds for any $\varepsilon, \delta > 0$ and bounded open $E \subset \mathbb{R}^n$. The key statement is that if $\mathcal{B}$ is the family of balls contained in E and of diameter less than δ, then there exists a subfamily $\mathcal{F}' = \{B_1, B_2, \ldots\} \subset \mathcal{B}$ of pairwise disjoint balls such that

$$\mathcal{L}^n\left(E \setminus \cup\, \mathcal{F}'\right) = 0, \quad \text{and hence } \mathcal{L}^n(E) = \sum_{F\in F'} \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n. \tag{1.62}$$

We construct B_m by induction on m where $B_1 \in \mathcal{B}$ is any ball. For $m \geqslant 2$, let d_m be the supremum of the diameters of the balls in E disjoint from $B_1, \ldots, B_{m-1}$, and let

$B_m \in \mathcal{B}$ be any ball of diameter at least $\min\{\frac{1}{2}\delta, \frac{1}{2}d_m\}$ and disjoint from $B_1, \dots, B_{m-1}$. To show that $\mathcal{F}' = \{B_m\}$ satisfies (1.62), let $B_m = z_m + r_m B^n$ for $r_m > 0$ where $\lim_{m\to\infty} \sum_{i=m}^{\infty} \mathcal{L}^n(B_i) = 0$ and $\lim_{m\to\infty} d_m = 0$ as E is bounded. Choose integer $m_0 \geqslant 2$ such that $d_m < \delta$ if $m \geqslant m_0$.

For $m \geqslant m_0$, if $x \in E \setminus \cup_{i=1}^{m-1} B_i$, then there exists some $k > m$ such that $B = x + d_k B^n \subset E \setminus \cup_{i=1}^{m-1} B_i$. If $j \in \{m, \dots, k-1\}$ is the smallest index such that B intersects $\cup_{i=1}^{j} B_i$, then $\operatorname{diam} B \leqslant d_j$ and $r_j \geqslant \frac{d_j}{4}$, and hence $B \cap B_j \neq \varnothing$ yields that $x \in z_j + 5r_j B^n$. We conclude that $\mathcal{L}^n(E \setminus \cup_{i=1}^{m-1} B_i) \leqslant 5^n \sum_{i=m}^{\infty} \mathcal{L}^n(B_i)$, which in turn implies (1.62).

It follows from (1.60) that also $\mathcal{H}^n(E \setminus \cup \mathcal{F}') = 0$, and hence there exists a countable covering $\widetilde{\mathcal{F}}$ of $E \setminus \cup \mathcal{F}'$ by subsets of $\mathbb{R}^n$ of diameter less than δ with the property that $\sum_{F\in\widetilde{\mathcal{F}}} \omega_n \left((\operatorname{diam} F)/2\right)^n < \varepsilon$. Therefore, the covering $\mathcal{F} = \widetilde{\mathcal{F}} \cup \mathcal{F}'$ of E satisfies

$$\mathcal{H}^n_\delta(E) \leqslant \sum_{F\in\mathcal{F}} \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n = \mathcal{L}^n(E) + \sum_{F\in\widetilde{\mathcal{F}}} \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n < \mathcal{L}^n(E) + \varepsilon,$$

proving (1.61). In turn, we conclude $\mathcal{H}^n = \mathcal{L}^n$ by (1.59). ∎

We note that (1.62) is a special case of Vitali's Covering Theorem (see Falconer [254]).

We observe that if $\delta > 0$, $t > s \geqslant 0$ and $\operatorname{diam} F < \delta$ for $F \subset \mathbb{R}^n$, then $(\operatorname{diam} F)^t \leqslant \delta^{t-s}(\operatorname{diam} F)^s$. We deduce that if $E \subset \mathbb{R}^n$ and $s > 0$, then

$$\begin{aligned} &\text{if } \mathcal{H}^s(E) < \infty, \quad \text{then } \mathcal{H}^t(E) = 0 \text{ for } \quad t > s; \\ &\text{if } \mathcal{H}^s(E) > 0, \quad \text{then } \mathcal{H}^t(E) = \infty \text{ for } \quad t \in [0, s). \end{aligned} \tag{1.63}$$

The following observation follows from Theorem 1.D.5 and (1.63).

Proposition 1.D.6. *If $s > n$, then $\mathcal{H}^s(E) = 0$ for any $E \subset \mathbb{R}^n$.*

It is ensured by (1.63) that the following definition makes sense:

Definition 1.D.7. For $E \subset \mathbb{R}^n$, the Hausdorff dimension $\dim_H(E)$ of E is defined by

$$\dim_H(E) := \inf\{s \in [0, \infty) \mid \mathcal{H}^s(E) = 0\}.$$

Remark. If $E \subset \mathbb{R}^n$, then

$$\mathcal{H}^s(E) = \begin{cases} 0 & \text{if } s > \dim_H(E), \\ \infty & \text{if } s < \dim_H(E). \end{cases}$$

Corollary 1.D.8. *For every $E \subset \mathbb{R}^n$, we have $\dim_H(E) \in [0, n]$.*

Proposition 1.D.9. *Given $X \subset \mathbb{R}^n$, let $f : X \to \mathbb{R}^m$ be Lipschitz, with Lipschitz constant $L \geqslant 0$; namely, $\|f(x) - f(y)\| \leqslant L\|x - y\|$ for $x, y \in X$. Then $\mathcal{H}^s(f(E)) \leqslant L^s \mathcal{H}^s(E)$ for $E \subset X$ and $s \geqslant 0$.*

Proof. Let $\mathcal{F}_\delta$ be an almost optimal covering for $\mathcal{H}^s_\delta(E)$ by subsets of $\mathbb{R}^n$ of diameter less than δ. Then $\{f(F)\}_{F\in\mathcal{F}_\delta}$ is a covering of $f(E)$ and $\operatorname{diam}(f(F)) \leqslant L\cdot\operatorname{diam}(F)$ for every $F\in\mathcal{F}_\delta$, and hence

$$\mathcal{H}^s_{L\delta}(f(E)) \leqslant \omega_s \sum_{F\in\mathcal{F}_\delta} \left(\frac{\operatorname{diam}(f(E))}{2}\right)^s \leqslant L^s\omega_s \sum_{F\in\mathcal{F}_\delta} \left(\frac{\operatorname{diam}(F)}{2}\right)^s.$$

Letting $\delta > 0$ tend to zero proves the claim. ∎

Let $X\subset\mathbb{R}^n$ be convex and closed. We recall that according to Lemma 1.2.13, the closest point map $\Pi_X : \mathbb{R}^n \to X$ is a contraction; namely, $\|\Pi_X(y) - \Pi_X(z)\| \leqslant \|y - z\|$ for $y, z \in \mathbb{R}^n$.

Corollary 1.D.10. *If $X\subset\mathbb{R}^n$ is convex and closed and $E\subset\mathbb{R}^n$, then the closest point map Π_X satisfies $\mathcal{H}^s(\Pi_X(E)) \leqslant \mathcal{H}^s(E)$ for $s \geqslant 0$.*

Chapter 2

Surface area, surface area measure and cone volume measure for convex bodies in $\mathbb{R}^n$

The notion of surface area of convex bodies needs some basic properties of Lipschitz surfaces from geometric measure theory. After discussing these properties, we verify the Isoperimetric and the Anisotropic Isoperimetric inequalities for convex bodies. We also introduce the surface area measure and cone volume measure on S^{n-1} associated to convex bodies, which encode many important geometric properties. We close the chapter with some basic estimates about polytopal approximation where local surface area estimates play an important role.

2.1 Some integral formulas involving exterior unit normals

In order to establish some of the fundamental properties of the surface area of convex bodies, we use the following basic notions on graphs of convex functions. The formula (2.1) in (ii) and (iv) are well-known properties of convex functions (see, for example, Federer [260] or Rockafellar [633]), (i) is a direct consequence of the definition of gradient, and (iii) follows from (ii).

Remark 2.1.1. Let $u \in S^{n-1}$, let $\varphi : \Omega \to \mathbb{R}$ be convex where $\Omega \subset u^\perp$ is a relatively open convex subset of $u^\perp$, and let $X = \{z + \varphi(z)u : z \in \Omega\} \subset \mathbb{R}^n$ be the graph of φ. The points of the form $x = z + \varphi(z)u \in X$ such that $D\varphi(z)$ exists for $z \in \Omega$ are denoted by X', and let $\nu(x) \in S^{n-1}$ be the unique normal at $x \in X'$ with $\langle \nu(x), u\rangle < 0$, and hence $\mathcal{H}^{n-1}(X \backslash X') = 0$ by Rademacher's theorem (cf. Lemma 10.4.7) or by Corollary 1.5.5.

(i) If $D\varphi(z)$ exists for $z \in \Omega$ and $x = z + \varphi(z)u \in X'$, then

$$\langle \nu(x), u\rangle = \frac{-1}{\sqrt{1+\|D\varphi(z)\|^2}} \text{ as } \nu(x) = \frac{D\varphi(z)}{\sqrt{1+\|D\varphi(z)\|^2}} + \frac{-1}{\sqrt{1+\|D\varphi(z)\|^2}} \cdot u.$$

(ii) If $f : X \to \mathbb{R}$ is a bounded measurable function on X, then

$$\int_X f \, d\mathcal{H}^{n-1} = \int_\Omega f(z + \varphi(z)u)\sqrt{1 + \|D\varphi(z)\|^2}\, d\mathcal{H}^{n-1}(z). \tag{2.1}$$

(iii) If $g : \Omega \to \mathbb{R}$ is a bounded measurable function on Ω, then

$$\int_\Omega g \, d\mathcal{H}^{n-1} = \int_{X'} g\,(\Pi_{u^\perp} x)\,\langle -\nu(x), u\rangle \, d\mathcal{H}^{n-1}(x). \tag{2.2}$$

(iv) If $\psi : \Omega \to \mathbb{R}$ convex and $D\varphi(z) = D\psi(z)$ for $\mathcal{H}^{n-1}$ a.e. $z \in \Omega$, then there exists $\gamma \in \mathbb{R}$ with $\psi(z) - \varphi(z) = \gamma$ for $z \in \Omega$.

We recall that, according to Theorem 1.5.2, if $K \subset \mathbb{R}^n$ is a convex body, then the supporting hyperplane is unique at $\mathcal{H}^{n-1}$ a.e. $x \in \partial K$. Such an $x \in \partial K$ is called *regular*, and $\nu_K(x)$ denotes the unique exterior unit normal at x. The set of regular points $x \in \partial K$ is denoted by $\partial' K$, thus $\mathcal{H}^{n-1}(\partial K \backslash \partial' K) = 0$. We note that the function $x \mapsto \nu_K(x)$ is continuous on $\partial' K$ (cf. Lemma 1.2.10), and hence it is measurable.

Lemma 2.1.2. *For a convex body $K \subset \mathbb{R}^n$, we have*

(i) $\displaystyle\int_{\partial' K} \langle \nu_K, u\rangle \, d\mathcal{H}^{n-1} = 0$ *for* $u \in \mathbb{R}^n$;

(ii) $\displaystyle\int_{\partial' K} |\langle \nu_K, u\rangle| \, d\mathcal{H}^{n-1} = 2\mathcal{H}^{n-1}(K|u^\perp)$ *for* $u \in S^{n-1}$.

Proof. We may assume that $u \in S^{n-1}$ also in case of (i), and let $\Omega = (\mathrm{int} K)\,|u^\perp$. In addition, let ψ_+ and ψ_- be the concave and convex functions on Ω such that $\mathrm{int}\, K = \{z + tu : z \in \Omega \ \&\ \psi_-(z) < t < \psi_+(z)\}$, and let us consider

$$\begin{array}{rcll} X_- & = & \{z + \psi_-(z)\, u : z \in \Omega\}, & \text{thus } \langle \nu_K(x), u\rangle < 0 \text{ for } x \in X_- \cap \partial' K; \\ X_+ & = & \{z + \psi_+(z)\, u : z \in \Omega\}, & \text{thus } \langle \nu_K(x), u\rangle > 0 \text{ for } x \in X_+ \cap \partial' K, \\ X_0 & = & \partial K \backslash (X_+ \cup X_-), & \text{thus } \langle \nu_K(x), u\rangle = 0 \text{ for } x \in X_0 \cap \partial' K. \end{array}$$

It follows from (2.2) that

$$\int_{X_- \cap \partial K'} \langle \nu_K, u\rangle \, d\mathcal{H}^{n-1} = -\int_{X_- \cap \partial K'} |\langle \nu_K, u\rangle| \, d\mathcal{H}^{n-1} = -\mathcal{H}^{n-1}(K|u^\perp);$$

$$\int_{X_+ \cap \partial K'} \langle \nu_K, u\rangle \, d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(K|u^\perp);$$

$$\int_{X_0 \cap \partial K'} \langle \nu_K, u\rangle \, d\mathcal{H}^{n-1} = 0,$$

proving both (i) and (ii). ∎

Corollary 2.1.3. *If $K \subset \mathbb{R}^n$ is a convex body, then* $\displaystyle\int_{\partial K} \nu_K \, d\mathcal{H}^{n-1} = o$.

It follows from Rademacher's theorem (cf. Lemma 10.4.7) that if $T = (T_1, \dots, T_n) : \Omega \to \mathbb{R}^n$ is locally Lipschitz for an open $\Omega \subset \mathbb{R}^n$, then the $n \times n$ derivative $DT(x)$ exists at $\mathcal{H}^{n-1}$ a.e. $x \in \Omega$ (cf. Lemma 4.1.2). At such an $x \in \Omega$, the divergence of T is

$$\mathrm{div}\, T(x) = \mathrm{tr}\, DT(x) = \sum_{i=1}^n \partial_i T_i(x),$$

which does not depend on the choice of the orthonormal basis of $\mathbb{R}^n$. The following fundamental divergence theorem is, for example, a special case of Federer [260], Theorem 4.5.6, but we provide the elementary argument for Theorem 2.A.1 in Section 2.A based on Remark 2.1.1 and the idea of the proof of Lemma 2.1.2.

Theorem 2.1.4 (Divergence Theorem). *If $K \subset \mathbb{R}^n$ is a convex body and $T : K \to \mathbb{R}^n$ is locally Lipschitz, then*

$$\int_K \operatorname{div} T = \int_{\partial K} \langle T, \nu_X \rangle \, d\mathcal{H}^{n-1}.$$

In particular, the Divergence Theorem 2.1.4 directly yields Lemma 2.1.2 (i). In addition, using the vector field $T(x) = x$ in Theorem 2.1.4 with $\operatorname{div} T(x) = n$, we obtain the following representation of the volume as a surface integral.

Theorem 2.1.5. *If $K \subset \mathbb{R}^n$ is a convex body, then*

$$|K| = \frac{1}{n} \int_{\partial K} \langle \nu_K(x), x \rangle \, d\mathcal{H}^{n-1}(x) = \frac{1}{n} \int_{\partial K} h_K(\nu_K(x)) \, d\mathcal{H}^{n-1}(x). \tag{2.3}$$

Example 2.1.6 (Volume of an n-polytope P in $\mathbb{R}^n$). If $F_1, \dots, F_k$ are the facets of P with exterior unit vectors $u_1, \dots, u_k$ (cf. Section 1.4), then considering the bounded "cones" $\operatorname{conv}\{o, F_i\}$ and (1.26) yield that

$$|P| = \frac{1}{n} \sum_{i=1}^{k} h_P(u_i) \, |F_i|. \tag{2.4}$$

More precisely, (2.4) readily holds if $o \in \operatorname{int} P$, which can be assumed by $h_{P-z}(u_i) = h_P(u_i) - \langle z, u_i \rangle$ for $z \in \operatorname{int} P$, Lemma 2.1.2 (i) and as $\nu_P(x) = u_i$ and $\langle \nu_P(x), x \rangle = h_P(u_i)$ for $x \in \operatorname{relint} F_i$. We observe that (2.4) is consistent with Theorem 2.1.5.

2.2 Parametrizing the boundary via the radial function

A central notion of this book is the spherical gradient (differential) of a Lipschitz function $\varphi : S^{n-1} \to \mathbb{R}^d$, which is the notion of differential with respect to a moving orthogonal frame used in differential geometry (see, for example, Definition 8.1.6, and the whole Chapter 8 and Chapter 9).

Definition 2.2.1 (Spherical gradient). For a Lipschitz function $\varphi : S^{n-1} \to \mathbb{R}^d$, $d \geqslant 1$, we consider its locally Lipschitz extension $\bar{\varphi}$ on $\mathbb{R}^n \backslash \{o\}$ defined by $\bar{\varphi}(tu) = \varphi(u)$ for $u \in S^{n-1}$ and $t > 0$. Since $\bar{\varphi}$ is homogeneous and $\mathcal{H}^n$ a.e. differentiable according to Rademacher's theorem (cf. Lemma 10.4.7), we can define the *spherical gradient*

$$\nabla \varphi(u) = D\bar{\varphi}(u)|_{u^\perp}$$

at $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$.

Example 2.2.2. Let us consider the function $\varphi(u) = u$ for any $u \in S^{n-1}$, and hence $\bar{\varphi}(x) = x/\|x\|$ for $x \in \mathbb{R}^n \backslash \{o\}$. For an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$, our goal is to

determine $\nabla\varphi(e_n)$. Calculating $D\bar{\varphi}(e_n)$ show that $\nabla\varphi(e_n)$ is the $n\times(n-1)$ matrix whose full zero last row is below the $(n-1)\times(n-1)$ identity matrix; or in other words,

$$\nabla\varphi(e_n)=\sum_{i=1}^{n-1} e_i\cdot\tilde{e}_i^t \tag{2.5}$$

where $\tilde{e}_i$ denotes e_i as an $(n-1)$-dimensional vector in $e_n^\perp$, $i=1,\dots,n-1$.

We recall that $r_K(u)=\varrho_K(u)\,u\in\partial K\subset\mathbb{R}^n$ for $u\in S^{n-1}$ for the radial function $\varrho_K>0$ of a convex body $K\subset\mathbb{R}^n$ with $o\in\operatorname{int}K$ (cf. Definition 1.9.5). Using the associated norm-type convex one-homogeneous function $\|x\|_K=\min\{t\geqslant 0:\ x\in tK\}$, we have $\varrho_K(u)=1/\|u\|_K$ for $u\in S^{n-1}$; therefore, ϱ_K is Lipschitz, and hence differentiable at $\mathcal{H}^{n-1}$ a.e. $u\in S^{n-1}$, and $\bar{\varrho}_K(x)=\|x\|/\|x\|_K$. Since ∂K consists of the points $x\in\mathbb{R}^n$ with $\|x\|_K=1$, and $x\in\partial K$ satisfies $x=r_K(u)$ for $u=x/\|x\|$, we deduce the following.

Lemma 2.2.3. *For a convex body $K\subset\mathbb{R}^n$ with $o\in\operatorname{int}K$, $x\in\partial K$ is a regular boundary point if and only if ϱ_K is differentiable at $x/\|x\|$. In addition, ∂K is a C^k manifold for an integer $k\geqslant 1$ - or in other words, ∂K is C^k - if and only if ϱ_K is C^k on S^{n-1}.*

To calculate the derivative of r_K at a $u\in S^{n-1}$ where ϱ_K is differentiable, let $e_1,\dots,e_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $e_n=u$ and $\nabla\varrho_K(u)=\|\nabla\varrho_K(u)\|\,\tilde{e}_1$ where $\tilde{e}_i$ denotes e_i as a vector in $u^\perp$, $i=1,\dots,n-1$. It follows from (2.5) that

$$\nabla r_K(u)=u\cdot(\nabla\varrho_K(u))^t+\varrho_K(u)\,\nabla u=\|\nabla\varrho_K(u)\|\,e_n\cdot\tilde{e}_1^t+\varrho_K(u)\sum_{i=1}^{n-1}e_i\cdot\tilde{e}_i^t. \tag{2.6}$$

On the one hand, we deduce that $-\|\nabla\varrho_K(u)\|e_1+\varrho_K(u)\,u$ is an exterior normal at $r_K(u)\in\partial K$, and hence we have the following.

Lemma 2.2.4. *For a convex body $K\subset\mathbb{R}^n$ with $o\in\operatorname{int}K$, if ϱ_K is differentiable at a $u\in S^{n-1}$, then $x=r_K(u)$ is a regular boundary point of ∂K, and*

$$\langle\nu_K(x),u\rangle=\frac{\varrho_K(u)}{\sqrt{\|\nabla\varrho_K(u)\|^2+\varrho_K(u)^2}}.$$

On the other hand, since $e_j^t\cdot e_i=0$ for $j\neq i$ and $e_j^t\cdot e_j=1$ in (2.6), it follows that $[(\nabla r_K(u))^t\nabla r_K(u)]$ has eigenvalue $\|\nabla\varrho_K(u)\|^2+\varrho_K(u)^2$ at e_1, and the other $n-2$ eigenvalues are $\varrho_K(u)^2$ if $n\geqslant 3$; therefore, the Jacobian of r_K at a $u\in S^{n-1}$ is

$$\sqrt{\det\left[(\nabla r_K(u))^t\nabla r_K(u)\right]}=\varrho_K(u)^{n-2}\sqrt{\|\nabla\varrho_K(u)\|^2+\varrho_K(u)^2}.$$

In turn, we conclude the main result of this section.

Proposition 2.2.5. *For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ and a $\mathcal{H}^{n-1}$ measurable function $\psi : \partial K \to [0, \infty)$, we have*

$$\int_{\partial K} \psi \, d\mathcal{H}^{n-1} = \int_{S^{n-1}} \psi(r_K) \cdot \varrho_K^{n-2} \sqrt{\|\nabla \varrho_K\|^2 + \varrho_K^2} \, d\mathcal{H}^{n-1}.$$

Finally, we discuss the connection between the support function and the radial function of a convex body. Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int} K$. As $h = h_K|_{S^{n-1}}$ is Lipschitz, it is differentiable at $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$. In addition, if h; or equivalently, h_K is differentiable at a $u \in S^{n-1}$, then $\nabla h(u) = Dh_K(u)|_{u^\perp}$. According Lemma 1.6.6, if h_K is differentiable at a $u \in S^{n-1}$, then $Dh_K(u) = x$ for the unique $x \in \partial K$ where u is an exterior normal; and in addition, the reverse radial Gauss image

$$\alpha_K^*(u) = x/\|x\| \tag{2.7}$$

satisfies $x = r_K(\alpha_K^*(u))$. In turn, we deduce the following.

Lemma 2.2.6. *Let $h = h_K|_{S^{n-1}}$ for a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, and let h be differentiable at a $u \in S^{n-1}$. Then*

(i) $x = \nabla h(u) + h(u) \cdot u$ for the unique $x \in \partial K$ where u is an exterior normal;

(ii) $\varrho_K(\alpha_K^*(u)) = \sqrt{\|\nabla h(u)\|^2 + h(u)^2}$.

2.3 Surface area, Cauchy formula, continuity, monotonicity, outer Minkowski content

In this section, we define the surface area of convex bodies, and establish the main properties.

Definition 2.3.1 (Surface area). The surface area of a compact and convex $K \subset \mathbb{R}^n$ is defined as follows.

(i) If $\dim K = n$, then $P(K) = \mathcal{H}^{n-1}(\partial K)$.

(ii) If $\dim K = n - 1$, then $P(K) = 2\mathcal{H}^{n-1}(K)$.

(iii) If $\dim K \leqslant n - 2$, then $P(K) = 0$.

The surface area of compact convex sets as defined in Definition 2.3.1 it is monotone (cf. Lemma 2.3.4) and continuous (cf. Lemma 2.3.5); moreover, it is also known as the outer Minkowski content of a compact convex $K \subset \mathbb{R}^n$ as (cf. Theorem 2.3.6)

$$P(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho B^n| - |K|}{\varrho}. \tag{2.8}$$

In addition, $P(K)$ is proportional with the mean $(n-1)$-dimensional projection according to the Cauchy formula (2.9).

We note that the surface area of a convex body $K \subset \mathbb{R}^n$ is frequently denoted by $S(K)$, but here we adopt the notation $P(K)$ because we also discuss sets of finite perimeter in Chapter 5. More precisely, if $K \subset \mathbb{R}^n$ is a convex body, then the surface area $P(K)$ coincides with the perimeter of K as defined in Section 5.1. If $\dim K \leqslant n-1$ for a compact convex set $K \subset \mathbb{R}^n$, then this correspondence does not hold anymore, but we preferred to use the same notation also in this case.

Example 2.3.2 (Surface area of n-polytopes). Let Q be a polytope with $\dim Q = n$ and facets $F_1, \dots, F_k$. Since $\beta(N_Q(F_i)) = \frac{1}{2}$ for $i = 1, \dots, k$, (1.11) and (1.10) yield

$$P(Q) = \sum_{i=1}^{k} \mathcal{H}^{n-1}(F_i) = \lim_{\varrho \to 0^+} \frac{|Q + \varrho B^n| - |Q|}{\varrho}.$$

Now we prove the classical integral representation of the surface area of compact convex sets. As we will soon see, integral formulas are very handy tools in order to prove monotonicity or continuity. We frequently write du for $d\mathcal{H}^{n-1}(u)$ when integrating on S^{n-1} in order to simplify the formulas.

Theorem 2.3.3 (Cauchy formula). *If $K \subset \mathbb{R}^n$ is a compact convex set, then*

$$P(K) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathcal{H}^{n-1}(K|u^\perp)\, du. \tag{2.9}$$

Proof. If $\dim K = n$, then Lemma 2.1.2 applied first to K, then to B^n yields

$$\begin{aligned} \int_{S^{n-1}} \mathcal{H}^{n-1}(K|u^\perp)\, du &= \int_{S^{n-1}} \frac{1}{2} \int_{\partial K} |\langle \nu_K, u \rangle|\, d\mathcal{H}^{n-1}\, du \\ &= \int_{\partial K} \frac{1}{2} \int_{\partial B^n} |\langle \nu_K(x), u \rangle|\, du\, d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial K} \mathcal{H}^{n-1}(B^n|\nu_K(x)^\perp)\, d\mathcal{H}^{n-1}(x) = \omega_{n-1} P(K). \end{aligned}$$

If $\dim K = n-1$, then we may assume that $K \subset v^\perp$ for a $v \in S^{n-1}$, and hence (2.2) and Lemma 2.1.2 applied to B^n yield

$$\int_{S^{n-1}} \mathcal{H}^{n-1}(K|u^\perp)\, d\mathcal{H}^{n-1}(u) = \int_{\partial B^n} |\langle v, u \rangle| \cdot \mathcal{H}^{n-1}(K)\, d\mathcal{H}^{n-1}(u) = 2\omega_{n-1} \mathcal{H}^{n-1}(K).$$

Finally, if $\dim K \leqslant n-2$, then $\mathcal{H}^{n-1}(K|u^\perp) = 0$ for every $u \in S^{n-1}$. ∎

The Lebesgue measure is readily monotone on measurable sets, and the Cauchy formula (2.9) and the monotonicity of the volume in $\mathbb{R}^{n-1}$ yield the monotonicity of the surface area.

Lemma 2.3.4. *Let $K \subset C \subset \mathbb{R}^n$ be compact convex sets.*

(i) $|K| \leqslant |C|$, *and if* $\dim C = n$ *and* $K \neq C$, *then* $|K| < |C|$.

(ii) $P(K) \leqslant P(C)$, *and if* $\dim C \geqslant n-1$ *and* $K \neq C$, *then* $P(K) < P(C)$.

Proof. Readily $|K| \leqslant |C|$ as it is a property of the Lebesgue measure. In addition, if $\dim C = n$ and $K \neq C$, then $(\operatorname{int} C) \backslash K \neq \varnothing$.

Next, since $\mathcal{H}^{n-1}(K|u^\perp) \leqslant \mathcal{H}^{n-1}(C|u^\perp)$ for $u \in S^{n-1}$ in the Cauchy formula (2.9), we deduce that $P(K) \leqslant P(C)$.

If $\dim C \geqslant n-1$ and $K \neq C$, then there exists a $z_0 \in (\operatorname{relint} C) \backslash K$ and $v_0 \in S^{n-1} \cap \operatorname{lin} C$ such that $\langle z_0, v_0 \rangle > h_K(v_0)$, and hence there exists a $\delta \in (0, \frac{\pi}{2})$ such that $\langle z_0, u \rangle > h_K(u)$ if $u \in S^{n-1}$ and $\angle(u, v_0) < \delta$. It follows from (i) that $\mathcal{H}^{n-1}(K|u^\perp) < \mathcal{H}^{n-1}(C|u^\perp)$ if $u \in S^{n-1}$ and $\angle(u, v_0) < \delta$; therefore, $P(K) < P(C)$. ∎

We recall from Section 1.7 that the Hausdorff distance between non-empty compact convex sets $K, C \subset \mathbb{R}^n$, $n \geqslant 1$, is

$$\delta_H(K, C) = \min\{\varrho \geqslant 0 : K \subset C + \varrho B^n \text{ and } C \subset K + \varrho B^n\}.$$

It is a metric, and we consider the space of compact convex sets in $\mathbb{R}^n$ equipped with the topology determined by δ_H.

Lemma 2.3.5. *The surface area $K \mapsto P(K)$ is continuous as a function of compact convex sets in $\mathbb{R}^n$, $n \geqslant 2$.*

Proof. Let $d = \dim K$. We may assume after translation that $o \in \operatorname{relint} K$, and hence there exists $r > 0$ such that $rB^n \cap L \subset K$ for the d-dimensional linear subspace $L = \operatorname{lin} K$. In particular, $rB^n \cap L \subset \frac{1}{\theta} K$. As we will see, the argument depends on the monotonicity of the surface area (cf. Lemma 2.3.4).

First let $d = n$. If $\delta_H(C, K) < \varepsilon$ for $\varepsilon \in (0, r)$, then

$$\left(1 - \frac{\varepsilon}{r}\right) K \subset C \subset \left(1 + \frac{\varepsilon}{r}\right) K;$$

therefore, $\left(1 - \frac{\varepsilon}{r}\right)^{n-1} P(K) \leqslant P(C) \leqslant \left(1 + \frac{\varepsilon}{r}\right)^{n-1} P(K)$ by Lemma 2.3.4, proving Lemma 2.3.5 in this case.

If $d \leqslant n-1$, then for $\varrho > 0$, we have

$$K + \varrho B^n \supset K + \varrho(B^n \cap L^\perp),$$
$$K + \varrho B^n \subset K + \varrho(B^n \cap L) + \varrho(B^n \cap L^\perp) \subset \left(1 + \frac{\varrho}{r}\right) K + \varrho(B^n \cap L^\perp)$$

where $B^n \cap L^\perp$ is $(n-d)$-dimensional. Thus

$$\mathcal{H}^d(K) \cdot \omega_{n-d} \varrho^{n-d} \leqslant |K + \varrho B^n| \leqslant \left(1 + \frac{\varrho}{r}\right)^d \mathcal{H}^d(K) \cdot \omega_{n-d} \varrho^{n-d}. \tag{2.10}$$

As $\omega_1 = 2$, we deduce from (2.10) that $P(K) = \lim_{\varrho \to 0^+} |K + \varrho\, B^n|/\varrho$. ∎

Finally, we prove that the surface area of a compact set coincides with its outer Minkowski content (cf. Theorem 2.3.6). We recall that according to the Coarea Formula (10.12), if $F : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, and $\varphi : \mathbb{R}^n \to [0, \infty)$ is measurable, then

$$\int_{\mathbb{R}^n} \varphi \cdot \|DF\| \, d\mathcal{H}^k = \int_{\mathbb{R}} \int_{F^{-1}(t)} \varphi \, d\mathcal{H}^{k-1} \, dt. \tag{2.11}$$

Theorem 2.3.6 (Outer Minkowski content). *If $K \subset \mathbb{R}^n$ is compact and convex, then*

$$P(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho B^n| - |K|}{\varrho}. \tag{2.12}$$

Proof. We write $K^{(r)} = K + r B^n$ for $r > 0$. Since $K^{(r)}$ tends to K as $r \to 0^+$, the continuity of the surface area (cf. Lemma 2.3.5) yields that the following statement implies Theorem 2.3.6: If $\varrho > 0$, then

$$\left| K^{(\varrho)} \backslash K \right| = \int_0^{\varrho} P(K^{(r)}) \, dr. \tag{2.13}$$

To prove (2.13), let $F(x) = \|x - \Pi_K x\| = \min_{z \in K} \|x - z\|$ where $\Pi_K x$ is the closest point of K to x (cf. Definition 1.2.12). We claim that

$$|F(x) - F(y)| \leqslant \|x - y\| \tag{2.14}$$

for $x, y \in \mathbb{R}^n$. As the triangle inequality yields that

$$F(x) - F(y) \leqslant \|x - \Pi_K y\| - \|y - \Pi_K y\| \leqslant \|x - y\|,$$

changing the role of x and y implies (2.14).

According to (2.14) and Rademacher's theorem (cf. Lemma 10.4.7), F is $\mathcal{H}^n$ a.e. differentiable on $\mathbb{R}^n$. Next, we prove that if F is differentiable at an $x \in \mathbb{R}^n \backslash K$, then

$$\|DF(x)\| = 1. \tag{2.15}$$

First, the definition of the derivative yields that if $y \in \mathbb{R}^n$ tends to x, then

$$F(y) - F(x) = \langle DF(x), y - x \rangle + o(\|y - x\|). \tag{2.16}$$

On the other hand, $u = \frac{x - \Pi_K x}{\|x - \Pi_K x\|}$ is a unit exterior normal to K at $\Pi_K x$, and hence $\Pi_K(x + tu) = \Pi_K x$ for $t > 0$; therefore, $F(x + tu) - F(x) = t$. Comparing the last formula to (2.16) shows that $\langle DF(x), u \rangle = 1$. As $\|DF(x)\| \leqslant 1$ by (2.14), we deduce $D(F(x)) = u$, and in turn (2.15).

Now we apply the Coarea formula (2.11) to F and $\varphi = \mathbf{1}_{K^{(\varrho)} \backslash K}$, and conclude that

$$\left| K^{(\varrho)} \backslash K \right| = \int_0^{\varrho} \mathcal{H}^{n-1} \left(F^{-1}(r) \right) \, dr.$$

As $F^{-1}(r) = \partial(K + rB^n)$ for $r > 0$, we deduce (2.13), and in turn (2.12). ∎

2.4 The Isoperimetric inequality and the Anisotropic Isoperimetric Inequality for convex bodies

First we verify the classical Isoperimetric inequality Theorem 2.4.1 using Steiner's symmetrization honoring the first really deep argument in convexity, and then we provide an actually much simpler proof of the more general Anisotropic Isoperimetric inequality based on the Brunn-Minkowski inequality.

Theorem 2.4.1 (Isoperimetric inequality for convex bodies). *If $K \subset \mathbb{R}^n$ is a convex body with $|K| = |rB^n|$ for $r > 0$, then $P(K) \geqslant P(rB^n)$ with equality if and only if K is a Euclidean ball.*

Remarks.

- Equivalently, $P(K) \geqslant n\omega_n^{\frac{1}{n}} |K|^{\frac{n-1}{n}}$, with equality if and only if K is a ball (see Theorem 8.6.2 and Corollary 8.6.3 for stability versions).
- The Isoperimetric inequality is proved for sets with Lipschitz boundary in Theorem 4.1.6, and even more generally, for sets of finite perimeter in Theorem 5.2.1.
- The Isoperimetric inequality for convex bodies is a direct consequence of the Brunn-Minkowski Inequality (cf. Lemma 1.12.2), including the characterization of the case of equality (see the proof of Theorem 2.4.4). However, first we show how Steiner's original method based on his symmetrizaton leads to the Isoperimetric inequality.

A key observation about the Steiner symmetrization (cf. Definition 1.10.1) is that it does not increase surface area.

Proposition 2.4.2. *If $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then $P(\Theta_{u^\perp} K) \leqslant P(K)$, with equality if and only if K is symmetric through a hyperplane parallel to $u^\perp$.*

Proof. We use Definition 1.10.1 (c) for Steiner symmetrization. Let $f, g : K|u^\perp \to \mathbb{R}$ be concave such that

$$K = \left\{ z + t\,u : z \in K|u^\perp \text{ and } -g(z) \leqslant t \leqslant f(z) \right\},$$

and

$$\Theta_{u^\perp} K = \left\{ z + t\,u : x \in K|u^\perp \text{ and } -\frac{f(z)+g(z)}{2} \leqslant t \leqslant \frac{f(z)+g(z)}{2} \right\}.$$

We write $\Omega = \operatorname{int} K|u^\perp$, $X = \{x \in \partial K : x|u^\perp \in \operatorname{relbd} \Omega\}$ and $\widetilde{X} = \{x \in \partial(\Theta_{u^\perp} K) : x|u^\perp \in \operatorname{relbd} \Omega\}$; therefore, $\mathcal{H}^{n-1}(\widetilde{X}) = \mathcal{H}^{n-1}(X)$ by Fubini's theorem. It follows

from (2.1) that

$$P(K) = \int_\Omega \sqrt{1 + \|Df(z)\|^2} + \sqrt{1 + \|Dg(z)\|^2}\, d\mathcal{H}^{n-1}(z) + \mathcal{H}^{n-1}(X);$$

$$P(\Theta_{u^\perp} K) = \int_\Omega 2 \cdot \sqrt{1 + \left\| \frac{Df(z) + Dg(z)}{2} \right\|^2}\, d\mathcal{H}^{n-1}(z) + \mathcal{H}^{n-1}(\widetilde{X}).$$

Since $\|(1,a)\| + \|(1,b)\| \geqslant 2 \cdot \left\|\left(1, \frac{a+b}{2}\right)\right\|$ for $a, b \in \mathbb{R}^{n-1}$ by the triangle inequality with equality if and only if $a = b$, we deduce that $P(\Theta_{u^\perp} K) \leqslant P(K)$.

Now, if $P(\Theta_{u^\perp} K) = P(K)$, then $Df(z) = Dg(z)$ for $\mathcal{H}^{n-1}$ a.e. $z \in \Omega$; therefore, $\exists\ \gamma \in \mathbb{R}$ such that $f(z) = g(z) + \gamma$ by Remark 2.1.1 (iv), and hence K is symmetric through $u^\perp + \frac{\gamma}{2} u$. ∎

Proof of Theorem 2.4.1*:* We may assume that $\sigma_K = o$, and hence any hyperplane of symmetry for K contains o by the affine equivariance of the centroid (cf. Lemma 1.11.2). We distinguish two cases: If for any $u \in S^{n-1}$, K is symmetric through some hyperperplane parallel to $u^\perp$, then $\varrho S^{n-1} \subset K$ for $\varrho > 0$ whenever $\varrho S^{n-1} \cap K \neq \varnothing$; therefore, K is a ball centered at o.

The other case is when there exists a $u \in S^{n-1}$ such that K is not symmetric through any hyperperplane parallel to $u^\perp$, and hence $|K_0| = |K|$ and $P(K_0) < P(K)$ for $K_0 = \Theta_{u^\perp} K$ by Proposition 2.4.2. Now starting from K_0, there exists a sequence of iterated Steined symmetrizations leading to rB^n by Theorem 1.10.7. Since the surface area is continuous, we deduce from Proposition 2.4.2 that $P(rB^n) \leqslant P(K_0) < P(K)$. ∎

Let us extend the classical notion $P(K)$ of surface area to the anisotropic one where the gauge body is a convex body instead of the unit ball.

Definition 2.4.3 (Anisotropic Surface area of compact convex sets). Given convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int} C$, if $K \subset \mathbb{R}^n$ is compact convex, then its anisotropic perimeter or anisotropic surface area is

$$P_C(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho\, C| - |K|}{\varrho}.$$

Remark. In particular, $P_{B^n}(K) = P(K)$.

As the volume is translation invariant, if $z \in \operatorname{int} C$, then

$$P_C(K) = P_{C-z}(K). \tag{2.17}$$

For given C, $P_C(K)$ is a continuous and monotonic function of the compact convex set $K \subset \mathbb{R}^n$ because it can be represented as a mixed volume $P_C(K) = nV(K, C; 1)$ (see Remark 7.4.5 or Theorem 8.3.5). If K is a convex body, then (see Schneider [661],

Chapter 5, or Chambolle, Lisini, Lussardi [184], or Lussardi, Villa [547])

$$P_C(K) = \int_{\partial K} h_C(\nu_K(x))\, d\mathcal{H}^{n-1}(x) = \int_{\partial K} \|\nu_K(x)\|_{C^*}\, d\mathcal{H}^{n-1}(x) \tag{2.18}$$

where C^* is the polar of C and $h_C(\nu_K(x)) = \|\nu_K(x)\|_{C^*}$ follows from Proposition 1.9.2.

We are ready to state and prove the Anisotropic Isoperimetric inequality for convex sets.

Theorem 2.4.4 (Anisotropic Isoperimetric inequality). *If $C, K \subset \mathbb{R}^n$ is a convex bodies with $o \in \operatorname{int} C$, then*

$$P_C(K) \geqslant n|C|^{\frac{1}{n}}|K|^{\frac{n-1}{n}},$$

with equality if and only if C and K are homothetic.

Proof. Recall that $f(t) = |K + tC|^{\frac{1}{n}}$ is concave for $t \in [0,1]$ by the Brunn-Minkowski inequality (cf. Lemma 1.12.2), and linear if and only if K and C are homothetic. It follows that

$$\frac{1}{n}|K|^{\frac{1-n}{n}} P_C(K) = f'(0) \geqslant f(1) - f(0) = |K + C|^{\frac{1}{n}} - |K|^{\frac{1}{n}} \geqslant |C|^{\frac{1}{n}} \tag{2.19}$$

using the form $|K + C|^{\frac{1}{n}} \geqslant |K|^{\frac{1}{n}} + |C|^{\frac{1}{n}}$ of the Brunn-Minkowski inequality at the end (cf. Lemma 1.12.2), thus $P_C(K) \geqslant n|C|^{\frac{1}{n}}|K|^{\frac{n-1}{n}}$.

If $P_C(K) = n|C|^{\frac{1}{n}}|K|^{\frac{n-1}{n}}$, then $f'(0) = f(1) - f(0)$, and hence f is linear on $[0,1]$, which in turn yields that C and K are homothetic. ∎

2.5 Surface area measure

In this section, we show how to encode various properties of a compact convex set into its surface area measure on S^{n-1} introduced by Aleksandrov [4, 11]. Besides the support function, the surface area measure is one of the most significant notions associated to a convex body because many other notions can be expressed with its aid, like mixed volumes (see Section 7.3) and L_p-surface area measures (see Section 9.3).

Remark 2.5.1. Let $K \subset \mathbb{R}^n$ be a convex body. We recall that the set $\partial' K$ of regular boundary points is Borel and satisfies $\mathcal{H}^{n-1}(\partial K \setminus \partial' K) = 0$ according to Theorem 1.5.2. In addition, the unique exterior unit normal $\nu_K(x)$ at an $x \in \partial K'$ is a continuous function on $\partial' K$ by Lemma 1.5.3. Therefore, $\nu_K^{-1}(\omega) \subset \partial' K$ is Borel as a subset of S^{n-1} for any Borel $\omega \subset S^{n-1}$.

Definition 2.5.2 (Surface area measure S_K on S^{n-1}). Let $K \subset \mathbb{R}^n$ be convex, compact.

- If K is convex body ($\dim K = n$), then for any for a measurable $\omega \subset S^{n-1}$, we set (cf. (1.12))

$$S_K(\omega) = \mathcal{H}^{n-1}\left(\nu_K^{-1}(\omega)\right) = \mathcal{H}^{n-1}\left(\bigcup_{u\in\omega} F(K,u)\right), \tag{2.20}$$

  which is well-defined according to Remark 2.5.1. In particular, if $g : S^{n-1} \to \mathbb{R}$ is a bounded measurable, then

$$\int_{S^{n-1}} g dS_K = \int_{\partial' K} g\circ\nu_K \, d\mathcal{H}^{n-1} = \int_{\partial K} g\circ\nu_K \, d\mathcal{H}^{n-1}. \tag{2.21}$$

- If $\dim K = n-1$ and $K \subset x + u^\perp$ for $u \in S^{n-1}$ and $x \in \mathbb{R}^n$, then $\operatorname{supp} S_K = \{u, -u\}$ and $S_K(\{u\}) = S_K(\{-u\}) = \mathcal{H}^{n-1}(K)$.
- If $\dim K \leqslant n-2$, then $S_K \equiv 0$.

The definition shows that the surface area measure S_K on S^{n-1} of a convex body $K \subset \mathbb{R}^n$ is the pushforward of the Hausdorff measure on $\partial' K$ by the Gauss map $x \mapsto \nu_K(x)$, as we now explain.

Definition 2.5.3 (Pushforward of a measure). If X, Y are topological spaces, μ Borel measure on X and $\varphi : X \to Y$ Borel measurable, then $\varphi_*\mu$ is a Borel measure on Y satisfying that $\varphi_*\mu(\omega) = \mu(\varphi^{-1}\omega)$ for any Borel set $\omega \subset Y$.

In particular, if $f : Y \to \mathbb{R}$ is Borel measurable, then

$$\int_Y f \, d\varphi_*\mu = \int_X f\circ\varphi \, d\mu. \tag{2.22}$$

Remark. $S_K = \nu_{K,*}\left(\mathcal{H}^{n-1}\llcorner\partial' K\right)$ for a convex body $K \subset \mathbb{R}^n$ where $\mathcal{H}^{n-1}\llcorner\partial' K$ is the restriction of $\mathcal{H}^{n-1}$ to $\partial' K$.

Example 2.5.4. (see Section 8.2 for examples of convex bodies with C^2_+ boundary when S_K is absolutely continuous)

- If K is an n-dimensional polytope with facets $F_1, \ldots, F_m$ and exterior unit normals $u_1, \ldots, u_m$, then S_K is a discrete measure,

$$\operatorname{supp} S_K = \{u_1, \ldots, u_m\} \text{ and } S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i),\ i = 1, \ldots, m.$$

- If $K = rB^n$ for $r > 0$, then $S_K(\omega) = r^{n-1}\cdot\mathcal{H}^{n-1}(\omega)$ for a Borel set $\omega \subset S^{n-1}$.

The following properties are immediate consequences of the definition of the surface area measure.

Lemma 2.5.5. *Let $K \subset \mathbb{R}^n$ be convex, compact.*

- *S_K is Borel measure on S^{n-1};*

- $S_K(S^{n-1}) = P(K)$;
- $S_{\lambda K} = \lambda^{n-1} \cdot S_K$ *for* $\lambda > 0$ *and* $S_{K+x} = S_K$ *for* $x \in \mathbb{R}^n$.

The significance of S_K for a convex body $K \subset \mathbb{R}^n$ is exhibited by the geometric properties of its support where $\operatorname{supp} S_K$ is the smallest closed $X \subset S^{n-1}$ such that $S_K(S^{n-1}\backslash X) = 0$. We need the following simple property of exterior normals (cf. Lemma 1.2.10): If $y_m \in \partial' K$ tends to $y \in \partial K$, then

$$u \in N_K(y) \text{ for any accumulation point } u \text{ of } \{\nu_K(y_m)\}. \tag{2.23}$$

Lemma 2.5.6. *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) $\operatorname{supp} S_K = \operatorname{cl}\{\nu_K(x) : x \in \partial' K\}$;

(ii) $$\begin{array}{rcl} K & = & \{x \in \mathbb{R}^n : \langle x, \nu_K(y)\rangle \leqslant h_K(\nu_K(y)) \text{ for } y \in \partial' K\} \\ & = & \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_K(u) \text{ for } u \in \operatorname{supp} S_K\} \end{array};$$

(iii) $\operatorname{supp} S_K$ *is not contained in any closed hemisphere;*

(iv) For a closed $\Omega \subset S^{n-1}$ *and* $\varphi : \Omega \to \mathbb{R}$, *if* $K = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant \varphi(u) \text{ for } u \in \Omega\}$, *then* $\operatorname{supp} S_K \subset \Omega$ *and* Ω *is not contained in any closed hemisphere.*

Proof. For $\Omega_0 = \operatorname{cl}\{\nu_K(x) : x \in \partial' K\}$, the definition of S_K (cf. (2.20)) and the almost everywhere differentiability of ∂K (cf. Theorem 1.5.2) yield $S_K(S^{n-1}\backslash\Omega_0) = 0$, thus $\operatorname{supp} S_K \subset \Omega_0$. On the other hand, for any $x_0 \in \partial' K$, if $U \subset S^{n-1}$ is open with $\nu_K(x_0) \in U$, then as ν_K is continuous on $\partial' K$ (cf. (2.23)), we deduce the existence of $\delta > 0$ such that $\nu_K(x) \in U$ if $x \in \partial' K \cap (x_0 + \delta B^n)$, and hence $S_K(U) > 0$. In turn, we conclude (i).

Turning to (ii), let

$$K_0 = \{x \in \mathbb{R}^n : \langle x, \nu_K(y)\rangle \leqslant h_K(\nu_K(y)) \text{ for } y \in \partial' K\}.$$

Since readily $K \subset K_0$, (ii) follows if $\partial K \subset \partial K_0$. For $y \in \partial K$, there exist $y_m \in \partial' K$ tending to $y \in \partial K$ as $\mathcal{H}^{n-1}$ a.e. point of ∂K is in $\partial' K$, and we may assume that $\nu_K(y_m)$ tends to a $u \in S^{n-1}$. Now $\langle x, u\rangle \leqslant \langle y, u\rangle$ follows for any $x \in K_0$ from the estimates $\langle x, \nu_K(y_m)\rangle \leqslant \langle y_m, \nu_K(y_m)\rangle$ for every y_m; therefore, $y \in \partial K_0$.

Next, (iii) is equivalent proving that for any $v \in S^{n-1}$, there exists $u \in \operatorname{supp} S_K$ with $\langle u, v\rangle > 0$. We consider an $x \in \operatorname{int} K$ and a $z = x + tv \notin K$ for large enough $t > 0$. It follows from (ii), that there exists $u \in \operatorname{supp} S_K$ such that $\langle z, u\rangle > \langle x, u\rangle$, and hence $\langle v, u\rangle > 0$.

For (iv), it is sufficient to prove that $\nu_K(y) \in \Omega$ for any $y \in \partial' K$ by (i) and (iii). The condition on φ yields that $\varphi(u) \geqslant h_K(u)$ for $u \in \Omega$, and for $k \geqslant 1$ and $z_k = y + \frac{1}{k}\nu_K(y) \notin K$, there exists $u_k \in \Omega$ such that $\langle z_k, u_k\rangle > \varphi(u_k)$. We deduce that $\langle z_k, u_k\rangle \geqslant h_K(u_k) \geqslant \langle y, u_k\rangle$ for $k \geqslant 1$. For a convergent subsequence $\{u_{k'}\} \subset \{u_k\}$, we have $u = \lim_{k'\to\infty} u_{k'} \in \Omega$, and $\lim_{k'\to\infty} z_{k'} = y$ implies $\langle y, u\rangle = h_K(u)$, and hence u is a normal vector at y. Since $y \in \partial' K$, we conclude that $\nu_K(y) = u \in \Omega$. ∎

Next we consider some basic properties of the surface area measure with respect to integration:

Lemma 2.5.7. *If $K \subset \mathbb{R}^n$ is compact, convex, then*

$$\int_{S^{n-1}} u \, dS_K(u) = o; \tag{2.24}$$

$$|K| = \frac{1}{n} \int_{S^{n-1}} h_K \, dS_K. \tag{2.25}$$

Proof. If K is a convex body, then Corollary 2.1.3, (2.3) and (2.21) yield the formulas. If $\dim K \leqslant n-1$, then the formulas directly follow from the definition of S_K. ∎

In the proof of Proposition 2.5.9, we will need the following version of the coarea formula Theorem 10.4.9:

Theorem 2.5.8 (Coarea formula). *For $m \leqslant k < q$ and Lipschitz embedded k-manifold $X \subset \mathbb{R}^q$, if $F : X \to \mathbb{R}^m$ is locally Lipschitz and $\varphi : X \to [0, \infty)$ measurable, then*

$$\int_X \varphi(x) \cdot J(F, x) \, d\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \int_{F^{-1}(y)} \varphi(x) \, d\mathcal{H}^{k-m}(x) \, d\mathcal{H}^m(y).$$

We recall that according to (2.18) about the anisotropic surface area, if $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in \operatorname{int} C$, then

$$\lim_{\varrho \to 0^+} \frac{|K + \varrho C| - |K|}{\varrho} = \int_{\partial K} h_C(\nu_K(x)) \, d\mathcal{H}^{n-1}(x). \tag{2.26}$$

We now extend this formula to compact convex sets:

Proposition 2.5.9. *If $K, C \subset \mathbb{R}^n$ are compact and convex, then*

$$\lim_{\varrho \to 0^+} \frac{|K + \varrho C| - |K|}{\varrho} = \int_{S^{n-1}} h_C \, dS_K. \tag{2.27}$$

Remark. The statement has been generalized to the case when K is a suitable compact set of finite perimeter by Kiderlen, Rataj [458].

Proof. If $\dim K \leqslant n-2$, then choose $R > 0$ such that $C \subset RB^n$, and hence Lemma 2.3.1 yields that

$$\lim_{\varrho \to 0^+} \frac{|K + \varrho C| - |K|}{\varrho} \leqslant \lim_{\varrho \to 0^+} \frac{|K + \varrho R B^n| - |K|}{\varrho} = R^n \lim_{\varrho \to 0^+} \frac{|\frac{1}{R} K + \varrho B^n| - |\frac{1}{R} K|}{\varrho} = 0.$$

Next, if $\dim K = n-1$, then we may assume that $o \in \operatorname{relint} K$ as the volume and S_K are translation invariant. Let $u \in S^{n-1}$ such that $u^\perp = \operatorname{lin} K$, and let $\tilde{x}, \tilde{y} \in C$ such that $\langle \tilde{x}, u \rangle = h_C(u)$ and $\langle \tilde{y}, -u \rangle = h_C(-u)$. According to the translation invariance

of the volume and (2.24), we may also assume that $\frac{1}{2}(\tilde{x}+\tilde{y})=o$; or in other words, $\tilde{y}=-\tilde{x}$. For the segment $\theta=\operatorname{conv}\{\tilde{x},\tilde{y}\}\subset C$, we have

$$\int_{S^{n-1}} h_C\,dS_K=\mathcal{H}^{n-1}(K)(h_C(u)+h_C(-u))=|K+\theta|. \tag{2.28}$$

On the one hand, $\theta\subset C$ yields that

$$\lim_{\varrho\to 0^+}\frac{|K+\varrho C|-|K|}{\varrho}\geqslant\lim_{\varrho\to 0^+}\frac{|K+\varrho\theta|}{\varrho}=|K+\theta|. \tag{2.29}$$

On the other hand, $o\in\operatorname{relint}K$ implies that $C\subset RK+\theta$ for some $R>0$, and hence

$$\begin{aligned}\lim_{\varrho\to 0^+}\frac{|K+\varrho C|-|K|}{\varrho}&\leqslant\lim_{\varrho\to 0^+}\frac{|K+\varrho RK+\varrho\theta|}{\varrho}\\&=\lim_{\varrho\to 0^+}(1+\varrho R)^{n-1}|K+\theta|=|K+\theta|.\end{aligned} \tag{2.30}$$

Combining (2.28), (2.29) and (2.30) yields Proposition 2.5.9.

Let $\dim K=n$. If also $\dim C=n$, then let $\widetilde{C}=C-z$ for a $z\in\operatorname{int}C$, and hence (2.26) and (2.21) yield that

$$\lim_{\varrho\to 0^+}\frac{|K+\varrho\widetilde{C}|-|K|}{\varrho}=\int_{S^{n-1}} h_{\widetilde{C}}\,dS_K.$$

As $h_{\widetilde{C}}(u)=h_C(u)-\langle z,u\rangle$ for $u\in S^{n-1}$, it follows from (2.24) that $\int_{S^{n-1}}h_C\,dS_K=\int_{S^{n-1}}h_{\widetilde{C}}\,dS_K$. Since the volume is translation invariant, we conlude (2.27).

Finally, we assume that $\dim K=n$ and $\dim C=d\leqslant n-1$ where we may also assume that $o\in\operatorname{relint}C$. Let $L=\operatorname{lin}C$, and we apply the previous case to $((x+L)\cap K)+\varrho C$ for $\mathcal{H}^{n-d}$ a.e. $x\in\operatorname{int}K|L^{\perp}$ such that the $\mathcal{H}^{d-1}$ measures of $(x+L)\cap\partial K$ and $(x+L)\cap\partial' K$ coincide (in this case, $(x+L)\cap K$ and $x+C$ are convex bodies in $x+L$). Integrating over such $x\in\operatorname{int}K|L^{\perp}$ the analogue of (2.27) in $x+L$ (where the exterior normals to $(x+L)\cap\partial' K$ lying in L are used), and using Theorem 2.5.8 for $X=(L+\operatorname{int}K)\cap\partial K$ and F being the orthogonal projection onto $L^{\perp}$, we obtain (2.27). ∎

We note that the Anisotropic Isoperimetric Inequality Theorem 2.4.4 is equivalent to the Minkowski inequality below as for convex bodies $K,C\subset\mathbb{R}^n$, both the expression $\lim_{\varrho\to 0^+}\frac{|K+\varrho C|-|K|}{\varrho}$ and (2.31) (cf. (2.24)) are invariant under translations of C (see also Corollary 7.4.3 and Remark 7.4.4 in the context of mixed volumes).

Theorem 2.5.10 (Minkowsi inequality). *If $K,C\subset\mathbb{R}^n$ convex bodies, then*

$$\int_{S^{n-1}} h_C\,dS_K\geqslant n\,|K|^{\frac{n-1}{n}}|C|^{\frac{1}{n}} \tag{2.31}$$

with equality if and only if K and C are homothetic.

This allows us, in turn, to deduce the characterization of the equality of two surface area measures.

Theorem 2.5.11. *For convex bodies $K, C \subset \mathbb{R}^n$, $S_K = S_C$ if and only if K and C are translates.*

Proof. We deduce from (2.25), $S_K = S_C$ and the Minkowski inequality (2.31) that

$$|K| = \frac{1}{n}\int_{S^{n-1}} h_K \, dS_K = \frac{1}{n}\int_{S^{n-1}} h_K \, dS_C \geqslant |C|^{\frac{n-1}{n}}|K|^{\frac{1}{n}}; \tag{2.32}$$

therefore, $|K| \geqslant |C|$. Reversing the role of K and C in (2.32) implies $|C| \geqslant |K|$, and hence $|C| = |K|$. In turn, equality in (2.32) implies equality in the Minkowski inequality (2.31), which fact combined with $|C| = |K|$ yields that K and C are translates. ∎

In order to understand the significance and the use of the surface area measure, we list the fundamental properties that will be proved only later when we have the necessary tools.

Remark 2.5.12 (Additional properties of S_K, Aleksandrov [4, 11]).

- S_K is weakly* continuous (see Proposition 2.6.13 if $\operatorname{int} K \neq \varnothing$, and Proposition 8.4.1 in general). In particular, if $g : S^{n-1} \to \mathbb{R}$ is continuous and $K_m \to K$ for compact convex $K_m, K \subset \mathbb{R}^n$, then $\lim_{m\to\infty} \int_{S^{n-1}} g \, dS_{K_m} = \int_{S^{n-1}} g \, dS_K$.
- S_K can be considered as the first variation of the volume (see (2.27) and Theorem 7.5.2);
- $\mu = S_K$ on S^{n-1} for a finite Borel measure μ on S^{n-1} if and only if $\int_{S^{n-1}} u \, d\mu(u) = o$ and $\operatorname{supp}\mu$ is not contained in any closed hemisphere (see Theorem 9.2.3 about the Minkowski problem, and note that the necessity of these conditions have been proved in Lemma 2.5.6 and Lemma 2.5.7).
- If ∂K is C^2_+, then $dS_K = f_K d\mathcal{H}^{n-1}$ where $f_K(\nu_K(x)) = 1/\kappa(x)$ for $x \in \partial K$ and the Gauss curvature $\kappa(x) > 0$ (see Section 8.2).

2.6 Cone Volume measure

The cone volume measure, introduced by Firey [287] in 1974, has the significant property that while the surface area measure only intertwines with orthogonal transformations, the cone volume measure intertwines with any linear transformation (cf. Proposition 2.6.15). In particular, it is an important tool in notions and problems intertwining with linear transformations like the affine isoperimetric inequality for the affine surface area measure (see Section 8.9), has a fundamental role in various Brunn-Minkowski type inequalities (see Section 8.7 and Section 8.8), and in various

versions of the Minkowki problem; namely, in certain Monge-Ampère equations on the sphere with geometric significance (see Section 9.3 and Section 9.4). These sections discuss the fundamental uniqueness and characterization results and conjectures about the cone volume measure. What is simple to show is that the cone volume measure is weakly* continuous (cf. Proposition 2.6.12), which fact in turn directly yields the otherwise non-trivial weak continuity of surface area on convex bodies (cf. Proposition 2.6.13).

Definition 2.6.1 (Cone volume measure). If $K \subset \mathbb{R}^n$ is convex compact with $o \in K$, then V_K is the Borel measure on S^{n-1} satisfying $dV_K = \frac{1}{n} h_K \, dS_K$.

Remark. $V_K(S^{n-1}) = \frac{1}{n} \int_{S^{n-1}} h_K \, dS_K = |K|$ according to (2.25), and if $\dim K \leqslant n-1$, then $V_K(\omega) = 0$ for measurable $\omega \subset S^{n-1}$.

The name cone volume measure stems from the case of full dimensional polytopes.

Example 2.6.2 (Cone volume measure of an n-polytope). Let $P \subset \mathbb{R}^n$ be an n-polytope with $o \in \operatorname{int} P$, and let $F_1, \dots, F_k$ be the facets of P, and $u_1, \dots, u_k$ be the corresponding unit exterior normals. Then $\operatorname{supp} V_P = \{u_1, \dots, u_k\}$, and $V_K(\{u_i\}) = \frac{1}{n} h_K(u_i) \mathcal{H}^{n-1}(F_i) = |\operatorname{conv}\{o, F_i\}|$ (cf. (1.26)); namely, the cone volume measure of the exterior normal of a facet is the volume of the corresponding "cone".

For many applications, one may assume that $o \in \operatorname{int} K$ when discussing the cone volume measure, which case is technically easier to handle. However, limits of convex bodies containing the origin in their interior might be a convex body that contains the origin on its boundary. Also, note that there exist convex bodies $K \subset \mathbb{R}^n$ such that $o \in \partial K$ and $dV_K = \varphi \, d\mathcal{H}^{n-1}$ for a positive $C^{0,\alpha}$ function φ on S^{n-1} (cf. Example 9.3.9). Such bodies are important for Monge-Ampère equations on S^{n-1} (see Chapter 9). Therefore, we extend the definition of radial functions of a convex body to allow the origin to be on the boundary.

Definition 2.6.3 (Radial function). For a convex body $K \subset \mathbb{R}^n$ with $o \in K$, if $u \in S^{n-1}$, then let $\varrho_K(u) = \max\{t \geqslant 0 : tu \in K\}$, and hence $\varrho_K(u) \geqslant 0$ and $\varrho_K(u)\, u \in \partial K$.

Remark. If $o \in \operatorname{int} K$, then ϱ_K is just the traditional radial function. If $o \in \partial K$, then we consider the open convex cone $\Sigma_K = \{(0, \infty)x : x \in \operatorname{int} K\}$. If $u \in \Sigma_K \cap S^{n-1}$, then $\varrho_K(u) = \max\{t \geqslant 0 : tu \in K\} > 0$, and ϱ_K is continuous on $\Sigma_K \cap S^{n-1}$. On the other hand, if $u \in S^{n-1} \backslash (\operatorname{cl}\Sigma_K)$, then $\varrho_K(u) = 0$, and hence ϱ_K is $\mathcal{H}^{n-1}$ measurable on S^{n-1}. However, ϱ_K might be a "wild" function on $\partial\Sigma_K \cap S^{n-1}$, and hence ϱ_K is possibly not Borel measurable if $o \in \partial K$.

In order to understand the cone volume measure V_K on the sphere S^{n-1}, it is practical to consider the intimately connected measure $\widetilde{V}_K$ on ∂K because it is equivariant

under volume preserving linear transformations (cf. Proposition 2.6.15), and satisfy some natural integral representations as in Lemma 2.6.6.

Definition 2.6.4 (Auxiliary Cone Volume measure). For a convex body $K \subset \mathbb{R}^n$ with $o \in K$, $\widetilde{V}_K$ is a Borel measure on ∂K such that if $\Xi \subset \partial K$ is Borel, then

$$\widetilde{V}_K(\Xi) = \left| \bigcup \{\mathrm{conv}\{o,x\} : x \in \Xi\} \right|.$$

Remark. We have $\widetilde{V}_K(\partial K) = |K|$. The Auxialiary Cone Volume measure is connected to the Cone Volume measure by the formula $V_K = \nu_{K,*}\widetilde{V}_K$; namely, if $\omega \subset S^{n-1}$ is measurable, then

$$V_K(\omega) = \widetilde{V}_K\left(\{x \in \partial' K : \nu_K(x) \in \omega\}\right) \tag{2.33}$$

$$= \left| \bigcup \{\mathrm{conv}\{o,x\} : x \in \partial K \text{ and } \nu_K(x) \cap \omega \neq \varnothing\} \right|. \tag{2.34}$$

Many statements related to the cone volume measure use the notion of radial projection onto the sphere.

Definition 2.6.5 (Radial projection). $\pi_{S^{n-1}}(x) = x/\|x\| \in S^{n-1}$ for $x \in \mathbb{R}^n \backslash \{o\}$.

Remark. Since the closest point map Π_{rB^n} is Lipschitz for any $r > 0$ (cf. Lemma 1.2.13), $\pi_{S^{n-1}}$ is locally Lipschitz on $\mathbb{R}^n \backslash \{o\}$.

Lemma 2.6.6. *If $K \subset \mathbb{R}^n$ is a convex body with $o \in K$, and $\widetilde{Z} = \pi_{S^{n-1}} Z$ for a Borel subset $Z \subset \partial K$, then*

$$\widetilde{V}_K(Z) = \frac{1}{n} \int_{\widetilde{Z}} \varrho_K^n \, d\mathcal{H}^{n-1} = \frac{1}{n} \int_Z \langle \nu_K(x), x \rangle \, d\mathcal{H}^{n-1}(x). \tag{2.35}$$

Proof. Integration using polar coordinates (cf. (10.13)) gives the first equality in (2.35).

To prove the second equality, we consider the open convex cone $\Sigma_K = \{(0,\infty)x : x \in \mathrm{int}\, K\}$, and hence $\Sigma_K = \mathbb{R}^n$ if $o \in \mathrm{int} K$. We also consider the Borel measure $\mu(Z) = \frac{1}{n}\int_Z \langle \nu_K(x), x\rangle \, d\mathcal{H}^{n-1}(x)$ for any measurable $Z \subset \partial K$ on ∂K, and hence the second equality in (2.35) is equivalent proving that $\mu(Z) = \widetilde{V}_K(Z)$ for any Borel set $Z \subset \partial K$. Since $\{x \in \partial' K : \langle \nu_K(x), x\rangle = 0\} = \partial \Sigma_K \cap \partial' K$, we have $\widetilde{V}_K(\partial K \cap \partial \Sigma_K) = 0$ if $\partial \Sigma_K \neq \varnothing$, and it is enough to consider the case when $Z \cap \partial \Sigma_K = \varnothing$ and $\bigcup\{[0,\infty)x : x \in Z\}$ is a closed convex cone; namely, $C = \bigcup\{\mathrm{conv}\{o,x\} : x \in Z\}$ is a convex body in $\mathbb{R}^n$.

Now $\widetilde{V}_K(Z) = |C| = \frac{1}{n}\int_Z \langle \nu_X(x), x\rangle \, d\mathcal{H}^{n-1}(x)$ by (2.3) because $\langle \nu_C(x), x\rangle = 0$ for $x \in (\partial' C)\backslash Z$, which in turn yields the second equality in (2.35). ∎

Lemma 2.6.6 directly yields the following formula.

Corollary 2.6.7. *If $K \subset \mathbb{R}^n$ is a convex body with $o \in K$ and $f : \partial K \to [0, \infty)$ is measurable, then*

$$\begin{aligned}\int_{\partial' K} f(x)\langle \nu_K(x), x\rangle \, d\mathcal{H}^{n-1}(x) &= n \int_{\partial K} f(x)\, d\widetilde{V}_K(x) \\ &= \int_{S^{n-1}} f\left(\varrho_K(u)\cdot u\right)\cdot \varrho_K(u)^n \, d\mathcal{H}^{n-1}(u).\end{aligned}$$

Corollary 2.6.7 will be used in Section 9.5.5 to provide a representation of dual curvature measures. Lemma 2.6.6 suggests to introduce the so-called radial Gauss map.

Definition 2.6.8 (Radial Gauss map). Let $K \subset \mathbb{R}^n$ be a convex body with $o \in K$, and let $\Sigma_K = \{(0,\infty)x : x \in \operatorname{int} K\}$ where Σ_K is an open convex cone. We define

$$\alpha_K(u) = \nu_K(\varrho_K(u)\cdot u) \;\text{ for } u \in \pi_{S^{n-1}}(\Sigma_K \cap \partial' K),$$

and $\alpha_K(u) = u$ if $u \in S^{n-1}\backslash(\mathrm{cl}\Sigma_K)$.

Remark. $\alpha_K(u) \in S^{n-1}$ is well-defined for a.e. $u \in S^{n-1}$, and is a measurable and bounded function on S^{n-1}. This follows as on the one hand,

$$\mathcal{H}^{n-1}(S^{n-1}\backslash\Theta) = 0 \;\text{ for }\; \Theta = \pi_{S^{n-1}}(\Sigma_K \cap \partial' K) \cup \left(S^{n-1}\backslash(\mathrm{cl}\Sigma_K)\right)$$

because $\pi_{S^{n-1}}$ is locally Lipschitz and $\mathcal{H}^{n-1}(\partial K\backslash\partial' K) = 0$, and on the other hand, α_K is continuous on $\Theta \cap \Sigma_K$ and on $\Theta\backslash\Sigma_K$.

The definition of the cone volume meausure, (2.33) and Lemma 2.6.6 yield the following.

Proposition 2.6.9. *If $K \subset \mathbb{R}^n$ is a convex body with $o \in K$, and $f : S^{n-1} \to \mathbb{R}$ is measurable such that f is bounded or non-negative, then*

$$\int_{S^{n-1}} f\, dV_K = \frac{1}{n}\int_{S^{n-1}} f\cdot h_K \, dS_K \tag{2.36}$$

$$= \frac{1}{n}\int_{\partial K} f(\nu_K(x))\langle \nu_K(x), x\rangle \, d\mathcal{H}^{n-1}(x) \tag{2.37}$$

$$= \frac{1}{n}\int_{S^{n-1}} f(\alpha_K(u))\cdot \varrho_K(u)^n \, d\mathcal{H}^{n-1}(u). \tag{2.38}$$

Proposition 2.6.9 allows us to express an integral with respect to the surface area measure in terms of the radial function and the radial Gauss map, as follows.

Corollary 2.6.10. *If $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, and $g : S^{n-1} \to \mathbb{R}$ is measurable where g is bounded or non-negative, then*

$$\int_{S^{n-1}} g\, dS_K = \int_{S^{n-1}} g(\alpha_K(u))\cdot \frac{\varrho_K(u)^n}{h_K(\alpha_K(u))}\, d\mathcal{H}^{n-1}(u). \tag{2.39}$$

Proof. Apply (2.38) to $f = \frac{g}{h_K}$. ∎

While (2.36) and (2.37) directly follow from the definition of the cone volume measure, the formulas (2.38) and (2.39) first appeared in Huang, Lutwak, Yang, Zhang [419].

It is easy to see that ν_K is continuous on $\partial' K$ for a convex body K (cf. Lemma 1.2.10). Lemma 2.6.11 below, needed in the proof of the weak convergence of the cone volume measure (cf. Proposition 2.6.12), extends this property:

Lemma 2.6.11. *For convex bodies $K_m, K \subset \mathbb{R}^n$ with $K_m \to K$, if $x \in \partial' K$ and $x_m \in \partial K_m$ satisfy $x_m \to x$, and $u_m \in S^{n-1}$ exterior normal to K_m at x_m, then $u_m \to \nu_K(x)$.*

Proof. As S^{n-1} is compact and $\nu_K(x)$ is the unique exterior unit normal to K at x, it is enough to prove that if $\lim_{m\to\infty} u_m = u$, then u is an exterior unit normal to K at x.

For $z \in \operatorname{int} K$, we have $z \in K_m$ for large m, and hence $\langle u_m, z - x_m\rangle \leqslant 0$ for large m. It follows that $\langle u, z - x\rangle \leqslant 0$; therefore, u is an exterior unit normal to K at x. ∎

We recall that for finite Borel measures μ_m, μ on S^{n-1}, μ_m tends weakly* to μ if and only if $\lim_{m\to\infty} \int_{S^{n-1}} f\, d\mu_m = \int_{S^{n-1}} f\, d\mu$ for any continuous function $f : S^{n-1} \to \mathbb{R}$.

Proposition 2.6.12. *If the convex compact sets K_m tend to a convex compact set K in $\mathbb{R}^n$ with $o \in K_m$, $o \in K$, then V_{K_m} tends weakly* to V_K.*

Proof. Let $f : S^{n-1} \to \mathbb{R}$ continuous. If $\dim K \leqslant n-1$, then $\lim_{m\to\infty} |K_m| = 0$ by the continuity of volume (cf. Lemma 1.7.5). Since $V_{K_m}(S^{n-1}) = |K_m|$ and f is bounded, we deduce that $\lim_{m\to\infty} \int_{S^{n-1}} f\, dV_{K_m} = 0 = \int_{S^{n-1}} f\, dV_K$.

Therefore, let K be a convex body, and hence K_m is convex body, as well, for large m. For $\Sigma_K = \{(0,\infty)x : x \in \operatorname{int} K\}$ and $\Theta = \pi_{S^{n-1}}(\Sigma_K \cap \partial' K) \cup \left(S^{n-1} \backslash (\operatorname{cl}\Sigma_K)\right)$, we have $\mathcal{H}^{n-1}(S^{n-1}\backslash\Theta) = 0$ (cf. the Remark after Definition 2.6.8). Since

$$\lim_{m\to\infty} \varrho_{K_m}(u) = \varrho_K(u) \text{ and } \lim_{m\to\infty} \alpha_{K_m}(u) = \alpha_K(u)$$

for $u \in \Theta \cap \Sigma_K$ by Lemma 2.6.11, and $\lim_{m\to\infty} \varrho_{K_m}(u) = 0 = \varrho_K(u)$ if $u \in \Theta \backslash \operatorname{cl}\Sigma_K$, we conclude Proposition 2.6.12 by (2.38) and the continuity of f. ∎

Proposition 2.6.13 (Aleksandrov [4, 11]). *If convex bodies K_m tend to a convex body K in $\mathbb{R}^n$, then S_{K_m} tends weakly* to S_K.*

Remark. Proposition 8.4.1 verifies this for the general case of convex compact sets.

Proof. We may assume that $o \in \operatorname{int} K$, and hence also $o \in \operatorname{int} K_m$ for large m.

Let $f : S^{n-1} \to \mathbb{R}$ continuous. It follows from Lemma 2.6.11 that if $\varrho_K(u) \cdot u \in \partial' K$ for $u \in S^{n-1}$, then $f(\alpha_{K_m}(u)) \cdot \frac{\varrho_{K_m}(u)^n}{h_{K_m}(\alpha_{K_m}(u))}$ tends to $f(\alpha_K(u)) \cdot \frac{\varrho_K(u)^n}{h_K(\alpha_K(u))}$, and hence (2.39) yields that $\lim_{m\to\infty} \int_{S^{n-1}} f\, dS_{K_m} = \int_{S^{n-1}} f\, dS_K$. ∎

Lemma 2.6.14 follows as $\langle \Phi^{-t}x, \Phi y\rangle = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$ and $\Phi \in \mathrm{GL}(n)$.

Lemma 2.6.14. *If $K \subset \mathbb{R}^n$ is a convex body and $\Phi \in \mathrm{GL}(n)$, then $\Phi^{-t}\nu_K(x)$ is an exterior normal at $\Phi x \in \partial(\Phi K)$ for $x \in \partial' K$, and $h_{\Phi K}(u) = h_K(\Phi^t u)$ for $u \in \mathbb{R}^n$.*

Now, we are ready to discuss how the cone volume measure interacts with linear transformations.

Proposition 2.6.15 (Intertwining with Linear Transformations). *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in K$, and let $\Phi \in \mathrm{GL}(n)$.*

(i) $\widetilde{V}_{\Phi K}(\Phi\Xi) = |\det \Phi| \cdot \widetilde{V}_K(\Xi)$ for measurable $\Xi \subset \partial K$, and hence if, in addition, we have $|\det \Phi| = 1$, then $\widetilde{V}_{\Phi K} = \Phi_* \widetilde{V}_K$;

(ii) $V_{\Phi K} = |\det \Phi| \cdot \Psi_* V_K$ where $\Psi : S^{n-1} \to S^{n-1}$ is defined by $\Psi(u) = \frac{\Phi^{-t}u}{\|\Phi^{-t}u\|}$, and hence if $f \geqslant 0$ is measurable on S^{n-1}, then

$$\int_{S^{n-1}} f \, dV_{\Phi K} = |\det \Phi| \cdot \int_{S^{n-1}} f\left(\frac{\Phi^{-t}u}{\|\Phi^{-t}u\|}\right) dV_K(u).$$

Proof. (i) holds by the very definition of $\widetilde{V}_K$ and by change of variables.

For (ii), if $\omega \subset \partial K$ is Borel, then (i), (2.33) and Lemma 2.6.14 yield for $\Xi = \{x \in \partial'(\Phi K) : \nu_{\Phi K}(x) \in \omega\}$ that

$$\begin{aligned} V_{\Phi K}(\omega) = \widetilde{V}_{\Phi K}(\Xi) &= |\det \Phi| \cdot \widetilde{V}_K(\Phi^{-1}\Xi) = |\det \Phi| \cdot V_K(\{\nu_K(\Phi^{-1}x) : x \in \Xi\}) \\ &= |\det \Phi| \cdot V_K\left(\left\{\frac{\Phi^t u}{\|\Phi^t u\|} : u \in \omega\right\}\right) = |\det \Phi| \cdot \Psi_* V_K(\omega). \end{aligned}$$

In turn, the integral formula in (ii) follows from (2.22). ∎

Remark 2.6.16 (Characterization and uniqueness of the cone volume measure). Concerning the characterization of cone volume measure, a problem proposed by Firey [287] in 1974, only partial results exist (see Section 9.3 for a detailed account). For example, any non-trivial absolutely continuous measure on S^{n-1} is a cone volume measure according to Chen, Li, Zhu [195], who actually verify that any finite Borel measure μ on S^{n-1} satisfying the "subspace concentration conditions" (i) and (ii) below is a cone volume measure (cf. Theorem 9.C.1). As a necessary condition (see (9.20) for a simplified statement), Böröczky, Hegedűs [130] characterized the restriction to a pair of antipodal points; namely, assuming that $o \in K$ for a convex body $K \subset \mathbb{R}^n$, $u \in S^{n-1}$, $\alpha = \frac{V_K(u)}{V_K(S^{n-1})}$ and $\beta = \frac{V_K(-u)}{V_K(S^{n-1})}$, if $n = 2$, then $\sqrt{\alpha} + \sqrt{\beta} \leqslant 1$, and if $n \geqslant 3$, then

$$\alpha + \beta + \min_{\varrho > 0} \sum_{i=1}^{n-1} (\alpha \varrho^{-2i} + \beta \varrho^{2i}) \leqslant 1. \tag{2.40}$$

For cone volume measures of polytopes, many important properties are described by Baumbach, Henk [77].

Following partial results by Chou, Wang [201], He, Leng, Li [383], Henk, Schürman, Wills [387], Stancu [687, 688], Xiong [728], the paper by Böröczky, Lutwak, Yang, Zhang [141] characterized even cone volume measures by the following so-called subspace concentration condition (i) and (ii) (cf. Theorem 9.B.4): A finite even non-trivial Borel measure μ on S^{n-1} is a cone volume measure if and only if

(i) $\mu(L \cap S^{n-1}) \leqslant \frac{\dim L}{n} \cdot \mu(S^{n-1})$ for any proper linear subspace $L \subset \mathbb{R}^n$;

(ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$ in (i) is equivalent with the existence of a complementary linear subspace $L' \subset \mathbb{R}^n$ with $\operatorname{supp} \mu \subset L \cup L'$.

Here the non-trivial linear subspaces L and L' are complementary, if $L + L' = \mathbb{R}^n$ and $L \cap L' = \{o\}$. We note that V_K for an o-symmetric convex body $K \subset \mathbb{R}^n$ satisfies (ii) if and only if $K = C + C'$ where $C \subset L^{\perp}$ and $C' \subset L'^{\perp}$ are o-symmetric compact convex sets (cf. Theorem 9.B.4). Böröczky, Henk [132] prove that the subspace concentration conditions (i) and (ii) hold for the cone volume measure of any centered convex body (i.e., for which its centroid is the origin), but in general, the cone volume measure of a centered convex body satisfies some additional conditions (however, a complete characterization is not known, not even in the plane). The cone volume measure of quadrilaterals is characterized by Liu, Lu, Sun, Xiong [527].

In fact, the main interest of Firey [287] was in the uniqueness of the cone volume measure. According to Chen, Li, Zhu [195], there exists an absolutely continuous measure on S^{n-1} that is the cone volume measure of two different convex bodies. However, in line with the "Worn Stone" problem initiated by Firey [287], if the cone volume measure of a convex body K is the Lebesgue measure on S^{n-1}, then K is a centered ball (see Brendle, Choi, Daskalopoulos [158], Ivaki, Milman [441] and Saroglou [648] for different arguments). More generally, even if a function f on S^{n-1} is $C^{0,\alpha}$ close to 1, then there exists a unique convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ such that $V_K = f\, d\mathcal{H}^{n-1}$ according to Chen, Feng, Liu [190] if $n = 3$ and Böröczky, Saroglou [147] if $n \geqslant 4$ (see Hu, Ivaki [414] for another argument). The question of uniqueness of the cone volume measure of o-symmetric convex bodies is a major open problem called the Logarithmic Minkowski Conjecture, asking, for example, whether $V_K = V_C$ for o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with C^1 boundaries implies that $K = C$ (see Section 8.7 and Section 9.4).

2.7 Well approximating polytopes in terms of the Hausdorff distance and the Banach-Mazur distance

We have already seen (cf. (1.17)) that a compact convex set can be arbitraily well approximated by polytopes in terms of the Hausdorff distance. This section discusses approximation by low-complexity polytopes where estimates on the cardinality of ε-nets have a key role. Here, if $X \subset \mathbb{R}^n$ and $\varepsilon > 0$, then an ε-net $\Xi \subset X$ is a discrete set

such that for any $x \in X$ there exists a $y \in \Xi$ with $\|x - y\| \leqslant \varepsilon$. This topic is included in this chapter because some local estimates for the surface area have important role.

We start with polytopal approximation of B^n by inscribed polytopes because in this case, we have the following direct correspondence with ε-nets in S^{n-1}.

Lemma 2.7.1. *If $\varepsilon \in (0, \sqrt{2})$ and $P = \operatorname{conv}\Xi$ for a discrete set $\Xi \subset S^{n-1}$, then the following three statements are equivalent.*

- *Ξ is an ε-net;*
- *$(1 - \frac{\varepsilon^2}{2})B^n \subset P$;*
- *$\delta_H(B^n, P) \leqslant \frac{\varepsilon^2}{2}$.*

Proof. For $y \in S^{n-1}$, we have $S^{n-1} \cap (y + \varepsilon B^n) = \{z \in S^{n-1} : \langle z, y\rangle \geqslant 1 - \frac{\varepsilon^2}{2}\}$. ∎

For ε-nets in S^{n-1} of minimal cardinality, a simple argument gives estimates of the right order in terms of ε.

Lemma 2.7.2. *Let $\varepsilon \in (0, \sqrt{2})$.*

(i) $\#\Xi < 3^n\sqrt{n} \cdot \varepsilon^{-(n-1)}$ for some ε-net $\Xi \subset S^{n-1}$;

(ii) $\#\Xi \geqslant 2(1 - \frac{\varepsilon^2}{2})\sqrt{n}\varepsilon^{-(n-1)}$ for any ε-net $\Xi \subset S^{n-1}$; in particular, $\#\Xi \geqslant \sqrt{n}\varepsilon^{-(n-1)}$ if $\varepsilon \leqslant 1$.

Remark. See Theorem 2.7.7 for improvements.

Proof. For any $z \in S^{n-1}$, projecting the cap $B_z(\varepsilon) = S^{n-1} \cap (z + \varepsilon B^n)$ to $z^\perp$ results in an $(n-1)$-ball of radius between $\varepsilon/\sqrt{2}$ and ε, and hence (2.2) yields

$$\frac{\omega_{n-1}\varepsilon^{n-1}}{\sqrt{2}^{n-1}} < \mathcal{H}^{n-1}(B_z(\varepsilon)) < \left(1 - \frac{\varepsilon^2}{2}\right)^{-1} \omega_{n-1}\varepsilon^{n-1}. \tag{2.41}$$

For (i), let $\Xi = \{z_1, \ldots, z_N\} \subset S^{n-1}$ be maximal with the property that $\|z_i - z_j\| \geqslant \varepsilon$ for $i \neq j$, and hence $\Xi \subset S^{n-1}$ is an ε-net, and the caps $B_{z_i}(\frac{\varepsilon}{2})$, $z_i \in \Xi$, only meet in boundary points. It follows from (2.41), the estimate $\frac{\omega_n}{\omega_{n-1}} < \sqrt{\frac{2\pi}{n}}$ (cf. (10.3)) and $\mathcal{H}^{n-1}(S^{n-1}) = n\omega_n$ that $N \leqslant \frac{n\omega_n}{\omega_{n-1}}\sqrt{2}^{n-1}(\frac{\varepsilon}{2})^{-(n-1)} < 3^n\sqrt{n} \cdot \varepsilon^{-(n-1)}$.

For (ii), if $\Xi \subset S^{n-1}$ is an ε-net, then the caps $B_{z_i}(\varepsilon)$, $z_i \in \Xi$ cover S^{n-1}, and hence (2.41) yields $N \geqslant (1 - \frac{\varepsilon^2}{2})\frac{n\omega_n}{\omega_{n-1}}\varepsilon^{-(n-1)} > 2(1 - \frac{\varepsilon^2}{2})\sqrt{n}\varepsilon^{-(n-1)}$ by $\frac{\omega_n}{\omega_{n-1}} > \sqrt{\frac{2\pi}{n+1}}$ (cf. (10.3)). ∎

Combining Lemmas 2.7.1 and 2.7.2, and using polarity for circumscribed polytopes (cf. Remark 1.9.4), we deduce the following estimates:

Corollary 2.7.3. *For $c = 16$ and $k \geqslant 4^n$, we have the following:*

(i) If $P_k \subset B^n$ is a polytope with k vertices minimizing $\delta_H(B^n, P_k)$, then

$$c^{-1}k^{\frac{-2}{n-1}} \leqslant \delta_H(B^n, P_k) \leqslant ck^{\frac{-2}{n-1}}.$$

(ii) If $P_{(k)} \supset B^n$ is a polytope with k facets minimizing $\delta_H(B^n, P_{(k)})$, then

$$c^{-1}k^{\frac{-2}{n-1}} \leqslant \delta_H(B^n, P_{(k)}) \leqslant ck^{\frac{-2}{n-1}}.$$

Given the estimates of Lemma 2.7.2 for ε-nets, we are ready to construct for any convex body a well approximating polytope of low complexity.

Theorem 2.7.4. *If $C \subset \mathbb{R}^n$ convex body and $\eta \in (0,1)$, then there exist a polytope $P_k \subset C$ with at most k vertices and a $P_{(k)} \supset C$ with at most k facets with $k \leqslant 6^n\sqrt{n} \cdot \eta^{-\frac{n-1}{2}}$ such that*

$$\delta_H(C, P_k),\ \delta_H(C, P_{(k)}) \leqslant \eta \cdot \operatorname{diam} C;$$

or in other words, for any $k > 6^n\sqrt{n}$, there exist a polytope $P_k \subset C$ with at most k vertices and a $P_{(k)} \supset C$ with at most k facets with

$$\delta_H(C, P_k),\ \delta_H(C, P_{(k)}) \leqslant 2^{12}\operatorname{diam} C \cdot k^{\frac{-2}{n-1}}.$$

Remarks. Bronshtein, Ivanov [160] proved the existence of a polytope containg C, having at most k vertices and satisfying a similar estimate. The estimates of Theorem 2.7.4 are of the right order in general by Corollary 2.7.3. We note that if C is a convex body with C^2 boundary, then the limits $\lim_{k\to\infty} k^{\frac{2}{n-1}}\delta_H(C, P_k)$ and $\lim_{k\to\infty} k^{\frac{2}{n-1}}\delta_H(C, P_{(k)})$ exist and are positive for the best approximating polytopes $P_k \subset C$ with at most k vertices and $P_{(k)} \supset C$ with at most k facets (see Section 8.10 and Böröczky [117]).

Proof. We may assume that $\operatorname{diam} C = 1$ and $C \subset B^n$, and hence $C + B^n \subset 2B^n$.

For $\varepsilon = \sqrt{\eta}$ and $k = \left\lfloor 3^n\sqrt{n}(\frac{\varepsilon}{2})^{-(n-1)} \right\rfloor \leqslant 6^n\sqrt{n} \cdot \eta^{-\frac{n-1}{2}}$ where $\lfloor t \rfloor$ is the largest integer not larger than t, Lemma 2.7.2 yields the existence of an ε-net $\widetilde{\Xi} = \{z_1, \dots, z_k\} \subset 2S^{n-1}$. In turn Lemma 1.2.13 yields that $\Xi = \{y_1, \dots, y_k\} \subset \partial(C + B^n)$ is an ε-net where $y_i = \Pi_{C+B^n} z_i$ is the closest point.

Next, let $x_i = \Pi_C y_i$, and hence $y_i = x_i + u_i$ for some $u_i \in S^{n-1}$ where $\langle u_i, x_i - p\rangle \geqslant 0$ for $p \in C$. It follows that $P_k = \operatorname{conv}\{x_1, \dots, x_N\} \subset C$, and for $\delta = \delta_H(C, P_k)$, there exists an $x \in \partial C$ such that $\|x - w\| = \delta$ for $w = \Pi_{P_k} x$ where

$$\langle u, x - p\rangle \geqslant \delta \text{ for } u = \tfrac{x-w}{\|x-w\|} \in S^{n-1} \text{ and } p \in P_k. \tag{2.42}$$

Since x is a farthest point of C from P_k, u is an exterior normal to C at x, thus $y = x + u \in \partial(C + B^n)$, which in turn yields that the existence of a $y_i \in \Xi$ with $\|x - y_i\| \leqslant \varepsilon$.

It follows using (2.42) at the end that

$$\begin{aligned}\varepsilon^2 &\geqslant \|y-y_i\|^2 = \|x-x_i\|^2 + \|u-u_i\|^2 + 2\langle u, x-x_i\rangle + 2\langle u_i, x_i - x\rangle \\ &\geqslant 2\langle u, x-x_i\rangle \geqslant 2\delta,\end{aligned} \tag{2.43}$$

and hence $\delta_H(C,P_k) \leqslant \frac{\varepsilon^2}{2} = \frac{\eta}{2}$.

Next let $P_{(k)} = \{p \in \mathbb{R}^n : \forall \langle p, u_i\rangle \leqslant \langle x_i, u_i\rangle\}$, and hence the facets of $P_{(k)}$ touch C at $x_1, \dots, x_N$. For $z \in P_{(k)} \backslash C$, we have $z = x + \delta u$ for $\delta > 0$ where $x = \Pi_C z$ and $u = \frac{z-x}{\|z-x\|} \in S^{n-1}$, and our task is to provide a good upper bound on δ. Since $y = x + u \in \partial(C + B^n)$, there exists $y_i \in \Xi$ with $\|x - y_i\| \leqslant \varepsilon$, and (2.43) yields that

$$\|u-u_i\|^2 + 2\langle u_i, x_i - x\rangle \leqslant \varepsilon^2.$$

We deduce that $\langle u_i, u\rangle = 1 - \frac{1}{2}\|u-u_i\|^2 \geqslant 1 - \frac{\varepsilon^2}{2} > \frac{1}{2}$, thus $\langle u_i, x_i\rangle \geqslant \langle u_i, z\rangle = \langle u_i, x\rangle + \delta\langle u_i, u\rangle$ yields $\frac{\varepsilon^2}{2} \geqslant \langle u_i, x_i - x\rangle \geqslant \delta\langle u_i, u\rangle > \frac{\delta}{2}$. As $z \in P_{(k)} \backslash C$ was an arbitrary point, we conclude that $\delta_H(C, P_{(k)}) \leqslant \varepsilon^2 = \eta$. ∎

Next we show that any compact convex set can be reasonably well approximated by n-dimensional polytopes containing the set.

Remark 2.7.5 (Approximating by n-polytopes containing the compact convex set). Given a compact convex set $C \subset \mathbb{R}^n$, the simple construction in (1.17) and the more sophisticated Theorem 2.7.4 provides a polytope $Q \subset C$ with at most m vertices such that $\delta = \delta_H(Q, C)$ is small (and it is possible to ensure that Q is o-symmetric if C is o-symmetric), and hence

$$\begin{aligned}C \subset Q + M \text{ for the } n\text{-polytope } Q + M \text{ with at most } 2nm \text{ vertices} \\ \text{and satisfying } \delta_H(Q+M, C) \leqslant \sqrt{n}\,\delta\end{aligned} \tag{2.44}$$

where for an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$, $M = \operatorname{conv}\{\pm\sqrt{n}\,\delta \cdot e_i\}_{i=1}^n$ is a cross-polytope with $2n$ vertices and $\delta B^n \subset M$. In particular, if $\dim C = d \geqslant 2$ and $k \geqslant 2n6^d\sqrt{d}$, then Theorem 2.7.4 and the estimate $(2d)^{\frac{2}{d-1}} \leqslant 16$ yield an n-polytope $\widetilde{P}_k \supset C$ (o-symmetric if C is o-symmetric) with at most k vertices and satisfying the estimates

$$\delta_H(C, \widetilde{P}_k) \leqslant 2^{16}\sqrt{n} \cdot \operatorname{diam} C \cdot k^{\frac{-2}{d-1}}. \tag{2.45}$$

Let us shortly discuss what Theorem 2.7.4 implies for the Banach-Mazur distance. For o-symmetric convex bodies $K, C \subset \mathbb{R}^n$, their Banach-Mazur distance $\delta_{\mathrm{BM}}(K, C)$ is the minimum of $\log\lambda$ where there exists $\Phi \in \mathrm{GL}(n)$ such that $K \subset \Phi C \subset \lambda K$, which is a metric on the equivalence classes of o-symmetric convex bodies with respect to linear transformations. This definition can be extended to a pair of any convex bodies

$K, C \subset \mathbb{R}^n$ as follows: The Banach-Mazur distance $\delta_{\mathrm{BM}}(K, C)$ of K and C is the minimum of $\log \lambda$ where there exist $\Phi \in \mathrm{GL}(n)$, $x \in K$ and $y \in C$ such that

$$K - x \subset \Phi(C - y) \subset \lambda(K - x).$$

According to Kannan, Lovász, Simonovits [454] (see Lemma 1.11.5), for any convex body $C \subset \mathbb{R}^n$ with centroid σ_C, there exist a linear transform $\Phi \in \mathrm{GL}(n)$ such that

$$B^n \subset \Phi(C - \sigma_C) \subset nB^n, \tag{2.46}$$

and hence the Banach-Mazur distance of C from any ellipsoid is at most $\log n$.

Corollary 2.7.6. *For any convex body $C \subset \mathbb{R}^n$ and $\varepsilon \in (0, 1)$, there exist a polytope P_k with at most k vertices and a polytope $P_{(k)}$ with at most k facets with $k \leqslant 12^n n^{n/2} \varepsilon^{-\frac{n-1}{2}}$ such that*

$$C - \sigma_C \subset P_k, P_{(k)} \subset (1 + \varepsilon)(C - \sigma_C),$$

and hence the Banach-Mazur distance of C from both P_k and $P_{(k)}$ is at most ε.

Proof. According to (2.46), we may assume that $\sigma_C = o$ and

$$B^n \subset C \subset nB^n,$$

and hence $\operatorname{diam} C \leqslant 2n$. Now Theorem 2.7.4 with $\eta = \frac{\varepsilon}{4n}$, the estimate $\eta \dim C \leqslant \frac{\varepsilon}{2}$ and $B^n \subset C$ yield the existence a polytope $P'_k \subset C$ with at most k vertices and a polytope $P_{(k)} \supset C$ with at most k facets with $k \leqslant 6^n \sqrt{n} \cdot \eta^{-\frac{n-1}{2}} \leqslant 12^n n^{\frac{n}{2}} \varepsilon^{-\frac{n-1}{2}}$ such that

$$\left(1 - \frac{\varepsilon}{2}\right) C \subset P'_k \subset C \subset P_{(k)} \subset \left(1 + \frac{\varepsilon}{2}\right) C,$$

and hence we choose $P_k = (1 + \varepsilon) P'_k$.

■

In the second half of this section, we discuss Hausdorff approximation of B^n by polytopes of low complexity in detail, since for that case very precise estimates and sometimes even exact solutions are known. We note that the inscribed regular cross-polytope $C_n = \operatorname{conv}\{\pm e_1, \ldots, \pm e_n\} \subset B^n$ for an orthonormal basis $e_1, \ldots, e_n$ is a polytope has $2n$ vertices and $\frac{1}{\sqrt{n}} B^n \subset C_n$. Therefore, first we consider the case when the number of vertices is at least $2n$. The following estimates were proved by Böröczky, Wintsche [150]:

Theorem 2.7.7. *If $n^{-1/2} \leqslant r < 1$, then there exists a polytope P_k with k vertices satisfying $rB^n \subset P_k \subset B^n$ (and hence $\delta_H(P_k, B^n) \leqslant 1 - r$) and*

$$k \leqslant c \cdot r(1 - r^2)^{\frac{1-n}{2}} \cdot n^{\frac{3}{2}} \log(1 + r^2 n)$$

where $c > 1$ is an absolute constant. In particular, if $0 < \varepsilon < 1$, then the minimal cardinality N_ε of an ε-net in S^{n-1} satisfies

$$n^{1/2}\varepsilon^{-(n-1)} \leqslant N_\varepsilon \leqslant cn^{3/2}\log n \cdot \varepsilon^{-(n-1)},$$

and even $N_\varepsilon \geqslant c^{-1}n^{3/2}\varepsilon^{-(n-1)}$ if $0 < \varepsilon < \frac{1}{\sqrt{n}}$.

Remarks. Lemma 2.7.2 (ii) implies that $k \geqslant c^{-1}r(1-r^2)^{\frac{1-n}{2}} \cdot n^{\frac{1}{2}}$.

Concerning the estimates in Theorem 2.7.8, Theorem 2.7.7 yields the lower bound in (2.47), and the volume estimates in Theorem 6.8.3 in Chapter 6 imply the upper bound in (2.47). In addition, (2.48) follows from (2.47) by polarity.

Theorem 2.7.8. *For an absolute constant $c > 1$ and $2n \leqslant k \leqslant 2^n$, if ϱ_k is the maximal $\varrho > 0$ with the property that there exists a polytope $P_k \subset B^n$ with k vertices and $\varrho B^n \subset P_k$, and R_k is the minimal $R > 0$ with the property that there exists a polytope $P_{(k)} \supset B^n$ with k faces and $P_{(k)} \subset RB^n$, then*

$$c^{-1}\sqrt{\frac{\log\frac{k}{n}}{n}} \leqslant \varrho_k \leqslant c\sqrt{\frac{\log\frac{k}{n}}{n}}; \tag{2.47}$$

$$c^{-1}\sqrt{\frac{n}{\log\frac{k}{n}}} \leqslant R_k \leqslant c\sqrt{\frac{n}{\log\frac{k}{n}}}. \tag{2.48}$$

Finally, let us discuss approximation by inscribed polytopes of at most $2n$ vertices.

Proposition 2.7.9 (Steiner). *If $P \subset B^n$ is a simplex and $rB^n \subset P$ for $r > 0$, then $r \leqslant \frac{1}{n}$, with equality if and only if P is an inscribed regular simplex.*

Remark. The proof of Proposition 2.7.9 yields that even $r(P) \leqslant \frac{1}{n}$ for the radius $r(P)$ of a largest ball in P.

Proof. If $Q \subset B^n$ is an inscribed regular simplex, then $r(Q) = \frac{1}{n}$, and the origin is the center of the inscribed ball. Therefore, it is sufficifient to prove that $r(P) \leqslant \frac{1}{n}$ for a simplex $P \subset B^n$ with equality if and only if P is an inscribed regular simplex.

We may assume that P has maximal inradius, and hence each vertex of P lies on S^{n-1}. We suppose that there exist vertices x_0, x_1, x_2 of P such that $\|x_1 - x_0\| \neq \|x_2 - x_0\|$, and seek a contradiction. We apply Steiner symmetrization with respect to $u^\perp$ for of $u = \frac{x_2 - x_1}{\|x_2 - x_1\|}$. Then $P' = \Theta_{u^\perp}P \subset B^n$ is a simplex according to Lemma 1.10.12, $r(P') \geqslant r(P)$, and the vertex $x_0|u^\perp$ of P' lies in $\operatorname{int}B^n$. Therefore the simplex $P' \subset B^n$ does not have maximal inradius, which is absurd. ∎

Proposition 2.7.10. *If $P \subset B^n$ is a polytope with at most $n+2$ vertices and $rB^n \subset P$ for $r > 0$, then $r \leqslant (\lceil\frac{n}{2}\rceil^2 + \lfloor\frac{n}{2}\rfloor^2)^{\frac{-1}{2}}$, with equality if and only if P is the convex hull of*

a $\lceil \frac{n}{2} \rceil$*-dimensional and a* $\lfloor \frac{n}{2} \rfloor$*-dimensional centered regular simplex of circumradius one whose affine hulls are orthogonal.*

Remark. The proof of Proposition 2.7.10 yields $r(P) \leqslant (\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2)^{\frac{-1}{2}}$.

Proof. If $Q \subset B^n$ is an optimal polytope as proposed by Proposition 2.7.10, then $r(Q) = (\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2)^{\frac{-1}{2}}$ and the origin is the center of the inscribed ball of Q. Therefore, it is sufficifient to prove that $r(P) \leqslant (\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2)^{\frac{-1}{2}}$ for a polytope $P \subset B^n$ with at most $n+2$ vertices with equality as in Proposition 2.7.10.

We may assume that P has maximal inradius, which is than larger the inradius of any simplex in B^n according to Proposition 2.7.9. Therefore, P has $n+2$ vertices, and for any ball of radius $r(P)$ in P, each vertex of P lies in a facet touching that ball, and hence each vertex of P lies on S^{n-1} by the maximality property of P. According to Radon's Theorem Proposition 1.1.8, there exist $m, k \geqslant 1$ with $m + k = n$ such that the vertices of P are $x_0, \dots, x_m, y_0, \dots, y_k$ and $S \cap S' \neq \varnothing$ for $S = \operatorname{conv}\{x_0, \dots, x_m\}$ and $S' = \{y_0, \dots, y_k\}$.

We suppose that there exist vertices x_i, x_j and y_ℓ such that $\|x_i - y_\ell\| \neq \|x_j - y_\ell\|$, and seek a contradiction. We apply Steiner symmetrization with respect to $u^\perp$ for of $u = \frac{x_i - x_j}{\|x_i - x_j\|}$. Then $P' = \Theta_u P \subset B^n$ is a polytope with $n+2$ vertices according to Lemma 1.10.12, $r(P') \geqslant r(P)$, and the vertex $y_\ell | u^\perp$ of P' lies in $\operatorname{int} B^n$. Therefore the simplex $P' \subset B^n$ does not have maximal inradius, which is absurd.

In particular, $\|x_i - y_\ell\| = \|x_j - y_\ell\|$, and similarly $\|x_i - y_\ell\| = \|x_i - y_p\|$ for any $i, j = 1, \dots, m$ and $\ell, p = 1, \dots, k$, and hence S is a centered regular m-simplex and S' is a centered regular k-simplex of circumradius one, and their affine hulls are orthogonal. As the relative indarius of S is $\frac{1}{m}$, and the relative inradius of S' is $\frac{1}{k}$, the symmetries of P yield that $r(P) = (k^2 + m^2)^{\frac{-1}{2}}$, which is maximal when $\{k, m\} = \{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor\}$. ∎

If the number k of vertices of the approximating polytope in $\mathbb{R}^n$ satisfies $n+1 \leqslant k \leqslant 2n$, then Tikhomirov [702] managed to find the essentially optimal estimates.

Theorem 2.7.11 (Tikhomirov). *For some absolute constant $c > 1$, if $n+1 \leqslant k \leqslant 2n$ and r_k is maximal with the property that there exists a polytope P_k with at most k vertices satisfying $r_k B^n \subset P_k \subset B^n$, then*

$$c^{-1} \cdot \frac{\sqrt{k-n}}{n} \leqslant r_k \leqslant c \cdot \frac{\sqrt{k-n}}{n}. \tag{2.49}$$

Remark. The lower bound follows from considering polytopes that are direct sums of lower dimensional centered regular simplices of circumradius one whose dimensions are either $\lceil \frac{n}{k-n} \rceil$ or $\lfloor \frac{n}{k-n} \rfloor$. This polytope might be extremal in (2.49).

2.8 Comments to Section 2

The Cauchy formula (2.9) was proved by Cauchy [181,182] in $\mathbb{R}^2$ and $\mathbb{R}^3$ in the middle of the 19th century. It was put in the modern setting in the 1920's, for example, by Kubota [489].

For throrough discussion of the relation between the surface area of a compact convex set and outer Minkowski content (cf. Theorem 2.3.6) see Ambrosio, Colesanti, Villa [23].

If a compact $X \subset \mathbb{R}^n$ has non-empty interior and Lipschitz boundary (cf. Definition 4.1.4), then the Divergence Theorem due to Federer [260], Theorem 4.5.6, says the following.

Theorem 2.8.1 (Divergence Theorem). *If a compact $X \subset \mathbb{R}^n$ has Lipschitz boundary and $T : X \to \mathbb{R}^n$ is locally Lipschitz, then*

$$\int_X \operatorname{div} T = \int_{\partial X} \langle T, \nu_X \rangle \, d\mathcal{H}^{n-1}.$$

For any compact $X \subset \mathbb{R}^n$ with Lipschitz boundary and locally Lipschitz $T : X \to \mathbb{R}^n$, the Divergence Theorem 2.8.1 yields the analogues of Lemma 2.1.2 (i) (taking $T(x) = u$) and Theorem 2.1.5 (taking $T(x) = x$), and hence Corollary 2.1.3, as well; namely;

- $\displaystyle\int_{\partial X} \langle \nu_X, u \rangle \, d\mathcal{H}^{n-1} = 0$ for $u \in \mathbb{R}^n$;
- $\displaystyle\int_{\partial X} \nu_X \, d\mathcal{H}^{n-1} = o$;
- $\displaystyle |X| = \frac{1}{n} \int_{\partial X} \langle \nu_X(x), x \rangle \, d\mathcal{H}^{n-1}(x)$.

The extremal property of balls with respect to the Isoperimetric problem was known to the ancient Greeks; for example, Zenodorus (circa 200 BC - 140 BC) suggested an argument using polygons in the plane, and even claimed that spheres are optimal in three dimensions (cf. Blasjö [98]). We refer the reader to Chapter 4 and the comments therein for more details on the interesting history of the Isoperimetric inequality.

Concerning the surface area measure, the basic properties were established by Aleksandrov [4–6, 11] in the 1930's extending Minkowski's ideas [590, 591] (see Section 8.4, Schneider [661], Chapter 4 or Hug, Weil [433] for more detailed discussions). Surface area measures, together with their uniqueness, were characterized by Minkowski [589, 590] if the measure is discrete (and hence the convex body is a polytope) or absolutely continuous. Minkowski's characterization was extended to any general measure by Aleksandrov [5,6,11] (see Remark 2.5.12 and Theorem 9.2.3).

Stability of the surface area measure in terms of the closeness of the convex bodies up to translation is discussed by Hug, Schneider [430, 432].

Livshyts [528] generalized the notion of surface area measure to the weighted case. Given a measure $d\mu = \varphi\, d\mathcal{H}^n$ for a continuous density function φ on $\mathbb{R}^n$, the *weighted surface area measure* $S_{(\mu,K)}$ of a convex body $K \subset \mathbb{R}^n$ is defined in a way such that

$$S_{(\mu,K)}(\omega) = \int_{\nu_K^{-1}(\omega)} \varphi\, \mathcal{H}^{n-1} \tag{2.50}$$

for a measurable $\omega \subset S^{n-1}$ (note that this notion is unrelated to the L_p surface area measure $S_{K,p}$, $p \in \mathbb{R}$, discussed in Section 9.3). For any convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int} C$, Livshyts [528] obtains the variational formula

$$\lim_{\varrho \to 0^+} \frac{\mu(K + \varrho C) - \mu(K)}{\varrho} = \int_{S^{n-1}} h_C \, dS_{(\mu,K)}. \tag{2.51}$$

Various results about the weighted surface area measure are proved by Kryvonos, Langharst [488] and Fradelizi, Langharst, Madiman, Zvavitch [301].

The cone volume measure was introduced by Firey [287], and has been a widely used tool since the paper Gromov, Milman [341]; see, for example, Barthe, Guédon, Mendelson, Naor [72], Ludwig [540], Naor [597], Paouris, Werner [613]. The still open Logarithmic Minkowski Problem about characterization of cone volume measure was posed by Firey [287], and it has been solved in the even case by Böröczky, Lutwak, Yang, Zhang [141] (see Chapter 9 for more details about the Cone Volume Measure).

Concerning best approximation of a convex body K with C^2 boundary by polytopes of low complexity, if $P_m \subset K$ is a polytope with at most m vertices, and $P_{(m)} \supset K$ is a polytope with at most m facets minimizing $\delta_H(K, P_m)$ and $\delta_H(K, P_{(m)})$, then Böröczky [117] determine the finite positive limits

$$\lim_{m\to\infty} m^{\frac{2}{n-1}} \delta_H(K, P_m) \quad \text{and} \quad \lim_{m\to\infty} m^{\frac{2}{n-1}} \delta_H(K, P_{(m)}).$$

For the earlier history of polytopal approximation of smooth convex bodies in terms of the Hausdorff metric, see Gruber [347, 349]. More recently, Naszódi, Nazarov, Ryabogin [599] prove an estimate for any centered convex body in terms of dilation distance that works for polytopal approximation using reasonably large number of vertices.

Concerning related polytopal approximation of a ball using smaller number of vertices, Bárány, Füredi [63] contains many estimates (see also Böröczky, Wintsche [150] and Galicer, Litvak, Merzbacher, Pinasco [323]). A related and even more intensively investigated topic is polytopal approximation in terms of volume difference that is discussed in Section 6.8 (see also Prochno, Schütt, Werner [628]).

2.A Supplement: Divergence Theorem for convex bodies

In this section, we present the elementary argument leading to the Divergence Theorem 2.1.4 in the case of convex bodies.

Theorem 2.A.1 (Divergence Theorem for Convex bodies). *If $K \subset \mathbb{R}^n$ is a convex body, and $T : K \to \mathbb{R}^n$ is locally Lipschitz, then*

$$\int_K \operatorname{div} T = \int_{\partial K} \langle T, \nu_K \rangle \, d\mathcal{H}^{n-1}.$$

Proof. Let $u_1, \dots, u_n$ be an orthonormal basis of $\mathbb{R}^n$, and let $T_i(x) = \langle T(x), u_i \rangle$ for $x \in K$ and $\nu_i(x) = \langle \nu_K(x), u_i \rangle$ for $x \in \partial' K$, $i = 1, \dots, n$. It is sufficient to prove that for any fixed $i \in \{1, \dots, n\}$, we have

$$\int_K \partial_i T_i = \int_{\partial K} T_i \cdot \nu_i \, d\mathcal{H}^{n-1}. \tag{2.52}$$

Let $\Omega = (\operatorname{int} K)\,|u_i^{\perp}$, and let ψ_+ and ψ_- be concave and convex functions on Ω such that $\operatorname{int} K = \{z + tu_i : z \in \Omega \ \&\ \psi_-(z) < t < \psi_+(z)\}$. It follows that

$$\begin{array}{rlll} \text{for } X_- = & \{z + \psi_-(z)\, u_i : z \in \Omega\}, & \nu_i(x) < 0 & \text{for} \quad x \in X_- \cap \partial' K \\ \text{for } X_+ = & \{z + \psi_+(z)\, u_i : z \in \Omega\}, & \nu_i(x) > 0 & \text{for} \quad x \in X_+ \cap \partial' K \\ \text{for } X_0 = & \partial K \backslash (X_+ \cup X_-), & \nu_i(x) = 0 & \text{for} \quad x \in X_0 \cap \partial' K. \end{array}$$

We deduce from the Fubini and the Newton-Leibniz theorems, and (2.2) that

$$\begin{aligned} \int_K \partial_i T_i &= \int_\Omega \int_{\psi_-(z)}^{\psi_+(z)} \partial_i T(z + tu_i)\, dt dz = \int_\Omega T_i(z + \psi_+(z)u_i) - T_i(z + \psi_-(z)u_i)\, dz \\ &= \int_{X^+} T_i \cdot \langle \nu_K, u_i \rangle \, d\mathcal{H}^{n-1} - \int_{X^-} T_i \cdot \langle \nu_K, -u_i \rangle \, d\mathcal{H}^{n-1} \\ &= \int_{\partial K} T_i \cdot \nu_i \, d\mathcal{H}^{n-1}. \end{aligned}$$

■

2.B The Projection Body and the Isoperimetric Inequality

Let $K \subset \mathbb{R}^n$ be a convex body. It follows from Proposition 1.6.9 and Lemma 2.1.2 that there exists a so-called *projection body* ΠK whose support function at $u \in \mathbb{R}^n$ is the area $\mathcal{H}^{n-1}(K|u^{\perp})$ of the projection onto $u^{\perp}$; namely,

$$h_{\Pi K}(u) = \mathcal{H}^{n-1}(K|u^{\perp}) = \frac{1}{2} \int_{\partial' K} |\langle \nu_K, u \rangle| \, d\mathcal{H}^{n-1} = \frac{1}{2} \int_{S^{n-1}} |\langle v, u \rangle| \, dS_K(v).$$

In particular, the projection body ΠK is an o-symmetric zonoid, which is a zonotope if K is a polytope (see Example 1.6.3). The existence of the projection body was probably known to Minkowski himself, and it is discussed in the 1934 classic Bonnesen, Fenchel [107]. In this section, we survey just a few properties of the projection body, see Gardner [314] and Schneider [661], Section 10.9 for more information in general, and, in addition, Kryvonos, Langharst [488] and Langharst, Roysdon, Zvavitch [492] for recent extensions of the notion.

A characteristic property of the projection body (cf. Ludwig [538]) proved by Petty [616] is that how it intertwines with linear transformations; namely, if $\Phi \in \mathrm{SL}(n)$, then

$$\Pi(\Phi K) = \Phi^{-t}\Pi K.$$

Partially due to this property, the projection body has a central role in affine invariant inequalities. For example, according to the Zhang projection body inequality in Zhang [733] (the lower bound) and the Petty projection inequality in Petty [617] (the upper bound), writing $\Pi^* K = (\Pi K)^*$ to denote the polar of the projection body, we have

$$\binom{2n}{n} n^{-n} \leqslant |K|^{n-1} |\Pi^* K| \leqslant \left(\frac{\omega_n}{\omega_{n-1}}\right)^n \tag{2.53}$$

where equality holds in the lower bound if and only if K is a simplex, and equality holds in the upper bound if and only if K is an ellipsoid.

What may sound surprising, the Petty projection inequality (the upper bound in (2.53)) for the affine invariant quantity $|K|^{n-1} |\Pi^* K|$ yields the Isoperimetric Inequality

$$P(K) \geqslant n\omega_n^{\frac{1}{n}} |K|^{\frac{n-1}{n}} \tag{2.54}$$

(cf. Theorem 2.4.1). To show this, we observe that combining the Petty projection inequality, Proposition 1.9.2 and Lemma 1.11.6 with the Jensen inequality (10.6) leads to the inequality

$$\left(\frac{|K|}{\omega_n}\right)^{\frac{n-1}{n}} \leqslant \frac{1}{\omega_{n-1}} \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \mathcal{H}^{n-1}(K|u^\perp)^p \, du\right)^{\frac{1}{p}} \tag{2.55}$$

for $p \geqslant -n$, where the $p = -n$ case is the Petty projection inequality, and the $p = 1$ case is the Isoperimetric Inequality (2.54) by the Cauchy formula (2.9).

Finally, we close the section by stating the fundamental 50-year-old conjecture by Petty [617], which states that

$$|\Pi K| \geqslant \frac{\omega_{n-1}^n}{\omega_n^{n-2}} \cdot |K|^{n-1}$$

where equality holds if K is an ellipsoid. While the conjecture is still wide open, related local estimates have been obtained by Saroglou, Zvavitch [650], Ivaki [437] and Ortega-Moreno, Schuster [609].

Chapter 3

The Brunn-Minkowski and the Prékopa-Leindler inequalities in the measurable case

The main goal of this chapter is to prove the Brunn-Minkowski inequality for measurable sets, and its functional analogue, the Prékopa-Leindler inequality. We also provide some typical applications of these inequalities.

3.1 The Brunn-Minkowski inequality for measurable sets

Since this chapter can be read essentially independently from the previous ones, we recall some definitions from Chapter 1 about convexity.

Definition 3.1.1 (Convex sets).

- $X \subset \mathbb{R}^n$ convex if $(1-\lambda)x + \lambda y \in X$ for $x, y \in X$ and $\lambda \in [0,1]$;
- $K \subset \mathbb{R}^n$ is a convex body if K convex, compact and int $K \neq \varnothing$;
- Convex bodies $K, C \subset \mathbb{R}^n$ are homothetic if $C = \gamma K + z$ for some $z \in \mathbb{R}^n$ and $\gamma > 0$.

Definition 3.1.2 (Minkowski combination). If $X, Y \subset \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, then

$$\alpha X + \beta Y = \{\alpha x + \beta y : x \in X \text{ and } y \in Y\}.$$

Remark. $\alpha X + \beta Y$ may not be measurable even if X and Y are measurable (but measurable, if X and Y are Borel, cf. Section 3.9). However, it is compact if X, Y are compact, it is convex if X, Y are convex, and it is open if either X or Y is open and $\alpha, \beta \neq 0$. We also note that if $X \subset \mathbb{R}^n$ is convex and $\lambda \in (0,1)$, then $(1-\lambda)X + \lambda X = X$.

We write $|X|_*$ to denote the inner Lebesgue measure of an $X \subset \mathbb{R}^n$; namely, it is the maximum of the Lebesgue measure of any measurable subset of X (see the Appendix Chapter 10). We have already provided two proofs of the core Brunn-Minkowski inequality below in the case of convex bodies in Chapter 1, and we are going to provide several other proofs in the general case - due to Lusternik [548] in 1935 - in this chapter.

Theorem 3.1.3 ("Classical" Brunn-Minkowski inequality). *If $X, Y \subset \mathbb{R}^n$ measurable and $\alpha, \beta \geqslant 0$, then*

$$|\alpha X + \beta Y|_*^{\frac{1}{n}} \geqslant \alpha |X|^{\frac{1}{n}} + \beta |Y|^{\frac{1}{n}}. \tag{3.1}$$

Equality holds assuming $\alpha, \beta, |X|, |Y| > 0$ if and only if there exist homothetic convex bodies $K \supset X$ and $C \supset Y$ with $|K \backslash X| = 0$ and $|C \backslash Y| = 0$, and hence $\alpha K + \beta C$ is homothetic to K and C, contains $\alpha X + \beta Y$ and $|\alpha X + \beta Y|_* = |\alpha K + \beta C|$.

While the Brunn-Minkowski inequality is one of the most widely used geometric inequalities, Theorem 1.11.7 shows that it is far from optimal if for example $|K| = |C|$ and $|K \cap C|$ is small for centered convex bodies $K, C \subset \mathbb{R}^n$ (see also Section 3.6).

Lemma 3.1.4 (Equivalent forms of the Brunn-Minkowski inequality). *The following statements are equivalent assuming that they hold for any measurable $X, Y \subset \mathbb{R}^n$ with $|X|, |Y| > 0$.*

(i) $|\alpha X + \beta Y|_*^{\frac{1}{n}} \geqslant \alpha |X|^{\frac{1}{n}} + \beta |Y|^{\frac{1}{n}}$ *for* $\alpha, \beta > 0$;

(ii) $|X + Y|_*^{\frac{1}{n}} \geqslant |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}$;

(iii) $|(1-\lambda) X + \lambda Y|_*^{\frac{1}{n}} \geqslant (1-\lambda)|X|^{\frac{1}{n}} + \lambda |Y|^{\frac{1}{n}}$ *for* $\lambda \in (0,1)$;

(iv) $|(1-\lambda) X + \lambda Y|_* \geqslant |X|^{1-\lambda} |Y|^{\lambda}$ *for* $\lambda \in (0,1)$.

Equality holds in (i), (ii) or (iii) if and only if there exist homothetic convex bodies $K \supset X$ and $C \supset Y$ with $|K \backslash X| = 0$ and $|C \backslash Y| = 0$. Equality holds in (iv) if and only if there exist $z \in \mathbb{R}^n$ and convex bodies $K \supset X$ and $C \supset Y$ with $|K \backslash X| = 0$, $|C \backslash Y| = 0$ and $K = C + z$.

Proof. (i), (ii) and (iii) are equivalent as $|\alpha X| = \alpha^n |X|$ for $\alpha > 0$, and (iii) yields (iv) by the AM-GM inequality.

To show that (iv) implies (i), set $\alpha_0 = |X|^{\frac{1}{n}}$, $\beta_0 = |Y|^{\frac{1}{n}}$, $X_0 = X/\alpha_0$, $Y_0 = Y/\beta_0$, and hence $|X_0| = |Y_0| = 1$. For $\lambda = \frac{\beta\beta_0}{\alpha\alpha_0 + \beta\beta_0}$, (iv) yields $|(1-\lambda) X_0 + \lambda Y_0|_* \geqslant 1$, and hence $|\alpha X + \beta Y|_*^{\frac{1}{n}} = (\alpha\alpha_0 + \beta\beta_0) \left| \frac{\alpha\alpha_0}{\alpha\alpha_0 + \beta\beta_0} X + \frac{\beta\beta_0}{\alpha\alpha_0 + \beta\beta_0} Y \right|_*^{\frac{1}{n}} \geqslant \alpha\alpha_0 + \beta\beta_0$. ∎

We provide a proof of the Brunn-Minkowski inequality (3.1) *via* the Prékopa-Leindler inequality Theorem 3.4.2 that is probably the simpliest way to also characterize the equality case, and various other arguments like the probably most elegant one due to Hadwiger and Ohman in Section 3.2, the one *via* Steiner symmetrization in Section 3.8, and another one *via* induction and the Prékopa-Leindler inequality in Section 3.A.

We note that recently Figalli, van Hintum, Tiba [273] have proved an essentially optimal stability version of the Brunn-Minkowski Inequality, generalizing the earlier results by van Hintum, Spink, Tiba [394, 395].

Theorem 3.1.5 (Stability of the Brunn-Minkowski Inequality, [273]). *For $n \geqslant 2$ and $t \in (0, 1/2]$, there exist $c_n, d_{n,t} > 0$ depending on n and on n, t, respectively, such that if the measurable sets $X, Y \subset \mathbb{R}^n$ satisfy that*

$$|(1-t)X + tY| \leqslant (1+\delta)|X| \text{ and } |X| = |Y| > 0 \tag{3.2}$$

for $\delta \in (0, d_{n,t})$, then there exists a convex body K such that $X, Y - y \subset K$ for some $y \in \mathbb{R}^n$, and

$$|K \backslash X| = |K \backslash (Y - y)| \leqslant c_n t^{-1/2} \delta^{1/2} |X|. \tag{3.3}$$

Remarks.

- The exponents of t and δ are optimal in (3.3). For example, take $X = [0,1]^n$ and $Y = [0, 1+a] \times \left[0, \frac{1}{1+a}\right] \times [0,1]^{n-2}$ for small $a > 0$, and hence $|(1-t)X + tY| \leqslant (1+\delta)|X|$ for $\delta = ta^2$, while $|X \backslash (Y - y)| \geqslant a/2$ for any $y \in \mathbb{R}^n$.
- The condition that $\delta < d_{n,t}$ in Theorem 3.1.5 for some $d_{n,t} > 0$ depending on n and t is necessary. For example, take $X = [0,1]^n \cup \{p\}$ and $Y = [0,1]^n$ where $\|p\| > 2n$. Then $|X| = |Y| = 1$ and $|(1-t)X + tY| = (1+t^n)|X|$ but $|\operatorname{conv} X|$ can be arbitrary large, and hence $d_{n,t} \leqslant t^n$. Actually, $d_{n,t} = t^n$ according to van Hintum, Keevash [392].
- Figalli, van Hintum, Tiba [273] verified an even stronger estimate if we compare the X and Y satisfying (3.2) to their respective convex hulls:

$$|(\operatorname{conv} X) \backslash X| \leqslant c_{n,t} \delta |X| \text{ and } |(\operatorname{conv} Y) \backslash Y| \leqslant c_{n,t} \delta |Y| \tag{3.4}$$

  where $c_{n,t} > 0$ depends on n, t.

As a direct application of the the Brunn-Minkowski inequality (3.1), we prove the Isodiametric inequality, that is Theorem 3.1.7 below.

Definition 3.1.6 (Diameter). If $X \subset \mathbb{R}^n$ is bounded, then $\operatorname{diam} X = \sup_{x,y \in X} \|x - y\|$.

Theorem 3.1.7 (Isodiametric inequality). *If $X \subset \mathbb{R}^n$ is bounded measurable with $|X| = |rB^n|$ for $r > 0$, then* $\operatorname{diam} X \geqslant 2r$*; or in other words,* $\operatorname{diam} X \geqslant 2\omega_n^{-1/n} |X|^{1/n}$.
Equality holds if and only if there exists $z \in \mathbb{R}^n$ with $X \subset z + rB^n$.

Proof. If $\operatorname{diam} X = D$, then $\frac{1}{2}(X - X) \subset \frac{D}{2} B^n$, and hence $|\frac{1}{2}(X - X)| \leqslant |\frac{D}{2} B^n|$, and $|\frac{1}{2}(X - X)|_* = |\frac{1}{2} X + \frac{1}{2}(-X)|_* \geqslant |X| = |rB^n|$ by the Brunn-Minkowski inequality (3.1); therefore, $D \geqslant 2r$.

If equality holds in the isodiametric inequality, then $\frac{1}{2} X + \frac{1}{2}(-X) \subset rB^n$ and $|\frac{1}{2} X + \frac{1}{2}(-X)|_* = |X| = |rB^n|$, thus the equality case of the Brunn-Minkowski inequality (3.1) implies that X is contained in a ball of radius r. ∎

3.2 The Hadwiger-Ohman proof of the Brunn-Minkowski inequality

In this section, we provide the probably simplest and most elegant proof of the Brunn-Minkowski inequality (3.1) due to Hadwiger, Ohman [379]. We sketch the argument for the characterization of the equality case in Section 3.B because that is more involved.

Theorem 3.2.1. *If $X, Y \subset \mathbb{R}^n$ measurable, then* $|X + Y|_*^{\frac{1}{n}} \geqslant |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}$

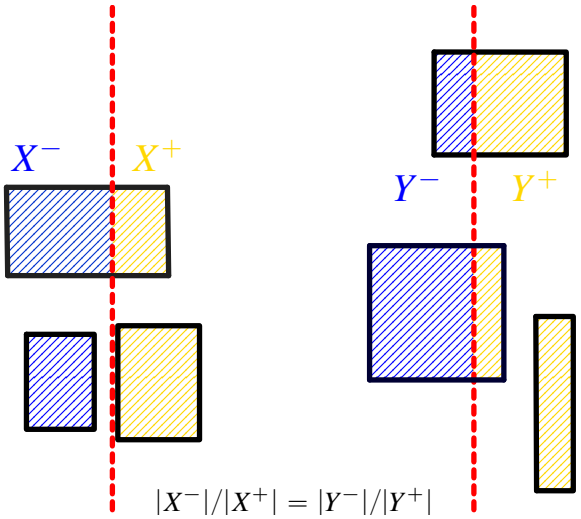
Proof. First we reduce the problem to the case when X and Y are unions of boxes.

We may assume that X and Y are compact by the regularity of the Lebesgue measure (see Section 10.1 in the Appendix Chapter 10), and hence $X + Y$ is compact.

Next let $G = \operatorname{int} B^n$. For $\varepsilon > 0$, there exists $\varrho > 0$ such that $|X + Y + \varrho\, G| < |X + Y| + \varepsilon$, and hence $\left|(X + \frac{\varrho}{2}\, G) + (Y + \frac{\varrho}{2}\, G)\right| < |X + Y| + \varepsilon$. In turn, we may assume that X and Y are bounded and open, thus we may assume that X and Y are unions of finitely many pairwise non-overlapping boxes with edges parallel to vectors in a fixed orthonormal basis $v_1, \ldots, v_n$.

If this is the case, then we prove the Brunn-Minkowski theorem by induction on the total number of boxes used for X and Y. First, let the total number of boxes be 2, and hence X is a box with edge-lengths $e_1, \ldots, e_n > 0$, and Y is a box with edge-lengths $f_1, \ldots, f_n > 0$. We deduce that $X + Y$ is a box with edge-lengths $e_1 + f_1, \ldots, e_n + f_n$; therefore, the AM-GM inequality yields

$$\frac{|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}}{|X + Y|^{\frac{1}{n}}} = \left(\prod_{i=1}^{n} \frac{e_i}{e_i + f_i}\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} \frac{f_i}{e_i + f_i}\right)^{\frac{1}{n}} \leqslant \frac{1}{n} \sum_{i=1}^{n} \frac{e_i}{e_i + f_i} + \frac{1}{n} \sum_{i=1}^{n} \frac{f_i}{e_i + f_i} = 1.$$



Finally, let us assume that the total number of boxes is $k \geqslant 3$. We may assume that X consists of $m \geqslant 2$ pairwise non-overlapping boxes $B^{(j)} = \oplus_{i=1}^{n} [a_i^{(j)}, b_i^{(j)}] v_i$, and hence there exists $\ell \in \{1, \ldots, n\}$ such that $(a_\ell^{(m-1)}, b_\ell^{(m-1)}) \cap (a_\ell^{(m)}, b_\ell^{(m)}) = \varnothing$. We may assume that $b_\ell^{(m-1)} \leqslant a_\ell^{(m)}$.

Let $X^- = \{x \in X : \langle x, v_\ell \rangle < a_\ell^{(m)}\}$ and $X^+ = \{x \in X : \langle x, v_\ell \rangle > a_\ell^{(m)}\}$, and hence both $\operatorname{cl} X^-$ and $\operatorname{cl} X^+$ are unions of at most $m - 1$ boxes as $B^{(m)} \cap X^- = \varnothing$ and $B^{(m-1)} \cap X^+ = \varnothing$. Choose $\gamma \in \mathbb{R}$ such that for $Y^- = \{y \in Y : \langle y, v_\ell \rangle < \gamma\}$ and $Y^+ = \{y \in Y : \langle y, v_\ell \rangle > \gamma\}$, we have $|X^+|/|X| = q = |Y^+|/|Y| \in (0, 1)$, thus

$|X+Y| \geqslant |X^+ + Y^+| + |X^- + Y^-|$ because if $z \in X^+ + Y^+$, then $\langle z, v_\ell \rangle > a_\ell^{(m)} + \gamma$ and if $w \in X^- + Y^-$, then $\langle w, v_\ell \rangle < a_\ell^{(m)} + \gamma$. Now we apply the induction hypothesis to $X^+ + Y^+$ and $X^- + Y^-$ as the total number of boxes for $\operatorname{cl} X^+$ and $\operatorname{cl} Y^+$ is at most $k-1$, and the same holds for $\operatorname{cl} X^-$ and $\operatorname{cl} Y^-$. We deduce that

$$\begin{aligned}\left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}\right)^n &= q\left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}\right)^n + (1-q)\left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}\right)^n \\ &= \left(|X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}}\right)^n + \left(|X^-|^{\frac{1}{n}} + |Y^-|^{\frac{1}{n}}\right)^n \\ &\leqslant |X^+ + Y^+| + |X^- + Y^-| \leqslant |X+Y|,\end{aligned}$$

proving the Brunn-Minkowski inequality. ∎

3.3 An application of the Brunn-Minkowski inequality: the Green-Tao theorem on sumsets

For finite $A \subset \mathbb{R}^n$, we write $\#A$ to denote its cardinality. Green, Tao [335] prove the following discrete version of the Brunn-Minkowski inequality.

Proposition 3.3.1. *If $A, B \subset \mathbb{R}^n$ are finite, then $\#(A + B + \{0,1\}^n) \geqslant 2^n \min\{\#A, \#B\}$.*

Proof. First we consider the case when $A, B \subset \mathbb{Z}^n$. Since $\{0,1\}^n + [0,1]^n = [0,2]^n = [0,1]^n + [0,1]^n$ and $A + B + \{0,1\}^n \subset \mathbb{Z}^n$, the Brunn-Minkowski inequality (3.1) yields

$$\begin{aligned}\#(A + B + \{0,1\}^n) &= |A + B + \{0,1\}^n + [0,1]^n| = |A + [0,1]^n + B + [0,1]^n| \\ &\geqslant \left(|A + [0,1]^n|^{\frac{1}{n}} + |B + [0,1]^n|^{\frac{1}{n}}\right)^n \\ &\geqslant 2^n \min\left\{|A + [0,1]^n|, |B + [0,1]^n|\right\} = 2^n \min\{\#A, \#B\}.\end{aligned}$$

In general, for finite sets $A, B \subset \mathbb{R}^n$, let the cosets of A with respect to $\mathbb{Z}^n$ be $A_1, \ldots, A_m$ and the cosets of B with respect to $\mathbb{Z}^n$ be $B_1, \ldots, B_k$, and hence $A_j \subset a_j + \mathbb{Z}^n$ for an $a_j \in A$, $j = 1, \ldots, m$, and $B_i \subset b_i + \mathbb{Z}^n$ for an $b_i \in B$, $i = 1, \ldots, k$ where $a_j - a_p \notin \mathbb{Z}^n$ and $b_i - b_q \notin \mathbb{Z}^n$ for $j \neq p$ and $i \neq q$. We may assume that $\#A_1 \geqslant \#B_i$ for $i = 1, \ldots, k$; therefore, the previous case on subsets of $\mathbb{Z}^n$ yields

$$\begin{aligned}\#(A + B + \{0,1\}^n) &\geqslant \#(A_1 + B + \{0,1\}^n) = \sum_{i=1}^k \#(A_1 + B_i + \{0,1\}^n) \\ &\geqslant \sum_{i=1}^k 2^n \min\{\#A_1, \#B_i\} \geqslant \sum_{i=1}^k 2^n \cdot \#B_i = 2^n \cdot \#B \\ &\geqslant 2^n \min\{\#A, \#B\}.\end{aligned}$$

∎

Remarks.

- The estimate in Proposition 3.3.1 is optimal: take $A = B = [0,k]^n \cap \mathbb{Z}^n$ for $k \in \mathbb{N}$.
- We recall (see for example Tao, Vu [698]) that if A, B are a finite subsets of a torsion free Abelian group, then $\#(A+B) \geqslant \#A + \#B - 1$, where assuming $\#A, \#B \geqslant 2$, equality holds if and only if A and B are arithmetic progressions of the same difference.
- Proposition 3.3.1 is a key tool in the improvement of Green, Tao [335] on the Freiman-Bilu theorem stating that if $\#(A+A) \leqslant K \cdot \#A$ for a finite subset A of a torsion free Abelian group and $K > 2$, then A can be covered by a few low dimensional progressions of size at most $\#A$ (where the meaning of "a few" and "low dimensional" depends on K).

3.4 The Prékopa-Leindler inequality

In this section, we introduce the Prékopa-Leindler inequality - which is a functional version of the Brunn-Minkowski inequality - in several equivalent forms. The Prékopa-Leindler inequality is proved together with the characterization of the equality case *via* optimal transport in Section 3.5, and using a more elementary argument - that does not characterize the equality case but more clearly shows the equivalence of the Brunn-Minkowski inequality and the Prékopa-Leindler inequality - in Section 3.A. Let us first introduce the fundamental notion of log-concave functions, which are essentially the extremizers in the Prékopa-Leindler inequality (see Section 10.9 in the Appendix for additional properties of log-concave functions).

Definition 3.4.1 (Log-concave functions). A function $f : \mathbb{R}^n \to [0,\infty)$ is *log-concave* if $f((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} f(y)^{\lambda}$ holds for any $x, y \in \mathbb{R}^n$ and any $\lambda \in (0,1)$.

Remarks.

- f is log-concave if and only if $f = e^{-\varphi}$ for a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$.
- For a log-concave function f, the level sets $\{f > t\}$ are convex for $t > 0$.
- If f is log-concave and positive on an open convex $\Omega \subset \mathbb{R}^n$, then f is continuous on Ω.
- Typical examples for log-concave functions are $e^{-\pi\|x\|^2}$ (Gaussian density), or $f(x) = e^{-\|x\|_K}$, or $f(x) = e^{-\|x\|_K^2}$ for $\|x\|_K = \min\{t \geqslant 0 : x \in tK\}$ (cf. Section 1.6) where $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$.
- $f = \mathbf{1}_X$ log-concave for $X \subset \mathbb{R}^n$ if and only if X is convex.

When not signaled, integration is always with respect to the Lebesgue measure on $\mathbb{R}^n$.

Theorem 3.4.2 (Prékopa-Leindler inequality, equality by Dubuc). *Given $\lambda \in (0,1)$ and measurable functions $f, g, h : \mathbb{R}^n \to [0,\infty)$, $n \geqslant 1$, with $h((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} g(y)^{\lambda}$ for any $x, y \in \mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} h \geqslant \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda}. \tag{3.5}$$

Equality holds assuming $0 < \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g < \infty$ if and only if for $a = \int_{\mathbb{R}} f / \int_{\mathbb{R}} g$, there exist a log-concave function ψ and $w \in \mathbb{R}^n$ such that

- *$h(x) = \psi(x)$ for a.e. x and $h(x) \geqslant \psi(x)$ for $x \in \operatorname{int} \operatorname{supp} \psi$;*
- *$f(x) = a^{\lambda} \psi(x + \lambda w)$ for a.e. x and $f(x) \leqslant a^{\lambda} \psi(x + \lambda w)$ for $x \in \mathbb{R}^n$;*
- *$g(x) = a^{-(1-\lambda)} \psi(x - (1-\lambda)w)$ for a.e. x and $g(x) \leqslant a^{-(1-\lambda)} \psi(x - (1-\lambda)w)$ for each $x \in \mathbb{R}^n$.*

Remarks.

- If $z \in \partial(\operatorname{supp} \psi)$ is an extreme point (cf. Definition 1.6.4), then it is possible that $f(z) = \psi(z) > h(z) > 0 = g(z)$.
- See Section 3.A for an elementary proof, not yielding the equality case, of the Prékopa-Leindler inequality (3.5).

Using $\int_{*,\mathbb{R}^n}$ to denote the inner integral (see Section 10.3 in the Appendix for properties of inner integral), the Prékopa-Leindler inequality can be written in the following form:

Theorem 3.4.3 (Prékopa-Leindler inequality for only f, g). *For $\lambda \in (0,1)$ and non-negative $f, g \in L^1(\mathbb{R}^n)$, $n \geqslant 1$,*

$$\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda} \, dz \geqslant \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda}. \tag{3.6}$$

Assuming $0 < \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g < \infty$, equality holds if and only if there exist $w \in \mathbb{R}^n$ and log-concave φ such that for $a = \int_{\mathbb{R}^n} f / \int_{\mathbb{R}^n} g$, we have $g(x) = \varphi(x)$ and $f(x) = a \cdot \varphi(x + w)$ for a.e. x, and $g(x) \leqslant \varphi(x)$ and $f(x) \leqslant a \cdot \varphi(x + w)$ for each $x \in \mathbb{R}^n$.

Remark. The Prékopa-Leindler inequality (3.6) yields the Brunn-Minkowski inequality Theorem 3.1.3 (including the characterization of equality) as if $f = \mathbf{1}_X$ and $g = \mathbf{1}_Y$, then $\sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda} = \mathbf{1}_{(1-\lambda)X+\lambda Y}(z)$.

While the function $z \mapsto \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$ may not be measurable even if f and g are measurable (see Sierpiński [683]), it is measurable if f, g are Borel, and it is log-concave, and hence measurable, if f and g are log-concave. We observe that Theorem 3.4.3 is slightly stronger than Theorem 3.4.2 because Theorem 3.4.2 yields directly (3.6) only for the outer integral.

Theorem 3.4.3 and induction on the number of functions yield the Prékopa-Leindler inequality for multiple functions.

Corollary 3.4.4 (Prékopa-Leindler inequality for multiple functions). *For non-negative $f_1, \dots, f_k \in L^1(\mathbb{R}^n)$ and $\lambda_1, \dots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$,*

$$\int_{*,\mathbb{R}^n} \sup_{z=\sum_{i=1}^k \lambda_i x_i} \prod_{i=1}^k f_i(x_i)^{\lambda_i}\, dz \geqslant \prod_{i=1}^k \left(\int_{\mathbb{R}^n} f_i \right)^{\lambda_i}. \tag{3.7}$$

Assuming that each $\int_{\mathbb{R}^n} f_i > 0$, equality holds in (3.7) if and only if there exist $a_1, \dots, a_k > 0$, $w_1, \dots, w_k \in \mathbb{R}^n$ and a log-concave function φ such that $\sum_{i=1}^k \lambda_i w_i = o$ and $f_i(x) = a_i \varphi(x + w_i)$ for a.e. x, and $f_i(x) \leqslant a_i \varphi(x + w_i)$ for each $x \in \mathbb{R}^n$.

Figalli, van Hintum, Tiba [274] have recently proved an optimal stability version of the Prékopa-Leindler Inequality:

Theorem 3.4.5 (Stability of the Prékopa-Lindler Inequality, [274]). *For $\lambda \in (0,1)$ and integer $n \geqslant 1$, there exist $c_{n,\lambda} > 0$ depending on n, λ such that if the integrable functions $f, g, h : \mathbb{R}^n \to [0, \infty)$ satisfy that*

- $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g > 0$,
- $h((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} g(y)^{\lambda}$ *for* $x, y \in \mathbb{R}^n$,
- $\int_{\mathbb{R}^n} h \leqslant (1+\delta) \int_{\mathbb{R}^n} f$ *for some* $\delta \geqslant 0$,

then there exist a log-concave function $\ell : \mathbb{R}^n \to [0, \infty)$ and a $b \in \mathbb{R}^n$ such that for $\ell_b(x) = \ell(x + b)$, we have

$$\int_{\mathbb{R}^n} |f - \ell| + \int_{\mathbb{R}^n} |g - \ell_b| \leqslant c_{n,\lambda} \sqrt{\delta} \int_{\mathbb{R}^n} f. \tag{3.8}$$

Remarks.

- The exponent $\frac{1}{2}$ of δ is optimal in (3.8), for example, let $n = 1$, $\lambda = \frac{1}{2}$ and $\delta \in (0,1)$, and consider the functions

$$f = (1+\sqrt{\delta})^{-1} \cdot \mathbf{1}_{[0, 1+\sqrt{\delta}]}, \qquad g = (1+\sqrt{\delta}) \cdot \mathbf{1}_{[0, (1+\sqrt{\delta})^{-1}]},$$
$$h = \mathbf{1}_{\left[0, \frac{(1+\sqrt{\delta}) + (1+\sqrt{\delta})^{-1}}{2}\right]}.$$

- Figalli, van Hintum, Tiba [274] verified an even stronger estimate if $g = f$; in this case, they prove the existence of a log-concave function $\ell : \mathbb{R}^n \to [0, \infty)$ such that

$$\int_{\mathbb{R}^n} |f - \ell| \leqslant c_{n,\lambda} \delta \int_{\mathbb{R}^n} f.$$

3.5 Proof of Prékopa-Leinder via optimal transport

In general, regularity theory of optimal maps is proved for densities supported on bounded domains. However, for application, it is useful to have statements that hold when the densities have unbounded support. The following property of the Brenier map can be found in [224] (see also [275]).

Theorem 3.5.1. *If f, g are positive, bounded, C^1 probability densities on $\mathbb{R}^n$, then there exists a C^2 convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $T = D\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism satisfying*

$$f(x) = g(T(x)) \cdot \det DT(x) = g(D\varphi(x)) \cdot \det D^2\varphi(x), \tag{3.9}$$

or in other words, $\int_A g = \int_{T^{-1}A} f$ for measurable $A \subset \mathbb{R}^n$.

Remarks.

- $DT = D^2\varphi$ is symmetric and positive definite (because φ is convex).
- $\det D^2\varphi = \dfrac{f}{g \circ T}$ is a Monge-Ampère equation for φ.
- If $n = 1$, then T can be simply chosen to be the monotone increasing function satisfying $\int_{-\infty}^{x} f = \int_{-\infty}^{T(x)} g$, and hence readily $\frac{f(x)}{g(T(x))} = T'(x)$.

Lemma 3.5.2. *If $\lambda \in (0,1)$, and A, B are symmetric positive definite $n \times n$ matrices, then*

$$\det((1-\lambda)A + \lambda B) \geqslant (\det A)^{1-\lambda}(\det B)^{\lambda}, \tag{3.10}$$

with equality if and only if $A = B$.

Proof. Since two symmetric positive definite matrices can be simultaneously diagonalized, there exists $M \in \mathrm{GL}(n)$ such that $I_n = M^t AM$ and $M^t BM$ is positive definite diagonal matrix; therefore, we may assume that $A = I_n$ and $B = \mathrm{diag}[b_1, \ldots, b_n]$. We deduce that $(1-\lambda)A + \lambda B = \mathrm{diag}[1-\lambda+\lambda b_1, \ldots, 1-\lambda+\lambda b_n]$, and in turn deduce (3.10) from the AM-GM inequality applied to the eigenvalues. ∎

Lemma 3.5.3. *If $a > 0$ and the C^1 map $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfies that $DT(x)$ is symmetric and all of its eigenvalues are at least a, then T is a diffeomorphism.*

Proof. For $x \neq y$, the function $f(s) = \langle y - x, T((1-s)x + s\,y)\rangle$ satisfies that $f'(s) \geqslant a\|y-x\|^2$, and hence $f(1) > f(0)$, which in turn yields that $T(y) \neq T(x)$. Therefore T is a diffeomorphism onto an open set $U \subset \mathbb{R}^n$.

Let $w_0 = T(o) \in U$. As U is open, $U = \mathbb{R}^n$ follows from the following claim: If $v \in S^{n-1}$ and $\tilde{s} > 0$, and $w_0 + sv \in U$ for $s \in [0, \tilde{s})$, then

$$w_0 + \tilde{s}v \in U. \tag{3.11}$$

We observe that the gradient of the inverse $T^{-1} : U \to \mathbb{R}^n$ of T is also always a symmetric matrix, and all of its eigenvalues are at most a^{-1}. It follows that $\|x_k\| \leqslant a^{-1}\tilde{s}$ for $x_k = T^{-1}(w_0 + (1 - \frac{1}{k})\tilde{s}v)$ and $k \geqslant 2$; therefore, a subsequence $\{x_{k'}\}$ tends to a x satisfying $Tx = w_0 + \tilde{s}v$, proving the claim (3.11), and in turn Lemma 3.5.3. ∎

We note that for measurable $f, g : \mathbb{R}^n \to [0, \infty)$ with $0 < \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g < \infty$ in the Prékopa-Leindler inequality, the functions f or g can be freely shifted, as if $w \in \mathbb{R}^n$ and $h(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$, then $h(z + \lambda w) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y + w)^{\lambda}$ satisfies

$$\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y + w)^{\lambda}\, dz = \int_{*,\mathbb{R}^n} h(z)\, dz. \tag{3.12}$$

Proposition 3.5.4 (Prékopa-Leindler inequality (3.6) for positive C^1 functions). *For positive C^1 functions $f, g \in L^1(\mathbb{R}^n)$ and $\lambda \in (0, 1)$,*

$$\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}\, dz \geqslant \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{3.13}$$

Equality holds if and only if f, g are log-concave and $f(x) = a \cdot g(x + w)$ for $a > 0$ and $w \in \mathbb{R}^n$.

Proof. We may assume that $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1$ and f, g are bounded as if f or g is unbounded, then $\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}\, dx = \infty$.

Let $T = D\varphi$ be the Brenier map satisfying (3.9). Since $D\left((1 - \lambda)I_n + \lambda T\right)$ is symmetric, and all of its eigenvalues are at least $1 - \lambda$, Lemma 3.5.3 yields that $(1 - \lambda)I_n + \lambda T$ is a diffeomorphism onto $\mathbb{R}^n$.

For $z \in \mathbb{R}^n$, we define $h(z) = f(x)^{1-\lambda} g(T(x))^{\lambda}$ if $z = (1 - \lambda)x + \lambda T(x)$ for $x \in \mathbb{R}^n$. We use Lemma 3.5.2 and the substitution $z = (1 - \lambda)x + \lambda T(x)$ that satisfies $dz = \det((1 - \lambda)I_n + \lambda DT)(x)\, dx$ to obtain (cf. (3.9))

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} f(x)\, dx = \int_{\mathbb{R}^n} f(x)^{1-\lambda} g(T(x))^{\lambda} \det DT(x)^{\lambda}\, dx \\ &\leqslant \int_{\mathbb{R}^n} h\Big((1 - \lambda)x + \lambda T(x)\Big) \det[(1 - \lambda)I_n + \lambda DT(x)]\, dx \\ &= \int_{\mathbb{R}^n} h(z)\, dz \leqslant \int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}\, dz. \end{aligned}$$

If equality holds in (3.13), then $DT(x) = I_n$ for $x \in \mathbb{R}^n$ by Lemma 3.5.2, and hence there exists $w \in \mathbb{R}^n$ such that $T(x) = x + w$. It follows that $g(x + w) = f(x)$ for $x \in \mathbb{R}^n$. We may assume $w = o$ by (3.12), thus $h = f = g$. Now equality in (3.13) yields $f(z) = h(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$ for any $z \in \mathbb{R}^n$. As $f = g$ is C^1, we deduce that $f((1 - t)x + ty) \geqslant f(x)^{1-t} f(y)^t$ for any $x, y \in \mathbb{R}^n$ and $t \in (0, 1)$, and hence f is log-concave. ∎

We recall that the convolution of $f, \tilde{f} \in L^1(\mathbb{R}^n)$ is $f * \tilde{f}(z) = \int_{\mathbb{R}^n} f(x)\tilde{f}(z-x)\,dx$ where $\int_{\mathbb{R}^n} f * \tilde{f} = (\int_{\mathbb{R}^n} f)(\int_{\mathbb{R}^n} \tilde{f})$ (see Section 10.3 in the Appendix).

Proof of Prékopa-Leindler inequality (3.6) for measurable f and g. We may assume $\int f, \int g > 0$. Approximating f and g by step functions from below (a step function is of the form $\sum_{i=1}^k t_i \mathbf{1}_{A_i}$ for $t_1, \dots, t_k > 0$ and pairwise disjoint measurable $A_1, \dots, A_k$), we may assume that $f = \sum_{i=1}^k s_i \mathbf{1}_{B_i}$ for $s_1, \dots, s_k > 0$ and pairwise disjoint compact $B_1, \dots, B_k$ and $g = \sum_{i=1}^m r_i \mathbf{1}_{C_i}$ for $r_1, \dots, r_m > 0$ and pairwise disjoint compact $C_1, \dots, C_k$, and hence $z \mapsto \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$ is measurable.

Next we approximate these f and g by decreasing sequences of positive C^∞ functions $\{f_m\}$ and $\{g_m\}$ from above. To construct such approximating sequences, it is enough to find a decreasing positive C^∞ sequence $\{\psi_m\}$ for $\mathbf{1}_X$ where X is compact. Let U_m be open sets such that $\bigcap_m U_m = X$ and $\operatorname{cl} U_m \subset U_{m-1}$, hence there exists $r_m > 0$ such that $U_m + r_m B^n \subset U_{m-1}$ and $U_{m+1} + r_m B^n \subset U_m$. Now we define $\phi_m = \mathbf{1}_{U_m} * h_m$ where $h_m : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is a C^1 function such that $\operatorname{supp} h_m \subset r_m B^n$ and $\int_{\mathbb{R}^n} h_m = 1$, and hence $\mathbf{1}_{U_{m-1}} \geqslant \phi_m \geqslant \mathbf{1}_{U_{m+1}}$. Therefore, we can choose $\psi_m = \phi_{2m}(x) + \frac{1}{m} e^{-\|x\|^2}$.

Since f and g have compact support, $\sup_{z=(1-\lambda)x+\lambda y} f_m(x)^{1-\lambda} g_m(y)^{\lambda}$ tends to $\sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$ for any $z \in \mathbb{R}^n$. Therefore Proposition 3.5.4 applied to f_m, g_m completes the proof of Prékopa-Leindler inequality (3.6). ∎

Definition 3.5.5 (Extremizers for the Prékopa-Leindler inequality). Non-negative $f_1, f_2 \in L^1(\mathbb{R}^n)$ are called extremizers for the Prékopa-Leindler inequality (3.6) for $\lambda \in (0,1)$ if $\int f_1, \int f_2 > 0$ and

$$\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f_1(x)^{1-\lambda} f_2(y)^{\lambda}\,dz = \left(\int_{\mathbb{R}^n} f_1\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} f_2\right)^{\lambda}.$$

Our next goal is to characterize extremizers for the Prékopa-Leindler inequality (3.6). For $F : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, we have $\int_{*,\mathbb{R}^n} F = \int_{\mathbb{R}^n} \widetilde{F}$ for measurable $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ if $F \geqslant \widetilde{F}$, and the set $\{F > \widetilde{F}\}$ has no measurable subset with positive measure. In this case, we say that $\widetilde{F}$ is a witness for F.

Lemma 3.5.6. *Given $\lambda \in (0,1)$, if $f_1, f_2 \in L^1(\mathbb{R}^n)$ are extremizers for the Prékopa-Leindler inequality (3.6), and the same holds for $g_1, g_2 \in L^1(\mathbb{R}^n)$, then the convolutions $f_1 * g_1, f_2 * g_2$ are also extremizers for (3.6).*

Proof. To simplify the formulas, we set $\lambda = c_1$ and $1 - \lambda = c_2$, and may assume that $\int_{\mathbb{R}^n} f_1 = \int_{\mathbb{R}^n} f_2 = \int_{\mathbb{R}^n} g_1 = \int_{\mathbb{R}^n} g_2 = 1$. We define (assuming that $x_i, y_i, z_i \in \mathbb{R}^n$)

$$F(x) = \sup_{x=\sum_{i=1}^2 c_i x_i} \prod_{i=1}^2 f_i(x_i)^{c_i} \text{ and } G(y) = \sup_{y=\sum_{i=1}^2 c_i y_i} \prod_{i=1}^2 g_i(y_i)^{c_i}.$$

Possibly F and G are not measurable but as f_1, f_2 and g_1, g_2 are extremizers, there exist measurable $0 \leqslant \widetilde{F} \leqslant F$ and $0 \leqslant \widetilde{G} \leqslant G$ such that $\int_{\mathbb{R}^n} \widetilde{F}(x)\,dx = \int_{\mathbb{R}^n} \widetilde{G}(x)\,dx = 1$, and neither $\{F > \widetilde{F}\}$ nor $\{G > \widetilde{G}\}$ contains a subset of $\mathbb{R}^n$ with positive measure. We write any point of $\mathbb{R}^{2n}$ in the form (x, y) for $x, y \in \mathbb{R}^n$, thus $(z, y) \mapsto \widetilde{F}(z)\widetilde{G}(y)$ is a witness for $(z, y) \mapsto F(z)G(y)$, and hence $(x, y) \mapsto \widetilde{F}(x - y)\widetilde{G}(y)$ is a witness for $(x, y) \mapsto F(x - y)G(y)$.

Setting $x_i = z_i + y_i$ in (3.14) , we have

$$
\begin{aligned}
1 &= \int_{\mathbb{R}^n} \widetilde{F} * \widetilde{G} = \int_{\mathbb{R}^{2n}} \widetilde{F}(x - y)\widetilde{G}(y)\,d(x, y) \\
&= \int_{*,\mathbb{R}^{2n}} \sup_{\substack{x-y=\sum_{i=1}^2 c_i z_i \\ z_i \in \mathbb{R}^n}} \sup_{\substack{y=\sum_{i=1}^2 c_i y_i \\ y_i \in \mathbb{R}^n}} \prod_{i=1}^2 f_i(z_i)^{c_i} \prod_{i=1}^2 g_i(y_i)^{c_i}\,d(x, y) \qquad (3.14) \\
&= \int_{*,\mathbb{R}^{2n}} \sup_{\substack{x=\sum_{i=1}^2 c_i x_i \\ x_i \in \mathbb{R}^n}} \sup_{\substack{y=\sum_{i=1}^2 c_i y_i \\ y_i \in \mathbb{R}^n}} \prod_{i=1}^2 f_i(x_i - y_i)^{c_i} \prod_{i=1}^2 g_i(y_i)^{c_i}\,d(x, y) \\
&\geqslant \int_{*,\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i x_i} \int_{*,\mathbb{R}^n} \sup_{y=\sum_{i=1}^k c_i y_i,\, y_i \in \mathbb{R}^n} \prod_{i=1}^2 \big(f_i(x_i - y_i)g_i(y_i)\big)^{c_i}\,dy dx,
\end{aligned}
$$

and hence using the Prékopa-Leindler inequality (3.6) yields

$$
\begin{aligned}
1 &\geqslant \int_{*,\mathbb{R}^n} \sup_{x=\sum_{i=1}^2 c_i x_i} \prod_{i=1}^k \left(\int_{E_i} f_i(x_i - y_i)g_i(y_i)\,dy_i \right)^{c_i} dx \\
&= \int_{*,\mathbb{R}^n} \sup_{x=\sum_{i=1}^2 c_i x_i} \prod_{i=1}^2 \big(f_i * g_i(x_i)\big)^{c_i} dx \geqslant \prod_{i=1}^2 \left(\int_{\mathbb{R}^n} f_i * g_i \right)^{c_i} = 1;
\end{aligned}
$$

therefore, $f_i * g_i$, $i = 1, 2$, are also extremizers. ∎

Characterizing equality in Prékopa-Leindler (3.6) for measurable f, g. For $h \in L^1(\mathbb{R}^n)$, let $\hat{h}$ be the Fourier transform $\hat{h}(z) = \int_{\mathbb{R}^n} h(x)e^{-2\pi i\langle x,z\rangle}\,dx$.

For $\psi(x) = e^{-\pi\|x\|^2}$, we have $\int_{\mathbb{R}^n} \psi = 1$ and the second partial derivatives $\partial_{\alpha\beta}\psi$ are bounded for $\alpha, \beta = 1, \ldots, n$. For $r > 0$, let $\psi_r(x) = r^{-n}\psi(x/r)$, and hence ψ_r is log-concave and equality holds for $f = g = \psi_r$ in the Prékopa-Leindler inequality (3.6). In addition, $\int_{\mathbb{R}^n} \psi_r = 1$, and $\widehat{\psi_r}(x) = \psi(rx)$, and for any $h \in L^1(\mathbb{R}^n)$, $h * \psi_r$ is C^1 for $r > 0$ (cf. Lemma 10.3.2 in the Appendix), and we have (cf. Lemma 10.3.3)

$$
\lim_{r \to 0^+} h * \psi_r(z) = h(z) \quad \text{if } z \in \mathbb{R}^n \text{ is a density point of } h. \qquad (3.15)
$$

Now, let f, g be measurable extremizers for the Prékopa-Leindler inequality (3.6) with $\int f, \int g > 0$. We may assume that $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$. Lemma 3.5.6 yields that for

$r > 0$, the positive C^1 functions $f * \psi_r, g * \psi_r \in L^1(\mathbb{R}^n)$ are extremizers, and $\int_{\mathbb{R}^n} f * \psi_r = \int_{\mathbb{R}^n} g * \psi_r$. We deduce from Proposition 3.5.4 that there exists $w \in \mathbb{R}^n$ such that $f * \psi_r(x) = g * \psi_r(x + w)$ log-concave.

We may assume that $w = 0$ (otherwise we can exchange g by $x \mapsto g(x + w)$ by (3.12)). It follows $f * \psi_r = g * \psi_r$, and hence $\hat{f} \cdot \widehat{\psi_r} = \widehat{f * \psi_r} = \widehat{g * \psi_r} = \hat{g} \cdot \widehat{\psi_r}$. We conlude that $\hat{f} = \hat{g}$; therefore, $f(x) = g(x)$ for a.e. $x \in \mathbb{R}^n$.

Finally, we have to show that f is a.e. log-concave. We deduce from Proposition 3.5.4 that $f * \psi_r$ log-concave for any $r > 0$, and hence for $t \in (0, 1)$ and $x, y \in \mathbb{R}^n$, we have $f * \psi_r((1-t)x + ty) \geqslant (f * \psi_r)(x)^{1-t}(f * \psi_r)(y)^t$. Thus for any $t \in (0, 1)$, the analogue of (3.15) yields

$$f((1-t)x + ty) \geqslant f(x)^{1-t} f(y)^t \quad \text{if } (1-t)x + ty, x, y \in Z \tag{3.16}$$

where Z is the set of density points of f satisfying that $|\mathbb{R}^n \backslash Z| = 0$ (cf. Remark 10.1.8).

For $x \in \mathbb{R}^n$, let $\varphi(x) = \limsup_{r \to 0^+} \frac{\int_{x + rB^n} f}{|rB^n|}$ where φ measurable and $\varphi(z) = f(z)$ for $z \in Z$, thus (3.16) implies that φ is log-concave. ∎

3.6 Coordinate-wise product of unconditional convex bodies

In this section, we show how the Prékopa-Leindler inequality (see Section 3.4) for log-concave functions yields a strengthening of the Brunn-Minkowski inequality (3.1.3) for unconditional convex bodies.

Definition 3.6.1. A convex body $K \subset \mathbb{R}^n$ is unconditional if $(\pm x_1, \ldots, \pm x_n) \in K$ for $(x_1, \ldots, x_n) \in K$.

The Minkowski linear combination $\frac{1}{2} X + \frac{1}{2} Y$ can be considered as an arithmetic mean of $X, Y \subset \mathbb{R}^n$. We now introduce an operation that can be considered as the geometric mean of two unconditional convex bodies.

Definition 3.6.2 (Coordinatewise product). If $K, C \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0, 1)$, then

$$K^{1-\lambda} \cdot C^{\lambda} = \Big\{ \big(\pm |x_1|^{1-\lambda} |y_1|^{\lambda}, \ldots, \pm |x_n|^{1-\lambda} |y_n|^{\lambda}\big) : (x_1, \ldots, x_n) \in K \text{ and } (y_1, \ldots, y_n) \in C \Big\}.$$

Remarks.

- $K^{1-\lambda} \cdot C^{\lambda} \subset (1-\lambda)K + \lambda C$.
- To show that $M = K^{1-\lambda} \cdot C^{\lambda}$ is an unconditional convex body, first we observe that if $0 \leqslant s_i \leqslant t_i$ for $i = 1, \ldots, n$, and $(t_1, \ldots, t_n) \in M$, then $(s_1, \ldots, s_n) \in M$. Using this property, the convexity of M easily follows.

Example 3.6.3. $K^{1-\lambda}\cdot C^{\lambda}$ might be much smaller than $(1-\lambda)K+\lambda C$. For example, if $K=[-a,a]\times[-\frac{1}{a},\frac{1}{a}]$ and $C=[-\frac{1}{a},\frac{1}{a}]\times[-a,a]$ for $a>1$, then

$$\begin{aligned} K^{\frac{1}{2}}\cdot C^{\frac{1}{2}} &= [-1,1]^2,\\ \tfrac{1}{2}K+\tfrac{1}{2}C &\supset [-\tfrac{a}{2},\tfrac{a}{2}]^2; \end{aligned}$$

namely, $|\frac{1}{2}K+\frac{1}{2}C|$ can be arbitrary large when a is large, while still $|K^{\frac{1}{2}}\cdot C^{\frac{1}{2}}|=4$.

Example 3.6.3 shows that the following inequality, proved independently by Uhrin [706], by Bollobás, Leader [106] and by Cordero-Erausquin, Fradelizi, Maurey [216], can be a substantial improvement on the Brunn-Minkowski inequality (3.1). We note that the case of equality in Theorem 3.6.4 was clarified by Saroglou [646].

Theorem 3.6.4. *If $K,C\subset\mathbb{R}^n$ are unconditional convex bodies and $\lambda\in(0,1)$, then*

$$\left|K^{1-\lambda}\cdot C^{\lambda}\right|\geqslant |K|^{1-\lambda}|C|^{\lambda}. \tag{3.17}$$

Equality holds in (3.17) if and only if $K=\Phi C$ for a positive definite diagonal matrix Φ, and hence $K^{1-\lambda}\cdot C^{\lambda}=\Phi^{1-\lambda}C$.

Proof. To prove (3.17), we apply the Prékopa-Leindler inequality (3.5) to the functions

$$\begin{aligned} f(x_1,\ldots,x_n) &= \mathbf{1}_K(e^{x_1},\ldots,e^{x_n})e^{x_1+\ldots+x_n},\\ g(x_1,\ldots,x_n) &= \mathbf{1}_C(e^{x_1},\ldots,e^{x_n})e^{x_1+\ldots+x_n},\\ h(x_1,\ldots,x_n) &= \mathbf{1}_{K^{1-\lambda}\cdot C^{\lambda}}(e^{x_1},\ldots,e^{x_n})e^{x_1+\ldots+x_n}. \end{aligned}$$

If equality holds in (3.17), then we may assume that $|K|=|C|$, and hence the equality conditions in the Prékopa-Leindler inequality (3.5) imply that $f(x_1,\ldots,x_n)=g(x_1+b_1,\ldots,x_n+b_n)$ for $(b_1,\ldots,b_n)\in\mathbb{R}^n$; therefore, $\Phi=\operatorname{diag}(e^{b_1},\ldots,e^{b_n})$. ∎

3.7 Hausdorff metric on the space of compact subsets

We have already seen in Section 1.7 that the Hausdorff distance between compact convex sets provides a natural topology on the space of compact convex subsets of $\mathbb{R}^n$. In this section, we show that the natural habitat of the Hausdorff distance is the space of compact subsets, as Hausdorff introduced it in [382] in 1914.

Definition 3.7.1. Let (Ξ,d) be a metric space where each closed ball is compact.

Diameter: If $X\subset\Xi$ compact, then $\operatorname{diam}X=\max\{d(x,y):x,y\in X\}$.

Parallel domain: For compact $X\subset\Xi$ and $\varrho\geqslant 0$, let

$$X^{(\varrho)}=\{z\in\Xi:\exists\, x\in X \text{ with } d(x,y)\leqslant\varrho\},$$

which is compact, as well, as each closed ball is compact.

Hausdorff distance: For compact $X, Y \subset \Xi$, their Hausdorff distance is

$$\delta_H(X,Y) = \min\left\{\varrho \geqslant 0 : X \subset Y^{(\varrho)} \text{ and } Y \subset X^{(\varrho)}\right\}.$$

It is equivalent to finding the minimal $\varrho \geqslant 0$ such that for any $x \in X$, there exists $y \in Y$ with $d(x,y) \leqslant \varrho$, and for any $y \in Y$, there exists $x \in X$ with $d(x,y) \leqslant \varrho$.

Remark. δ_H is readily a metric on the space of compact subsets of Ξ. For compact $X_m, Y \subset \Xi$, we write $\lim_{m\to\infty} X_m = Y$ (X_m tends to Y), if $\lim_{m\to\infty} \delta_H(X_m, Y) = 0$.

The following properties directly follow from the definition.

Lemma 3.7.2. *Let (Ξ, d) be a metric space where each closed ball is compact. For compact $X_m, Y \subset \Xi$, $\lim_{m\to\infty} X_m = Y$ if and only if*

- *the sequence $\{X_m\}$ is bounded (contained in a fixed ball), and*
- *for any $y \in Y$, there exists $x_m \in X_m$ such that $\lim_{m\to\infty} x_m = y$, and*
- *for any sequence $x_m \in X_m$, any accumulation point y of $\{x_m\}$ lies in Y.*

Theorem 3.7.3 (Hausdorff). *If (Ξ, d) is a metric space where each closed ball is compact, and $X_m \subset \Xi$ is a bounded sequence of compact subsets, then there exists a subsequence $\{X_{m'}\} \subset \{X_m\}$ and a compact $Z \subset \Xi$ with $\lim_{m'\to\infty} X_{m'} = Z$.*

Proof. We may assume that (Ξ, d) is compact, and for $z \in \Xi$ and $r > 0$, we set $B(z,r) = \{x \in \Xi : d(x,y) \leqslant r\}$ to denote the compact metric ball of center z and radius r.

For any integer $k \geqslant 1$, we consider an $1/k$-net $Y_k \subset \Xi$, and hence Y_k is finite and for any $z \in \Xi$, one finds a $y \in Y_k$ with $d(z,y) \leqslant \frac{1}{k}$. We take subsequences of $\{X_m\}$ for $k = 1, 2, \ldots$ by induction on k to eventually construct a subsequence $\{X_{m'}\} \subset \{X_m\}$ with the following property: For any $k \geqslant 1$, there exists a threshold N_k and $Y_k' \subset Y_k$ such that if $m' > N_k$, then $B(y, \frac{1}{k}) \cap X_{m'} \neq \varnothing$ for $y \in Y_k'$ and $B(z, \frac{1}{k}) \cap X_{m'} = \varnothing$ for $z \in Y_k \backslash Y_k'$, and hence $X_{m'} \subset \bigcup_{y\in Y_k'} B(y, \frac{1}{k}) = Z_k$. It follows that $Z = \bigcap_{k=1}^{\infty} Z_k$ is a non-empty compact set, and $\lim_{m'\to\infty} X_{m'} = Z$. ∎

Lemma 3.7.4. *Let (Ξ, d) be a metric space where each closed ball is compact, and let μ be a Borel measure on Ξ such that $\mu(X) < \infty$ and $\mu(X) = \inf\{\mu(U) : U \supset X \text{ open}\}$ for any compact $X \subset \Xi$. If $\lim_{m\to\infty} X_m = Y$ for compact $X_m, Y \subset \Xi$, then*

(i) $\operatorname{diam} Y = \lim_{m\to\infty} \operatorname{diam} X_m$;

(ii) $\mu(Y) \geqslant \limsup_{m\to\infty} \mu(X_m)$;

(iii) $\mu(Y^{(\varrho)}) \leqslant \liminf_{m\to\infty} \mu(X_m^{(\varrho+\varepsilon)})$ *for* $\varrho, \varepsilon > 0$.

Proof. (i) follows from the definition of δ_H and sequential compactness.

For (ii), choose any open $U \subset \Xi$ with $Y \subset U$, and choose $\varrho > 0$ with $Y^{(\varrho)} \subset U$. Since $X_m \subset Y^{(\varrho)} \subset U$ for large m, we deduce that $\mu(X_m) \leqslant \mu(U)$.

For (iii), the set $U = \left\{z \in \Xi : \exists y \in Y \text{ with } d(z,y) < \varrho + \frac{\varepsilon}{2}\right\}$ is open and $Y^{(\varrho)} \subset U \subset X_m^{(\varrho+\varepsilon)}$ for large m. ∎

3.8 Steiner symmetrization of compact sets and the Brunn-Minkowski inequality

We have seen in Section 1.10 how to use Steiner symmetrization to prove the Brunn-Minkowski inequality for convex bodies. In this section, we extend the method to compact subsets of $\mathbb{R}^n$. First, we prove the Brunn-Minkowski inequality for compact sets in $\mathbb{R}$.

Lemma 3.8.1 (One-dimensional Brunn-Minkowski). *If $A, B \subset \mathbb{R}$ are compact, then*

$$\mathcal{H}^1(A+B) \geqslant \mathcal{H}^1(A) + \mathcal{H}^1(B). \tag{3.18}$$

Proof. Assume $\max A = \min B = 0$, and hence $A, B \subset A+B$ and $A \cap B = \{0\}$. ∎

Definition 3.8.2 (Steiner symmetrization of compact sets). If $X \subset \mathbb{R}^n$ is compact, and $u \in S^{n-1}$, then

$$\Theta_{u^\perp} X = \bigcup \left\{ x + [-q,q]u : x \in X|u^\perp \text{ and } \mathcal{H}^1\Big(X \cap (x + \mathbb{R}u)\Big) = 2q \right\}.$$

Lemma 3.8.3. *Let $X, Y \subset \mathbb{R}^n$ be compact, and let $u \in S^{n-1}$.*

(i) $\Theta_{u^\perp} X$ is symmetric through $u^\perp$, and $\Theta_{u^\perp} X = rB^n$ if $X = rB^n$ for $r > 0$;

(ii) $\Theta_{u^\perp} X$ is compact, and $\Theta_{u^\perp} X$ is convex if X is convex;

(iii) $\operatorname{diam} \Theta_{u^\perp} X \leqslant \operatorname{diam} X$;

(iv) $|\Theta_{u^\perp} X| = |X|$;

(v) $|\alpha \Theta_{u^\perp} X + \beta \Theta_{u^\perp} Y)| \leqslant |\alpha X + \beta Y|$ *for* $\alpha, \beta > 0$.

Proof. (i) follows by definition, and (iv) by Fubini's theorem. For (ii), $\Theta_{u^\perp} X$ is compact by Lemma 3.7.4 (ii), and if X is convex, then Proposition 1.10.3 (i) yields that $\Theta_{u^\perp} X$ convex.

For (iii), as $\Theta_{u^\perp} X$ is symmetric through $u^\perp$, there exist $a_1, a_2 \in X|u^\perp$ and $t_1, t_2 \geqslant 0$ such that $a_i \pm t_i u \in \Theta_{u^\perp} X$ and $\operatorname{diam} \Theta_{u^\perp} X = \|(a_1 + t_1 u - (a_2 - t_2 u)\| = \|(a_1 - a_2) + (t_1 + t_2)u\|$, thus there exist $r_i \geqslant s_i$ with $r_i \geqslant s_i + 2t_i$ and $a_i + r_i u, a_i + s_i u \in X$ for $i = 1, 2$. Assuming $r_1 - s_2 \geqslant r_2 - s_1$ (otherwise switch a_1 and a_2), we have $r_1 - s_2 \geqslant t_1 + t_2$, and hence $\operatorname{diam} X \geqslant \|(a_1 + r_1 u - (a_2 + s_2 u)\| = \sqrt{\|a_1 - a_2\|^2 + (r_1 - s_2)^2} \geqslant \operatorname{diam} \Theta_{u^\perp} X$.

For (v), we may assume that $\alpha = \beta = 1$, and it is enough to prove that $\Theta_{u^\perp} X_1 + \Theta_{u^\perp} X_2 \subset \Theta_{u^\perp}(X_1 + X_2)$ for compact $X_1, X_2 \subset \mathbb{R}^n$ by (iv) where $\Theta_{u^\perp} X_1 + \Theta_{u^\perp} X_2$ is symmetric through $u^\perp$.

Let $a_i + t_i u \in \Theta_{u^\perp} X_i$ for $a_i \in X_i | u^\perp$ and $t_i \geqslant 0$, and hence $\mathcal{H}^1\Big(X_i \cap (a_i + \mathbb{R}u)\Big) \geqslant 2t_i$, which in turn yields that $\mathcal{H}^1\Big((X_1 + X_2) \cap (a_1 + a_2 + \mathbb{R}u)\Big) \geqslant 2(t_1 + t_2)$ by the one-dimensional Brunn-Minkowski inequality (3.18); therefore, $a_1 + a_2 + (t_1 + t_2)u \in \Theta_{u^\perp}(X_1 + X_2)$. ∎

First we show that n "orthogonal" Steiner symmetrizations in $\mathbb{R}^n$ lead to an o-symmetric set.

Lemma 3.8.4. *If $u_1, \ldots, u_n$ orthonormal basis of $\mathbb{R}^n$ and $X \subset \mathbb{R}^n$ compact, then $\Theta_{u_n^\perp} \circ \ldots \circ \Theta_{u_1^\perp} X$ is symmetric through $u_1^\perp, \ldots, u_n^\perp$, and hence is o-symmetric.*

Proof. If $Y \subset \mathbb{R}^n$ compact, $1 \leqslant i < j \leqslant n$ and Y is symmetric through $u_i^\perp$, then $\Theta_{u_j^\perp} Y$ is also symmetric through $u_i^\perp$. ∎

For any compact $X \subset \mathbb{R}^n$ with $|X| > 0$, it is well-known that there exists a sequence of Steiner symmetrizations that tend to a ball (see Theorem 3.8.7 below). Here we only prove a weaker version that is sufficient for our purposes, and whose proof is much simpler.

Theorem 3.8.5 (Iterated Steiner symmetrizations). *If $X \subset \mathbb{R}^n$ is compact with $|X| = |rB^n|$ for $r > 0$, and $\varepsilon \in (0, \frac{1}{2})$, then there exist finitely many Steiner symmetrizations starting with X producing a X' with $|X' \cap rB^n| \geqslant (1 - \varepsilon)|rB^n|$. In particular, there exists a sequence $\{X_m\}$ of compact sets tending to a compact set $\widetilde{X}$ with $rB^n \subset \widetilde{X}$ where $X_0 = X$ and $X_{m+1} = \Theta_{u_m^\perp} X_m$, $u_m \in S^{n-1}$.*

Proof. According to Lemma 3.8.4, we may assume that X is o-symmetric, and let $X \subset RB^n$ for $R > 0$. Writing $\mathcal{F}_X$ to denote the family of compact sets resulting from finitely many iterated Steiner symmetrizations starting from X, we have $|C| = |rB^n|$ for $C \in \mathcal{F}_X$ by Lemma 3.8.3, and Theorem 3.8.5 is equivalent to proving that

$$\Xi = \sup\{|C \cap rB^n| : C \in \mathcal{F}_X\} = |rB^n|. \tag{3.19}$$

The argument is indirect, so we suppose that $\Xi < |rB^n|$, and seek a contradiction. For $k \geqslant 2$, let $C_k \in \mathcal{F}_X$ such that $|C_k| > \Xi - \frac{1}{k}$, and hence

$$\lim_{k \to \infty} |C_k \cap rB^n| = \Xi. \tag{3.20}$$

As $C_k \subset RB^n$, we may assume that $C_k \to C_0$ for an o-symmetric compact set $C_0 \subset RB^n$ by Theorem 3.7.3. In addition, we may assume that the compact sets $C_k' = C_k \backslash \mathrm{int}(rB^n)$ tend to some compact set, and the compact sets $C_k \cap rB^n$ tend to some compact set as k tends to infinity. We claim that

$$|C_0| = |rB^n|. \tag{3.21}$$

As $|C_k| = |X|$ for $k \geqslant 2$ by Lemma 3.8.3, Lemma 3.7.4 (ii) and (3.20) yield that $|C_0| \geqslant |rB^n|$, and hence (3.21) follows if $|C_0| \leqslant |rB^n| + \eta$ for any $\eta > 0$. Choose $\alpha > 0$ such that $|X + \alpha B^n| < |X| + \eta$, and hence $|C_k + \alpha B^n| < |X| + \eta$ for any $k \geqslant 2$ by Lemma 3.8.3 (v), which estimate implies $|C_0| \leqslant |rB^n| + \eta$, and in turn (3.21) by Lemma 3.7.4 (iii).

Next we claim that

$$|C_0 \cap rB^n| = \Xi. \tag{3.22}$$

Since $\lim_{k\to\infty} C'_k \subset C_0 \backslash \mathrm{int}(rB^n)$ and $\lim_{k\to\infty} |C'_k| = |rB^n| - \Xi$ by (3.20), we deduce from Lemma 3.7.4 (ii) that $|C_0 \backslash \mathrm{int}(rB^n)| \geqslant |rB^n| - \Xi$. Moreover Lemma 3.7.4 (ii), (3.20) and $\lim_{k\to\infty} C_k \cap rB^n \subset C_0 \cap rB^n$ imply $|C_0 \cap rB^n| \geqslant \Xi$, which estimate combined with the already verified ones $|C_0 \backslash \mathrm{int}(rB^n)| \geqslant |rB^n| - \Xi$ and (3.21) implies (3.22).

As $|C_0 \cap rB^n| = \Xi < |rB^n|$, there exist $z_0 \in \mathrm{int}(rB^n) \backslash C_0$ and $\varrho > 0$ such that $z_0 + \varrho B^n \subset \mathrm{int}(rB^n) \backslash C_0$ and $|(r + 3\varrho)B^n| - |rB^n| < \frac{1}{2}(|rB^n| - \Xi)$. It follows from $|C_k| = |rB^n|$ and $|C_k \cap rB^n| \leqslant \Xi$ that for

$$|\widetilde{C}_k| \geqslant \frac{1}{2}(|rB^n| - \Xi) \ \text{ for } \ \widetilde{C}_k = C_k \backslash (r + 3\varrho)B^n, \ k \geqslant 2.$$

For any $k \geqslant 2$, considering the averge of the integral of $|(z + \varrho B^n) \cap \widetilde{C}_k|$ for $z \in (R + \varrho)B^n \backslash (r + 2\varrho)B^n$, we deduce that existence of $z_k \in (R + \varrho)B^n \backslash (r + 2\varrho)B^n$ such that

$$|(z_k + \varrho B^n) \cap \widetilde{C}_k| \geqslant \gamma |\varrho B^n| \ \text{ for } \gamma = \frac{\frac{1}{2}(|rB^n| - \Xi)}{|(R + \varrho)B^n \backslash (r + 2\varrho)B^n|}, \tag{3.23}$$

while $(z_k + \varrho B^n) \cap rB^n = \varnothing$ as $z_k \notin (r + 2\varrho)B^n$.

There exists k large enough such that $\frac{1}{k} < \gamma\, |\varrho B^n|$ and $z_0 + \varrho B^n \subset (rB^n) \backslash C_k$; therefore, if $u_k = \frac{z_0 - z_k}{\|z_0 - z_k\|}$ and $\|x - x_k\| < \varrho$ for $x \in u_k^{\perp}$ and $x_k = z_k | u_k^{\perp}$, then $\ell_x = x + \mathbb{R}u_k$ satifies

$$\begin{aligned} \mathcal{H}^1\left(\ell_x \cap rB^n \cap \Theta_{u_k^{\perp}} C_k\right) &= \min\left\{\mathcal{H}^1\left(\ell_x \cap rB^n\right), \mathcal{H}^1\left(\ell_x \cap C_k\right)\right\} \\ &\geqslant \mathcal{H}^1\left(\ell_x \cap rB^n \cap C_k\right) + \mathcal{H}^1\left(\ell_x \cap (z_k + \varrho B^n) \cap C_k\right), \end{aligned}$$

and hence (3.23) implies

$$\left|rB^n \cap \Theta_{u_k^{\perp}} C_k\right| \geqslant |C_k| + \gamma\, |\varrho B^n| > \Xi.$$

This contradiction finally verify (3.19). ∎

We are ready to show how the Steiner symmetrization can be used to prove the Brunn-Minkowski inequality.

Theorem 3.8.6 (Brunn-Minkowski via Steiner symmetrization).
If $X, Y \subset \mathbb{R}^n$ are measurable and $\alpha, \beta \geqslant 0$, then

$$|\alpha X + \beta Y|_*^{\frac{1}{n}} \geqslant \alpha |X|^{\frac{1}{n}} + \beta |Y|^{\frac{1}{n}}.$$

Proof. As the Lebesgue measure is regular, we may assume that X, Y are compact with $|X| = |rB^n|$ and $|Y| = |RB^n|$ for $r, R > 0$, and as the Lebesgue measure is n homogeneous, we may assume that $\alpha = \beta = 1$. It is sufficient to prove that for any $\varepsilon > 0$, we have

$$|X + Y| + \varepsilon \geqslant \left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}} \right)^n. \tag{3.24}$$

Choose $\varrho > 0$ such that $|X + Y + \varrho B^n| < |X + Y| + \varepsilon$.

We consider the sequences X_m, Y_m of compact sets provided by Theorem 3.8.5 such that $X = X_0$ and $Y = Y_0$, X_{m+1} and Y_{m+1} are Steiner symmetrials of X_m and Y_m, respectively, and X_m and Y_m tend to compact sets $\widetilde{X}$ and $\widetilde{Y}$, respectively, satisfying $rB^n \subset \widetilde{X}$ and $RB^n \subset \widetilde{Y}$. It follows from Lemma 3.7.2 that $X_m + Y_m$ tends to $\widetilde{X} + \widetilde{Y}$. Applying Lemma 3.7.4 (iii), and then Lemma 3.8.3 (v) yields that

$$\begin{aligned} \left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}} \right)^n &= \left(|rB^n|^{\frac{1}{n}} + |RB^n|^{\frac{1}{n}} \right)^n = |rB^n + RB^n| \leqslant \left| \widetilde{X} + \widetilde{Y} \right| \\ &\leqslant \liminf_{m \to \infty} |X_m + Y_m + \varrho B^n| \leqslant |X + Y + \varrho B^n| < |X + Y| + \varepsilon, \end{aligned}$$

proving (3.24). ∎

Actually, there is a stronger version of Theorem 3.8.5 where one uses a fixed set of hyperplanes independent of the convex body.

Theorem 3.8.7 (Bianchi, Gardner, Gronchi [93, 94]). *Let $v_1, \dots, v_n \in S^{n-1}$ be independent such that $\langle v_i, v_j \rangle \neq 0$ for $i \neq j$, and $\angle(v_1, v_2) = \alpha\pi$ for irrational $\alpha \in (0, 1)$, and let $u_{kn+i} = v_i$ for $k \in \mathbb{N}$ and $i \in \{1, \dots, n\}$.*

If $X \subset \mathbb{R}^n$ is a compact set with $|X| = |rB^n|$ for $r > 0$, then X_m tends to rB^n where $X_0 = X$ and $X_{m+1} = \Theta_{u_m^\perp} X_m$.

3.9 Comments to Chapter 3

For surveys/books on related material, see, for example, Artstein-Avidan, Giannopoulos, Milman [38], Chapter 1, Colesanti, Hug [210], Gardner [313] and Schneider [661].

The Brunn-Minkowski inequality for convex bodies was proved by Brunn [164] in dimensions $n = 2, 3$ in 1889, and by Minkowski in any dimensions around 1900 (see Section 1.12 for their argument). It was Minkowski's work [591] where the importance of the inequality was recognized, and it has found its place within a whole, as now

called, Brunn-Minkowski theory. The Brunn-Minkowski inequality for measurable subsets of $\mathbb{R}^n$ is proved by Lusternik [548] in 1935, and the equality case is clarified by Henstock, Macbeath [388] much later. The beautiful proof of the general Brunn-Minkowski inequality in Section 3.2 is due to Hadwiger, Ohmann [379]. Their idea also leads to a simple proof of the Brunn-Minkowski inequality with exponent $\frac{1}{n}$ in a simply connected nilpotent Lie group of topological dimension n (see Pozuelo [625] and (3.25)), and is also extensively used in the study of log-concave measures by Borell [112].

There is a trivial way to construct examples of measurable subsets $X, Y \subset \mathbb{R}^2$ such that $X + Y$ is not measurable; namely, for an orthornormal basis $e_1, e_2 \in \mathbb{R}^2$, let $X \subset \mathbb{R}e_1$ and $Y \subset \mathbb{R}e_2$ be non-measurable subsets with respect to the one-dimensional Lebesgue measure. To construct X and Y spanning $\mathbb{R}^2$, one can use the measurable subsets $A, B \subset \mathbb{R}$ constructed by Sierpiński [683] such that $A + B$ is non-measurable, and take $X = A \times [0, 1]$ and $Y = B \times [0, 1]$.

However, Minkowski linear combination of Borel sets is analytic (in the sense of Descriptive Set Theory, see Kechris [456]), and hence measurable. Here a set is analytic if it is the continuous image of a Borel subset of a complete separable metric space, and if $X, Y \in \mathbb{R}^n$ Borel and $\alpha, \beta > 0$, then $\alpha X + \beta Y$ is the image of $X \times Y$ by the map $(x, y) \mapsto \alpha x + \beta y$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$.

One of the early arguments proving the Brunn-Minkowski inequality for convex bodies is due to Hilbert, and is based on a spectral gap estimate for a differential operator (see Bonnesen, Fenchel [107] for the original argument of Hilbert). This approach was further developed by Aleksandrov [5, 11] leading to the generalization Aleksandrov-Fenchel inequality of the Brunn-Minkowski inequality (see Section 8.5.2 for an argument based on spectral gap estimates and additional ideas by Shenfeld, van Handel [678]), and also to the L_p-Minkowski inequality Theorem 8.8.6 improving the Brunn-Minkowski inequality for origin symmetric convex bodies where the local version using spectral gap estimates is due to Kolesnikov, Milman [480], and the "global" Theorem 8.8.6 is proved by Chen, Huang, Li, Liu [192].

Another fundamental approach proving the Brunn-Minkowski inequality for convex bodies is initiated by Gromov's influential appendix to Milman, Schechtman [587] (extending ideas by Knothe [474]) providing a proof of the Isoperimetric Inequality Theorem 2.4.1 using optimal transport, and the argument can be readily extended to the Brunn-Minkowski inequality and the Prékopa-Leindler inequality, as in Section 3.5. This approach led even to the optimal stability versions Theorem 8.6.4 and Corollary 8.6.5 of the Brunn-Minkowski inequality for convex bodies by Figalli, Maggi, Pratelli [277, 278]. We note that the original argument of Brunn and Minkowski (see Section 1.12) can be also considered as a version of the optimal transport approach.

The application of the Brunn-Minkowski inequality in Section 3.3 is due to Green, Tao [335] (see also Tao, Vu [698]). There seems to be no "true" discrete analogue of the Brunn-Minkoski inequality, but various discrete versions of Brunn-Minkoski

inequality in $\mathbb{R}^n$ have been proved, for example, by Gardner, Gronchi [315], Hernández Cifre, Iglesias, Yepes Nicolás [389] and van Hintum, Keevash, Tiba [393].

Let us discuss some versions of the Brunn-Minkowski inequality in various groups. For example, the main result by van Hintum, Keevash, Tiba [393] confirms Ruzsa's conjecture: For $n \geqslant 2$ and $\varepsilon \in (0,1)$, there exists $N_{n,\varepsilon} \geqslant 1$ such that if $A, B \subset \mathbb{Z}^n$ are finite and B is not contained in $N_{n,\varepsilon}$ parallel hyperplanes, then

$$\sqrt[n]{\#(A+B)} \geqslant \sqrt[n]{\#A} + (1-\varepsilon)\sqrt[n]{\#B}.$$

Next, let G be a compact Lie group with Haar probability measure μ, and let d denote the minimal codimension of a proper subgroup. According to Machado [563], for any $\tau \in (0,1]$, there exists $C > 0$ depending on G and τ such that if $\tau \leqslant \frac{\mu(A)}{\mu(B)} \leqslant \tau^{-1}$ for compact $A, B \subset G$, then

$$\mu(AB)^{\frac{1}{d}} \geqslant \left(1 - C\mu(A)^{\frac{2}{d}}\right)\left(\mu(A)^{\frac{1}{d}} + \mu(B)^{\frac{1}{d}}\right).$$

Finally, let G be a simply connected nilpotent group with Haar probability measure μ and of topological dimension n. According to Tao [697], if $X \cdot Y \subset Z$ for measurable $X, Y, Z \subset G$, then

$$\mu(Z)^{\frac{1}{n}} \geqslant \mu(X)^{\frac{1}{n}} + \mu(Y)^{\frac{1}{n}}. \tag{3.25}$$

The Prékopa-Leindler inequality was proved by Prékopa [626,627], Leindler [507], Borell [112], and the case of equality was first characterized by Dubuc [239], and later by Balogh, Kristály [61] who provided an argument using displayment convexity in optimal transport. The paper F. Barthe, D. Cordero-Erausquin [70] provides a proof of the Prékopa-Leindler inequality, and hence in turn of the Brunn-Minkowski inequality, based on the heat equation. The argument characterizing equality in the Prékopa-Leindler inequality in Section 3.4 is new. Essentially optimal stability version of the Prékopa-Leindler inequality is proved by Figalli, van Hintum, Tiba [274] (for much more elementary arguments in the case at least two of the three functions f, g, h involved are log-concave, see Figalli, Ramos [282] for an optimal estimate if $n = 1$, and Böröczky, De [123] for a somewhat weaker stability result if $n \geqslant 2$).

A generalization of the Prékopa-Leindler inequality is the Borell-Brascamp-Lieb inequality proved by Borell [112] and Brascamp-Lieb [154] (see Bobkov, Ledoux [103] for a simple argument based on the Brunn-Minkowski inequality, and Balogh, Kristály [61] for a generalization to Riemannian manifolds). For $p \in \mathbb{R}$, $a, b > 0$ and $\lambda \in (0,1)$, let $\mathcal{M}_\lambda^p(a,b) = ((1-\lambda)a^p + \lambda b^p)^{\frac{1}{p}}$ and let $\mathcal{M}_\lambda^{-\infty}(a,b) = \min\{a,b\}$. Now the Borell-Brascamp-Lieb inequality says that if $p \geqslant -\frac{1}{n}$, $\lambda \in (0,1)$ and measurable $f, g, h : \mathbb{R}^n \to [0,\infty)$ with positive integrals satisfy that $h((1-\lambda)x + \lambda y) \geqslant \mathcal{M}_\lambda^p(f(x), g(y))$ whenever $f(x), g(y) > 0$, then the Borell-Brascamp-Lieb inequality says

$$\int_{\mathbb{R}^n} h \geqslant \mathcal{M}_\lambda^{\frac{p}{1+pn}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right). \tag{3.26}$$

If $p = 0$, then we get back the Prékopa-Leindler inequality.

Equality in the Borell-Brascamp-Lieb inequality (3.26) for any $p \geqslant -\frac{1}{n}$ is characterized by Dubuc [239], and later by Balogh, Kristály [61] using displayment convexity in optimal transport. We note that the equality condition involves the so-called p-concave functions where a function $\ell : \mathbb{R}^n \to [0, \infty)$ is p-concave if $\ell((1-\lambda)x + \lambda y) \geqslant \mathcal{M}_\lambda^p(\ell(x), \ell(y))$ whenever $\ell(x), \ell(y) > 0$. Recently, Figalli, van Hintum, Tiba [274] proved an optimal stability version of the Borell-Brascamp-Lieb inequality (similar to Theorem 3.4.5) when $p > -\frac{1}{n}$.

A harmonic version of the Prékopa-Leindler inequality is provided by Artstein-Avidan, Florentin, Segal [37].

The Prékopa-Leindler inequality, and even the Borell-Brascamp-Lieb inequality, were generalized to Riemannian manifolds by Cordero-Erausquin, Schmuckenschläger, McCann [220] (see Cordero-Erausquin [215] for the spherical and hyperbolic case) where equality is characterized by Balogh, Kristály [61]. In these papers, the expression $h((1-\lambda)x + \lambda y)$ in the condition of the Prékopa-Leindler inequality (or in the more general Borell-Brascamp-Lieb inequality) is replaced by $h(z_\lambda(x, y))$ for $\lambda \in (0, 1)$ where $z_\lambda(x, y)$ is the point of the geodesic segment between x and y dividing the length of the geodesic segment in ratio $\lambda : (1-\lambda)$. Other applications of optimal transport to prove geometric inequalities are discussed by Artstein-Avidan [36].

Tao [697] also verified the following analogue of the Prékopa-Leindler inequality for a simply connected nilpotent group G of topological dimension n similarly to the argument in Section 3.A. Let μ be a Haar measure on G (μ is unimodular in this case; namely, both left and right invariant) and let $\lambda \in (0, 1)$. If non-negative measurable functions f, g, h on G satisfy that $h(xy) \geqslant f(x)^{1-\lambda} g(y)^\lambda$ for x, y, then

$$\int_G h \, d\mu \geqslant \frac{1}{(1-\lambda)^{n(1-\lambda)} \lambda^{n\lambda}} \left(\int_G f \, d\mu \right)^{1-\lambda} \left(\int_G g \, d\mu \right)^{\lambda}.$$

Actually, this unusual Prékopa-Leindler type inequality still yields the Brunn-Minkowski inequality (3.25) in G according to Tao [697].

In the proof of the Prékopa-Leindler inequality in Section 3.5, one can also use the analogous Knothe map initiated by Knothe [474] instead of the Brenier map (see, for instance, Maggi [557]). For positive C^1 probability densities f, g on $\mathbb{R}^n$, the Knothe map $T(x_1, \ldots, x_n) = (T_1(x_1), T_2(x_1, x_2), \ldots, T_n(x_1, \ldots, x_n))$ is defined as follows:

$$\int_{\{(z_1,\ldots,z_n):\, z_1 \leqslant T_1(x_1)\}} f = \int_{\{(y_1,\ldots,y_n):\, y_1 \leqslant x_1\}} g; \tag{3.27}$$

$$\int_{\{(z_1,\ldots,z_n):\, z_1 \leqslant T_1(x_1),\ z_2 \leqslant T_2(x_1,x_2)\}} f = \int_{\{(y_1,\ldots,y_n):\, y_1 \leqslant x_1,\ y_2 \leqslant x_2\}} g; \tag{3.28}$$

and so on. In particular, we have again (3.9). Since DT is lower triangular, one can use the analogue of Lemma 3.5.2 for lower triangle matrices.

For in depth studies on Steiner symmetrization and Schwarz symmetrization of compact sets, see Bianchi, Gardner, Gronchi [93, 94].

3.A Supplement: Common elementary proof of the Brunn-Minkowski inequality and the Prékopa-Leindler inequality by induction

We present the probably most elementary proof of the Brunn-Minkowski inequality and the Prékopa-Leindler inequality due to Keith Ball [52] (using outer measures, in opposition to our use of the inner measure throughout this chapter). While this argument exhibits the equivalence of the Brunn-Minkowski inequality and the Prékopa-Leindler inequality, it does not seem to lead to a characterization of the equality case.

Theorem 3.A.1. *Let $\lambda \in (0,1)$ and $\alpha, \beta \geqslant 0$.*

Brunn-Minkowski inequality *For measurable $X, Y, Z \subset \mathbb{R}^n$, if $\alpha X + \beta Y \subset Z$, then*

$$|Z|^{\frac{1}{n}} \geqslant \alpha |X|^{\frac{1}{n}} + \beta |Y|^{\frac{1}{n}}; \tag{3.29}$$

Prékopa-Leindler inequality *for measurable $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, if $h((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} g(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} h \geqslant \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda}. \tag{3.30}$$

Proof. **Step 1.** Brunn-Minkowski inequality when $n = 1$.

We may assume that X and Y are compact by the regularity of the Lebesgue measure, and $\max X = \min Y = 0$. Then $\alpha X, \beta Y \subset Z$, and $\alpha X \cap \beta Y = \{0\}$, yielding (3.29).

Step 2. Prékopa-Leindler inequality when $n = 1$.

We may assume that $f, g \in L^1(\mathbb{R})$, are bounded and $\sup f = \sup g = 1$. As the condition on f, g, h yields that $(1-\lambda)\{f > t\} + \lambda\{g > t\} \subset \{h > t\}$ for $t \in (0,1)$; therefore, the Layer Cake formula (cf. Lemma 4.2.3), the Brunn-Minkowski inequality (3.29) when $n = 1$ and AM-GM inequality imply

$$\begin{aligned} \int_R h &\geqslant \int_0^1 \mathcal{H}^1(\{h > t\})\, dt \geqslant \int_0^1 (1-\lambda)\mathcal{H}^1(\{f > t\}) + \lambda \mathcal{H}^1(\{g > t\})\, dt \\ &= (1-\lambda) \int_{\mathbb{R}} f + \lambda \int_{\mathbb{R}} g \geqslant \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda}. \end{aligned}$$

Step 3. Prékopa-Leindler inequality for $n \geqslant 1$ by induction.

Assume that the inequality holds in $\mathbb{R}^{n-1}$ for $n \geqslant 2$. For a $u \in S^{n-1}$ and a.e. $t \in \mathbb{R}$ (whenever they defined), let

$$F(t) = \int_{u^\perp + tu} f\, d\mathcal{H}^{n-1}, \quad G(t) = \int_{u^\perp + tu} g\, d\mathcal{H}^{n-1}, \quad H(t) = \int_{u^\perp + tu} h\, d\mathcal{H}^{n-1}.$$

Since $h((1-\lambda)x+\lambda y+tu) \geqslant f(x+ru)^{1-\lambda}g(y+su)^{\lambda}$ for $x, y \in u^{\perp}$ and $r, s, t \in \mathbb{R}$ with $(1-\lambda)r+\lambda s = t$, the Prékopa-Leindler inequality in $u^{\perp}$ implies that $H((1-\lambda)r+\lambda s) \geqslant F(r)^{1-\lambda}G(s)^{\lambda}$ for $r, s \in \mathbb{R}$. In turn, the Fubini theorem and the one-dimensional Prékopa-Leindler inequality for F, G, H yield

$$\int_{\mathbb{R}^n} h = \int_{\mathbb{R}} H \geqslant \left(\int_{\mathbb{R}} F\right)^{1-\lambda}\left(\int_{\mathbb{R}} G\right)^{\lambda} = \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g\right)^{\lambda}.$$

We note that F, G, H may not defined for every t, but are defined for a.e. $t \in \mathbb{R}$. To deal with this issue, one may assume that $f(x) \leqslant M \cdot e^{-\pi\|x\|^2}$ and $g(x) \leqslant M \cdot e^{-\pi\|x\|^2}$ for some suitable $M > 1$, and hence also that $h(x) \leqslant M \cdot e^{-\pi\|x\|^2}$. Now whenever F or G are undefined, we set them 0, and whenever $H(t)$ is undefined, we set it to be $M \cdot e^{-\pi\|t\|^2}$.

Step 4. Brunn-Minkowski inequality for $n \geqslant 1$.

If $\lambda \in (0,1)$ and measurable $X, Y, Z \subset \mathbb{R}^n$ satisfy $(1-\lambda)X + \lambda Y \subset Z$, then the Prékopa-Leindler inequality for $f = \mathbf{1}_X$, $g = \mathbf{1}_Y$ and $h = \mathbf{1}_Z$ implies that $|Z| \geqslant |X|^{1-\lambda}|Y|^{\lambda}$. The general Brunn-Minkowski inequality follows by the n-homogeneity of the Lebesgue measure (see Lemma 3.1.4). ∎

3.B Supplement: Equality in the Brunn-Minkowski inequality for measurable sets *via* Henstock-Macbeath and Hadwiger-Ohmann

This section sketches the argument to classify equality in the Brunn-Minkowski inequality (3.1.3) based on the papers Henstock, Macbeath [388] and Hadwiger, Ohmann [379]. One of the equivalent forms of the Brunn-Minkowski inequality (cf. Lemma 3.1.4) is that if $X, Y \subset \mathbb{R}^n$ are measurable, then

$$|X+Y|_*^{\frac{1}{n}} \geqslant |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}.$$

Theorem 3.B.1 (Equality in Brunn-Minkowski inequality). *If $X, Y \subset \mathbb{R}^n$ measurable with $|X|, |Y| \in (0,\infty)$ and $|X+Y|_*^{\frac{1}{n}} = |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}$, then there exist homothetic convex bodies $K \supset X$ and $C \supset Y$ with $|K \backslash X| = 0$ and $|C \backslash Y| = 0$.*

In order to reduce the equality case of Brunn-Minkowski inequality to compact sets, we use the notion of inner density points following Henstock, Macbeath [388]. For bounded $Z \subset \mathbb{R}^n$, we say that $z \in \mathbb{R}^n$ is an inner density point (cf. Remark 10.1.8 in the Appendix) if

$$\lim_{\varrho \to 0^+} \frac{|Z \cap (z + \varrho B^n)|_*}{|\varrho B^n|} = 1.$$

Let Z_* be the set of inner density points associated to Z, and hence $|Z_*| = |Z_* \cap Z| = |Z|_*$ follows from applying Lebesgue's density theorem to a measurable set of maximal

measure in Z. We also observe that if Z is compact, then $Z_* \subset Z$. The following simple observation is due to Henstock, Macbeath [388]:

Lemma 3.B.2 (Henstock, Macbeath). *If $X, Y \subset \mathbb{R}^n$ are bounded measurable, then $X_* + \operatorname{cl} Y \subset (X+Y)_*$.*

To handle the equality case of Brunn-Minkowski for compact sets, we also need the following estimates due to Hadwiger, Ohmann [379]. Here, we say that H^+, H^- are complementrary closed half-spaces if $H^+ \cap H^-$ is the common boundary.

Lemma 3.B.3. *If $|X|, |Y| > 0$ for compact $X, Y \subset \mathbb{R}^n$, and $z \in \mathbb{R}^n$ and complementary closed half-space H^+, H^- satisfy that $0 < |X^+|/|X| = |Y^+|/|Y| < 1$ for $X^+ = X \cap H^+$, $X^- = X \cap H^-$, $Y^+ = Y \cap (z + H^+)$ and $Y^- = Y \cap (z + H^-)$, then*

(i) $\dfrac{|X+Y|^{\frac{1}{n}}}{|X|^{\frac{1}{n}}+|Y|^{\frac{1}{n}}} \geqslant \min\left\{ \dfrac{|X^+ + Y^+|^{\frac{1}{n}}}{|X^+|^{\frac{1}{n}}+|Y^+|^{\frac{1}{n}}}, \dfrac{|X^- + Y^-|^{\frac{1}{n}}}{|X^-|^{\frac{1}{n}}+|Y^-|^{\frac{1}{n}}} \right\}$

(ii) Equality holds in (i) if and only if $|X+Y| = |X^+ + Y^+| + |X^- + Y^-|$ and $\dfrac{|X^+ + Y^+|^{\frac{1}{n}}}{|X^+|^{\frac{1}{n}}+|Y^+|^{\frac{1}{n}}} = \dfrac{|X^- + Y^-|^{\frac{1}{n}}}{|X^-|^{\frac{1}{n}}+|Y^-|^{\frac{1}{n}}}$

Proof. We may assume that $o \in \partial H^+$ and $z = o$, and hence $X^+ + Y^+ \subset H^+$ and $X^- + Y^- \subset \operatorname{cl}(\mathbb{R}^n \backslash H^+)$. It follows that $|X+Y| \geqslant |X^+ + Y^+| + |X^- + Y^-|$ and

$$\begin{aligned}
\frac{|X+Y|}{(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}})^n} &= \frac{|X+Y|}{(|X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}})^n + (|X^-|^{\frac{1}{n}} + |Y^-|^{\frac{1}{n}})^n} \\
&\geqslant \frac{|X^+ + Y^+| + |X^- + Y^-|}{(|X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}})^n + (|X^-|^{\frac{1}{n}} + |Y^-|^{\frac{1}{n}})^n} \\
&\geqslant \min\left\{ \frac{|X^+ + Y^+|}{\left(|X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}}\right)^n}, \frac{|X^- + Y^-|}{\left(|X^-|^{\frac{1}{n}} + |Y^-|^{\frac{1}{n}}\right)^n} \right\}.
\end{aligned}$$

The characterization of the case of equality in (i) readily follows. ∎

Sketch of the proof of Theorem 3.B.1. X and Y are bounded because otherwise $|X + Y|_* = \infty$. We may replace X and Y by their closures by Lemma 3.B.2, and hence we may assume that X and Y are compact.

We claim that

$$X \subset \operatorname{cl}\operatorname{conv} X_* \text{ and } Y \subset \operatorname{cl}\operatorname{conv} Y_*. \tag{3.31}$$

The argument is indirect, we suppose that there exist for example an $x_0 \in X \backslash \widetilde{X}$ for $\widetilde{X} = \operatorname{cl}\operatorname{conv} X_*$ and $\widetilde{Y} = \operatorname{cl}\operatorname{conv} Y_*$. For the closest point $\tilde{x} \in \widetilde{X}$ to x_0 (cf. Lemma 1.2.2), let $\tilde{y} \in \partial \widetilde{Y}$ be a boundary point with exterior normal $x_0 - \tilde{x}$, and let $y \in Y_*$ with $\|y - \tilde{y}\| < \frac{1}{2}\|x_0 - \tilde{x}\|$. Then $x_0 + y \in X + Y$ is a density point not contained in $\widetilde{X} + \widetilde{Y}$, which

fact leads to a contradiction as $|\widetilde{X}+\widetilde{Y}|^{\frac{1}{n}} \geqslant |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}$ by the Brunn-Minkowski inequality, verifying (3.31).

Next we observe that

$$x_1, x_2 \in X_* \text{ and } \operatorname{conv}\{x_1, x_2\} \subset X \text{ yield } \operatorname{conv}\{x_1, x_2\} \subset X_* \tag{3.32}$$

by the Brunn-Minkowski inequality, and the analogous statement holds for Y.

Now we prove that

$$X = \operatorname{cl}\operatorname{conv} X_* \text{ and } Y = \operatorname{cl}\operatorname{conv} Y_*. \tag{3.33}$$

The argument is again indirect: we suppose that, for example, $X \neq \operatorname{cl}\operatorname{conv} X_*$. It follows from (3.31) and (3.32) that there exist $x_1, x_2 \in X_*$ and $\lambda \in (0,1)$ with $w = (1-\lambda)x_1 + \lambda x_2 \notin X$, and hence writing $e_1, \dots, e_n$ to denote an orthonomal basis of $\mathbb{R}^n$ with $e_n = \frac{x_1 - x_2}{\|x_1 - x_2\|}$, there exists $a > 0$ such that $(w + [-a,a]^n) \cap X = \varnothing$ and $w + [-a,a]^n \subset \operatorname{int}\operatorname{conv} X_*$. By translating X, we may assume that $w = o$.

Setting $X_0 = X$ and $Y_0 = Y$, we construct $X_1, \dots, X_{2n-2}$ and $Y_1, \dots, Y_{2n-2}$ by induction on $i = 1, \dots, 2n-2$ such that $X_i = X_{i-1} \subset H_i^+$ for $H_i^+ = \{x : \langle x, e_i\rangle \leqslant a\}$ if $i = 1, \dots, n-1$ and $H_i^+ = \{x : \langle x, -e_i\rangle \leqslant a\}$ if $i = n, \dots, 2n-2$, and compact $Y_1, \dots, Y_{2n-2}$ are constructed using Lemma 3.B.3 in a way such that $0 < |X_i|/|X| = |Y_i|/|Y| < 1$ and $|X_i + Y_i|^{\frac{1}{n}} = |X_i|^{\frac{1}{n}} + |Y_i|^{\frac{1}{n}}$ for $i = 1, \dots, 2n-2$.

Set $\overline{X} = X_{2n-2}$, $\overline{Y} = X_{2n-2}$, and hence $X^+ = H^+ \cap \overline{X}$ and $X^- = H^- \cap \overline{X}$ for $H^+ = \{x : \langle x, e_n\rangle \geqslant 0\}$ and $H^- = \{x : \langle x, e_n\rangle \leqslant 0\}$ satisfy that $\alpha^+ = \min\{\langle x, e_n\rangle : x \in X^+\} \geqslant a$, $\alpha^- = \max\{\langle x, e_n\rangle : x \in X^-\} \leqslant -a$, $|X^+| > 0$ and $|X^-| > 0$ as $x_1 \in X^+$ and $x_2 \in X^-$. Let us translate Y in a way such that $0 < |X^+|/|X| = |Y^+|/|Y| < 1$ for $Y^+ = Y \cap H^+$ and $Y^- = Y \cap H^-$, and let $\beta^+ = \min\{\langle y, e_n\rangle : y \in Y^+\} \geqslant 0$ and $\beta^- = \max\{\langle y, e_n\rangle : y \in Y^-\} \leqslant 0$. We deduce from Lemma 3.B.3 that $|\overline{X} + \overline{Y}| = |X^+ + Y^+| + |X^- + Y^-|$ and $|X^+ + Y^+|^{\frac{1}{n}} = |X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}}$.

Since $Y^+ \subset \operatorname{cl}\operatorname{conv} Y_*^+$ by (3.31), there exists $y \in Y_*^+$ with $\beta_+ < \langle y, e_n\rangle < \beta_+ + a$. Choose $x \in X^-$ with $\langle x, e_n\rangle = \alpha_-$, and hence

$$\max\{\langle z, e_n\rangle : z \in X^- + Y^-\} \leqslant \alpha^- < \langle x + y, e_n\rangle < \beta_+ + a \leqslant \min\{\langle z, e_n\rangle : z \in X^+ + Y^+\}.$$

Therefore, $x + y$ is a density point of $\overline{X} + \overline{Y}$ contained neither in $X^- + Y^-$ nor in $X^+ + Y^+$, contradicting $|\overline{X} + \overline{Y}| = |X^+ + Y^+| + |X^- + Y^-|$, and verifying (3.33).

Finally, (3.33) implies that X and Y are convex bodies. We deduce from the equality case of the Brunn-Minkowski inequality for convex bodies (see Theorem 1.12.3) that X and Y are homothetic. ∎

Chapter 4

The Isoperimetric inequality in the case of Lipschitz boundary

The classical isoperimetric inequality in $\mathbb{R}^n$, which states that the surface area of a body of given volume is minimized by balls, is probably the most fundamental geometric inequality. However, while the question of which notion of volume to use has been essentially clear for millennia - that is the Lebesgue measure in today's terms -, the suitable notions of surface area have been understood only in the last decades. In the most part of the book, surface area in $\mathbb{R}^n$ denotes the $(n-1)$-dimensional Hausdorff measure of Lipschitz hypersurfaces, except for Chapter 5 where the more technical (but more general, and in some sense more natural) notion of sets of finite perimeter is considered. We note that even if Chapter 4 concentrates on sets with Lipschitz boundary, it does uses results from Chapter 5 on sets of finite perimeter to understand equality in the Isoperimetric inequality and when relating Minkowski content to surface area.

We discuss the classical isoperimetric inequality and its anisotropic form in $\mathbb{R}^n$, and their relation to Sobolev type inequalities. We also consider the isoperimetric problem on the sphere S^n and the hyperbolic space H^n, the isoperimetric problem for the Gaussian density on $\mathbb{R}^n$ (where extremal sets are half-spaces, and the argument uses the isoperimetric inequality on the sphere), and finally, the isoperimetric problem for any log-concave measure on $\mathbb{R}^n$ in terms of the Cheeger constant.

4.1 The Isoperimetric inequality in the Euclidean space

First, we recall from Section 1.D and the Section 10.4 in the Appendix the notions we need in order to speak about surface area in this chapter.

Definition 4.1.1 (Hausdorff measure $\mathcal{H}^\alpha$). To define the Hausdorff measure $\mathcal{H}^\alpha_\Xi$ for $\alpha > 0$ on a metric space (Ξ, d), for $\varepsilon > 0$ and $X \subset \Xi$, we consider

$$\mathcal{H}^\alpha_{(\varepsilon)}(X) = \inf\left\{ \frac{\omega_\alpha}{2^\alpha} \sum_{i=1}^{\infty} (\operatorname{diam} Z_i)^\alpha \,:\, X \subset \cup_{i=1}^\infty Z_i \ \text{ and } \ \forall \operatorname{diam} Z_i < \varepsilon \right\}.$$

The Hausdorff outer measure is $\mathcal{H}^{*,\alpha}(X) = \lim_{\varepsilon \to 0^+} \mathcal{H}^\alpha_{(\varepsilon)}(X)$, and $\mathcal{H}^\alpha_\Xi$ is the corresponding Borel measure (here $\omega_\alpha = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}+1\right)}$ and $\omega_n = |B^n|$, cf. Remark 10.1.7).

Remark. Concerning normalization, $\mathcal{H}^n(X) = \mathcal{H}^n_{\mathbb{R}^n}(X) = |X|$ holds for a Borel set $X \subset \mathbb{R}^n$ (cf. Section 1.D), which follows via the Isodiametric Inequality (cf. Theorem 3.1.7). For a subset $X \subset \mathbb{R}^n$, we frequently write $\mathcal{H}^n(X)$ instead of $\mathcal{H}^n_{\mathbb{R}^n}(X)$.

Let us recall some key properties of Lipschitz functions (see Section 10.4).

Lemma 4.1.2 (Lipschitz function and Rademacher's theorem).
For $L > 0$ and metric spaces (X, d_X) and (Z, d_Z), let $f : X \to Z$ be surjective and satisfy $d_Z(f(x), f(y)) \leqslant L \cdot d_X(x, y)$ for $x, y \in X$.

- *$\mathcal{H}^\alpha_Z(Z) \leqslant L^\alpha \cdot \mathcal{H}^\alpha_X(X)$ for any $\alpha > 0$. In particular, $\mathcal{H}^\alpha_Z(Z) = 0$ if $\mathcal{H}^\alpha_X(X) = 0$.*
- *If $X \subset \mathbb{R}^n$ is open and $Z \subset \mathbb{R}^m$, then f is $\mathcal{H}^n$ a.e differentiable; namely, there exists $m \times n$ matrix $Df(y)$ at a.e. $y \in X$ with*

$$f(x) = f(y) + Df(y)(x - y) + o(\|x - y\|),$$

and hence $\lim_{x \to y} \frac{f(x) - f(y) - Df(y)(x-y)}{\|x-y\|} = 0$.

For graphs of Lipschitz functions, we have the following observation.

Remark 4.1.3 (Graphs of Lipschitz functions). Let $u \in S^{n-1}$, let $\varphi : \Omega \to \mathbb{R}$ be Lipschitz where $\Omega \subset u^\perp$ is a relatively open subset of $u^\perp$, and let $X = \{z + \varphi(z)u : z \in \Omega\}$ be the graph of φ. The points of the form $x = z + \varphi(z)u \in X$ such that $D\varphi(z)$ exists for $z \in \Omega$ are denoted by X', and let $\nu(x) \in S^{n-1}$ be the unique normal at $x \in X'$ with $\langle \nu(x), u \rangle < 0$. We note that $\mathcal{H}^{n-1}(X \backslash X') = 0$ by Rademacher's theorem (cf. Lemma 4.1.2). Now if $x = z + \varphi(z)u \in X'$ for $z \in \Omega$, then

$$\langle \nu(x), u \rangle = \frac{-1}{\sqrt{1 + \|D\varphi(z)\|^2}} \quad \text{as} \quad \nu(x) = \frac{D\varphi(z)}{\sqrt{1 + \|D\varphi(z)\|^2}} + \frac{-1}{\sqrt{1 + \|D\varphi(z)\|^2}} \cdot u.$$

We are ready to define the main objects of study in this chapter.

Definition 4.1.4 (Sets with Lipschitz boundary and their perimeter). We say that a non-empty $X \subset \mathbb{R}^n$ has *Lipschitz boundary* if X is the closure of its non-empty interior, and for any $x \in \partial X$, there exists $r > 0$ (depending on x, X) such that $\partial X \cap \operatorname{int}(x + rB^n)$ is the graph of a Lipschitz function as in Remark 4.1.3.

In this case, for $\mathcal{H}^{n-1}$ a.e. $x \in \partial X$, the hypersurface ∂X is differentiable at x by Lemma 4.1.2 and Remark 4.1.3, and there exists an exterior unit normal $\nu_X(x) \in S^{n-1}$ such that if $\langle v, \nu_X(x) \rangle > 0$ for $v \in \mathbb{R}^n$, then $x - tv \in \operatorname{int} X$ for small $t > 0$, and $x + tv \notin X$ for small $t > 0$.

If $X \subset \mathbb{R}^n$ is compact with Lipschitz boundary, then $0 < \mathcal{H}^{n-1}(\partial X) < \infty$, and we define the perimeter (or surface area) of X to be

$$P(X) = \mathcal{H}^{n-1}(\partial X).$$

Remarks.

- Compact sets with Lipschitz boundary are sets of finite perimeter (cf. Example 5.1.5).
- Convex bodies are sets with Lipschitz boundary (cf. Section 2.1).

- If compact $X \subset \mathbb{R}^n$ has Lipschitz boundary, then $P(\gamma X) = \gamma^{n-1} P(X)$ for $\gamma > 0$ and $P(\Phi X + z) = P(X)$ for $\Phi \in O(n)$ and $z \in \mathbb{R}^n$.

In this book, with the notable exception of Chapter 5 about sets of finite perimeter, we target to use notions of surface area that coincide with the outer Minkowski content in a suitable sense. The "Euclidean" outer Minkowski content is derived from volume of the *parallel domain* $X^{(\varrho)} = X + \varrho B^n$ of a compact $X \subset \mathbb{R}^n$. Now the lower outer Minkowski content - which we typically simply call lower Minkowski content - of a compact $X \subset \mathbb{R}^n$ is

$$\mathcal{M}_-(X) = \liminf_{\varrho \to 0^+} \frac{|X^{(\varrho)}| - |X|}{\varrho}. \tag{4.1}$$

Here possibly $\mathcal{M}_-(X) = \infty$. The advantage of considering the surface area as an outer Minkowski content is the opportunity to use the Brunn-Minkowski inequality (3.1) stating that if $X, Y \subset \mathbb{R}^n$ are compact sets and $\alpha, \beta > 0$, then

$$|\alpha X + \beta Y|^{\frac{1}{n}} \geqslant \alpha |X|^{\frac{1}{n}} + \beta |Y|^{\frac{1}{n}}. \tag{4.2}$$

The following statement is proved by Ambrosio, Colesanti, Villa [23].

Theorem 4.1.5 (Existence of Minkowski content). *If $X \subset \mathbb{R}^n$ is compact with Lipschitz boundary, then its outer Minkowski content is*

$$\mathcal{M}_-(X) = \lim_{\varrho \to 0^+} \frac{|X^{(\varrho)}| - |X|}{\varrho} = \mathcal{H}^{n-1}(\partial X) = P(X). \tag{4.3}$$

Remark. Theorem 4.1.5 yields for example that $P(B^n) = n|B^n| = n\omega_n$.

Theorem 4.1.6 (Isoperimetric Inequality in $\mathbb{R}^n$). *If $|X| = |rB^n|$ holds for a compact $X \subset \mathbb{R}^n$ and $r > 0$, then $\mathcal{M}_-(X) \geqslant P(rB^n) = n\omega_n^{\frac{1}{n}} |X|^{\frac{n-1}{n}}$.*

If, in addition, X has Lipschitz boundary, then

$$P(X) \geqslant n\omega_n^{\frac{1}{n}} |X|^{\frac{n-1}{n}} \tag{4.4}$$

where equality holds in (4.4) if and only if X is a Euclidean ball.

Remarks.

- According to Theorem 5.2.1, the Isoperimetric Inequality (4.4) holds even if $X \subset \mathbb{R}^n$ has finite perimeter, and we will handle the case of equality in (4.4) only in the proof of Theorem 5.2.1 in Chaper 5.
- The Isoperimetric Inequality for convex bodies is also discussed in Theorem 2.4.1.

Proof of the Isoperimetric inequality without equality. If $|X| = |rB^n|$ for the compact $X \subset \mathbb{R}^n$ and $r > 0$, then the Brunn-Minkowski inequality (4.2) yields

$$\begin{aligned}\mathcal{M}_-(X) &= \liminf_{\varrho\to 0^+} \frac{|X+\varrho B^n| - |X|}{\varrho} \geqslant \lim_{\varrho\to 0^+} \frac{\left(|X|^{\frac{1}{n}} + \varrho|B^n|^{\frac{1}{n}}\right)^n - |X|}{\varrho} \\ &= \lim_{\varrho\to 0^+} \frac{|(r+\varrho)B^n| - |rB^n|}{\varrho} = P(rB^n).\end{aligned}$$

■

After numerous attempts, a definitive, sharp stability version of the Isoperimetric Inequality Theorem 4.1.6 was provided by Fusco, Maggi, Pratelli [311]. Indeed, it has been shown even in the more general framework of sets of finite perimeter, cf. Theorem 5.2.2. This estimate was then improved and generalised to the non-isotropic context by Figalli, Maggi, Pratelli [278] with optimal transport techniques. In order to state the result for compact sets $X, Y \subset \mathbb{R}^n$, let $\alpha = |X|^{\frac{-1}{n}}$ and $\beta = |Y|^{\frac{-1}{n}}$, and let

$$A(X,Y) = \min\left\{|\alpha X \Delta (z+\beta Y)| : z \in \mathbb{R}^n\right\}.$$

Theorem 4.1.7 (Figalli, Fusco, Maggi, Pratelli). *For $\theta_n = 2^{-16}n^{-17}$, if a compact $X \subset \mathbb{R}^n$ has Lipschitz boundary (cf. Definition 4.1.4), then*

$$P(X) \geqslant n\omega_n^{\frac{1}{n}} |X|^{\frac{n-1}{n}} \left[1 + \theta_n \cdot A(B^n, X)^2\right]. \tag{4.5}$$

Remark. Here the exponent 2 in the right-hand side is optimal, and constant θ_n cannot be larger than $36n^{-2}$ (see Remark 8.6.6).

4.2 Sobolev inequality

The Sobolev inequality on the gradient of a Lipschitz function, established by Brunn [163] in 1887 when $n = 2, 3$, is essentially equivalent to the Isoperimetric Inequality (see Theorem 5.3.1 for a more general version).

Theorem 4.2.1 (Sobolev inequality). *If $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 2$, is a Lipschitz function with compact support and $n' = \frac{n}{n-1}$, then*

$$\int_{\mathbb{R}^n} \|Df\| \geqslant n\omega_n^{\frac{1}{n}} \|f\|_{n'}. \tag{4.6}$$

Remark 4.2.2.

(i) The factor $n\omega_n^{\frac{1}{n}}$ is optimal, as if $\varrho > 0$, and

$$f_\varrho(x) = \begin{cases} \mathbf{1}_{B^n}(x) & \text{if } \|x\| \leqslant 1 \text{ or } \|x\| \geqslant 1+\varrho; \\ 1 - \frac{1}{\varrho}(\|x\| - 1) & \text{if } 1 \leqslant \|x\| \leqslant 1+\varrho, \end{cases}$$

then $\|Df_\varrho(x)\| = \frac{1}{\varrho}$ for $x \in (1+\varrho)B^n \backslash B^n$, and $\|Df_\varrho(x)\| = 0$ otherwise; therefore,

$$\lim_{\varrho \to 0^+} \int_{\mathbb{R}^n} \|Df_\varrho\| = \mathcal{H}^{n-1}(\partial B^n) = n\omega_n^{\frac{1}{n}} |B^n|^{1/n'} = n\omega_n^{\frac{1}{n}} \lim_{\varrho \to 0^+} \left(\int_{\mathbb{R}^n} |f_\varrho|^{n'} \right)^{1/n'}.$$

(ii) Sobolev inequality holds for the larger class of BV functions (functions of bounded variations) (cf. Theorem 5.3.1) where equality holds if and only if $f = a\mathbf{1}_{x+rB^n}$ a.e. on $\mathbb{R}^n$ for some $a, r > 0$ and $x \in \mathbb{R}^n$.

(iii) Sobolev inequality yields the Isoperimetric Inequality (4.4) for any convex body $X \subset \mathbb{R}^n$. For $\varrho > 0$, let $f_\varrho(x) = \mathbf{1}_X(x)$ if $x \in X$ or $x \notin X^{(\varrho)}$, and let $f_\varrho(x) = 1 - \frac{d(x,X)}{\varrho}$ if $x \in X^{(\varrho)} \backslash X$ where $d(x, X)$ is the distance of x from X. Then $\|Df_\varrho(x)\| = \frac{1}{\varrho}$ if $x \in X^{(\varrho)} \backslash X$, and $\|Df_\varrho(x)\| = 0$ otherwise; therefore,

$$\mathcal{H}^{n-1}(\partial X) = \lim_{\varrho \to 0^+} \int_{\mathbb{R}^n} \|Df_\varrho\| \geqslant n\omega_n^{\frac{1}{n}} \lim_{\varrho \to 0^+} \left(\int_{\mathbb{R}^n} |f_\varrho|^{n'} \right)^{1/n'} = n\omega_n^{\frac{1}{n}} |X|^{\frac{n-1}{n}}.$$

(iv) For $1 < p < n$ and $q > 1$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$, the L_p Sobolev inequality says that if $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function with compact support, then

$$\|f\|_q \leqslant C_{n,p} \|Df\|_p \tag{4.7}$$

where $C_{n,p} > 0$ depends on p and n (see (5.50) for a more general version). With a non-optimal factor $C(n, p)$, (4.7) follows from the Sobolev inequality (4.6) applied to $|f|^r$ where $\frac{nr}{n+r-1} = p$, using $\|D|f|^r\| = r|f|^{r-1}\|Df\|$ and the Hölder inequality (10.4). For the optimal factor in (4.7), see, for example, Cordero-Erausquin, Nazaret, Villani [217] using optimal transport, or Bobkov, Ledoux [104] using the Brunn-Minkowski inequality.

A technical problem concerning the Sobolev inequality is that even if the f in Theorem 4.2.1 is C^1 besides having compact support, the typical level set $f^{-1}(t)$ may not have Lipschitz boundary to apply the Isoperimetric Inequality to $\mathcal{H}^{n-1}(f^{-1}(t))$. Therefore, one either assumes that f is C^n; namely, n times differentiable (see Remark 4.2.5, and especially (4.10)), or use more advanced geometric measure theory to ensure that a.e. level sets are suitably nice, as we do it below.

In order to prove the Sobolev inequality (4.6), we use Lemma 4.2.4 and the three fundamental results listed in Lemma 4.2.3 (the first formula about level sets is Theorem 10.4.12 in this book due to Ambrosio, Di Marino, Gigli [24], and the second Coarea formula is (10.12) due to Federer [260], Theorem 3.2.12.).

Lemma 4.2.3. *Let $f : \mathbb{R}^n \to [0,\infty)$ be Lipschitz with compact support and $\int_{\mathbb{R}^n} f > 0$.*
Level sets: $\mathcal{H}^{n-1}\left(f^{-1}(t)\right) = \mathcal{M}_-\left(\{f \geqslant t\}\right)$ *for a.e. $t > 0$ in the image of f.*
Coarea formula: $\displaystyle\int_{\mathbb{R}^n} \|Df\| = \int_{\mathbb{R}} \mathcal{H}^{n-1}\left(f^{-1}(t)\right)\, dt.$
Layer Cake formula: $\displaystyle\int_{\mathbb{R}^n} f = \int_0^\infty \Big|\{f > t\}\Big|\, dt.$

Lemma 4.2.4. *If $F : [0,\infty) \to [0,\infty)$ is a decreasing function, and $\alpha > 1$, then*

$$\alpha \int_0^\infty t^{\alpha-1} F(t)^\alpha dt \leqslant \left(\int_0^\infty F(t)\, dt\right)^\alpha. \tag{4.8}$$

Proof. Since F is decreasing, we have $tF(t) \leqslant \int_0^t F(s)\, ds$, and hence

$$\alpha\big(tF(t)\big)^{\alpha-1} F(t) \leqslant \alpha\left(\int_0^t F(s)\, ds\right)^{\alpha-1} F(t) = \frac{d}{dt}\left(\int_0^t F(s)\, ds\right)^\alpha.$$

Integrating the inequality above over $(0,\infty)$ yields the inequality (4.8). ∎

Proof of Theorem 4.2.1. We can assume that f is not identically zero, and observe that $|f|$ is a non-negative Lipschitz function with compact support. We deduce combining the first formula in Lemma 4.2.3, and the Isoperimetric Inequality (Theorem 4.1.6) that for a.e. $t > 0$ in the image of $|f|$ and for $n' = \frac{n}{n-1}$, we have

$$\mathcal{H}^{n-1}\left(|f|^{-1}(t)\right) = \mathcal{M}_-\left(|f|^{-1}(t)\right) \geqslant n\omega_n^{\frac{1}{n}} |\{|f| \geqslant t\}|^{1/n'}. \tag{4.9}$$

We observe that $\|Df\| = \|D|f|\,\|$ holds at the points of $\{f \neq 0\}$ where Df exists, and $\|D|f|\,\| = 0$ holds at the points of $\{f = 0\}$ where $D|f|$ exists. Therefore, combining the Coarea Formula in Lemma 4.2.3 with (4.9) and Lemma 4.2.4 (with $F(t) = |\{|f| > t\}|^{1/n'}$ and $\alpha = n'$), and using the substitution $s = t^{n'}$ lead to

$$\begin{aligned}
\int_{\mathbb{R}^n} \|Df\| \geqslant \int_{\mathbb{R}^n} \|D|f|\,\| &= \int_0^\infty \mathcal{H}^{n-1}\left(|f|^{-1}(t)\right)\, dt \geqslant n\omega_n^{\frac{1}{n}} \int_0^\infty |\{|f| \geqslant t\}|^{1/n'}\, dt \\
&\geqslant n\omega_n^{1/n}\left(n' \int_0^\infty t^{n'-1} |\{|f| \geqslant t\}|\, dt\right)^{1/n'} = n\omega_n^{1/n}\left(\int_0^\infty |\{|f|^{n'} \geqslant s\}|\, ds\right)^{1/n'} \\
&= n\omega_n^{1/n} \|f\|_{n'},
\end{aligned}$$

where the last equality follows from the Layer Cake formula (cf. Lemma 4.2.3). ∎

Remark 4.2.5. If the $f : \mathbb{R}^n \to \mathbb{R}$ of compact support in the Sobolev inequality (4.6) is assumed to be C^n (n times differentiable), then $F = |f|$ is Lipschitz and F is C^n

on the bounded open set $\Omega = \{F > 0\}$. Therefore, the Sard theorem can be applied to $F|_\Omega$ (see Figalli [271] for a simple proof); namely,

$$\mathcal{H}^1\left(\left\{t > 0 : \exists x \in F^{-1}(t) \text{ such that } Df(x) = 0\right\}\right) = 0,$$

and hence for $\mathcal{H}^1$ a.e. t in the image of F,

$$F^{-1}(t) \text{ is a compact } (n-1)\text{-dimensional } C^n \text{ manifold.} \tag{4.10}$$

In turn, combining (4.3) and (4.10) yields that for $\mathcal{H}^1$ a.e. t in the image of F, we have

$$\mathcal{M}_-(\{F \geqslant t\}) = \mathcal{H}^{n-1}\left(F^{-1}(t)\right). \tag{4.11}$$

Therefore, applying (4.11) instead of the first formula in Lemma 4.2.3, the argument to prove Theorem 4.2.1 yields the Sobolev inequality (4.6) for compactly supported C^n functions.

We note that if $n \geqslant 2$ and one only assumes that F is C^1 besides having compact support, then it is possible that (4.11) does not hold for a set t of positive $\mathcal{H}^1$ measure in the image of F (cf. Ambrosio, Di Marino, Gigli [24]); therefore, one has to apply the first formula in Lemma 4.2.3 even in the case when the f in the Sobolev inequality (4.6) is C^1.

4.3 The Anisotropic Isoperimetric inequality

In order to define the Anisotropic Perimeter of a compact set with Lipschitz boundary (cf. Definition 4.1.4), we need some notions in convexity. For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, and for $x \in \mathbb{R}^n$, Section 1.6 introduces the convex and one homogeneous support function $h_K(x) = \max\{\langle y, x\rangle : y \in K\}$ on $\mathbb{R}^n$, and Section 1.9 discusses the associated norm $\|x\|_K = \min\{t \geqslant 0 : x \in t\,K\}$, the polar convex body $K^* = \{z \in \mathbb{R}^n : \langle z, y\rangle \leqslant 1 \ \forall\ y \in K\}$, and the relation $h_K(x) = \|x\|_{K^*}$. According to Definition 4.1.4, if a compact $X \subset \mathbb{R}^n$ has Lipschitz boundary, then $0 < \mathcal{H}^{n-1}(x) < \infty$, and for $\mathcal{H}^{n-1}$ a.e. $x \in \partial X$, there exists a unique exterior normal $\nu_X(x) \in S^{n-1}$.

Given a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ and $\varrho > 0$, the Minkowski sum $X + \varrho K$ can be considered as the anisotropic parallel domain of a compact $X \subset \mathbb{R}^n$. Extending the ideas of Ambrosio, Colesanti, Villa [23] about the outer Minkowski content, the following statement is due to Chambolle, Lisini, Lussardi [184] and Lussardi, Villa [547].

Theorem 4.3.1 (Anisotropic Minkowski content). *For a compact $X \subset \mathbb{R}^n$ with Lipschitz boundary, if $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$P_K(X) = \lim_{\varrho \to 0^+} \frac{|X + \varrho K| - |X|}{\varrho} = \int_{\partial X} h_K(\nu_X)\, d\mathcal{H}^{n-1} = \int_{\partial X} \|\nu_X\|_{K^*}\, d\mathcal{H}^{n-1}.$$

Remarks. $P_K(X)$ is called anisotropic perimeter or anisotropic Minkowski content.

- Theorem 4.3.1 yields that $P_K(K) = n|K|$.
- $P_K(\gamma X) = \gamma^{n-1} P_K(X)$ for $\gamma > 0$.
- As the Lebesgue measure is translation invariant, $P_K(X)$ does not depend on translating K, and hence $\int_{\partial X} \langle u, \nu_X \rangle \, d\mathcal{H}^{n-1} = 0$ for $u \in \mathbb{R}^n$.

The Anisotropic Isoperimetric Inequality is a consequence of the Brunn-Minkowski inequality (4.2).

Theorem 4.3.2 (Anisotropic Isoperimetric Inequality). *For a compact $X \subset \mathbb{R}^n$ with Lipschitz boundary (cf. Definition 4.1.4), if $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, and $|X| = |rK|$ for $r > 0$, then $P_K(X) \geqslant P_K(rK)$. Equivalently,*

$$P_K(X) \geqslant n|K|^{\frac{1}{n}} |X|^{\frac{n-1}{n}}. \tag{4.12}$$

Remark. Equality holds if and only if $X = z + rK$, as it follows from Theorem 5.2.1 even dealing with the case of sets of finite perimeter. The case when X is a convex body has been already considered in Theorem 2.4.4.

Proof. The Brunn-Minkowski inequality (4.2) yields that

$$\begin{aligned} \frac{|X + \varrho K| - |X|}{\varrho} &\geqslant \frac{\left(|X|^{\frac{1}{n}} + \varrho |K|^{\frac{1}{n}}\right)^n - |X|}{\varrho} = \frac{\left(|rK|^{\frac{1}{n}} + |\varrho K|^{\frac{1}{n}}\right)^n - |rK|}{\varrho} \\ &= \frac{|rK + \varrho K| - |rK|}{\varrho} = P_K(rK). \end{aligned}$$

■

A satisfactory stability version of the Anisotropic Isoperimetric Inequality Theorem 4.3.2 was provided by Figalli, Maggi, Pratelli [278] (actually, even in the more general framework of sets of finite perimeter, cf. Theorem 5.2.2). We recall that for compact sets $X, Y \subset \mathbb{R}^n$, if $\alpha = |X|^{\frac{-1}{n}}$ and $\beta = |Y|^{\frac{-1}{n}}$, then

$$A(X, Y) = \min \left\{ |\alpha X \Delta (z + \beta Y)| : z \in \mathbb{R}^n \right\}.$$

Theorem 4.3.3 (Figalli, Maggi, Pratelli). *For $\theta_n = 2^{-16} n^{-17}$, if $X \subset \mathbb{R}^n$ is compact with Lipschitz boundary, and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$P_K(X) \geqslant n|K|^{\frac{1}{n}} |X|^{\frac{n-1}{n}} \left[1 + \theta_n \cdot A(K, X)^2\right]. \tag{4.13}$$

Remark. Here the exponent 2 of $A(K, X)^2$ is optimal, and θ_n can't be larger than $36n^{-2}$ (see Remark 8.6.6).

To verify the anisotropic version of the Sobolev inequality, we need a special case Lemma 4.3.4 of the anisotropic version of the coarea formula, which actually follows from the classical version (cf. (10.12)); namely,

$$\int_{\mathbb{R}^k} \varphi \cdot \|DF\| \, d\mathcal{H}^k = \int_{\mathbb{R}} \int_{F^{-1}(t)} \varphi \, d\mathcal{H}^{k-1} \, dt \tag{4.14}$$

where $F : \mathbb{R}^n \to [0, \infty)$ is locally Lipschitz and $\varphi : \mathbb{R}^n \to [0, \infty)$ measurable.

While the Anisotropic Coarea formula holds for functions of bounded variation (see Theorem 5.1.17), here, we prove it only for a positive C^n function F defined in an open bounded $\Omega \subset \mathbb{R}^n$, because in this case, Sard's theorem says (cf. Figalli [271] for a simple proof)

$$\mathcal{H}^1 \left(\{ t > 0 : \exists x \in F^{-1}(t) \text{ such that } DF(x) = 0 \} \right) = 0,$$

and hence for $\mathcal{H}^1$ a.e. t in the image of F,

$$F^{-1}(t) \text{ is a compact } (n-1)\text{-dimensional } C^1 \text{ manifold.} \tag{4.15}$$

It follows that the right hand side makes sense in Lemma 4.3.4 (i).

Lemma 4.3.4 (Anisotropic Coarea formula). *Let $n \geqslant 1$, let $K \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int} K$ and let $F : \mathbb{R}^n \to (0, \infty)$ be Lipschitz with compact support such that F is C^n on $\Omega = \{F > 0\}$.*

(i) $$\int_{\mathbb{R}^n} \|DF\|_{K^*} \, d\mathcal{H}^n = \int_0^\infty P_K(\{F \geqslant t\}) \, dt.$$

(ii) If $\psi : \mathbb{R}^n \to [0, \infty)$ is measurable, then

$$\int_{\mathbb{R}^n} \psi \cdot \|DF\|_{K^*} \, d\mathcal{H}^n = \int_0^\infty \int_{F^{-1}(t)} \psi \cdot \|\nu_{\{\{F \geqslant t\}}\|_{K^*} \, d\mathcal{H}^{n-1} \, dt.$$

Remark. We set $P_K(\varnothing) = 0$.

Proof. We prove (ii), which in turn yields (i). If $DF(x) \neq o$ for $x \in \Omega$, and $F(x) = t$, then an open neighbourhood of x of the boundary of $\{F \geqslant t\}$ is a C^n manifold, and hence there exists an exterior unit normal $\nu_{\{\{F \geqslant t\}}(x) \in S^{n-1}$ to $\{F \geqslant t\}$ at x satisfying that

$$DF(x) = \|DF(x)\| \cdot \nu_{\{\{F \geqslant t\}}(x),$$

which in turn yields that

$$\|DF(x)\|_{K^*} = \|DF(x)\| \cdot \|\nu_{\{\{F \geqslant t\}}(x)\|_{K^*}. \tag{4.16}$$

On the other hand, the Sard theorem (4.10) says that $F^{-1}(t)$ is a compact $(n-1)$-dimensional C^n manifold for $\mathcal{H}^1$ a.e. t in the image of F. Therefore, applying (10.12) to

$$\varphi(x) = \begin{cases} \psi(x) \cdot \|\nu_{\{\{F \geqslant t\}}(x)\|_{K^*} & \text{if } x \in \Omega \text{ and } DF(x) \neq o, \\ \psi(x) & \text{otherwise,} \end{cases}$$

implies (ii). ∎

Analogously to the classical Sobolev inequality, the following Anisotropic Sobolev inequality is, in some sense, the functional version of the Anisotropic Isoperimetric inequality.

Theorem 4.3.5 (Anisotropic Sobolev inequality). *If $n' = \frac{n}{n-1}$, the function $f \in C^n(\mathbb{R}^n)$ has compact support and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$\int_{\mathbb{R}^n} \|Df\|_{K^*} \geqslant n|K|^{\frac{1}{n}} \|f\|_{n'}.$$

Idea of the proof. The proof of the Anisotropic Sobolev inequality runs as the argument for the Sobolev inequality Theorem 4.2.1 as it is sketched in Remark 4.2.5, only the Coarea formula has to be changed into the Anisotropic Coarea formula Lemma 4.3.4, and the classical Isoperimetric inequality is replaced by the Anisotropic one (cf. (4.12)). ∎

Remark 4.3.6.

- This proof yields the Anisotropic Sobolev inequality for any Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ with compact support such that for a.e. $t > 0$ in the image of $|f|$, the level sets $\{|f| \geqslant t\}$ have Lipschitz boundary (the condition that f is C^n was only used to ensure this property).
- The factor $n|K|^{\frac{1}{n}}$ is optimal, and the Anisotropic Sobolev inequality yields the Anistropic Isoperimetric Inequality.
- According to Theorem 5.3.1 (see also Figalli, Maggi, Pratelli [279]), the Anistropic Sobolev inequality has a natural extension to functions of bounded variation where equality holds if and only if $f = a\mathbf{1}_{x+rK}$ for $a, r > 0$ and $x \in \mathbb{R}^n$. Moreover, [279] also proves a sharp stability version of that inequality.

4.4 Wulff shape and minimizing surface energy

For a thorough discussion of properties of Wulff shapes, whose main properties were developed by Aleksandrov [9, 11], see Schneider [661], Section 7.5 (see also Section 7.5 for the Aleksandrov lemma for Wulff shapes). We note that for a convex body

$K \subset \mathbb{R}^n$, a closed $\Omega \subset S^{n-1}$ and $\varphi : \Omega \to \mathbb{R}$, if $K = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant \varphi(u) \text{ for } x \in \Omega\}$, then Lemma 2.5.6 yields that Ω is not contained in any closed hemisphere and

$$\{\nu_K(x) : x \in \partial' K\} \subset \operatorname{supp} S_K \subset \Omega. \tag{4.17}$$

For a closed $\Omega \subset S^{n-1}$, we recall that a function $\varrho : \Omega \to (0, \infty)$ is lower semi-continuous if $u_k \to u$ for $u, u_k \in \Omega$ yields that $\varrho(u) \leqslant \liminf_{k\to\infty} \varrho(u_k)$. In this case, $\inf \varrho > 0$.

Definition 4.4.1 (Wulff shape). Given a closed $\Omega \subset S^{n-1}$ not contained in a closed hemisphere, and a bounded lower semi-continuous function $\varrho : \Omega \to (0, \infty)$, the corresponding Wulff shape is given by

$$W_\varrho = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant \varrho(u)\ \forall u \in \Omega\}.$$

Remark. Since for any $v \in S^{n-1}$, one finds $u \in \Omega$ such that $\langle u, v\rangle > 0$, W_ϱ is a convex body with $o \in \operatorname{int} W_\varrho$, and $h_{W_\varrho}(u) \leqslant \varrho(u)$ for $u \in \Omega$.

Wulff shapes occur for example in the solution of Minkowski-type Monge-Ampère equations *via* the variational method (see Chapter 9), like the Minkowski Problem (cf. Section 9.2) or its generalizations, such as the L_p-Minkowski problem (cf. Section 9.3). In these applications, it frequently happens that the Ω in Definition 4.4.1 is finite, and hence W_ϱ is automatically a polytope. However, originally, the Wulff shape was considered as a model for possible shapes of a crystal *via* minimizing a certain surface tension given the volume.

Example 4.4.2 (Modelling crystal growth). Given a lattice $\Lambda = \sum_{i=1}^n \mathbb{Z}\, v_i$ for independent $v_1, \dots, v_n \in \mathbb{R}^n$, and constants $\tau, c > 0$ such that $\|v\| > \frac{c}{\tau}$ for $v \in \Lambda \backslash o$, if $u \in S^{n-1}$, then the "surface tension" is given by

$$\varrho(u) = \begin{cases} \tau & \text{if } \mathbb{R}\, u \cap \Lambda = \{o\}; \\ \tau - \frac{c}{\alpha} & \text{if } \alpha u \in \Lambda \text{ for } \alpha > 0 \text{ and } (0, \alpha)\, u \cap \Lambda = \varnothing. \end{cases}$$

Then Diophantine approximation yields that the corresponding Wulff shape W_ϱ is a polytope; namely, there exist "short" $w_1, \dots, w_k \in \Lambda \backslash o$ such that

$$W_\varrho = \{x \in \mathbb{R}^n : \langle x, u_i\rangle \leqslant \varrho(u_i),\ i = 1, \dots, k\} \ \text{ for } \ u_i = \frac{w_i}{\|w_i\|}.$$

Actually, W_ϱ is a possible shape for a crystal whose underlying crystallographic structure is the dual lattice $\Lambda^* = \{z \in \mathbb{R}^n : \langle z, y\rangle \in \mathbb{Z} \text{ for } y \in \Lambda\}$.

We note that for the Wulff shape W_ϱ defined as in Definition 4.4.1, it can happen that $h_{W_\varrho}(u) < \varrho(u)$ for the typical $u \in \Omega$. However, these two functions coincide on the exterior normals at the regular boundary points of the Wulff shape, and hence the support function can be used instead of ϱ in calculating the surface tension of W_ϱ where the integrals in Lemma 4.4.3 make sense by $\partial' W_\varrho \subset \nu_K^{-1}(\Omega)$ (cf. (4.17)).

Lemma 4.4.3 (Aleksandrov [9, 11]). *If $\Omega \subset S^{n-1}$ is closed and not contained in a closed hemisphere, and $\varrho : \Omega \to (0,\infty)$ is bounded and lower semi-continuous, then*

$$\int_{\partial W_\varrho} \varrho(\nu_{W_\varrho})\, d\mathcal{H}^{n-1} = \int_{\partial W_\varrho} h_{W_\varrho}(\nu_{W_\varrho})\, d\mathcal{H}^{n-1} = n|W_\varrho|,$$
$$\varrho(u) = h_{W_\varrho}(u) \text{ for } u \in \operatorname{supp} S_{W_\varrho}.$$

Proof. Set $W = W_\varrho$. It is sufficient to prove by Theorem 1.5.2, Theorem 2.1.5 and Lemma 2.5.6, and by the lower semi-continuity of ϱ that

$$\varrho(\nu_W(x)) = h_W(\nu_W(x)) \text{ for } x \in \partial' W. \tag{4.18}$$

If $x \in \partial' W$, and hence $\nu_K(x) \in \Omega$ according to (4.17), then there exists $x_k \notin W$ with $x_k \to x$, thus there exists $v_k \in \Omega$ satisfying $\langle v_k, x_k\rangle > \varrho(v_k)$ by $x_k \notin W$. We may assume that $v_k \to v \in \Omega$, thus the lower semi-continuity of ϱ implies that

$$\varrho(v) \leqslant \liminf_{k\to\infty} \varrho(v_k) \leqslant \liminf_{k\to\infty} \langle v_k, x_k\rangle = \langle v, x\rangle$$

where $x \in W$ yields $\langle v, x\rangle \leqslant h_W(v) \leqslant \varrho(v)$. We deduce that $v = \nu_K(x)$ with $h_W(v) = \varrho(v)$, and in turn (4.18) follows. ∎

For measurable and bounded $\varrho : S^{n-1} \to (0,\infty)$ and non-empty compact $X \subset \mathbb{R}^n$ with Lipschitz boundary (cf. Definition 4.1.4), the associated surface energy of X is

$$\mathcal{E}_\varrho(\partial X) = \int_{\partial X} \varrho(\nu_X)\, d\mathcal{H}^{n-1},$$

where $\nu_X(x)$ is the unique exterior unit normal at $\mathcal{H}^{n-1}$ a.e. $x \in \partial X$ (see Theorem 5.2.5 for the version for sets of finite perimeter).

Theorem 4.4.4 (Minimizing surface energy). *Let $\varrho : S^{n-1} \to (0,\infty)$ be bounded and lower semi-continuous. If a compact $X \subset \mathbb{R}^n$ has Lipschitz boundary (cf. Definition 4.1.4), and $|X| = |rW_\varrho|$ for $r > 0$, then*

$$\mathcal{E}_\varrho(\partial X) \geqslant \mathcal{E}_\varrho\left(\partial(rW_\varrho)\right) = n\,|W_\varrho|^{\frac{1}{n}} \cdot |X|^{\frac{n-1}{n}}. \tag{4.19}$$

Equality holds in (4.19) if and only if $X = z + rW_\varrho$ for $z \in \mathbb{R}^n$.

Proof. For $W = W_\varrho$, $\varrho \geqslant h_W$, the Anisotropic Isoperimetric Inequality and Lemma 4.4.3 yield

$$\begin{aligned}\mathcal{E}_\varrho(\partial X) &= \int_{\partial X} \varrho \circ \nu_X\, d\mathcal{H}^{n-1} \geqslant \int_{\partial X} h_W \circ \nu_X\, d\mathcal{H}^{n-1} \\ &\geqslant \int_{\partial(rW)} h_W \circ \nu_{rW}\, d\mathcal{H}^{n-1} = \int_{\partial(rW)} \varrho \circ \nu_{rW}\, d\mathcal{H}^{n-1} = \mathcal{E}_\varrho\left(\partial(rW)\right)\end{aligned}$$

where $\int_{\partial(rW)} h_W \circ \nu_{rW}\, d\mathcal{H}^{n-1} = n\,|W_\varrho|^{\frac{1}{n}} \cdot |X|^{\frac{n-1}{n}}$.

Equality in (4.19) yields equality in the Anisotropic Isoperimetric Inequality; therefore, X is a translate of rW. In this case, we do have equality in (4.19) by Lemma 4.4.3. ∎

Combining the argument above with the stability version of the Anisotropic Isoperimetric Inequality Theorem 4.3.3 due to Figalli, Maggi and Pratelli [278] yields the following stability version of Wulff's theorem (see Theorem 4.3.3 for notation).

Theorem 4.4.5 (Figalli, Maggi, Pratelli). *Let $\theta_n = 2^{-16}n^{-17}$, and $\varrho : S^{n-1} \to (0,\infty)$ be bounded and lower semi-continuous. If a compact $X \subset \mathbb{R}^n$ has Lipschitz boundary, then*

$$\mathcal{E}_\varrho(\partial X) \geqslant n|W_\varrho|^{\frac{1}{n}}|X|^{\frac{n-1}{n}}\left[1 + \theta_n \cdot A(W_\varrho, X)^2\right]. \tag{4.20}$$

Remark. Figalli, Zhang [281] proved an even stronger stability estimate in the crystalline case when W_ϱ is a polytope.

4.5 Isodiametric and Isoperimetric inequalities in the Hyperbolic and Spherical spaces

The main goal of this section is to prove the Isoperimetric inequality and the Isodiametric inequality in the Spherical and Hyperbolic spaces. Equality in the Isodiametric inequality will be characterized in Section 4.B, but we do not consider the equality case in the Isoperimetric inequality in this monograph. For the fundamental properties of the spherical space S^n and the hyperbolic space H^n, we refer the reader to Section 4.A.

Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n. First, we discuss some fundamental notions that are needed in order to discuss surface area in $\mathcal{M}^n$. The k-dimensional Hausdorff measure with respect to the intrinsic metric of $\mathcal{M}^n$ (see Section 10.4) is denoted by $\mathcal{H}^k_{\mathcal{M}^n}(\cdot)$, for example, $\mathcal{H}^n_{\mathcal{M}^n}(\cdot) = |\cdot|$ is the volume on $\mathcal{M}^n$. We note that for the standard embedding $S^n \subset \mathbb{R}^{n+1}$, we have $\mathcal{H}^k_{S^n}(X) = \mathcal{H}^k_{\mathbb{R}^{n+1}}(X)$ for Borel set $X \subset S^n$ and $0 < k \leqslant n$. We write $d(x,y) = d_{\mathcal{M}^n}(x,y)$ to denote the geodesic distance of $x, y \in \mathcal{M}^n$, and write $B(z,r) = \{x \in \mathcal{M}^n : d(x,z) \leqslant r\}$ to denote the metric ball of radius r centered at x where we assume $r < \pi$ if $\mathcal{M}^n = S^n$. For measurable $X \subset \mathcal{M}^n$ and $\varrho > 0$, the corresponding parallel domain is

$$X^{(\varrho)} = \{y \in \mathcal{M}^n : \exists\, x \in X \text{ with } d(x,y) \leqslant \varrho\} = \bigcup \{B_{\mathcal{M}^n}(x,\varrho) : x \in X\},$$

which is compact if X is compact.

Following Remark 10.4.11, we say that a non-empty compact $X \subset \mathcal{M}^n$ has Lipschitz boundary if $X = \operatorname{cl}\operatorname{int} X$, $\partial X \neq \varnothing$, and there exists a C^2 diffeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ where $\Omega \subset \mathcal{M}^n$ is an open neighbourhood of X and $\widetilde{\Omega} \subset \mathbb{R}^n$ is open such that the compact set $\varphi(X) \subset \mathbb{R}^n$ has Lipschitz boundary. We note that it is equivalent saying that

$X = \operatorname{cl}\operatorname{int} X$, and for any $x \in \partial X$ where $\partial X \neq \varnothing$, there exists an open neighbourhood U of x on ∂X, and a hyperplane $H \subset \mathcal{M}^n$ such that the orthogonal projection from U into H is bi-Lipschitz onto its open image in H. We note that $0 < \mathcal{H}^{n-1}_{\mathcal{M}^n}(\partial X) < \infty$, and (10.14) yields that the perimeter of X coincides with the outer Minkowski content; namely,

$$P(X) = \mathcal{H}^{n-1}_{\mathcal{M}^n}(\partial X) = \lim_{\varrho \to 0^+} \frac{\left|X^{(\varrho)}\right| - |X|}{\varrho}. \tag{4.21}$$

In order to state Theorem 4.5.1, we fix a $z_0 \in \mathcal{M}^n$. The following statement in the spherical case is proved by Lévy [508] already in 1922, and the hyperbolic version is due to Schmidt [653, 654] in the 1940's.

Theorem 4.5.1. *Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n, and let $X \subset \mathcal{M}^n$ be bounded and measurable with $|X| = |B(z_0, r)|$ for $r > 0$ (where $r < \pi$ if $\mathcal{M}^n = S^n$).*

(a) Isodiametric Inequality: $\operatorname{diam} X \geqslant 2r$ *(where $r \leqslant \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$) where equality holds if and only if $X \subset B(y, r)$ for some $y \in \mathcal{M}^n$.*

(b) Isoperimetric Inequality (Parallel domains): $\left|X^{(\varrho)}\right| \geqslant |B(z_0, r + \varrho)|$ *for $\varrho > 0$.*

(c) Isoperimetric Inequality (classical): *Assuming that X is compact and has Lipschitz boundary, we have $P(X) \geqslant P(B(z_0, r))$ where equality holds if and only if $X = B(y, r)$ for some $y \in \mathcal{M}^n$.*

Remarks.

- In the two dimensional case, the Isoperimetric Inequality has the following nice form based on formulas of the perimeter and area of circular discs in Section 4.A: If $\mathcal{M}^2$ is either H^2, $\mathbb{R}^2$ or S^2 with curvature $\kappa = -1, 0, 1$, respectively, and $X \subset \mathcal{M}^2$ is compact with Lipschitz boundary, then

$$P(X)^2 \geqslant 4\pi|X| - \kappa\,|X|^2.$$

- In this monograph, the equality case is only considered for the Isodiametric Inequality in H^n, S^n and $\mathbb{R}^n$ but assuming $r < \frac{\pi}{4}$ if $\mathcal{M}^n = S^n$ (see Section 4.B). For the case of equality, and even for stability version of the Isoperimetric Inequality in terms of the volume difference, see Bögelein, Duzaar, Fusco [108] in the spherical case, and Bögelein, Duzaar, Scheven [109] in the hyperbolic case.

The main method to prove Theorem 4.5.1 is the so-called two-point symmetrization (sometimes called polarization). For a closed half-space H^+ in $\mathcal{M}^n$, in this section we write $H = \partial H^+$ to denote the bounding hyperplane, and H^- to denote the complementary closed half-space where $H = H^+ \cap H^-$. The reflected image of an $x \in \mathcal{M}^n$ through H is denoted by $\xi_H x$ (see Section 4.A).

Definition 4.5.2 (Two-point symmetrization). Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n. For compact $X \subset \mathcal{M}^n$ and closed half-space where $H^+ \subset S^n$, the two-point symmetrial

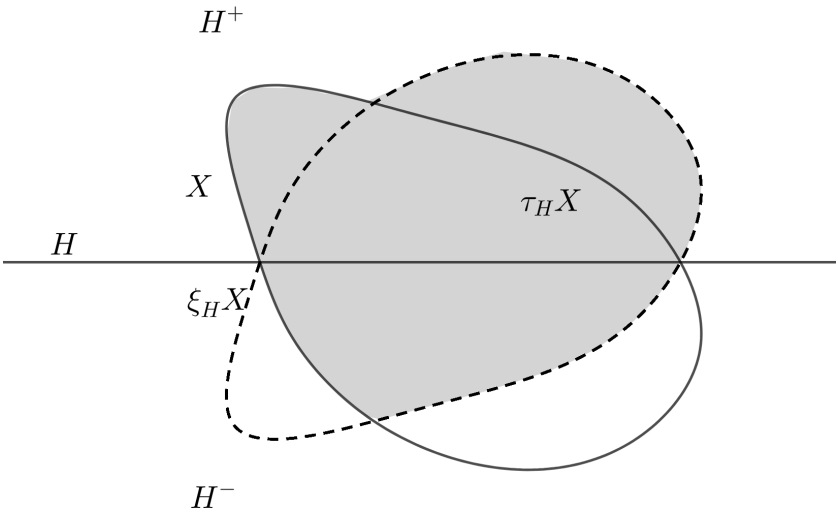


Figure 4.1

of X is

$$\tau_{H^+}X = \left((H^+ \cap X) \cup (H^+ \cap \xi_H X)\right) \bigcup \left((H^- \cap X) \cap (H^- \cap \xi_H X)\right).$$

Remark. We frequently write only $\tau_H X$ to denote $\tau_{H^+}X$ because $\tau_{H^+}X$ and $\tau_{H^-}X$ are symmetric through H.

While $\tau_H X$ may not look more symmetric than X, the two-point symmetrization actually preserves balls. We note that if $K \subset \mathcal{M}^n$ is a convex body such that $\tau_H K$ is convex for any hyperplane H, then K is a ball (see Proposition 4.B.2).

Lemma 4.5.3. *If $\mathcal{M}^n$ is either $\mathbb{R}^n$, H^n or S^n, and $H^+ \subset \mathcal{M}^n$ is a closed half-space, then*

(i) $\tau_{H^+}B(z,r)$ is a ball, and if $z \in H^+$, then $\tau_{H^+}B(z,r) = B(z,r)$;

(ii) $\tau_{H^+}Y \subset \tau_{H^+}Z$ for compact $Y \subset Z$.

The following properties follow from the very definition of the two-point symmetrization, where we also use Lemma 4.A.6 (ii) in the case of (v):

Lemma 4.5.4. *If $\mathcal{M}^n$ is either $\mathbb{R}^n$, H^n or S^n, $H^+ \subset \mathcal{M}^n$ is a closed half-space and $X \subset \mathcal{M}^n$ is compact where* $\operatorname{diam}_{\mathcal{M}^n}(X) < \pi$ *provided $\mathcal{M}^n = S^n$, then*

(i) $\tau_{H^+}X = \tau_{H^+}\left(\xi_H X\right)$ is compact;

(ii) $(\tau_{H^+}X) \cap H^+ = \left(X \cup \xi_H X\right) \cap H^+$;
(iii) $(\tau_{H^+}X) \cap H^- = \left(X \cap \xi_H X\right) \cap H^-$;
(iv) $|\tau_{H^+}X| = |X|$;
(v) $\operatorname{diam}_{\mathcal{M}^n}(\tau_{H^+}X) \leqslant \operatorname{diam}_{\mathcal{M}^n}(X)$.

The reason why two-point symmetrization is so useful for us is that it does not increase the volume of the paralleldomain, as stated in the lemma below.

Lemma 4.5.5 (Benyamini). *If $\varrho > 0$, $\mathcal{M}^n$ is either $\mathbb{R}^n$, H^n or S^n, $H^+ \subset \mathcal{M}^n$ is a closed half-space and $X \subset \mathcal{M}^n$ is compact, then $\left|(\tau_H X)^{(\varrho)}\right| \leqslant \left|X^{(\varrho)}\right|$.*

Proof. It follows from Lemma 4.5.4 (iv) applied to $X^{(\varrho)}$ that it is sufficient to prove

$$(\tau_{H^+}X)^{(\varrho)} \subset \tau_{H^+}\left(X^{(\varrho)}\right). \tag{4.22}$$

Therefore, let $z \in (\tau_{H^+}X)^{(\varrho)}$, thus there exists some $y \in \tau_{H^+}X$ such that $d(y,z) \leqslant \varrho$. Since the role of X and $\xi_H X$ are symmetric in the definition of two-point symmetrization, we may assume that $y \in X$, and hence

$$y \in X \cap \tau_{H^+}X;\ d(y,z) \leqslant \varrho,\ \text{thus } z \in X^{(\varrho)}. \tag{4.23}$$

If $z \in H^+$, then Lemma 4.5.4 (ii) applied to $X^{(\varrho)}$ yields $z \in \tau_{H^+}(X^{(\varrho)})$.

Finally, we assume that $z \notin H^+$, and claim that

$$z \in \xi_H\left(X^{(\varrho)}\right) = (\xi_H X)^{(\varrho)}. \tag{4.24}$$

Here (4.24) readily holds if $y \in \xi_H X$, while if $y \notin \xi_H X$, then $y \in H^+$ by Lemma 4.5.4 (ii), thus $d(z, \xi_H y) \leqslant d(z,y) \leqslant \varrho$ by Lemma 4.A.6 (i), verifying (4.24).

We deduce from (4.23), (4.24), and Lemma 4.5.4 (iii) applied to $X^{(\varrho)}$ that $z \in H^- \cap X^{(\varrho)} \cap \xi_H(X^{(\varrho)}) \subset \tau_{H^+}(X^{(\varrho)})$, completing the proof of Lemma 4.5.5. ∎

We also need the property that starting from any $X \subset \mathcal{M}^n$, there is a sequence of two-point symmetrizations that leads to a ball of equal volume. More precisely, we prove a slightly weaker, but equally useful property.

Lemma 4.5.6 (Benyamini). *Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n. Let $H^+ \subset \mathcal{M}^n$ be a closed half-space and $X \subset \mathcal{M}^n$ be compact with $|X| = |B(z_0,r)|$ for $r > 0$. If $\mathcal{F}$ is the smallest family of compact sets containing X and closed under two-point symmetrization, taking limit and isometries of $\mathcal{M}^n$, then*

(i) $\operatorname{diam} Y \leqslant \operatorname{diam} X$ *for* $Y \in \mathcal{F}$;
(ii) $\left|Y^{(\varrho)}\right| \leqslant \left|X^{(\varrho)}\right|$ *for* $\varrho > 0$ *and* $Y \in \mathcal{F}$;
(iii) $|Y| = |X|$ *for* $Y \in \mathcal{F}$;

(iv) there exists $Z_0 \in \mathcal{F}$ *with* $B(z_0, r) \subset Z_0$.

Proof. We may asume that $z_0 \in X$, and it is enough to prove (i)-(iv) for $\mathcal{F}_0 = \{Z \in \mathcal{F} : B(z_0, r) \cap Z \neq \varnothing\}$.

Now (i) and (ii) follow from Lemmas 3.7.4, 4.5.4 and 4.5.5, which, combined with (ii), also yield (iii).

For (iv), Lemma 3.7.4 yields that there $Z_0 \in \mathcal{F}_0$ with $|B(z_0, r) \cap Z_0|$ is maximal.

Indirectly, we suppose that $B(z_0, r) \not\subset Z_0$, and hence $|B(z_0, r) \cap Z_0| < |B(z_0, r)|$. Since $|Z_0 \backslash B(z_0, r)| = |B(z_0, r) \backslash Z_0| > 0$ by (iii), there exist density points $x \in Z_0 \backslash B(z_0, r)$ and $y \in (\mathrm{int} B(z_0, r)) \backslash Z_0$. We define the closed half-space H^+ by the properties $\xi_H x = y$ and $y \in H^+$, and hence $|B(z_0, r) \cap \tau_{H^+} Z_0| > |B(z_0, r) \cap Z_0|$ by Lemma 4.5.3, which is a contradiction verifying Lemma 4.5.6. ∎

Proof of Theorem 4.5.1 without the equality cases. As for the Z_0 in Lemma 4.5.6 (iv), we have $\mathrm{diam}\, Z_0 \geqslant 2r$ and $|Z_0^{(\varrho)}| \geqslant |B(z_0, r + \varrho)|$ for $\varrho > 0$, we deduce (i) and (ii) in Theorem 4.5.1 (the Isodiametric inequality and the Isoperimetric inequality for parallel domains) from Lemma 4.5.6 (i) and (ii). In turn, we conclude Theorem 4.5.1 (iii) (the classical Isoperimetric inequality) by $|Z_0| = |X| = |B(z_0, r)|$ and (4.21). ∎

4.6 The Gaussian Isoperimetric and the log-Sobolev inequalities

In this section, we show how the Spherical Isoperimetric inequality Theorem 4.5.1 (ii) yields the Gaussian Isoperimetric inequality. In particular, we consider the version

$$\gamma_n(X) = \int_X e^{-\pi \|x\|^2}\, dx \text{ for measurable } X \subset \mathbb{R}^n$$

of the Gaussian probability measure because the formulas are simpler than in the case of the standard normal Gaussian distribution. Actually, the Gaussian Isoperimetric inequality Theorem 4.6.1 holds whatever Gaussian density function we use (see Artstein-Avidan, Giannopoulos, V. Milman [39], Chapter 2 for a more detailed study on relatives of the Gaussian Isoperimetric inequality). We recall that for $X \subset \mathbb{R}^n$ and $\varrho > 0$, $X^{(\varrho)} = X + \varrho\, B^n$ is the parallel domain; namely, the set of points whose distance from X is at most ϱ. The Gaussian Isoperimetric inequality is due to Borell [111] and to Sudakov, Tsirelson [694] around 1975.

Theorem 4.6.1 (Gaussian Isoperimetric inequality). *If* $\varrho > 0$, $X \subset \mathbb{R}^n$ *is measurable and* H_+ *is a closed half-space with* $\gamma_n(H_+) = \gamma_n(X) \in (0, 1)$, *then*

$$\gamma_n\left(X^{(\varrho)}\right) \geqslant \gamma_n\left(H_+^{(\varrho)}\right). \tag{4.25}$$

Remark. For $u \in S^{n-1}$ and $a > 0$, the parallel domain of the half-space $H_+ = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant a\}$ with radius $\varrho > 0$ is $H_+^{(\varrho)} = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant a + \varrho\}$. In addition, the Fubini Theorem yields that

$$\gamma_n(H_+) = \Psi(a) \tag{4.26}$$

for the cumulative distribution function $\Psi(s) = \int_{-\infty}^{s} e^{-\pi\tau^2}\, d\tau$ of the one-dimensional Gaussian γ_1. Therefore, an equivalent form of the Gaussian Isoperimetric inequality is that if $X \subset \mathbb{R}^n$ is measurable with $\gamma_n(X) \in (0,1)$, then

$$\Psi^{-1}\left(\gamma_n\left(X^{(\varrho)}\right)\right) \geqslant \Psi^{-1}\left(\gamma_n(X)\right) + \varrho. \tag{4.27}$$

Let us formulate the Gaussian Isoperimetric Inequality in terms of some notions of "Gaussian surface area". For example, the (lower) Gaussian outer Minkowski content of a measurable $X \subset \mathbb{R}^n$ is

$$\gamma_{n,+}(X) = \liminf_{\varrho \to 0^+} \frac{\gamma_n(X^{(\varrho)}) - \gamma_n(X)}{\varrho}$$

that is a special case of the upcoming Definition 4.7.1 concerning log-concave densities. In particular, if $X \subset \mathbb{R}^n$ is compact and has Lipschitz boundary (a neighbourhood on ∂X of any boundary point is the graph of a Lipschitz function, cf. Definition 4.1.4), then the upcoming formula (5.56) in Chapter 5 says that

$$\gamma_{n,+}(X) = \lim_{\varrho \to 0^+} \frac{\gamma_n(X^{(\varrho)}) - \gamma_n(X)}{\varrho} = \int_{\partial X} e^{-\pi\|x\|^2}\, d\mathcal{H}^{n-1}(x). \tag{4.28}$$

Here the last integral in (4.28) is the Gaussian perimeter of X. We also deduce from (4.26) and the Fubini Theorem that if $H_+ = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant s\}$ is a half-space for $s \in \mathbb{R}$ and $u \in S^{n-1}$, then

$$\gamma_{n,+}(H^+) = \lim_{\varrho \to 0^+} \frac{\gamma_n(H_+^{(\varrho)}) - \gamma_n(H_+)}{\varrho} = \int_{\partial H_+} e^{-\pi\|x\|^2}\, d\mathcal{H}^{n-1}(x) = e^{-\pi s^2}. \tag{4.29}$$

For Theorem 4.6.2, we conclude (i) directly from the Gaussian Isoperimetric inequality (4.25) and (4.29), and (ii) follows from combining (i), (4.28) and Corollary 5.4.6.

Theorem 4.6.2. *Let $X \subset \mathbb{R}^n$ be measurable, and let H_+ be a closed half-space such that $\gamma_n(X) = \gamma_n(H_+) \in (0,1)$.*

(i) $\gamma_{n,+}(X) \geqslant \gamma_{n,+}(H^+) = \int_{\partial H_+} e^{-\pi\|x\|^2}\, d\mathcal{H}^{n-1}(x)$.

(ii) If $X \subset \mathbb{R}^n$ has Lipschitz boundary, then

$$\int_{\partial X} e^{-\pi\|x\|^2}\, d\mathcal{H}^{n-1}(x) \geqslant \int_{\partial H_+} e^{-\pi\|x\|^2}\, d\mathcal{H}^{n-1}(x). \tag{4.30}$$

Remark 4.6.3. Theorem 4.6.1 and Theorem 4.6.2 also hold for the standard Gaussian measure $d\tilde{\gamma}_n(x) = (2\pi)^{\frac{-n}{2}} e^{-\frac{\|x\|^2}{2}} dx$, that has zero mean and its covariance matrix is the identity (see Definition 4.7.12), only $\tilde{\gamma}_{n,+}$ is defined in terms of $\tilde{\gamma}_n$. In particular, for the cumulative distribution function $\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\tau^2/2}\, d\tau$ of the one-dimensional Gaussian $\tilde{\gamma}_1$, we consider the density $\varphi(s) = \Phi'(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}$ of $\tilde{\gamma}_1$, and hence we have the following analogues of (4.27) and (4.30): If $X \subset \mathbb{R}^n$ is measurable with $\tilde{\gamma}_n(X) \in (0,1)$ and $\varrho > 0$, then

$$\Phi^{-1}\left(\tilde{\gamma}_n\left(X^{(\varrho)}\right)\right) \geqslant \Phi^{-1}\left(\tilde{\gamma}_n(X)\right) + \varrho \tag{4.31}$$

where equality holds if X is a half-space. In addition, if $X \subset \mathbb{R}^n$ is has (non-empty) Lipschitz boundary, and $\tilde{\gamma}_n(X) = \tilde{\gamma}_n(H_+) = \Phi(s)$ for a closed half-space H_+ and $s \in \mathbb{R}$, then

$$(2\pi)^{\frac{-n}{2}} \int_{\partial X} e^{-\frac{\|x\|^2}{2}}\, d\mathcal{H}^{n-1}(x) \geqslant (2\pi)^{\frac{-n}{2}} \int_{\partial H_+} e^{-\frac{\|x\|^2}{2}}\, d\mathcal{H}^{n-1}(x) = \varphi(s). \tag{4.32}$$

The idea of the proof of the Gaussian Isoperimetric inequality is to embed $\mathbb{R}^n$ into $\mathbb{R}^{k+1}$ as a linear subspace where $k \to \infty$, and consider the embedded submanifold $r_k S^k \subset \mathbb{R}^{k+1}$, and the orthogonal projection $\pi_k : r_k S^k \to \mathbb{R}^n$ satisfying $\|\pi_k(x) - \pi_k(y)\| \leqslant d_{r_k S^k}(x,y)$ for

$$r_k = \sqrt{\frac{k}{2\pi}}.$$

Noting that the density function of the uniform probability measure on $r_k S^k$ is $\frac{1}{\mathcal{H}^k(r_k S^k)}$, the core claim is that

$$\gamma_n(Z) = \lim_{k\to\infty} \frac{1}{\mathcal{H}^k(r_k S^k)} \cdot \mathcal{H}^k\left(\pi_k^{-1}(Z)\right) \quad \text{for any } Z \subset \mathbb{R}^n \text{ with } |Z| > 0. \tag{4.33}$$

Here if $H_+ \subset \mathbb{R}^n$ is a half-space, then $\pi_k^{-1}(H_+)$ is a spherical cap - that is, a spherical ball - for large k, which is the extremal body for the Spherical Isoperimetric Inequality.

Now the claim (4.33) yields Theorem 4.6.1 via the following chain of equalities and inequalities (see the explanations after (4.37)):

$$\begin{aligned}
\gamma_n\left(X^{(\varrho)}\right) &= \lim_{k\to\infty} \frac{1}{\mathcal{H}^k(r_k S^k)} \cdot \mathcal{H}^k\left(\pi_k^{-1}\left(X^{(\varrho)}\right)\right) && (4.34)\\
&\geqslant \lim_{k\to\infty} \frac{1}{\mathcal{H}^k(r_k S^k)} \cdot \mathcal{H}^k\left(\left(\pi_k^{-1} X\right)^{(\varrho)}\right) && (4.35)\\
&\geqslant \lim_{k\to\infty} \frac{1}{\mathcal{H}^k(r_k S^k)} \cdot \mathcal{H}^k\left(\left(\pi_k^{-1} H_+\right)^{(\varrho)}\right) && (4.36)\\
&= \gamma_n\left(H_+^{(\varrho)}\right). && (4.37)
\end{aligned}$$

Before explaning how to get from (4.34) to (4.37) based on (4.33), we remark that if $z \in S^k$ and $x = \pi_k(z)$ with $\|x\| < r_k$, and $v \in T_z$, then

$$D\pi_k v = v|\mathbb{R}^n. \tag{4.38}$$

Firstly, (4.33) directly yields (4.34). In turn (4.34) implies (4.35) as π_k contraction, and hence $\left(\pi_k^{-1}X\right)^{(\varrho)} \subset \pi_k^{-1}\left(X^{(\varrho)}\right)$. Next (4.36) follows from (4.35) and the Spherical Isoperimetric inequality Theorem 4.5.1 (ii), as if $H_+ = \{x \in \mathbb{R}^n : \langle x, w\rangle \geqslant s\}$ for $w \in S^{n-1}$ and $s \in \mathbb{R}$, and for large k, $H_{+,k} = \{x \in \mathbb{R}^n : \langle x, w\rangle \geqslant s_k\}$ is the half-space for $s_k \in \mathbb{R}$ such that $\mathcal{H}^k\left(\pi_k^{-1}X\right) = \mathcal{H}^k\left(\pi_k^{-1}H_{+,k}\right)$, then (4.33) yields $\lim_{k\to\infty} s_k = s$.

In order to show that (4.36) implies (4.37), writing $\left(\pi_k^{-1}H_+\right)^{(\varrho)}$ to denote the parallel domain of $\pi_k^{-1}H_+$ on S^k of spherical radius ϱ, we prove that for given $\varepsilon \in (0, \varrho)$, if k is large, then

$$\pi_k^{-1}\left(H_+^{(\varrho-\varepsilon)}\right) \subset \left(\pi_k^{-1}H_+\right)^{(\varrho)} \subset \pi_k^{-1}\left(H_+^{(\varrho)}\right) \tag{4.39}$$

where we the second inequality in (4.39) follows as π_k is a contraction. To verify the first inequality in (4.39), let $z \in \partial\pi_k^{-1}H_+$, $x = \pi_k z$ and $v \in T_z$ be the exterior unit normal to $\pi_k^{-1}H_+$, and hence $z_0 = z\cos\frac{\varrho}{r_k} + vr_k\sin\frac{\varrho}{r_k} \in \left(\pi_k^{-1}H_+\right)^{(\varrho)}$ and if k is large, then

$$\langle \pi_k z_0, w\rangle = \langle z_0, w\rangle = s\cdot\cos\frac{\varrho}{r_k} + \sqrt{1-\frac{s^2}{r_k^2}}\cdot r_k\sin\frac{\varrho}{r_k} > s + \varrho - \varepsilon,$$

completing the proof of (4.39). In turn, combining our core claim (4.33) with (4.39) implies (4.37).

Therefore all we are left to do is to prove the core claim (4.33). First we recall the coarea formula in the following form (cf. Corollary 10.4.10).

Remark 4.6.4 (Coarea formula). For $n \leqslant k < q$ and a C^2 embedded Riemannian k-manifold $X^k \subset \mathbb{R}^q$, if $F : X^k \to \mathbb{R}^n$ is locally Lipschitz and $\psi : \mathbb{R}^n \to \mathbb{R}$ is measurable, then

$$\int_{X^k} \psi(F(z))\cdot J(F,z)\, d\mathcal{H}^k(z) = \int_{\mathbb{R}^n} \psi(y)\cdot\mathcal{H}^{k-n}\left(F^{-1}(y)\right)\, d\mathcal{H}^n(y). \tag{4.40}$$

The following statement collects some facts about the projection π_k:

Claim 4.6.5. *Let $x \in \operatorname{int}(r_k B^n)$ with $x \neq o$, and let and $x^\perp = \{y \in \mathbb{R}^k : \langle x, y\rangle = 0\}$.*

(a) *$\pi_k^{-1}(x) = x^\perp \cap S^k$ is a $(k-n)$-sphere of radius $\sqrt{r_k^2 - \|x\|^2}$;*

(b) *for $z \in \pi_k^{-1}(x)$, and $v \in T_z$, we have $\|v|\mathbb{R}^n\| = \sqrt{1-\frac{\|x\|^2}{r_k^2}}\cdot\|v\|$ if v is normal to $x^\perp \cap T_z$, and $\|D\pi_k v\| = \|v|\mathbb{R}^n\| = \|v\|$ if $v \in T_z \cap x^\perp \cap \mathbb{R}^n$, and $\|D\pi_k v\| =$*

$\|v|\mathbb{R}^n\| = 0$ *if* $v \in T_z \cap (\mathbb{R}^n)^\perp$ *(cf.* (4.38)*). In particular,* $J(\pi_k, z) = \left(1 - \frac{\|x\|^2}{r_k^2}\right)^{\frac{1}{2}}$ *in the Coarea formula* (4.40) *with* $F = \pi_k$.

As the final preparation for the proof of the core claim (4.33), we note that for $m \geqslant 3$, the estimate (10.3) stating $\sqrt{\frac{m}{2\pi}} < \frac{\omega_{m-1}}{\omega_m} < \sqrt{\frac{m+1}{2\pi}}$ yields

$$\frac{\mathcal{H}^{m-1}(S^{m-1})}{\mathcal{H}^m(S^m)} = \frac{(m-1)\omega_{m-1}}{m\omega_m} = \left(1 + O\left(\frac{1}{m}\right)\right)\sqrt{\frac{m}{2\pi}}. \tag{4.41}$$

Proof of the core claim (4.33)*, and in turn of Theorem* 4.6.1. For $Z \subset \mathbb{R}^n$ with $|Z| > 0$, $k \geqslant 3n$, $r_k = \sqrt{\frac{k}{2\pi}}$, we deduce from Claim 4.6.5 that first using $\psi(x) = 1/J(\pi_k, z)$ for $x \in Z \cap \operatorname{int}(r_k B^n)$ and $z \in \pi_k^{-1} Z$ and $\psi(x) = 0$ if $x \notin Z \cap \operatorname{int}(r_k B^n)$ in the Coarea formula (4.40) with $F = \pi_k$, and later applying (4.41), we have

$$\begin{aligned}
\frac{\mathcal{H}^k\left(\pi_k^{-1}(Z)\right)}{\mathcal{H}^k(r_k S^k)} &= \frac{\mathcal{H}^{k-n}(S^{k-n})}{\mathcal{H}^k(r_k S^k)} \cdot \int_{Z \cap r_k B^n} \left(1 - \frac{\|x\|^2}{r_k^2}\right)^{\frac{-1}{2}} \left(r_k^2 - \|x\|^2\right)^{\frac{k-n}{2}} dx \\
&= \frac{\mathcal{H}^{k-n}(S^{k-n})}{r_k^n \mathcal{H}^k(S^k)} \cdot \int_{Z \cap r_k B^n} \left(1 - \frac{\|x\|^2}{r_k^2}\right)^{\frac{k-n-1}{2}} dx \\
&= \left(1 + O\left(\frac{n}{k}\right)\right) \int_{Z \cap r_k B^n} \left(1 - \frac{2\pi\|x\|^2}{k}\right)^{\frac{k-n-1}{2}} dx.
\end{aligned} \tag{4.42}$$

Now, we use the simple estimates

$$1 + t \leqslant e^t, \quad \text{and if } |t| \leqslant \tfrac{1}{2}, \text{ then } 1 + t = e^{t + O(t^2)} \text{ and } e^t = 1 + O(t). \tag{4.43}$$

We deduce for the $Z \subset \mathbb{R}^n$ with $|Z| > 0$ from (4.42) and (4.43) that

$$\lim_{k\to\infty} \frac{\mathcal{H}^k\left(\pi_k^{-1}(Z \backslash k^{\frac{1}{8}} B^n)\right)}{\mathcal{H}^k(r_k S^k)} = 0; \tag{4.44}$$

$$\lim_{k\to\infty} \gamma_n\left(Z \backslash k^{\frac{1}{8}} B^n\right) = 0, \tag{4.45}$$

and hence combining (4.42), (4.44) and (4.45) leads to

$$\begin{aligned}
\lim_{k\to\infty} \frac{\mathcal{H}^k\left(\pi_k^{-1}(Z)\right)}{\mathcal{H}^k(r_k S^k)} &= \lim_{k\to\infty} \left(1 + O\left(\frac{n}{k}\right)\right) \int_{Z \cap k^{\frac{1}{8}} B^n} e^{-\pi\|x\|^2 + O\left(\frac{n}{k^{3/4}}\right) + O\left(\frac{1}{k^{1/2}}\right)} dx \\
&= \gamma_n(Z).
\end{aligned}$$

Therefore, we conclude the core claim (4.33), and in turn of Theorem 4.6.1. ∎

4.6.1 The log-Sobolev inequality and the functional version of the Gaussian isoperimetric inequality

In this section, first we describe Ehrhard's and Bobkov's functional analogue of the Gaussian isoperimetric inequality, and then we show how it implies the log-Sobolev inequality (4.48) or (4.50) proved by Gross [344]. The standard Gaussian measure $d\tilde{\gamma}_n(x) = (2\pi)^{\frac{-n}{2}} e^{-\frac{\|x\|^2}{2}} dx$ has zero mean and its covariance matrix is the identity (see Definition 4.7.12). As it is discussed in Barthe, Maurey [73] or Latała [495] in more detail, there are intimate connections between certain fundamental functional inequalities and isoperimetric type inequalities for the standard Gaussian.

For the cumulative distribution function $\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\tau^2/2}\, d\tau$ of the one-dimensional Gaussian $\tilde{\gamma}_1$, we consider the density $\varphi(s) = \Phi'(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}$ of $\tilde{\gamma}_1$, and the "Gaussian isoperimetric function"

$$I(t) = \varphi \circ \Phi^{-1}(t) \quad \text{for } \tau \in [0,1].$$

We note that $I(t) = I(1-t)$ for $t \in [0,1]$ by the evenness of φ.

Using this notation, the version (4.32) of the Gaussian Isoperimetric inequality says that if $X \subset \mathbb{R}^n$ has non-empty interior, and non-empty and Lipschitz boundary, then

$$(2\pi)^{-\frac{n}{2}} \int_{\partial X} e^{-\frac{\|x\|^2}{2}}\, d\mathcal{H}^{n-1}(x) \geqslant I\left(\tilde{\gamma}_n(X)\right) \tag{4.46}$$

where equality holds if X is a closed half-space.

The Bobkov-Ehrhard inequality proved independently by Ehrhard [242] and Bobkov [100] is the functional form of the Gaussian isoperimetric inequality.

Theorem 4.6.6 (Bobkov-Ehrhard). *If $f : \mathbb{R}^n \to [0,1]$ is locally Lipschitz, then*

$$I\left(\int_{\mathbb{R}^n} f\, d\gamma_n\right) \leqslant \int_{\mathbb{R}^n} \sqrt{I(f)^2 + \|Df\|^2}\, d\gamma_n. \tag{4.47}$$

Remark. For any $X \subset \mathbb{R}^n$ with locally Lipschitz boundary, applying (4.47) to a suitable sequence $f_n : \mathbb{R}^n \to [0,1]$ of locally Lipschitz functions approximating $\mathbf{1}_X$ implies the Gaussian isoperimetric inequality (4.46).

Proof. To prove (4.47), we apply the $(n+1)$-dimensional version of the Gaussian isoperimetric inequality (4.46) to the set $X = \{(x,s) \in \mathbb{R}^n \times \mathbb{R} : \Phi(s) \leqslant f(x)\} \subset \mathbb{R}^{n+1}$ and use (2.1) to calculate the integral on ∂X using the formula $D(\Phi^{-1} \circ f) = \sqrt{2\pi} \cdot e^{\frac{(\Phi^{-1} \circ f)^2}{2}} \cdot Df$. ∎

Next we describe how Ehrhard's and Bobkov's functional analogue (4.47) of the Gaussian isoperimetric inequality implies the log-Sobolev inequality (4.48) or (4.50) proved originally by Gross [344] in 1976.

Theorem 4.6.7 (log-Sobolev inequality). *For a locally Lipschitz $f : \mathbb{R}^n \to [0, \infty)$, we have*

$$\int_{\mathbb{R}^n} f \log f \, d\tilde{\gamma}_n - \int_{\mathbb{R}^n} f \, d\tilde{\gamma}_n \cdot \log \left(\int_{\mathbb{R}^n} f \, d\tilde{\gamma}_n \right) \leqslant \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|Df\|^2}{f} \, d\tilde{\gamma}_n. \tag{4.48}$$

Unlike in the case of the Sobolev inequality (4.6), an important feature of the log-Sobolev inequality (of the form either (4.48) or (4.50) below) is that the factor involved on the right hand side is independent of the dimension, which is a very useful property when dealing with infinite dimensional spaces.

In order to prove Theorem 4.6.7, first we verify Proposition 4.6.8. We note that by integrating by parts twice based on the formula $e^{-\frac{\tau^2}{2}} = (e^{-\frac{\tau^2}{2}})' \cdot \frac{-1}{\tau}$, one easily gets (see, for example, Gordon [329]) the series expansion

$$\Phi(s) = \frac{1}{\sqrt{2\pi}} \frac{e^{-s^2/2}}{|s|} \left(1 - \frac{1}{s^2} + O\left(\frac{1}{s^4} \right) \right) \qquad \text{for } s \leqslant -1. \tag{4.49}$$

Let $g : \mathbb{R}^n \to [0, \infty)$ be a bounded locally Lipschitz function. Following Barthe, Maurey [73] and Latała [495], we show that applying (4.47) to $f = \varepsilon g^2$ as $\varepsilon \to 0^+$ and using (4.49) yield the following form of the log-Sobolev inequality proved by Gross [344].

Proposition 4.6.8 (log-Sobolev inequality). *For a locally Lipschitz $g : \mathbb{R}^n \to [0, \infty)$, we have*

$$\int_{\mathbb{R}^n} g^2 \log g^2 \, d\tilde{\gamma}_n - \int_{\mathbb{R}^n} g^2 \, d\tilde{\gamma}_n \cdot \log \left(\int_{\mathbb{R}^n} g^2 \, d\tilde{\gamma}_n \right) \leqslant 2 \int_{\mathbb{R}^n} \|Dg\|^2 \, d\tilde{\gamma}_n. \tag{4.50}$$

Remark. Substituting $f = g^2$ into (4.50) yields the equivalent version (4.48) in Theorem 4.6.7 of the log-Sobolev inequality.

Proof. We set $F(t) = \sqrt{2}t|\log t|$, and notice that as s tends to $-\infty$, we have

$$\begin{aligned}
\frac{\varphi(s)}{F(\Phi(s))} &= \frac{|s|}{\sqrt{2}\left(1 - \frac{1}{s^2} + O\left(\frac{1}{s^4}\right)\right) \sqrt{\left| \log \left[\frac{1}{\sqrt{2\pi}} \frac{e^{-s^2/2}}{|s|} \left(1 + O\left(\frac{1}{s^2}\right)\right) \right] \right|}} \\
&= \frac{|s|}{\sqrt{2}\left(1 - \frac{1}{s^2} + O\left(\frac{1}{s^4}\right)\right) \sqrt{\frac{s^2}{2} + \log\sqrt{2\pi} + \log|s| + O\left(\frac{1}{s^2}\right)}} \\
&= \frac{1}{\left(1 - \frac{1}{s^2} + O\left(\frac{1}{s^4}\right)\right) \left(1 + \frac{\log\sqrt{2\pi}}{\sqrt{2}|s|} + \frac{\log|s|}{\sqrt{2}|s|} + O\left(\frac{1}{s^3}\right)\right)} \\
&= 1 - \frac{\log\sqrt{2\pi}}{\sqrt{2}|s|} - \frac{\log|s|}{\sqrt{2}|s|} + \frac{1}{s^2} + o\left(\frac{1}{s^2}\right).
\end{aligned}$$

Hence, if we define $R(t)$ so that $I(t) = t\sqrt{2|\log t|}\big(1 - R(t)\big)$ for $t \in (0,1)$, it follows that

$$R(\Phi(s)) = \rho(s) + \frac{1}{s^2} + o\left(\frac{1}{s^2}\right), \qquad \rho(s) := \frac{\log\sqrt{2\pi}}{\sqrt{2}|s|} + \frac{\log|s|}{\sqrt{2}|s|} \tag{4.51}$$

as s tends to $-\infty$.

Also, given $\varepsilon > 0$ small and $a \in [1/M, M]$ for some fixed $M > 1$, if we define $\tau_{\varepsilon a} := \Phi^{-1}(\varepsilon a)$ and $\tau_\varepsilon := \Phi^{-1}(\varepsilon)$, then

$$\tau_{\varepsilon a} \sim \tau_\varepsilon \sim -\sqrt{|\log\varepsilon|}, \qquad |\tau_{\varepsilon a} - \tau_\varepsilon| = O\left(\frac{1}{\sqrt{|\log\varepsilon|}}\right),$$

where the implicit constant in $O(\cdot)$ depends on M but not on a. Hence, by (4.51) and a Taylor expansion, it holds

$$\begin{aligned} R(\varepsilon a) - R(\varepsilon) &= R(\Phi(\tau_{\varepsilon a})) - R(\Phi(\tau_\varepsilon)) \\ &= O\left(\sup_{s\in[\tau_\varepsilon,\tau_{\varepsilon a}]} |\rho'(s)| \cdot |\tau_{\varepsilon a} - \tau_\varepsilon|\right) + o\left(\frac{1}{\tau_{\varepsilon a}^2}\right) + +o\left(\frac{1}{\tau_\varepsilon^2}\right) \\ &= O\left(\frac{\log|\log\varepsilon|}{|\log\varepsilon|^{3/2}}\right) + o\left(\frac{1}{|\log\varepsilon|}\right), \end{aligned} \tag{4.52}$$

where the expression in (4.52) depends on M and ε but not on a. In particular,

$$R(\varepsilon a) = R(\varepsilon) + o\left(\frac{1}{|\log\varepsilon|}\right), \tag{4.53}$$

from which it follows that

$$\begin{aligned} I(\varepsilon a) &= \sqrt{2}\varepsilon a\sqrt{|\log\varepsilon| - \log a}\big(1 - R(\varepsilon a)\big) \\ &= \sqrt{2}\varepsilon a\sqrt{|\log\varepsilon| - \log a}\big(1 - R(\varepsilon)\big) + o\left(\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\right) \qquad (4.54) \\ &= \sqrt{2}\varepsilon\sqrt{|\log\varepsilon|}\big(1 - R(\varepsilon)\big)a - \frac{1}{\sqrt{2}}\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}a\log a + o\left(\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\right), \qquad (4.55) \end{aligned}$$

where (4.54) follows from (4.53), while in (4.55) we used the Taylor expansion $\sqrt{1+s} \sim 1 + \frac{1}{2}s$, and the expression $o\left(\frac{\varepsilon}{\sqrt{|\log\varepsilon|}}\right)$ in (4.55) depends on M and ε but not on a.

Now, given a locally Lipschitz function, $g : \mathbb{R}^n \to [1/M, M]$, using (4.55), we get

$$I\left(\varepsilon \int_{\mathbb{R}^n} g^2 \, d\tilde{\gamma}_n\right) = \sqrt{2}\varepsilon\sqrt{|\log \varepsilon|}\big(1 - R(\varepsilon)\big) \int_{\mathbb{R}^n} g^2 \, d\tilde{\gamma}_n + o\left(\frac{\varepsilon}{\sqrt{|\log \varepsilon|}}\right) - \frac{1}{\sqrt{2}} \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \int_{\mathbb{R}^n} g^2 \, d\tilde{\gamma}_n \log\left(\int_{\mathbb{R}^n} g^2 \, d\tilde{\gamma}_n\right), \tag{4.56}$$

while (4.54) and a Taylor expansion yield

$$\begin{aligned} &\sqrt{I(\varepsilon g^2)^2 + \varepsilon^2 \|D(g^2)\|^2} \\ &= \sqrt{2\varepsilon^2 g^4 \left(|\log \varepsilon| - \log g^2\right)\left(1 - R(\varepsilon)\right)^2 + o\left(\varepsilon^2\right) + 4\varepsilon^2 g^2 \|Dg\|^2} \\ &= \sqrt{2}\varepsilon\sqrt{|\log \varepsilon|}\big(1 - R(\varepsilon)\big) g^2 - \frac{1}{\sqrt{2}} \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} g^2 \log g^2 \\ &\quad + \sqrt{2} \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \|Dg\|^2 + o\left(\frac{\varepsilon}{\sqrt{|\log \varepsilon|}}\right). \end{aligned} \tag{4.57}$$

Integrating (4.57) with respect to $\tilde{\gamma}_n$ and combining it with (4.47) and (4.56), we obtain the validity of (4.50) for every locally Lipschitz function $g : \mathbb{R}^n \to [1/M, M]$. By approximation, (4.50) holds also for g bounded, locally Lipschitz, and nonnegative. ∎

4.7 Log-concave measures: The Cheeger (isoperimetric) constant, a Poincaré inequality and the Kannan-Lovász-Simonovits conjecture

After solving the isoperimetric problem with respect to a Gaussian density in $\mathbb{R}^n$, we consider the version with respect to any log-concave density $p : \mathbb{R}^n \to [0, \infty)$; namely, if $p = e^{-\varphi}$ for a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$. Naturally, we do not expect an exact solutions in this generality, but rather only approximate ones. The "classical" results in this field (up to 2020) are thoroughly discussed in Artstein-Avidan, Giannopoulos, V. Milman [39], Chapter 2. As typical level sets even of a C^1 function may not have locally Lipschitz boundary, even if our statements are about Lipschitz functions, we will need to use the theory of sets weighted finite perimeter summarized in Section 5.4.

First, we define the so-called Cheeger constant that is a version of the isoperimetric ratio, then we review the necessary tools to show how the isoperimetric type problem relates to integral inequalities in terms of the gradient, like Theorem 4.7.8 and the Poicaré type inequality Theorem 4.7.10. This is followed by a discussion of bounds for the Cheeger constant; in particular, the Kannan-Lovász-Simonovits conjecture. A

central theme of this section is the notion of log-concave functions (cf. Section 10.9): A function $p : \mathbb{R}^n \to [0, \infty)$ is log-concave if $p((1-\lambda)x + \lambda y) \geqslant p(x)^{1-\lambda} p(y)^{\lambda}$ holds for any $x, y \in \mathbb{R}^n$ and any $\lambda \in (0,1)$; or in other words, $p = e^{-\varphi}$ for a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$. The support of p is the closed convex set

$$\operatorname{supp} p = \operatorname{cl}\{p > 0\},$$

which is also the support of the corresponding log-concave measure $d\mu = p\, d\mathcal{H}^n$ on $\mathbb{R}^n$ if $\operatorname{supp} p$ is not contained in a hyperplane (cf. Definition 10.9.3). We note that p is locally Lipschitz, and hence continuous and a.e. differentiable in $\operatorname{int} \operatorname{supp} p$ (see Section 10.9). Assuming that $\operatorname{int} \operatorname{supp} p \neq \varnothing$, Lemma 10.9.6 yields that $\int_\Omega p < \infty$ if and only if the non-empty level sets $\{p > t\}$ for $t > 0$ are bounded.

However, we start the discussion of the Cheeger constant using rather general functions as the probability density p in order to show where log-concavity enters the picture. Let $\Omega \subset \mathbb{R}^n$ be a non-empty open set, and let $p : \mathbb{R}^n \to [0, \infty)$ be such that

$$\begin{aligned} p|_\Omega &\quad \text{is positive, continous and bounded,} \\ p(x) &= 0 \quad \text{if } x \notin \operatorname{cl} \Omega, \\ \textstyle\int_{\mathbb{R}^n} p &= 1. \end{aligned} \tag{4.58}$$

To handle level sets of Lipschitz functions, we will need some results from Chapter 5 on sets of finite perimeter in this section.

First, we consider the corresponding weighted lower outer Minkowski content.

Definition 4.7.1. For the measure $d\mu = p\, d\mathcal{H}^n$ on $\mathbb{R}^n$ where $p : \mathbb{R}^n \to [0, \infty)$ satisfies (4.58), if $X \subset \mathbb{R}^n$ is measurable, then

$$\mu_+(X) = \liminf_{\varrho \to 0^+} \frac{\mu(X^{(\varrho)}) - \mu(X)}{\varrho}. \tag{4.59}$$

Remark. If $X \subset \mathbb{R}^n$ is compact and has Lipschitz boundary (cf. Definition 4.1.4), and $\mathcal{H}^{n-1}(\partial X \cap \partial \Omega) = 0$, then (5.56) in Chapter 5 says that

$$\mu_+(X) = \int_{\partial X} p\, d\mathcal{H}^{n-1}. \tag{4.60}$$

Definition 4.7.2 (Cheeger constant). For the measure $d\mu = p\, d\mathcal{H}^n$ on $\mathbb{R}^n$ where $p : \mathbb{R}^n \to [0, \infty)$ satisfies (4.58), the Cheeger constant $C_{\mathrm{Che}}(\mu) > 0$ is defined to be minimal such that for every measurable $X \subset \mathbb{R}^n$, we have

$$\min\{\mu(X), 1 - \mu(X)\} \leqslant C_{\mathrm{Che}}(\mu) \cdot \mu_+(X). \tag{4.61}$$

Remark. We may assume that $X \subset \Omega$. If $0 < \mu_+(X) < \infty$, then (5.54) in Chapter 5 says that the weighted perimeter $\operatorname{Per}_\omega(X) \leqslant \mu_+(X)$ of X with respect to $\omega = p|_\Omega$

is finite, and hence X is a set of locally finite perimeter by Remark 5.54 (cf. Section 5.1), and Theorem 5.4.3 yields the existence of measurable sets $X_k \subset \Omega$ such that $\lim_{k\to\infty} \mu(X_k \Delta X) = 0$, and

$$\lim_{k\to\infty} \mu_+(X_k) = \mathrm{Per}_\omega(X) = \int_{\Omega\cap\partial^* X} p \, d\mathcal{H}^{n-1}.$$

Here $\partial^* X$ is the reduced boundary of X (cf. Section 5.1). We deduce that the Cheeger constant $C_{\mathrm{Che}}(\mu) > 0$ is the minimal number such that for every set $X \subset \Omega$ of locally finite perimeter, we have

$$\min\{\mu(X), 1-\mu(X)\} \leqslant C_{\mathrm{Che}}(\mu) \cdot \int_{\Omega\cap\partial^* X} p \, d\mathcal{H}^{n-1}. \tag{4.62}$$

If $X \subset \mathbb{R}^n$ has Lipschitz boundary, then we obtain the following result estimate from combining (4.60) and (5.61) from Chapter 5 with the definition (4.61) of the Cheeeger constant.

Proposition 4.7.3. *Let $\Omega \subset \mathbb{R}^n$ be open such that $\mathcal{H}^{n-1}(RB^n \cap \partial\Omega) < \infty$ for any $R > 0$, and let $d\mu = p \, d\mathcal{H}^n$ be a probability measure on $\mathbb{R}^n$ where $p : \mathbb{R}^n \to [0,\infty)$ satisfies (4.58). If $X \subset \mathbb{R}^n$ has Lipschitz boundary and satisfies $\mathcal{H}^{n-1}(\partial X \cap \partial\Omega) = 0$, then*

$$\min\{\mu(X), 1-\mu(X)\} \leqslant C_{\mathrm{Che}}(\mu) \cdot \int_{\partial X} p \, d\mathcal{H}^{n-1}.$$

If p is log-concave, we have some additional information on the Cheeger constant.

Remark 4.7.4. (i) If p is log-concave, then it is enough to consider the case when $\mu(X) = \frac{1}{2}$ in the definition (4.61) or (4.62) of the Cheeger constant $C_{\mathrm{Che}}(\mu)$ according to E. Milman [577].

(ii) *(Gaussian measure)* If $p(x) = e^{-\pi\|x\|^2}$, then $C_{\mathrm{Che}}(\mu) = \frac{1}{2}$ by (i) and the Gaussian Isoperimetric inequality (4.32).

(iii) *(Uniform measure on the unit ball)* If $d\mu = \omega_n^{-1} \mathbf{1}_{B^n} \, d\mathcal{H}^n$, then

$$C_{\mathrm{Che}}(\mu) = \frac{\omega_n}{2\omega_{n-1}} < \sqrt{\frac{\pi}{2n}}. \tag{4.63}$$

To prove (4.63), one may assume that $|X| = \frac{1}{2}|B^n|$ for the measurable $X \subset B^n$ by (i), and the corresponding isoperimetric-type problem was solved by Almgren [18] and Bokowski, Sperner [105]. The optimal X is a half ball (intersection of a ball and a half-space), yielding (4.63) (see (10.3) for the estimate on $\frac{\omega_n}{\omega_{n-1}}$).

We note that in some references, what is called Cheeger constant is the reciprocal $1/C_{\mathrm{Che}}(\mu)$ in our notation. We choose our normalization because it fits better with the integral inequalities Theorem 4.7.8 and Theorem 4.7.10. Now we collect the tools

we need to prove Theorem 4.7.8, that is the integral version of the definition of the Cheeger constant. We need the following version of the Coarea formula (cf. (10.12)):

Lemma 4.7.5 (Coarea formula). *If $f \in L^1(\mathbb{R}^n)$ is Lipschitz and $p : \mathbb{R}^n \to [0, \infty)$ is measurable, then*

$$\int_{\mathbb{R}^n} \|Df\| \cdot p = \int_{\mathbb{R}} \int_{f^{-1}(t)} p \, d\mathcal{H}^{n-1} \, dt. \tag{4.64}$$

Definition 4.7.6 (Median). If μ is an absolutely continuous probability measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, then a median $m \in \mathbb{R}$ satisfies $\mu(\{f > m\}) \leqslant \frac{1}{2}$ and $\mu(\{f < m\}) \leqslant \frac{1}{2}$.

According to Theorem 5.4.7 based on the paper Ambrosio, Di Marino, Gigli [24], the weighted lower Minkowski content (or actually the weighted perimeter) of the typical level sets of a Lipschitz function coincide with integral of the density function p on the boundary of the level set - but here we have to assume (which we will do from now on) that p is log-concave. In particular, Theorem 5.4.7 yields the following.

Lemma 4.7.7. *If $d\mu = p \, d\mathcal{H}^n$ is a probability density on $\mathbb{R}^n$ for a log-concave density $p : \mathbb{R}^n \to [0, \infty)$, and $f \in L^1(\mathbb{R}^n, \mu)$ is Lipschitz and not a constant function on* supp p*, then*

$$\mu_+ (\{f > t\}) = \int_{f^{-1}(t)} p \, d\mathcal{H}^{n-1} \tag{4.65}$$

for $\mathcal{H}^1$ a.e. $t \in \mathbb{R}$ in the image of supp p *of f.*

Remark. The reason why we can't handle density functions more general than log-concave is that the boundary of a typical level set of some general Lipschitz function on $\mathbb{R}^n$ is only countably rectifiable; namely, it is the disjoint union of a set of $\mathcal{H}^{n-1}$ measure zero, and countable many sets that are C^1 images of bounded subsets of $\mathbb{R}^{n-1}$. Therefore, the equality of the weighted lower Minkowski content and the integral of the density function p on the boundary of the level set is not guarantied (see the discussion in Ambrosio, Di Marino, Gigli [24]).

Theorem 4.7.8. *If $d\mu = p \, d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for log-concave p, and $f \in L^1(\mathbb{R}^n, \mu)$ is Lipschitz with median m, then*

$$\int_{\mathbb{R}^n} |f - m| \, d\mu \leqslant C_{\mathrm{Che}}(\mu) \cdot \int_{\mathbb{R}^n} \|Df\| \, d\mu. \tag{4.66}$$

Remark. Since Lipschitz functions on supp p can be extended to $\mathbb{R}^n$ (cf. Theorem 10.4.1), Theorem 4.7.8 also holds for a Lipschitz function $f :$ supp $p \to \mathbb{R}$ where the integration is on supp p.

Proof. We may assume that f is not a constant function on supp p. It follows from the Layer Cake formula (cf. Lemma 4.2.3), the definition of the Cheeger constant, the

formula (4.65) and the Coarea formula (4.64) that

$$\begin{aligned}\int_{\mathbb{R}^n} |f-m|\,d\mu &= \int_{m(f)}^{\infty} \mu(\{f>t\})\,dt + \int_{-\infty}^{m(f)} \mu(\{f<t\})\,dt \\ &= \int_{\mathbb{R}} \min\{\mu(\{f>t\}), 1-\mu(\{f>t\})\}\,dt \\ &\leqslant C_{\mathrm{Che}}(\mu) \int_{\mathbb{R}} \mu_+\left(\{f>t\}\right)\,dt \\ &= C_{\mathrm{Che}}(\mu) \int_{\mathbb{R}} \int_{f^{-1}(t)} p\,d\mathcal{H}^{n-1}\,dt = C_{\mathrm{Che}}(\mu) \int_{\mathbb{R}^n} \|Df\|\,d\mu.\end{aligned}$$

■

The Poincaré-type inequality (Theorem 4.7.10), one of our next goals, uses the the notion of variance, that we recall below for the reader's convenience.

Definition 4.7.9. If $d\mu = p\,d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for a log-concave p, and $f \in L^2(\mathbb{R}^n, \mu)$ is Lipschitz, then

$$\mathrm{Var}_\mu(f) = \int_{\mathbb{R}^n} f^2\,d\mu - \left(\int_{\mathbb{R}^n} f\,d\mu\right)^2.$$

Theorem 4.7.10 (Poincaré inequality). *If $d\mu = p\,d\mathcal{H}^n$ probability measure on $\mathbb{R}^n$ for a log-concave p, and $f \in L^2(\mathbb{R}^n, \mu)$ is Lipschitz, then*

$$\mathrm{Var}_\mu(f) \leqslant 4C_{\mathrm{Che}}(\mu)^2 \cdot \int_{\mathbb{R}^n} \|Df\|^2\,d\mu. \tag{4.67}$$

Remark. It is equivalent to saying that if $\int_{\mathbb{R}^n} f\,d\mu = 0$, then

$$\int_{\mathbb{R}^n} f^2\,d\mu \leqslant 4C_{\mathrm{Che}}(\mu)^2 \cdot \int_{\mathbb{R}^n} \|Df\|^2\,d\mu.$$

Proof. We may assume that f is not a constant function on $\mathrm{supp}\,p$. We observe that if m is a median for f, then 0 is a the median for $\tilde{f} = (f-m)^2\mathrm{sgn}\,(f-m)$ with respect to μ where $\mathrm{sgn}\,t$ is $+1, 0, -1$ provided $t>0$, $t=0$ or $t<0$, respectively. Therefore, (4.66), the product rule applied to $D(f-m)^2$, the formula $D(f-m) = Df$ and the Hölder inequality (10.4) yield

$$\begin{aligned}\int_{\mathbb{R}^n} |f-m|^2\,d\mu = \int_{\mathbb{R}^n} |\tilde{f}| &\leqslant C_{\mathrm{Che}}(\mu) \int_{\mathbb{R}^n} \|D\tilde{f}\|\,d\mu = C_{\mathrm{Che}}(\mu) \int_{\mathbb{R}^n} \|D(f-m)^2\|\,d\mu \\ &= 2C_{\mathrm{Che}}(\mu) \int_{\mathbb{R}^n} |f-m| \cdot \|Df\|\,d\mu \\ &\leqslant 2C_{\mathrm{Che}}(\mu) \sqrt{\int_{\mathbb{R}^n} |f-m|^2\,d\mu} \cdot \sqrt{\int_{\mathbb{R}^n} \|Df\|^2\,d\mu}.\end{aligned}$$

We conclude (4.67) because $\mathrm{Var}_\mu(f) \leqslant \int_{\mathbb{R}^n} |f-m|^2\, d\mu$ as $\int_{\mathbb{R}^n} |f-t|^2\, d\mu$ is minimized by the mean $t = \int_{\mathbb{R}^n} f\, d\mu$. ∎

Remark 4.7.11 (Poincaré inequality as a spectral gap estimate). Let $p = e^{-V}$ for suitable convex $V : \mathbb{R}^n \to (0,\infty]$ such that $\int_{\mathbb{R}^n} p = 1$.

Consider the heat equation $\partial_t u = Lu$ on $[0,\infty) \times \mathbb{R}^n$ where

$$Lf = \Delta f - \langle Df, DV\rangle$$

for suitable $f \in L^2(\mathbb{R}^n,\mu)$. Then L can be extended to a (non-positive) self-adjoint elliptic operator on $L^2(\mathbb{R}^n,\mu)$ (cf. Section 10.7), satisfying

$$\int_{\mathbb{R}^d} (-Lf)\cdot f\, d\mu = \int_{\mathbb{R}^d} \|Df\|^2\, d\mu$$

for Lipschitz $f \in L^2(\mathbb{R}^n,\mu)$. Readily, the constant functions form the kernel $\mathrm{Ker}\, L$. If $\int_{\mathbb{R}^n} f\, d\mu = 0$, then f is orthogonal to $\mathrm{Ker}\, L$, and hence the smallest positive eigenvalue λ_μ of $-L$ satisfies that λ_μ is the largest with the property

$$\lambda_\mu \mathrm{Var}_\mu(f) \leqslant \int_{\mathbb{R}^n} \|Df\|^2\, d\mu \quad \text{for locally Lipschitz } f \in L^2(\mu). \tag{4.68}$$

Now (4.67) yields that

$$\lambda_\mu \geqslant C_{\mathrm{Che}}(\mu)^{-2}/4, \tag{4.69}$$

which is known as Cheeger's inequality (see Ledoux [497,498]). As p is log-concave, De Ponti, Mondino [233] verified a matching bound on λ_μ, which in turn yields

$$\tfrac{1}{4} \leqslant \lambda_\mu \cdot C_{\mathrm{Che}}(\mu)^2 \leqslant \pi. \tag{4.70}$$

Now we turn to bounds on the Cheeger constant. We observe that if $d\mu = p\, d\mathcal{H}^n$ where $p = \mathbf{1}_{R_a}$ for $a > 1$ and rectangular box $R_a = [0, \frac{1}{a}]^{n-1} \times [0, a^{n-1}]$, then $C_{\mathrm{Che}}(\mu) \geqslant a^{n-1}/2$. In particular, even if p is log-concave, we can only expect an upper bound on $C_{\mathrm{Che}}(\mu)$ depending only on the dimension n after certain normalization.

Definition 4.7.12 (Covariance matrix, Isotropic measure).

(i) If $d\mu = p\, d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ with log-concave p with mean $\int_{\mathbb{R}^n} x\, d\mu(x) = o$, then the *covariance matrix* is $\mathrm{Cov}(\mu) = [m_{ij}]$ where the coefficients are $m_{ij} = \int_{\mathbb{R}^n} x_i x_j\, d\mu(x_1,\ldots,x_n)$.

(ii) μ is *isotropic* if in addition $\mathrm{Cov}(\mu) = I_n$; or equivalently, if $\int_{\mathbb{R}^n} \langle v, x\rangle^2\, d\mu(x) = \|v\|^2$ for $v \in \mathbb{R}^n$.

Remarks.

(i) For any probability measure $d\mu = p\, d\mathcal{H}^n$ on $\mathbb{R}^n$ with log-concave p, there exists $\Phi \in \mathrm{GL}(n)$ such that the probability measure $\tilde{\mu}(X) = |\det \Phi| \cdot \mu(\Phi(X+b))$ is isotropic where $b = \int_{\mathbb{R}^n} x\, d\mu(x)$.

(ii) As $\mu_+(\partial X)$ in (4.59) and $\|Df\|$ in (4.66) are defined in terms of the given Euclidean structure, and the importance of the measure being isotropic is that the measure "matches" the Euclidean structure in that case.

(iii) The standard Gaussian $d\mu(x) = (2\pi)^{\frac{-n}{2}} e^{-\frac{\|x\|^2}{2}} dx$ is the isotropic Gaussian measure, and its Cheeger constant is

$$C_{\mathrm{Che}}(\mu) = \sqrt{\frac{\pi}{2}}$$

by the Gaussian Isoperimetric inequality (4.32).

According to the fundamental conjecture by Kannan, Lovász, Simonovits [454] in 1995, the Cheeger constant of any isotropic log-concave measure on $\mathbb{R}^n$ is essentially same as of the standard Gaussian (see (4.80) for a lower bound on the Cheeger constant). We note that according to (4.80), $C_{\mathrm{Che}}(\mu) \geqslant \frac{\sqrt{3/2}}{e}$ holds for any isotropic log-concave measure μ on $\mathbb{R}^n$.

Conjecture 4.7.13 (KLS conjecture). *If $d\mu = p\, d\mathcal{H}^n$ is an isotropic probability measure on $\mathbb{R}^n$ for a log-concave p, then $C_{\mathrm{Che}}(\mu) \leqslant c$ for an absolute constant c.*

After intense research for more than two decades (see the Comments Section 4.8), Klartag [468] proved the currently best upper bound on $C_{\mathrm{Che}}(\mu)$ in the KLS Conjecture 4.7.13.

Theorem 4.7.14 (Klartag). *If $d\mu = p\, d\mathcal{H}^n$ is an isotropic probability measure on $\mathbb{R}^n$ for a log-concave p, then $C_{\mathrm{Che}}(\mu) \leqslant c\sqrt{\log n}$ for an absolute constant c.*

Remark. It follows that if $d\mu = p\, d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for a log-concave p, then

$$C_{\mathrm{Che}}(\mu) \leqslant c\sqrt{\log n}\sqrt{\|\mathrm{Cov}(\mu)\|_{\mathrm{op}}},$$

where c is an absolute constant and $\|\mathrm{Cov}(\mu)\|_{\mathrm{op}}$ is the ℓ_2 operator norm of the symmetric matrix $\mathrm{Cov}(\mu)$ (the largest eigenvalue).

We deduce from the Poincaré inequality (4.67) and from Klartag's Theorem 4.7.14 the following estimate.

Corollary 4.7.15 (Poincaré inequality for isotropic measures). *If $d\mu = p\, d\mathcal{H}^n$ is an isotropic probability measure on $\mathbb{R}^n$ for a log-concave p, and $f \in L^2(\mu)$ is Lipschitz, then*

$$\mathrm{Var}_\mu(f) \leqslant C \log n \cdot \int_{\mathbb{R}^n} \|Df\|^2 \, d\mu \tag{4.71}$$

for some absolute constant $C > 0$.

In Definition 4.7.2, we have no information on what X to take to have a good approximation of the Cheeger constant. However, for algorithms, one would take half-spaces as X. The main motivation of Kannan, Lovász, Simonovits [454] to state the KLS conjecture 4.7.13 was the hope is that half-spaces are reasonably close to being optimal (see Remark 4.7.18 below).

Lemma 4.7.16. *Let $d\mu = p\, d\mathcal{H}^n$ be a probability measure on $\mathbb{R}^n$ for a log-concave function p on $\mathbb{R}^n$, $n \geqslant 1$, and let $\Omega = \operatorname{int}\operatorname{supp}\mu$. For a $u \in S^{n-1}$, $t \in \mathbb{R}$ and half-space $H^+ = \{x \in \mathbb{R}^n : \langle x, u\rangle \geqslant t\}$ such that $(tu + u^\perp) \cap \Omega \neq \varnothing$, we claim that*

$$\mu_+(H^+) = \int_{\partial H^+} p\, d\mathcal{H}^{n-1}. \tag{4.72}$$

Proof. To prove (4.72), let $y < z$ such that $\Omega|(\mathbb{R}u) = (y, z)u$ where possibly $y = -\infty$ or $z = \infty$. For $s \in (y, z)$, we consider the marginal $f(s) = \int_{su+u^\perp} p\, d\mathcal{H}^{n-1}$, that is a positive continuous log-concave probability density function on (y, z) by the Prékopa-Leindler inequality (3.5). If $t \in (y, z)$ and $\varrho \in (0, t - y)$, then for $X = H^+$, we have

$$\mu\left(X^{(\varrho)}\right) - \mu(X) = \int_{t-\varrho}^{t} f(s)\, ds,$$

which in turn yields (4.72). ∎

The following results about how half-spaces divide a log-concave measure with zero mean were proved by Lovász, Vempala [533] and Fradelizi [293].

Theorem 4.7.17 (Lovász, Vempala, Fradelizi). *If $d\mu = p\, d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for a log-concave p with mean $\int_{\mathbb{R}^n} x\, p(x)\, dx = o$, $n \geqslant 1$, and $H^+ = \{x \in \mathbb{R}^n : \langle x, u\rangle \geqslant 0\}$ for a fixed $u \in S^{n-1}$, then*

$$\int_{H^+} p \geqslant \frac{1}{e}; \tag{4.73}$$

$$\mu_+(\tau u + H^+) \leqslant e \cdot \mu_+(H^+) \text{ for } \tau \in (y, z); \tag{4.74}$$

$$\frac{1}{150} < \mu_+(H^+)^2 \cdot \int_{\mathbb{R}^n} \langle u, x\rangle^2 p(x)\, dx < \frac{2}{3} \tag{4.75}$$

where $y < 0 < z$ such that $(y, z)u = (\operatorname{int}\operatorname{supp}\mu)|\mathbb{R}u$ (possibly $y = -\infty$ or $z = \infty$).

Remark. (4.73) and (4.74) are optimal (for example, consider $p(x) = e^{-x-1}$ if $x \geqslant -1$, and $p(x) = 0$ if $x < -1$ for $n = 1$). Instead of (4.75), Fradelizi [293] proves the optimal estimates

$$\frac{1}{12} \leqslant \mu_+(H^+)^2 \cdot \int_{\mathbb{R}^n} \langle u, x\rangle^2 p(x)\, dx \leqslant \frac{1}{2}, \tag{4.76}$$

$$\max_{\tau \in (y,z)} \mu_+(\tau u + H^+) \int_{\mathbb{R}^n} \langle u, x\rangle^2 p(x)\, dx \leqslant 1. \tag{4.77}$$

Proof. We observe that the marginal $f(t) = \int_{tu+u^\perp} p\, d\mathcal{H}^{n-1}$ is long-concave for $t \in \mathbb{R}$ by the Prékopa-Leindler inequality (3.5), and satisfies $\int_{\mathbb{R}} f = 1$ and $\int_{\mathbb{R}} t\, f(t)\, dt = 0$. According to (4.72), our problem is essentially one-dimensional: it is equivalent to prove that if $\tau \in (y, z)$, then

$$\int_0^\infty f \geqslant \frac{1}{e} \text{ and } \int_{-\infty}^0 f \geqslant \frac{1}{e} \text{ and } f(\tau) \leqslant e \cdot f(0); \tag{4.78}$$

$$\frac{1}{150} < f(0)^2 \cdot \int_{\mathbb{R}} t^2 f(t)\, dt < \frac{2}{3}. \tag{4.79}$$

For (4.78), let $F(x) = \int_y^x f$ be the cumulative distribution function for $x \in (y, z)$, and hence $F'(t) = f(t)$. It follows from the Prékopa-Leindler inequality (3.5) that $\log F$ is concave on (y, z), and hence the Jensen inequality Lemma 10.2.4 and later using the substitution $s = F(t)$ yield that

$$\begin{aligned} \log F(0) &= \log F\left(\int_y^z t f(t)\, dt\right) \geqslant \int_y^z (\log F(t)) \cdot f(t)\, dt \\ &= \int_y^z (\log F(t)) \cdot F'(t)\, dt = \int_0^1 \log s\, ds = -1, \end{aligned}$$

proving the first inequality in (4.78), and in turn, the second inequality in (4.78).

To continue, we may assume that $f(0) = 1$, and $\int_0^\infty f = a = \int_0^\infty e^{-t/a}\, dt$ where

$$e^{-1} \leqslant a \leqslant 1 - e^{-1}$$

by the first two inequalities in (4.78).

To prove the third estimate in (4.78), our goal is to verify that $f(\tau) \leqslant e$ for $\tau \in (y, z)$ where may assume that $f(\tau) > 1$ and $\tau < 0$. Since f is log-concave, and $f(\tau) = e^{-\gamma\tau}$ for $\gamma = \frac{\log f(\tau)}{|\tau|}$ and $f(0) = e^{-\gamma \cdot 0}$, we deduce that $f(s) \geqslant e^{-\gamma s}$ if $s \in [\tau, 0]$ and $f(s) \leqslant e^{-\gamma s}$ if $s \geqslant 0$, and hence

$$f(\tau) = \frac{e^{-\gamma\tau}}{e^{-\gamma \cdot 0}} = \frac{\int_\tau^\infty e^{-\gamma s}\, ds}{\int_0^\infty e^{-\gamma s}\, ds} = \frac{\int_\tau^0 e^{-\gamma s}\, ds}{\int_0^\infty e^{-\gamma s}\, ds} + 1 \leqslant \frac{\int_\tau^0 f}{\int_0^\infty f} + 1 \leqslant \frac{1-a}{a} + 1 \leqslant e.$$

For the upper bound in (4.79), as f is log-concave, and the value at 0 and the integral on $[0, \infty)$ agree for f and $e^{-t/a}$, there exists some $t_0 > 0$ such that $f(t) \geqslant e^{-t/a}$ if $t \in [0, t_0]$, and $f(t) \leqslant e^{-t/a}$ if $t \geqslant t_0$. It follows that

$$\begin{aligned} \int_0^\infty t^2 f(t)\, dt - 2a^3 &= \int_0^\infty t^2 f(t)\, dt - \int_0^\infty t^2 e^{-t/a}\, dt \\ &= \int_0^{t_0} t^2 \left(f(t) - e^{-t/a}\right) dt - \int_{t_0}^\infty t^2 \left(e^{-t/a} - f(t)\right) dt \\ &\leqslant t_0^2 \int_0^{t_0} \left(f(t) - e^{-t/a}\right) dt - t_0^2 \int_{t_0}^\infty \left(e^{-t/a} - f(t)\right) dt = 0. \end{aligned}$$

Since $\int_{-\infty}^{0} f = 1-a$, we similarly obtain that $\int_{-\infty}^{0} t^2 f(t)\,dt \leqslant 2(1-a)^3$. As $\min\{a, 1-a\} \geqslant 1/e$, we deduce that $\int_{\mathbb{R}} t^2 f(t)\,dt \leqslant 2a^3 + 2(1-a)^3 \leqslant 2\left(\frac{1}{e^3} + (1-\frac{1}{e})^3\right) < \frac{2}{3}$, verifying the upper bound in (4.79).

For the lower bound in (4.79), we use that $f = e^{-V}$ for a convex function V with $V(0) = 0$. Let $b \in \partial V(0)$ from the subdifferential of V at 0 (cf. Definition 1.5.4), and hence $V(t) \geqslant bt$ where we may assume that $b \geqslant 0$. It follows that $f(t) \leqslant e^{-bt}$, thus $\int_0^\infty e^{-bt}\,dt \geqslant \int_0^\infty f = a \geqslant e^{-1}$ implies that $b \leqslant \frac{1}{a} \leqslant e$. We deduce from $e^{-bt} \leqslant 1$ for $t \geqslant 0$ that there exists $q \in \left[\frac{1}{e}, \infty\right]$ with $\int_0^q e^{-bt} = a$, and hence $f(t) \leqslant e^{-bt}$ for $t \geqslant 0$ yields

$$\int_0^\infty t^2 f(t)\,dt \geqslant \int_0^q t^2 e^{-bt}\,dt \geqslant \int_0^{1/e} t^2 e^{-et}\,dt = \frac{2 - 5\cdot e^{-1}}{e^3} > \frac{1}{150},$$

completing the proof of (4.79). ∎

Remark 4.7.18 (Half-spaces are efficient for the Cheeger constant). Let $d\mu = p\,d\mathcal{H}^n$ be an isotropic probability measure for a log-concave function p on $\mathbb{R}^n$. On the one hand, writing $\mathcal{F}_\mu$ to denote the family of $X \subset \mathbb{R}^n$ with Lipschitz boundary and $\mu(X) \in (0,1)$, we have

$$\sup_{X \in \mathcal{F}_\mu} \frac{\min\{\mu(X), 1-\mu(X)\}}{\mu_+(X)} \leqslant C_{\mathrm{Che}}(\mu) \leqslant c\sqrt{\log n}$$

for an absolute constant $c > 0$ according to Theorem 4.7.14 by Klartag [468].

On the other hand, if H^+ is any half-space with $o \in \partial H^+$, then (4.73) and (4.75) yield that

$$C_{\mathrm{Che}}(\mu) \geqslant \frac{\min\{\mu(H^+), 1-\mu(H^+)\}}{\mu_+(H^+)} \geqslant \frac{\sqrt{3/2}}{e}. \tag{4.80}$$

Therefore, one can use hyperplanes to subdivide a measure almost as efficiently as general hypersurfaces. This is very important for algorithms, and this was the original motivation of Kannan, Lovász and Siminovits to study the problem (see Alonso-Gutiérrez, Bastero [19] and Lee, Vempala [500]).

Finally, we discuss the so-called Variance conjecture (also known as Thin-shell conjecture). First, we observe that for an isotropic probability measure $d\mu = p\,d\mathcal{H}^n$ in the case of a log-concave function p on $\mathbb{R}^n$, and for the smallest positive eigenvalue λ_μ in the Poincaré inequality (4.68), plugging the test function $f(x) = \|x\|^2$ into (4.68) shows that

$$\lambda_\mu \mathrm{Var}_\mu(\|x\|^2) \leqslant \int_{\mathbb{R}^n} \|2x\|^2\,d\mu(x) = 4\sum_{i=1}^n \int_{\mathbb{R}^n} \langle e_i, x\rangle^2\,d\mu(x) = 4n \tag{4.81}$$

using an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Confirming the long standing Variance (or Thin-shell) conjecture due to Anttila, Ball, Perissinak [33] and Bobkov, Koldobsky [102], Klartag, Lehec [471] proved the following.

Theorem 4.7.19 (Klartag, Lehec). *If μ is an isotropic log-concave probability measures on $\mathbb{R}^n$, then*

$$\mathrm{Var}_\mu(\|x\|^2) \leqslant Cn \quad \textit{and} \quad \mathbb{E}_\mu(\|x\| - \sqrt{n})^2 \leqslant C$$

for an absolute constant $C > 1$.

Remark. As Klartag, Lehec [471] explains, it follows the existence of absolute constants $c, C > 0$ such that

$$\mathbb{P}_\mu(|\|x\| - \sqrt{n}| \geqslant t) \leqslant Ce^{-c\sqrt{t}}$$

holds for any $t > 1$; therefore, the mass with respect to μ is concentrated in a "thin shell" around the sphere of radius $\sqrt{n}$.

We deduce from Cheeger's inequality (4.69) and (4.81) that the KLS conjecture Conjecture 4.7.13 would yield Theorem 4.7.19.

4.8 Comments to Chapter 4

The extremal property of balls with respect to the isoperimetric problem was known to the ancient Greeks; for example, Zenodorus (circa 200 BC - 140 BC) suggested an argument first proving that regular polygons are optimal in the plane, and even claimed that spheres are optimal in three dimensions (cf. Blasjö [98]). In higher dimensional spaces, the Isoperimetric Inequality for convex bodies was proved by the work of Steiner, Schwarz, Weierstrass and Minkowski in the 19th century (see Gruber [349]).

For properties of sets with Lipschitz boundary (cf. Definition 4.1.4), see Federer [260] and Ambrosio, Colesanti, Villa [23] in the classical setting, and Chambolle, Lisini, Lussardi [184] and Lussardi, Villa [547] concerning the Anisotropic Perimeter.

The right framework for the isoperimetric inequality is sets of finite perimeter (see Chapter 5, or for example Ambrosio, Fusco, Pallara [25], Maggi [557]). Talenti [696] provided a proof of the isoperimetric inequality for sets of finite perimeter based on Steiner Symmetrization, also characterizing the equality case. Fusco, Maggi, Pratelli [311] even managed to prove a stability version of the isoperimetric inequality using Steiner symmetrization.

The optimal factor in Sobolev's inequality Theorem 4.2.1 is verified by Federer, Fleming [259] using symmetrization, and the stability version of the Sobolev inequality of optimal order for functions of bounded variation is due to Figalli, Maggi, Pratelli

[279]. For the L_p version (4.7) of the Sobolev inequality for $1 < p < n$, the optimal factor and the extremizers were determined by Talenti [695] and Aubin [45] (see D. Cordero-Erausquin, B. Nazaret, C. Villani [217] for a simplified argument using optimal transport), and the stability version where the error term has optimal order is due to Bianchi, Egnell [92] if $p = 2$, and to Figalli, Zhang [280] if $1 < p < n$.

As the Anisotropic Isoperimetric inequality for convex bodies is a direct consequence of the Brunn-Minkowski theorem, it was already known to Minkowski (see [590, 591]) even if not using this specific terminology.

The notion of Wulff shape originates from the paper Wulff [721] related to Crystallography; see Maggi [557] for a dicussion of Wulff's theorem for sets of finite perimeter, and the papers by Taylor [699], Miracle-Sole [592] and Figalli, Maggi [276] for the role of Wulff shape within crystallography; moreover, Figalli, Zhang [281] for a strong stability version of the Wulff inequality. Wulff-type isoperimetric inequalities within a convex cone are discussed by Cabré, Ros-Oton,· Serra [168]. Many examples of Wulff shapes and the relation to the underlying periodic and quasi-periodic structure are discussed in Böröczky, Schnell, Wills [148]. In particular, Wulff shapes are also successful models of certain quasi-crystals.

Many properties of the Wulff-shape have been established by Aleksandrov in the 1930s (see Aleksandrov [11]), like Lemma 4.4.3 and Aleksandrov's Lemma (cf. Theorem 7.5.2).

For in depth studies on Steiner symmetrization and Schwarz symmetrization of compact sets, see Bianchi, Gardner, Gronchi [93, 94]. They provide a broder class of n hyperplanes for the iterated Steiner Symmetrization than Theorem 1.C.3; namely, $v_1, \dots, v_n \in S^{n-1}$ are independent in a way such that $\langle v_i, v_j \rangle \neq 0$ for $i \neq j$, and $\angle(v_1, v_2) = \alpha\pi$ for irrational $\alpha \in (0, 1)$.

For the fundamental properties of hyperbolic and spherical spaces, see Berger [81] Chapter 19, Ratcliffe [630] Chapters 3 and 4, or Vinberg [712]. The Isoperimetric Inequality in the spherical case is proved by Lévy [508] in 1922, and in the hyperbolic spaces is due to E. Schmidt [653]. We provide the elegant argument by Benyamini [79] because it works simultaneously in all spaces of constant curvature, and yields also the Isodiametric Inequality. Stability versions of the Isoperimetric Inequality in terms of the volume difference were proved by Bögelein, Duzaar, Fusco [108] in the spherical case, and by Bögelein, Duzaar, Scheven [109] in the hyperbolic case.

The Isodiametric Inequality in $\mathbb{R}^n$ is due to P. Urysohn [707], in the spherical case is proved by Lévy [508] in 1922, and the hyperbolic version is due to Schmidt [654, 655] in the 1940's (see also Böröczky, Sagmeister [145] for discussion of equality, and [146] for a stability version).

The two-point symmetrization appeared first in Wolontis [719] in the framework of conformal functions. It is applied to prove the isoperimetric inequality in the spherical space by Benyamini [79], whose argument is adapted to the hyperbolic space by

Böröczky, Sagmeister [146]. Two-point symmetrization leads to the spherical analogue of the Blaschke-Santaló inequality (6.31) by Gao, Hug, Schneider [312].

A hyperbolic version of the Brunn-Minkowski inequality has been proved by Assouline, Klartag [44] in H^2. A horocycle Σ in H^2 has constant curvature one, and is an orthogonal trajectory of a pencil of pairwise parallel lines, which is the boundary of an ellipse touching the boundary of the hyperbolic plane in the Bertrami-Cayley-Klein model; therefore, the orientation of H^2 induces an orientation of Σ. For any $x \neq y \in H^2$, there exists exactly one horocycle arc connecting x and y where x comes first before y according to the orientation of the horocycle. For measurable $X, Y \subset H^2$ and $\lambda \in (0,1)$, [44] defines their "hyperbolic Minkowski combination" analogously to the Euclidean case as

$$(1-\lambda)\, X : \lambda\, Y \quad = \quad \{\sigma(\lambda) : \sigma : [0,1] \to H^2 \text{ oriented constant-speed horocycle arc with } \sigma(0) \in X \text{ and } \sigma(1) \in Y\},$$

and proves

$$\sqrt{|(1-\lambda)\, X : \lambda\, Y|_*} \geqslant (1-\lambda)\sqrt{|X|} + \lambda\sqrt{|Y|}.$$

For a C^3 even convex function $g : \mathbb{R} \to (0,\infty)$, let $f(x) = e^{g(\|x\|)}$ for $x \in \mathbb{R}^n$. According to the Log-Convex Density Theorem by Chambers [183], if $X \subset \mathbb{R}^n$ is of finite perimeter and $\int_X f = \int_{rB^n} f$ for $r > 0$, then

$$\int_{\partial^* X} f \, d\mathcal{H}^{n-1} \geqslant \int_{rS^{n-1}} f \, d\mathcal{H}^{n-1}.$$

Fusco, La Manna [310] proved a stability version of Chambers' Log-Convex Density Theorem if g" is positive, and Silini [685] verified the hyperbolic version of the Log-Convex Density Theorem in H^n.

We note that the isoperimetric inequality in the complex hyperbolic space $H^n_{\mathbb{C}}$ is still open (see J.R. Parker [610] for a beautiful introduction into complex hyperbolic geometry). It has been verified for the so-called Hopf-symmetric sets by Silini [685] using the hyperbolic Log-Convex Density Theorem.

Another famous open isoperimetric problem is the Isoperimetric Inequality in the Heisenberg group $\mathbb{H}^d$. It is the "simplest" simply connected nilpotent Lie-group whose product structure is the easiest to describe on $\mathbb{C}^d \times \mathbb{R}$, and hence its topological dimension is $n = 2d + 1$ (see Franceschi, Leonardi, Monti [305] for the known results about the Isoperimetric Problem in the Heisenberg group $\mathbb{H}^d$). We note that the Brunn-Minkowski (3.25) with exponent $\frac{1}{n}$ does not help in this case because (3.25) does not hold with exponent $\frac{1}{Q}$ for $Q = 2d + 2 = n + 1$, and the isoperimetric problem asks for the minimal $C > 0$ such that $\mu(X)^{\frac{Q-1}{Q}} \leqslant C \cdot P(X)$ where μ is a Haar measure and $P(X)$ is the corresponding finite perimeter.

The Gaussian Isoperimetric Inequality was proved independently by Borell [111] and Sudakov, Tsirelson [694] using the Spherical Isoperimetric inequality as in Section 4.6. The significance of the Gaussian Isoperimetric Inequality is shown also by the fact that many different methods may be used to prove it; see, for example, Bobkov [101], Bakry, Ledoux [50], Barthe, B. Maurey [73]. In addition, Cianchi, Fusco, Maggi, Pratelli [203] verified a stability version of the Gaussian Isoperimetric inequality of optimal order using Ehrhard's symmetrization method leading to (4.82) (see also Barchiesi, Brancolini, Julin [65] for a stability version of the Gaussian Isoperimetric inequality).

Both the Spherical and the Gaussian Isoperimetric inequalities can be considered as results about measure concentration: If X has at least half of the measure of the space, then the complement of $X^{(\varrho)}$ has "small" measure (cf. Section 4.D).

An important Brunn-Minkowski type generalization of the Gaussian Isoperimetric inequality is the Ehrhard inequality, proved by Ehrhard [241, 243] for convex bodies using a symmetrization method, and by Borell [115] (see also van Handel [709]) for measurable sets. Equality in the Ehrhard inequality is clarified by Shenfeld, van Handel [677]. Actually, the Ehrhard inequality says that any measurable sets $X, Y \subset \mathbb{R}^n$ satisfy

$$\psi^{-1}\left(\gamma_n\left((1-\lambda)X+\lambda Y\right)\right) \geqslant (1-\lambda)\psi^{-1}\left(\gamma_n(X)\right)+\lambda\,\psi^{-1}\left(\gamma_n(Y)\right) \tag{4.82}$$

where $d\gamma_n(x) = e^{-\pi\|x\|^2}dx$, $\lambda \in (0,1)$ and $\psi(s) = \int_{-\infty}^{s} e^{-\pi t^2}\,dt$.

A famous open problem, posed by Barthe [68], is the Gaussian Isoperimetric inequality for origin symmetric sets. Here the conjectured optimal set depends on the fixed "volume" (Gaussian measure) $V \in (0,1)$ of the origin symmetric set. The only known case is due to the recent paper by Barchiesi, Julin [66] when the fixed volume is close to one. More precisely, there exists an absolute constant $c \in (0,1)$ (actually very close to one) such that if $V \in (c,1)$, then among origin symmetric closed sets X with Lipschitz boundary (cf. Definition 4.1.4) and satisfying $\gamma_n(X) = V$, the origin symmetric slabs bounded by two parallel hyperplanes minimize the Gaussian perimeter. For the current state of art of the problem, see Barchiesi, Julin [66] and Livshyts [531].

Possible strengthenings of the Ehrhard inequality for symmetric sets are discussed by Livshyts [531], stating explicit conjectures.

A Brunn-Minkowski type inequality for the Gaussian measure that only holds for origin symmetric convex sets $X, Y \subset \mathbb{R}^n$ is due to Eskenazis, Moschidis [249]; namely,

$$\gamma_n\left((1-\lambda)X+\lambda Y\right)^{\frac{1}{n}} \geqslant (1-\lambda)\gamma_n(X)^{\frac{1}{n}}+\lambda\,\gamma_n(Y)^{\frac{1}{n}} \tag{4.83}$$

holds for $\lambda \in (0,1)$. Both conditions - convexity and symmetry with respect to the origin - are important; for example, if X is a fixed origin symmetrix convex body and Y is a translate by a vector whose length tends to infinity, then $\gamma_n\left((1-\lambda)X+\lambda Y\right)$ can be arbitrarily small.

The Gaussian Correlation conjecture (independent of the Gaussian isoperimetric inequality or (4.83)) was verified by Royen [638] in 2014 after various failed attempts (see R. Latała, D. Matlak [496] and E. Milman [580] for surveys on the history of the conjecture and simplified proofs, and Latała [495] for similar problems). The conjecture was concerned with the standard Gaussian measure $d\tilde{\gamma}_n(x) = (2\pi)^{\frac{-n}{2}} e^{-\frac{\|x\|^2}{2}} dx$ that has zero mean and its covariance matrix is the identity (see Definition 4.7.12).

Theorem 4.8.1 (Gaussian Correlation inequality). *If $X, Y \subset \mathbb{R}^n$ are o-symmetric convex sets, then*

$$\tilde{\gamma}_n(X \cap Y) \geqslant \tilde{\gamma}_n(X)\tilde{\gamma}_n(Y).$$

Remarks.

- It follows that if $f, g : \mathbb{R}^n \to [0, \infty)$ are even log-concave functions, then

$$\int_{\mathbb{R}^n} fg \, d\tilde{\gamma}_n \geqslant \int_{\mathbb{R}^n} f \, d\tilde{\gamma}_n \int_{\mathbb{R}^n} g \, d\tilde{\gamma}_n.$$

- The Strengthened Gaussian Correlation conjecture stating that

$$\tilde{\gamma}_n(X + Y)\tilde{\gamma}_n(X \cap Y) \geqslant \tilde{\gamma}_n(X)\tilde{\gamma}_n(Y)$$

  holds for o-symmetric convex sets $X, Y \subset \mathbb{R}^n$ is still open (cf. Assouline, Chor, Sadovsky [43]).

- Nakamura, Tsuji [596] prove the generalized version of Gaussian Correlation inequality that if $X_1, \ldots, X_m \subset \mathbb{R}^n$ are convex sets such that the Gaussian centroids are the origin; namely, $\int_{X_i} x \, d\tilde{\gamma}_n(x) = o$ holds for $i = 1, \ldots, m$, then

$$\tilde{\gamma}_n(X_1 \cap \ldots \cap X_m) \geqslant \tilde{\gamma}_n(X_1) \cdot \ldots \cdot \tilde{\gamma}_n(X_m).$$

The KLS conjecture about the Cheeger constant $C_{\mathrm{Che}}(\mu)$ was stated in the groundbreaking paper Kannan, Lovász, Simonovits [454] in 1995. The paper [454] proved that $C_{\mathrm{Che}}(\mu) \leqslant c\sqrt{n}$ holds for some absolute constant c and any isotropic log-concave measure μ on $\mathbb{R}^n$. Even after 20 years of intense research (see Alonso-Gutiérrez, Bastero [19], Artstein-Avidan, Giannopoulos, Milman [39], Chapter 2, and Klartag [468]), the best upper bound still stayed at $cn^{\frac{1}{4}}$. The breakthrough came by Yuansi Chen in 2020 verifying that the bound $cn^{\frac{1}{4}}$ can be replaced by $n^{o(1)}$, and the currently best upper bound is the $c\sqrt{\log n}$ of Theorem 4.7.14 due to Klartag [468].

A related problem is discussed by Alter, Caselles [21]. For the uniform probability measure on a convex body $K \subset \mathbb{R}^n$, [21] considers the infimum of $\mathcal{H}^{n-1}(\partial X)/|X|$ for subsets of $X \subset K$ with Lipschitz boundary, more precisely, they consider sets of finite perimeter (see Chapter 5). Alter, Caselles [21] prove that there is essentially a unique optimal set $X \subset K$, and it is a convex body.

4.A Supplement: The Spherical Space and the Hyperpolic Space

In this section, we summarize the basic notions related to the spherical space and the hyperbolic space. We recall that $\operatorname{pos}\{x, y\} = \{\alpha x + \beta y : \alpha, \beta \geqslant 0\}$ for $x, y \in \mathbb{R}^{n+1}$.

First we introduce the spherical space $S^n \subset \mathbb{R}^{n+1}$. Fix an $e \in \mathbb{R}^{n+1}$ with $\|e\| = 1$.

Remark 4.A.1 (Notions related to the spherical space $S^n \subset \mathbb{R}^{n+1}$).

Spherical Space: $S^n = \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = 1\}$.

Distance: $d_{S^n}(x, y) = \arccos\langle x, y\rangle$ for $x, y \in S^n$.

Isometries: $O(n+1)$ - acts transitively (any $x \in S^n$ can be mapped to any other $y \in S^n$ by an isometry), stabilizer of a point is isomorphic to $O(n)$.

Tangent space at $z \in S^n$: $T_z = \{y \in \mathbb{R}^{n+1} : \langle y, z\rangle = 0\}$, T_z is equipped with the scalar product $\langle \cdot, \cdot \rangle$.

Geodesic segment: (also called geodesic arc) $[x, y]_{S^n} = \operatorname{pos}\{x, y\} \cap S^n$ for $x, y \in S^n$ provided $y \neq \pm x$.
If $x, y \in S^n$ with $y \neq \pm x$, then there exists unique "directional vector" $v \in T_x$ with $\langle v, v\rangle = 1$ and $y = x\cos r + v \sin r$ where $r = d_{S^n}(x, y) \in (0, \pi)$, and we have $[x, y]_{S^n} = \{x \cos t + v \sin t : t \in [0, r]\}$.

Angle: If $x, y, z \in S^n$ with $y \neq \pm x$, $z \neq \pm x$, and $y = x\cos r + v \sin r$, $z = x \cos s + w \sin s$ for $r, s \in (0, \pi)$ and $v, w \in T_x$ with $\langle v, v\rangle = \langle w, w\rangle = 1$, then the spherical angle $\angle(y, x, z) = \arccos\langle v, w\rangle = \alpha$, which is also the angle of $[x, y]_{S^n}$ and $[x, z]_{S^n}$, and $\cos d_{S^n}(x, y) = \cos r \cos s + \sin r \sin s \cos\alpha$ according to the *Spherical Law of Cosines for sides*.

"Hyperplane": (or Great subsphere) $H = \{x \in S^n : \langle x, u\rangle = 0\}$ for a $u \in S^n$.

"Closed half-space": (or closed hemisphere) $H^+ = \{x \in S^n : \langle x, u\rangle \geqslant 0\}$ for a $u \in S^n$, and the corresponding open hemisphere is $\{x \in S^n : \langle x, u\rangle > 0\}$.

Reflection through the hyperplane H: If $H = \{x \in S^n : \langle x, u\rangle = 0\}$ for $u \in S^n$, then the reflected image of an $x \in S^n$ through H is $\xi_H x = x - 2\langle x, u\rangle \cdot u$.

- $\xi_H x = x$ for $x \in H^n$ if and only if $x \in H$;
- ξ_H is an isometry such that $\xi_H(\xi_H x) = x$ for $x \in S^n$;
- for any $y, z \in S^n$, there exists a hyperplane H such that $\xi_H y = z$ ("perpendicular bisector" of $[y, z]_{S^n}$ provided $y \neq \pm z$).

Spherically convex sets: $X \subset S^n$ is convex, if X contained in a closed hemisphere, and $[x, y]_{S^n} \subset X$ for any non-antipodal $x, y \in X$ ($x \neq -y$); or equivalently, $X = C \cap S^n$ for a Euclidean convex cone $C \subset \mathbb{R}^{n+1}$ with $C \neq \mathbb{R}^{n+1}$ (cf. Definition 1.2.8).
$X \subset S^n$ is a convex compact set if and only if X is the intersection of closed hemispheres (cf. Corollary 1.2.8).
We call a compact convex $X \subset S^n$ a *convex body* if $\operatorname{int} X \neq \varnothing$.

Bi-Lipschitz map to Euclidean geometry:
$S^n_+ = \{x \in S^n : \langle x, e\rangle > 0\}$ fixed open hemisphere. The radial projection $x \mapsto \frac{1}{\langle x,e\rangle} x$ is a bi-Lipschitz map $S^n_+ \to e^\perp + e$ (actually, C^∞ diffeomorphism) mapping spherical segments and convex sets in S^n_+ onto Euclidean segments and convex sets in $e^\perp + e$.

Now we turn to basic properties of the hyperbolic space H^n. One of its characteristic properties is the way how parallel lines occur. We observe that on S^2, any two lines (great circles) intersect in two (antipodal) points, and in $\mathbb{R}^2$, if $\ell \subset \mathbb{R}^2$ is a line an $x \in \mathbb{R}^2$ does not lie on ℓ, then there exists exactly one line through x not intersecting ℓ, which is called parallel. Now, if ℓ is a line in the hyperbolic plane H^2, and $x \in H^2$ does not lie on ℓ, then there exist exactly two lines ℓ_1 and ℓ_2 through x which do not intersect ℓ, but which get asymptotically arbitrary close to ℓ (see the Beltrami-Cayley-Klein model below). In addition, any line through x between ℓ_1 and ℓ_2 (in a suitable sense) not simply avoids ℓ, but diverges away at both ends.

The hyperbolic space has various useful models, and we discuss two of them in what follows. We start with the Hyperboloid model, which exhibits the metric properties of H^n analogously to S^n. After that, we briefly discuss Beltrami-Cayley-Klein model, whose advantage is that hyperbolic lines are Euclidean segments in the model, and it is very transparent how ideal points of the Hyperbolic Space and parallelism work.

To define the Hyperboloid model of H^n in $\mathbb{R}^{n+1}$, fix an $e \in \mathbb{R}^{n+1}$ with $\|e\| = 1$. We consider the following symmetric bilinear form $\mathcal{B}(\cdot,\cdot)$ on $\mathbb{R}^{n+1}$ with eigenvalues $1, -1, \ldots, -1$: If $x = \tilde{x} + te \in \mathbb{R}^{n+1}$ and $y = \tilde{y} + se \in \mathbb{R}^{n+1}$ for $\tilde{x}, \tilde{y} \in e^\perp$ and $t, s \in \mathbb{R}$, then

$$\mathcal{B}(x, y) = ts - \langle \tilde{x}, \tilde{y}\rangle.$$

Remark 4.A.2 (Hyperboloid Model of the hyperbolic space H^n in $\mathbb{R}^{n+1}$).

Hyperbolic Space: $H^n \subset \mathbb{R}^{n+1}$.
$H^n = \{x = \tilde{x} + te : \mathcal{B}(x, x) = 1 \text{ and } \tilde{x} \in e^\perp \text{ and } t > 0\}$.

Distance: $d_{H^n}(x, y) = \operatorname{arccosh} \mathcal{B}(x, y)$ for $x, y \in H^n$ where $\mathcal{B}(x, y) \geqslant 1$ for $x, y \in H^n$ (this inequality is sometimes called the reverse Cauchy-Schwarz inequality).

Isometries: $\{\Phi \in \mathrm{GL}(n+1) : \mathcal{B}(\Phi x, \Phi y) = \mathcal{B}(x, y) \text{ and } \langle \Phi e, e\rangle > 0\}$.
The isometry group is transitive, and the stabilizer of a point is isomorphic to $O(n)$.

Tangent space at $z \in H^n$: $T_z = \{y \in \mathbb{R}^{n+1} : \mathcal{B}(y, z) = 0\}$. T_z is equipped with the scalar product $-\mathcal{B}(\cdot,\cdot)$.

Geodesic segment: $[x, y]_{H^n} = \operatorname{pos}\{x, y\} \cap H^n$ for $x, y \in H^n$ with $y \neq x$. There exists a unique "directional vector" $v \in T_x$ with $\mathcal{B}(v, v) = -1$ and $y = x \cosh r + v \sinh r$ for $r = d_{H^n}(x, y) > 0$, and $[x, y]_{H^n} = \{x \cosh t + v \sinh t : t \in [0, r]\}$.

Hyperbolic line: $\ell = \Pi \cap H^n$ where $\Pi \subset \mathbb{R}^{n+1}$ is a two-dimensional linear subspace where $\Pi \cap H^n$ has at least two points.
If $x \in \ell$ and $v \in T_x \cap \Pi$ with $\mathcal{B}(v,v) = -1$, then $\ell = \{x\cosh r + v \sinh r : r \in \mathbb{R}\}$. In particular, $w_1 = x + v \in \Pi$ and $w_2 = x - v \in \Pi$ satisfy $\mathcal{B}(w_i, w_i) = 0$, and $\mathbb{R}w_1$ and $\mathbb{R}w_2$ represent the two ideal points of ℓ where $\mathbb{R}w_i$ is part of the "asymptotic cone" $\{w \in \mathbb{R}^{n+1} : \mathcal{B}(w,w) = 0\}$ of the hyperboloid in $\mathbb{R}^{n+1}$ for $i = 1, 2$.

Parallel lines: Lines ℓ_1 and ℓ_2 are parallel if there exists $w \in \mathbb{R}^{n+1}$ with $\mathcal{B}(w,w) = 0$ such that $\mathbb{R}w$ is an ideal point of both lines.

Angle: If $x, y, z \in H^n$ with $y \neq x$, $z \neq x$, then $y = x\cosh r + v \sinh r$, $z = x \cosh s + w \sinh s$ for $r, s > 0$ and $v, w \in T_x$ with $\mathcal{B}(v,v) = \mathcal{B}(w,w) = -1$, and the hyperbolic angle is $\angle(y,x,z) = \arccosh(-\mathcal{B}(v,w)) = \alpha$, which is also the angle of $[x,y]_{H^n}$ and $[x,z]_{H^n}$, and satisfies $\cosh d_{H^n}(x,y) = \cosh r \cosh s - \sinh r \sinh s \cos\alpha$ according to the *Hyperbolic Law of Cosines for sides*

Horosphere: Let $w \in \mathbb{R}^{n+1}$ with $\mathcal{B}(w,w) = 0$ and $\langle w, e\rangle > 0$. For $t > 0$, $\Sigma = \{x \in H^n : \mathcal{B}(w,x) = t\}$ is a horosphere centered at the ideal point $\mathbb{R}w$. If $x \in \Sigma$, then $v = x - \frac{1}{t}w \in T_x$ with $\mathcal{B}(v,v) = -1$, and hence $\mathbb{R}w$ ideal point of the line $\ell = \{x\cosh r + v \sinh r : r \in \mathbb{R}\}$ orthogonal to Σ; or in other words, Σ is the orthogonal trajectory to the pencil of parallel lines at $\mathbb{R}w$.

Hyperplane: $H = \{x \in H^n : \mathcal{B}(x,u) = 0\}$ where $u \in \mathbb{R}^{n+1}$ with $\mathcal{B}(u,u) = -1$.

Closed half-space: $H^+ = \{x \in H^n : -\mathcal{B}(x,u) \geqslant 0\}$ where $u \in \mathbb{R}^{n+1}$ with $\mathcal{B}(u,u) = -1$.

Reflection through the hyperplane H: $\xi_H x = x + 2\mathcal{B}(x,u)\cdot u$ for $x \in H^n$ provided $H = \{x \in H^n : \mathcal{B}(x,u) = 0\}$ where $u \in \mathbb{R}^{n+1}$ with $\mathcal{B}(u,u) = -1$.

- ξ_H is an isometry such that $\xi_H(\xi_H x) = x$ for $x \in H^n$;
- $\xi_H x = x$ for $x \in H^n$ if and only if $x \in H$;
- if $y \neq z$, then there exists a unique hyperplane H with $\xi_H y = z$ (called the "perpendicular bisector" of $[y,z]_{H^n}$).

Hyperbolic convex sets: $X \subset H^n$ is convex if $[x,y]_{H^n} \subset X$ for any $x, y \in X$; or equivalenty, if $X = C \cap H^n$ for a Euclidean convex cone $C \subset \mathbb{R}^{n+1}$ (cf. Definition 1.2.8).

- $X \subset H^n$ is a convex compact set if and only if X is the intersection of closed half-spaces (cf. Corollary 1.2.8).
- We call a compact convex $X \subset S^n$ a *convex body* if $\operatorname{int} X \neq \varnothing$.

Bi-Lipschitz map between Hyperbolic and Euclidean geometry: The radial projection $x \mapsto \frac{1}{\langle x,e\rangle}x$ is a bi-Lipschitz map $H^n \to e^\perp + e$ (actually, C^∞ diffeomorphism) mapping hyperbolic segments and convex sets in H^n onto Euclidean segments and convex sets in $e^\perp + e$.

For the Beltrami-Cayley-Klein model of H^n, we only discuss a few notions to indicate how ideal points and parallelism work in the Hyperbolic World.

Remark 4.A.3 (Beltrami-Cayley-Klein model of H^n).

Beltrami-Cayley-Klein model: $\operatorname{int} B^n$ is set of points of the model.

Hyperbolic segment: For $x, y \in \operatorname{int} B^n$, the Euclidean segment $\operatorname{conv}\{x, y\}$ coincides with the hyperbolic segment $[x, y]_{H^n}$.

Ideal points: ∂B^n is the set of ideal points.

Hyperbolic line: $\operatorname{conv}\{w_1, w_2\}\backslash\{w_1, w_2\}$ for the ideal points $w_1, w_2 \in \partial B^n$.

Parallel Hyperbolic lines: Hyperbolic lines $\ell_1, \ell_2 \subset \operatorname{int} B^n$ are parallel if ℓ_1 and ℓ_2 have common ideal point $w \in \partial B^n$.

Horosphere: $\partial E\backslash\{w\}$ for certain Euclidean ellipsoid $E \subset B^n$ touching B^n from inside at a $w \in \partial B^n$.

Isometry with the Hyperboloid model: $\tilde{\pi} : H^n \to \operatorname{int} B^n$, $\tilde{\pi}(x) = \frac{1}{\langle x,e\rangle} x - e$.

Finally, we discuss properties of the three geometries that work similarly in all three of them. Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n. The following properties readily follow from the definition of convexity in $\mathcal{M}^n$, and from properties of convex sets in $\mathbb{R}^n$ in Chapter 1 using radial projections to a tangent hyperplane to the "canonical" model of the spherical space (S^n) or hyperbolic space (hyperboloid model) in $\mathbb{R}^{n+1}$. The metric ball centered at $x \in \mathcal{M}^n$ and of radius $r > 0$ (where $r < \pi$ if $\mathcal{M}^n = S^n$) is $B_{\mathcal{M}^n}(x, r) = \{y \in \mathcal{M}^n : d_{\mathcal{M}^n}(x, y) \leqslant r\}$.

Lemma 4.A.4. *Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n.*

(i) Intersection of convex sets is convex.

(ii) Closed half-spaces are convex.

(iii) If $K \subset \mathcal{M}^n$ is closed and convex, then at any boundary point $x \in K$, there exists a supporting hyperplane H; namely, $x \in H$ and K lies on one side of H.

(iv) If $K \subset \mathcal{M}^n$, $K \neq \mathcal{M}^n$, is closed and convex with non-empty interior, then $\mathcal{H}^{n-1}_{\mathcal{M}^n}$ a.e. $x \in \partial K$ is regular; namely, there is a unique supporting hyperplane to K at x, which is also the unique supporting hyperplane to $B_{\mathcal{M}^n}(x, r) \cap K$ for any $r > 0$.

Remark 4.A.5 (Convex hull in $\mathcal{M}^n$). Let $X \subset \mathcal{M}^n$ where $\mathcal{M}^n$ is either H^n, $\mathbb{R}^n$ or S^n, and X is contained in a closed hemisphere if $\mathcal{M}^n = S^n$. Then the convex hull $\operatorname{conv}_{\mathcal{M}^n} X$ is the smallest convex set containing X; namely, the intersection of all convex sets containing X. We note that $\operatorname{conv}_{\mathcal{M}^n} X$ is compact if X is compact, which follows from Proposition 1.1.7 in $\mathbb{R}^n$, and from using the models above in the case of S^n and H^n.

For compact $X \subset \mathcal{M}^n$ and $\varrho > 0$, the corresponding parallel domain is

$$X^{(\varrho)} = \{y \in \mathcal{M}^n : \exists\, x \in X \text{ with } d_{\mathcal{M}^n}(x, y) \leqslant \varrho\} = \bigcup \{B_{\mathcal{M}^n}(x, \varrho) : x \in X\}.$$

Lemma 4.A.6. *Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n, let H^+ be a closed half-space in $\mathcal{M}^n$ bounded by the hyperplane H, and let H^- be the complementary closed half-space with $H^+ \cap H^- = H$.*

(i) (Triangle inequality) $d_{\mathcal{M}^n}(x,y) + d_{\mathcal{M}^n}(y,z) \geqslant d_{\mathcal{M}^n}(x,z)$ for $x, y, z \in \mathcal{M}^n$, with equality if and only if $y \in [x,z]_{\mathcal{M}^n}$.

(ii) Given $x \in H^+$, $y \in H^+$ if and only if $d_{\mathcal{M}^n}(y,x) \leqslant d_{\mathcal{M}^n}(y, \xi_H x)$.

(iii) If $H \cap \mathrm{int} B_{\mathcal{M}^n}(z,r) \neq \varnothing$ and $z \in H^+$, then $\xi_H(H^- \cap B_{\mathcal{M}^n}(z,r)) \subset H^+ \cap B_{\mathcal{M}^n}(z,r)$ where we assume that $r < \pi$ if $\mathcal{M}^n = S^n$.

(iv) $B_{\mathcal{M}^n}(x,r)$ convex for $r > 0$ provided that $r \leqslant \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$.

(v) If X compact and $\varrho > 0$, then $\operatorname{diam} X^{(\varrho)} = 2\varrho + \operatorname{diam} X$ where we assume that $2\varrho + \operatorname{diam} X < \pi$ if $\mathcal{M}^n = S^n$.

Remark. $B_{S^n}(x,r)$ is not convex if $\frac{\pi}{2} < r < \pi$.

Proof. In the case of $\mathbb{R}^n$, these properties are well-known or proved in Chapter 1; therefore, we only present the argument when $\mathcal{M}^n$ is either H^n or S^n.

(i) follows from the Law of Cosines for sides, and in turn (i) yields (ii), and (ii) implies (iii). For (iv), we may assume that $x = e$, and for $B(e,r)_{\mathcal{M}^n}$, and we use the radial projection $\tilde{\pi}_{\mathcal{M}^n}$ and the convexity of Euclidean balls.

For (v), $\operatorname{diam} X^{(\varrho)} \leqslant 2\varrho + \operatorname{diam} X$ follows from (i). If $\operatorname{diam} X = d(x,y)$ for $x, y \in X$, choose $x', y' \in X^{(\varrho)}$ such that $x, y \in [x', y']$ and $d(x,x') = d(y,y') = \varrho$, and hence $d(x',y') = 2\varrho + \operatorname{diam} X$. ∎

Remark 4.A.7 ("Volume" in $\mathcal{M}^n$). The "canonical" measure on $\mathcal{M}^n$ is just the n-dimensional Hausdorff measure $\mathcal{H}^n_{\mathcal{M}^n}$ determined by the metric of $\mathcal{M}^n$, which coincides with the Euclidean $\mathcal{H}^n$ in $\mathbb{R}^{n+1}$ in the case of S^n (but does not in the hyperbolic case). For measurable $X \subset \mathcal{M}^n$, its "canonical" measure is denoted by $|X|$. This measure is also the suitably normalized Haar measure on $\mathrm{Iso}(\mathcal{M}^n)/O(n)$ for the isometry group $\mathrm{Iso}(\mathcal{M}^n)$ of $\mathcal{M}^n$. The "canonical" measure is regular (cf. Appendix Chapter 10); namely, if $X \subset \mathcal{M}^n$ is meaurable with $|X| < \infty$, then

$$|X| = \inf\{|U| : X \subset U \ \&\ U \text{ open}\} = \sup\{|C| : C \subset X \ \&\ C \text{ compact}\}.$$

Remark 4.A.8 (Surface area in $\mathcal{M}^n$). Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n. Following Remark 10.4.11, we say that a non-empty compact $X \subset \mathcal{M}^n$ has Lipschitz boundary if $X = \mathrm{cl}\,\mathrm{int} X$, $\partial X \neq \varnothing$, and there exists a C^2 diffeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ where $\Omega \subset \mathcal{M}^n$ is an open neighbourhood of X and $\widetilde{\Omega} \subset \mathbb{R}^n$ is open such that the compact set $\varphi(X) \subset \mathbb{R}^n$ has Lipschitz boundary. We note that it is equivalent saying that $X = \mathrm{cl}\,\mathrm{int} X$ with $\partial X \neq \varnothing$, and for any $x \in \partial X$, there exists an open neighbourhood U of x on ∂X, and a hyperplane $H \subset \mathcal{M}^n$ such that the orthogonal projection from U into H is bi-Lipschitz onto its open image in H. We note that $0 < \mathcal{H}^{n-1}_{\mathcal{M}^n}(\partial X) < \infty$, and (10.14)

yields that the perimeter of X coincides with the outer Minkowski content; namely,

$$P(X) = \mathcal{H}^{n-1}_{\mathcal{M}^n}(\partial X) = \lim_{\varrho \to 0^+} \frac{\left|X^{(\varrho)}\right| - |X|}{\varrho}. \tag{4.84}$$

For $x \in \mathcal{M}^n$ and $r > 0$, we note that

$$P\left(B_{\mathcal{M}^n}(x,r)\right) = \begin{cases} n\omega_n r^{n-1} & \text{if } \mathcal{M}^n = \mathbb{R}^n; \\ n\omega_n (\sinh r)^{n-1} & \text{if } \mathcal{M}^n = H^n; \\ n\omega_n (\sin r)^{n-1} & \text{if } \mathcal{M}^n = S^n \text{ and } r < \pi, \end{cases}$$

which formulas follow by considering $B_{\mathcal{M}^n}(e,r)$ in the models above if $\mathcal{M}^n$ is either S^n or H^n. It follows that the volume of the ball is $\left|B_{\mathbb{R}^n}(x,r)\right| = \omega_n r^n$, $\left|B_{H^n}(x,r)\right| = n\omega_n \int_0^r (\sinh r)^{n-1}$ and if $r < \pi$, then $\left|B_{S^n}(x,r)\right| = n\omega_n \int_0^r (\sin r)^{n-1}$. We observe that if $\mathcal{M}^n$ is either $\mathbb{R}^n$, H^n or S^n, then locally the geometry is close to being Euclidean; in particular,

$$\lim_{r \to 0^+} \frac{\left|B_{\mathcal{M}^n}(x,r)\right|}{\omega_n r^n} = 1 \text{ and } \lim_{r \to 0^+} \frac{P\left(B_{\mathcal{M}^n}(x,r)\right)}{n\omega_n r^{n-1}} = 1.$$

We finish this section with a discussion of some interesting facts which show how the curvature $\kappa = +1, 0, -1$ of S^n, $\mathbb{R}^n$ and H^n, respectively, influences the geometry. For example, S^n is compact ($\kappa = 1 > 0$), the surface area of a large Euclidean ball is negligible compared to the volume ($\kappa = 0$); namely, $\lim_{r \to \infty} \frac{P(B_{\mathbb{R}^n}(x,r))}{|B_{\mathbb{R}^n}(x,r)|} = 0$, while the hyperbolic space "expands" ($\kappa = -1 < 0$); namely, $\lim_{r \to \infty} \frac{P(B_{H^n}(x,r))}{|B_{H^n}(x,r)|} = 1$. In addition, if T is a triangle in $\mathcal{M}^2$; namely, it is the convex hull of three non-collinear points x, y, z, then ∂T consists of the three segments determined by x, y, z, and the three angles α, β, γ of T at x, y, z satisfy

$$\alpha + \beta + \gamma = \pi + \kappa \cdot |T|.$$

4.B Supplement: Equality in the Isodiametric Inequality in H^n and S^n

In this section, we characterize equality in the Isodiametric Inequality Theorem 4.5.1 in H^n and S^n. We note that the argument works in the Euclidean case, as well, even if a simpler proof is available in $\mathbb{R}^n$ based on the Brunn-Minkowski inequality (cf. Theorem 1.12.5). Therefore, let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n, and let $S^n_+ = \{x \in S^n : \langle x, e \rangle \geqslant 0\}$ be a closed hemisphere for a fixed $e \in \mathbb{R}^{n+1}$ in the spherical case. We first relate the diameter of a compact set to its convex hull (cf. Remark 4.A.5). In order to make the formulas simpler, we drop the index $\mathcal{M}^n$ from the notation $\operatorname{conv} X$ and $\operatorname{diam} X$.

Lemma 4.B.1. *Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n, and let $X \subset \mathcal{M}^n$ be compact where $X \subset S^n_+$ if $\mathcal{M}^n = S^n$.*

(i) $\operatorname{diam}\operatorname{conv} X = \operatorname{diam} X$ *provided* $\operatorname{diam} X \leqslant \frac{\pi}{2}$ *if* $\mathcal{M}^n = S^n$.

(ii) If $|X| > 0$ *and* $\operatorname{conv} X \neq X$, *then* $|\operatorname{conv} X| > |X|$.

Remark. (i) does not hold if $\mathcal{M}^n = S^n$ and $\operatorname{diam} X > \frac{\pi}{2}$.

Proof. For (i), let $D = \operatorname{diam} X$. Lemma 4.A.4 (i) and Lemma 4.A.6 (iv) yield that it is sufficient to prove the claim that $d(p,q) \leqslant D$ for $p, q \in \widetilde{X}$ where $\widetilde{X} = \bigcap\{B(y,D) : X \subset B(y,D)\}$.

To verify the claim, we observe that $p \in B(x,D)$ for any $x \in X$, and hence $d(p,x) \leqslant D$ for any $x \in X$, which in turn yields that $X \subset B(p,D)$. It follows that $q \in B(p,D)$, thus $d(p,q) \leqslant D$, proving the claim, and in turn (i).

For (ii), if $|X| > 0$ and $\operatorname{conv} X \neq X$, then $(\operatorname{int}\operatorname{conv} X) \setminus X \neq \varnothing$, implying that $|\operatorname{conv} X| > |X|$. ∎

For the two-point symmetrization (cf. Definition 4.5.2), we prove at least in the case of a convex body K with C^1 boundary that if $\tau_H K$ is convex for any hyperplane H, then K is a ball.

Proposition 4.B.2. *If $K \subset \mathcal{M}^n$ convex body such that every boundary point is regular and $\tau_H K$ is convex for any hyperplane H, then K is a ball.*

Remark. Proposition 4.B.2 holds even if K is any convex body that may have non-regular boundary points, as well (see Aubrun, Fradelizi [46] and Böröczky, Sagmeister [145]).

Proof. For any $p \in \partial K$, we write H_p to denote the unique supporting hyperplane to K at p. Let $x, y \in K$ satisfy that $d(x,y) = \operatorname{diam} K$, and hence H_x and H_y are ortogonal to $[x,y]$, and $w \in \operatorname{int} K$ for the midpoint w of $[x,y]$.

Let $r > 0$ be maximal such that $B(w,r) \subset K$. In particular, $r \leqslant d(w,x) = d(w,y)$, and there exists $z \in \partial K \cap \partial B(w,r)$ such that $z \neq x, y$ as if $r = d(w,x)$, then $K = B(w,r)$ follows from $\operatorname{diam} K = 2r$. Since $B(w,r) \subset K$, the unique supporting hyperplane H_z at z is orthogonal to $[w,z]$.

Let H be the perpendicular bisector hyperplane of the segment $[x,z]$, and let H^+ be the corresponding closed half-space with $z \in H^+$, and hence $\xi_H x = z$. Since a supporting hyperplane H' at z to the convex body $\tau_{H^+} K$ should support both $H^+ \cap K$ and $H^+ \cap \xi_H K$, and x and z are regular boundary points of K, we deduce that $H_z = H' = \xi_H H_x$. Now, the lines of $[w,x]$ and $[w,z]$ are orthogonal to H_x and H_z, respectively; therefore, the lines of the segments $[w,x]$ and $[w,z]$ are mapped onto each other by ξ_H. We deduce that $\xi_H w = w$, thus $d(z,w) = d(x,w) = r$, and in turn $B(w,r) = K$. ∎

Now, we are ready to characterize equality in the Isodiametric inequality.

Theorem 4.B.3 (Isodiametric Inequality with equality). *Let $\mathcal{M}^n$ be either H^n or S^n, let $r > 0$ where $r < \frac{\pi}{4}$ if $\mathcal{M}^n = S^n$, and let $z_0 \in \mathcal{M}^n$. If $X \subset \mathcal{M}^n$ is bounded and measurable with $|X| \geqslant |B(z_0, r)|$ and $\operatorname{diam} X \leqslant 2r$, then $X \subset B(y, r)$ for some $y \in \mathcal{M}^n$.*

Proof. Since $\operatorname{cl} X$ has the same diameter as X, we may assume that X is compact, and hence Lemma 4.B.1 yields that X is a convex body.

Fix $\varrho > 0$ where in the spherical case also assume that $r + \varrho < \frac{\pi}{4}$, thus $2\varrho + \operatorname{diam} X < \frac{\pi}{2}$, and the Isoperimetric Theorem 4.5.1 and Lemma 4.A.6 (v) yield

$$\left|X^{(\varrho)}\right| \geqslant |B(z_0, r + \varrho)| \text{ and } \operatorname{diam} X^{(\varrho)} \leqslant 2r + 2\varrho.$$

For any half-space H^+, τ_{H^+} keeps the volume and does not increase the diameter of $X^{(\varrho)}$ according to Lemma 4.5.4, and hence combining the Isodiametric Theorem 4.5.1 and Lemma 4.B.1 yields that $\tau_{H^+} X^{(\varrho)}$ is a convex body. Since any boundary point of the convex body $X^{(\varrho)}$ is regular, Proposition 4.B.2 implies that $X^{(\varrho)} = B(y, r + \varrho)$ for some $y \in \mathcal{M}^n$, thus $X = B(y, r)$. ∎

4.C Supplement: Isoperimetric inequality for k-gons in H^2, S^2 or $\mathbb{R}^2$

Since one of the earliest the isoperimetric type problem is the case of polygons of given number of sides considered by Zenodorus (circa 200 BC - 140 BC), we briefly discuss it. Let $\mathcal{M}^2$ be either S^2, H^2 or $\mathbb{R}^2$ (see Section 4.A for related notions in the spherical and hyperbolic space). The optimality of regular polygons in $\mathbb{R}^2$ was only proved rigorously by Weierstrass at the end of the 19th century (see Weierstrass [714]), and the spherical and hyperbolic cases are due to Fejes Tóth [264], p. 214 and Bezdek [88].

Theorem 4.C.1 (Isoperimetric inequality for k-gons). *If $\mathcal{M}^2$ is either S^2, H^2 or $\mathbb{R}^2$, and $k \geqslant 3$, then among convex k-gons of given area in $\mathbb{R}^2$, the regular k-gons are the ones having minimal perimeter.*

Remark. The "discrete isoperimetric inequality" Theorem 4.C.1 extends to non-convex k-gons, as well. Let γ be a simple closed polygonal path in $\mathcal{M}^2$ where if $\mathcal{M}^2 = S^2$, then we also assume that $\mathcal{H}^1(\gamma) < 2\pi$ (and hence γ is contained in an open hemisphere). According to Jordan's theorem (cf. Hatcher [381]), $\mathcal{M}^2 \backslash \gamma$ has two open path connected components. If $\mathcal{M}^2 = \mathbb{R}^2, H^2$, then one of these components is bounded, which we denote by X. If $\mathcal{M}^2 = S^2$, then one of the two components lies in open hemisphere containing γ, which denote by X in this case. We call $Q = \operatorname{cl} X$ with $\partial Q = \gamma$ a possibly non-convex k-gon.

We observe that the convex hull of $\mathrm{conv}Q$ is a convex polygon with at most k vertices. If X (and hence Q) is not convex, then $P(\mathrm{conv}Q) < \mathcal{H}^1(\gamma)$ by the triangle inequality, and $|\mathrm{conv}Q| > |Q|$ by containment. Therefore, if Q_k is the regular k-gon with $|Q| = |Q_k|$, then $P(Q) \geqslant P(Q_k)$ by Theorem 4.C.1, with equality if and only if Q is congruent to Q_k.

We write $[x, y]$ to denote the geodesic segment between different $x, y \in \mathcal{M}^2$ where x are y are not antipodal if $\mathcal{M}^2 = S^2$, and $[x, y, z]$ to denote the triangle (the convex hull of x, y, z) with vertices $x, y, z \in \mathcal{M}^2$ if x, y, z are not contained in a line. The geodesic distance of $x, y \in \mathcal{M}^2$ is $d(x, y)$. In order to prove Theorem 4.C.1, we need the notion of equidistant curves in $\mathcal{M}^2$ where $\mathcal{M}^2$ is either S^2, H^2 or $\mathbb{R}^2$.

Remark 4.C.2 (Equidistant curves and area in $\mathcal{M}^2$). Let $\mathcal{M}^2$ be either S^2, H^2 or $\mathbb{R}^2$, let $\ell \subset \mathcal{M}^2$ be a line, and let $\varrho > 0$ where $\varrho < \frac{\pi}{2}$ if $\mathcal{M}^2 = S^2$. Then the set of points of $\mathcal{M}^2$ whose distance from ℓ is ϱ is the union of two congruent curves symmetric through ℓ, where one of this curves is called an *equidistant curve*, and is a

line if $\mathcal{M}^2 = \mathbb{R}^2$;

hypercycle if $\mathcal{M}^2 = H^2$;

circle, so-called Lexell-circle if $\mathcal{M}^2 = S^2$.

If $x_0, y_0, z_0 \in \mathcal{M}^2$ are not contained in a line, then let p and q be the midpoints of the segments $[x_0, z_0]$ and $[y_0, z_0]$, respectively. Let ℓ be the line containing $[p, q]$, and let α be the equidistant curve corresponding to ℓ and passing through z_0. Simple cut and past argument shows that if $z \in \alpha$, then the triangles $[x_0, y_0, z_0]$ and $[x_0, y_0, z]$ satisfy

$$\big|[x_0, y_0, z_0]\big| = \big|[x_0, y_0, z]\big|. \tag{4.85}$$

Next let s be the closest point of α to the midpoint of $[x_0, y_0]$, and hence $d(s, x_0) = d(s, y_0)$. We claim that if $z_0 \neq s$, and $z \neq z_0$ is contained in the arc of α between z_0 and s (the shorter arc of α between z_0 and s in the case of $\mathcal{M}^2 = S^2$), then

$$d(z, x_0) + d(z, y_0) < d(z_0, x_0) + d(z_0, y_0). \tag{4.86}$$

To verify (4.86), we may assume that $d(z_0, x_0) < d(z_0, y_0)$. Then for the reflected image y_1 of y_0 through the line of z_0 and z, we have $[z, x_0, y_1] \subset [z_0, x_0, y_1]$, and hence $S([z, x_0, y_1]) < S([z_0, x_0, y_1])$ yields that

$$d(z,x_0) + d(z,y_0) = d(z,x_0) + d(z,y_1) < d(z_0,x_0) + d(z_0,y_1) = d(z_0,x_0) + d(z_0,y_0).$$

Here we provide the elegant argument due to L. Fejes Tóth [264], p. 214 verifying Theorem 4.C.1 (originally presented in the spherical case). It was Bezdek [88] who adapted the argument to the hyperbolic case.

Proof of Theorem 4.C.1. We prove the reverse statement: For $\aleph > 0$ (where $\aleph < 2\pi$ if $\mathcal{M}^2 = S^2$), among convex k-gons of perimeter at most $\aleph$ in $\mathcal{M}^2$, the regular k-gons of perimeter $\aleph$ are the ones having maximal area. Since the diameter of a polygon is at most the perimeter, there exists a convex polygon Q_0 of perimeter $\aleph$ with at most k vertices and of maximal area.

First, we show that all sides of Q_0 are equal. Otherwise, there exist consecutive vertices v_1, v_2, v_3 of Q_0 such that $d(v_2, v_1) < d(v_2, v_3)$. We write ℓ to denote the line passing through the midpoints of $[v_2, v_1]$ and $[v_2, v_3]$, α to denote the equidistant curve corresponding to ℓ and passing through v_2 and s to denote the closest point of α to the midpoint of $[v_1, v_3]$, and hence $d(s, v_1) = d(s, v_3)$. We move v_2 into a position $\tilde{v}_2$ along α towards s in a way such that replacing v_2 by $\tilde{v}_2$ among the vertices of Q_0, we obtain a polygon $\widetilde{Q}$ of the same number of vertices as Q_0. We deduce from (4.85) and (4.86) that $P(\widetilde{Q}) < P(Q)$ and $|\widetilde{Q}| = |Q_0|$ for the polygon $\widetilde{Q}$ of at most k vertices. We can enlarge $\widetilde{Q}$ into a k-gon Q' satisfying $P(Q') < P(Q) = \aleph$ and $|Q'| > |Q_0|$, contradicting the maximality of $|Q_0|$, and proving that all sides of Q_0 are equal. This fact actually settles Theorem 4.C.1 if $k = 3$.

Similar indirect argument yields that Q_0 has exactly k vertices. We suppose that Q_0 has less than k vertices. Let v_0, v_2 and v_3 be consecutive vertices of Q_0, and let $v_1 \in [v_0, v_2]$, $v_1 \neq v_0, v_2$ satisfy that $d(v_2, v_1) < d(v_2, v_3)$. The argument as above shows that Q_0 can be deformed into a polygon $\widetilde{Q}$ such that $v_2 \notin \widetilde{Q}$, but v_1 and the new position $\tilde{v}_2$ of v_2 are vertices of $\widetilde{Q}$ besides all vertices of Q_0 but v_2. In addition, $P(\widetilde{Q}) < P(Q)$ and $|\widetilde{Q}| = |Q_0|$. Since $\widetilde{Q}$ still has at most k vertices, it can be enlarged in order to contradict the maximality of $|Q_0|$, and to prove that Q_0 is a k-gon.

Finally, we prove that if $k \geqslant 4$, then all angles of Q_0 are the same. Indirectly, we suppose that there exist consecutive vertices v_1, v_2, v_3, v_4 of Q_0 such that $\angle(v_1, v_2, v_3) < \angle(v_2, v_3, v_4)$. For the rest of the proof, we distinguish the cases when $\mathcal{M}^2 = \mathbb{R}^2, S^2$ or $\mathcal{M}^2 = H^2$.

Case 1. $\mathcal{M}^2 = \mathbb{R}^2, S^2$

Let $B(z, r)$, $z \in \mathcal{M}^2$ and $r > 0$, be the unique circular disc touching P_0 from the outside at a point w of the side $[v_2, v_3]$, and also the line of v_1, v_2 at w_2, and the line of v_4, v_3 at w_3. In particular, $[z, v_2]$ halves $\angle(w_2, v_2, v_3)$, and $[z, v_3]$ halves $\angle(w_3, v_3, v_2)$, and hence $\angle(v_1, v_2, v_3) < \angle(v_2, v_3, v_4)$ yields that

$$d(w, v_2) < d(w, v_3). \tag{4.87}$$

Since the two tangent segments to the circular disc $B(z, r)$ from either v_2 or from v_3 are equal, we deduce that

$$d(v_1, v_2) + d(v_2, v_3) + d(v_3, v_4) = d(v_1, w_2) + d(v_4, w_3). \tag{4.88}$$

Now we choose $v_2' \in [v_2, v_1] \backslash \{v_2\}$ close to v_2 and $v_3' \in [v_3, w_3] \backslash \{v_3\}$ close to v_3 in a way such that $[v_2', v_3']$ also touches $B(z, r)$, and the intersection point q of $[v_2, v_3]$ and

$[v_2', v_3']$ satisfies (cf. (4.87))

$$d(q, v_2) < d(q, v_3) \text{ and } d(q, v_2') < d(q, v_3'). \tag{4.89}$$

Analogously to (4.88), we have

$$d(v_1, v_2') + d(v_2', v_3') + d(v_3', v_4) = d(v_1, w_2) + d(v_4, w_3);$$

therefore, $P(Q_0) = P(Q')$ for the k-gon Q' obtained by replacing the vertices v_2, v_3 of Q_0 by v_2', v_3'. On the other hand, it follows from (4.89) and $\angle(v_2, q, v_2') = \angle(v_3, q, v_3')$ that for the triangles $[q, v_3, v_3']$ and $[q, v_2, v_2']$, we have

$$|Q'| - |Q_0| = \left|[q, v_3, v_3']\right| - \left|[q, v_2, v_2']\right| > 0,$$

contradicting the maximality of $|Q_0|$.

Case 2. $\mathcal{M}^2 = H^2$

In this case, the argument is similar, only it may happen that the circle $\partial B(z, r)$ is replaced by a horocycle or a hypercycle. Let $\tilde{w}_2$ and $\tilde{w}_3$ be the points such that v_2 is the midpoint of the segment $[v_1, \tilde{w}_2]$, and v_3 is the midpoint of the segment $[v_4, \tilde{w}_3]$. In addition, let h_2 and h_3 be the half-lines emanating from v_2 and v_3, respectively, and halving the "outer" angles $\angle(v_3, v_2, \tilde{w}_2)$ and $\angle(v_2, v_3, \tilde{w}_3)$. We observe that

$$\text{both } h_2 \text{ and } h_3 \text{ enclose an acute angle with } [v_2, v_3]. \tag{4.90}$$

If h_2 and h_3 intersect in a $z \in H^2$, then according to (4.90), the orthogonal projection of z onto the line of v_2, v_3 lands in $[v_2, v_3]$. For $r = d(z, w)$, the rest of the argument runs exactly as in the case of $\mathbb{R}^2$.

If h_2 and h_3 do not intersect, then (4.90) yields that there exist two possibilities:

(a) either h_2 and h_3 are parallel,

(b) or there exists a line ℓ that is orthogonal to both h_2 and h_3.

In either case, there exists a $w \in [v_2, v_3]$ such that the half-line $\tilde{h}_w$ emanating from w and orthogonal to $[v_2, v_3]$ is parallel to h_2, h_3 in the case of (a), and orthogonal to ℓ in the case of (b). For $i = 2, 3$, let w_i and $\tilde{h}_{w_i}$ be the reflected image of w and $\tilde{h}_w$ through h_i where $\tilde{h}_{w_i}$ is orthogonal to the line of v_1, v_2 if $i = 2$, and to the line of v_3, v_4 if $i = 3$. In particular, $\tilde{h}_{w_i}$ for $i = 2, 3$ is parallel to h_w in the case of (a), and orthogonal to ℓ in the case of (b). In the case of (a), let γ be the horocycle orthogonal to h_w at w, and in the case of (b), let γ be the hypercycle corresponding to ℓ and passing though w. In both case, γ touches P_0 at w from the outside, and touches the line of v_1, v_2 at w_2 and the line of v_4, v_3 at w_3. Therefore, the argument can be finished as in the case of $\mathbb{R}^2$ using γ instead of $\partial B(z, r)$. ∎

Since polygons can approximate a general convex body in $\mathcal{M}^2$ arbitrarily well (use the argument in (1.17)), the discrete isoperimetric inequality Theorem 4.C.1 yields the classical isoperimetric inequality for convex bodies in $\mathcal{M}^2$ (cf. Theorem 4.5.1).

4.D Supplement: Concentration of measure via Isoperimetric type inequalities

In this section, we discuss the "concentration of measure phenomenon", which holds in certain metric probability spaces Ξ; namely, if $\mathbb{P}(X) \geqslant \frac{1}{2}$ for measurable $X \subset \Xi$ implies that $\mathbb{P}\left(X^{(t)}\right)$ is "exponentially close to 1" for $t > 0$ where $X^{(t)}$ is the set of points of Ξ of distance at most t from X. This phenomenon is naturally related to the isoperimetric problem in Ξ (see the monograph Ledoux [497], or Chapter 1 of Artstein-Avidan, Giannopoulos, V. Milman [39]).

Let us specify what we mean by concentration of a probability measure. For a Borel probability measure μ on a metric space (Ξ, d), and $t > 0$, let

$$\alpha_\mu(t) = \alpha_{\mu,\Xi,d}(t) = \sup\{1 - \mu(A^{(t)}) : \mu(A) \geqslant \tfrac{1}{2} \text{ for a measurable } A \subset \Xi\},$$

and the phenomenon "concentration of measure" occurs when $\alpha_\mu(t)$ decreases very fast (at least exponentially fast) as t increases. As it turns out, concentration of the measure is intimately related to concentration of 1-Lipschitz functions where $f : \Xi \to \mathbb{R}$ is 1-*Lipschitz* when

$$|f(x) - f(y)| \leqslant d(x, y) \text{ for } x, y \in \Xi.$$

We note that for any non-empty $A \subset \Xi$, the triangle inequality yields that the function

$$x \mapsto d(x, A) \text{ is 1-Lipschitz for } x \in \Xi \tag{4.91}$$

where $d(x, A) = \inf\{d(x, z) : z \in A\}$. We recall that if μ is a probability measure on Ξ and $f \in L^1(\mu)$, then a median $m \in \mathbb{R}$ satisfies $\mu(\{f > m\}) \leqslant \frac{1}{2}$ and $\mu(\{f < m\}) \leqslant \frac{1}{2}$. In addition, we write $\mathbb{E}f = \int_\Xi f \, d\mu$, that is the expectation of f with respect to μ.

Proposition 4.D.1. *Let μ be a Borel probability measure on a metric space (Ξ, d).*

(i) *If $t > 0$ and m is the median of a 1-Lipschitz function $f : \Xi \to \mathbb{R}$, then*

$$\mu(\{|f - m| > t\}) \leqslant 2\alpha_\mu(t).$$

(ii) *If the function $\beta(t) > 0$ of $t > 0$ satisfies that $\mu(\{|f - m| > t\}) \leqslant \beta(t)$ for any 1-Lipschitz function $f : \Xi \to \mathbb{R}$ with median m, then $\alpha_\mu(t) \leqslant \beta(t)$.*

(iii) *If $\beta(t) > 0$ is a monotone decreasing continuous function of $t > 0$ such that $\mu(\{|f - \mathbb{E}f| > t\}) \leqslant \beta(t)$ for any 1-Lipschitz function $f : \Xi \to \mathbb{R}$, then*

$$\alpha_\mu(t) \leqslant 2\beta(\tfrac{t}{2}). \tag{4.92}$$

Proof. For (i), we have $\mu(A) \geqslant \frac{1}{2}$ for $A = \{f \leqslant m\}$. Thus f being 1-Lipschitz function implies that $\mu(\{f > m + t\}) \leqslant 1 - \mu\left(A^{(t)}\right) \leqslant \alpha_\mu(t)$, and we deduce similarly that $\mu(\{f < m - t\}) \leqslant \alpha_\mu(t)$.

For (ii), let $\mu(A) \geqslant \frac{1}{2}$ for a measurable $A \subset \Xi$. Since $f(x) = d(x, A)$ is 1-Lipschitz by (4.91), and $m = 0$ is the median, we have $\mu\left(\{|f - m| \leqslant t\}\right) = \mu(A^{(t)}) \geqslant 1 - \beta(t)$.

For (iii), let $\mu(A) \geqslant \frac{1}{2}$ for a measurable $A \subset \Xi$, let $f(x) = d(x, A)$, that is 1-Lipschitz by (4.91), and let $E = \mathbb{E}f \geqslant 0$. If $t > 2E$, then $\{|f - E| \leqslant \frac{t}{2}\} \subset A^{(t)}$, and hence $\mu(A^{(t)}) \geqslant 1 - \beta(\frac{t}{2})$, verifying (4.92) in this case. Therefore, we may assume that $E > 0$. Now, $\beta(E) \geqslant \frac{1}{2}$ follows from the continuity of β as if $s \in (0, E)$, then $A \subset \{|f - E| > s\}$. Therefore, (4.92) also holds if $t \leqslant 2E$ by the monotonicity of β and $\alpha_\mu(t) \leqslant 1$.

∎

Among the probability measures discussed in detail in this section, first we consider the uniform probability measure on the sphere S^n based on the Spherical Isoperimetric inequality.

Theorem 4.D.2. *Let $\mathbb{P}$ be the uniform probability measure on the sphere S^n; namely, $\mathbb{P}(X) = \mathcal{H}^n(X)/\mathcal{H}^n(S^n)$ for measurable $X \subset S^n$. If $t \in (0, \frac{\pi}{2})$, and $\mathbb{P}(A) \geqslant \frac{1}{2}$ for a measurable $A \subset S^n$, then*

$$\mathbb{P}\left(A^{(t)}\right) \geqslant 1 - \frac{1}{2} \cdot e^{-\frac{nt^2}{2}}.$$

Proof. Fixing a $v \in S^n \subset \mathbb{R}^{n+1}$, we may assume that A is the closed hemisphere $\{u \in S^n : \langle u, v\rangle \leqslant 0\}$ by the Spherical Isoperimetric inequality Theorem 4.5.1, and hence $A^{(t)} = \{u \in S^n : \langle u, v\rangle \leqslant \sin t\}$. It follows that $S^n \backslash A^{(t)}$ is contained in the solid hemisphere (half-ball) of $(\sin t)v + (\cos t)B^{n+1}$ bounded by the hyperplane $(\sin t)v + v^\perp$; therefore, the monotonicity of the surface area of convex bodies Lemma 2.3.4 implies

$$\mathcal{H}^n\left(S^n \backslash A^{(t)}\right) \leqslant \frac{1}{2}\,\mathcal{H}^n\Big((\cos t)S^n\Big) = \frac{(\cos t)^n}{2} \cdot \mathcal{H}^n(S^n).$$

Now $e^{\frac{t^2}{2}} \cos t \leqslant 1$ follows from $t \leqslant \tan t = \frac{\sin t}{\cos t}$ by differentiation, and, in turn, we conclude that $\mathcal{H}^n\left(S^n \backslash A^{(t)}\right) \leqslant \frac{1}{2}\, e^{-\frac{nt^2}{2}} \cdot \mathcal{H}^n(S^n)$. ∎

For the standard normal multivariate distribution $d\tilde{\gamma}_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\|x\|^2/2}\, dx$, the following estimate follows from the Gaussian Isoperimetric inequality.

Theorem 4.D.3. *If $t > 0$, and a measurable $A \subset \mathbb{R}^n$ satisfies that $\tilde{\gamma}_n(A) \geqslant \frac{1}{2}$, then*

$$\tilde{\gamma}_n\left(A^{(t)}\right) > 1 - \frac{1}{2} \cdot e^{-\frac{t^2}{2}}.$$

Proof. According to the Gaussian Isoperimetric inequality (4.25), we may assume that $A = H^+$ for a half-space $H^+ = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant 0\}$ where $u \in S^{n-1}$. In this

case, $A^{(t)} = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant t\}$, and hence using the substitution $s = r + t$ and $(r+t)^2 > r^2 + t^2$ for $r, t > 0$ at the end, we deduce that

$$1 - \tilde{\gamma}_n\left(A^{(t)}\right) = (2\pi)^{-\frac{n}{2}} \int_t^\infty \int_{u^\perp} e^{-\frac{\|y\|^2+s^2}{2}}\, dyds = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{s^2}{2}}\, ds$$
$$< \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{r^2+t^2}{2}}\, dr = \frac{1}{2} \cdot e^{-\frac{t^2}{2}},$$

proving Theorem 4.D.3. ∎

Next, the concentration of measure property for the Gaussian measure (cf. Theorem 4.D.3) yields the analogous property for the uniform probability measure on a cube.

Theorem 4.D.4. *If $t > 0$, and a measurable $A \subset [0,1]^n$ satisfies that $|A| \geqslant \frac{1}{2}$, then*

$$\left|A^{(t)}\right| > 1 - \frac{1}{2} \cdot e^{-\pi t^2}.$$

Proof. The main idea is that both the Lebesgue measure on $[0,1]^n$ and the standard normal multivariate distribution $\tilde{\gamma}_n$ on $\mathbb{R}^n$ are product measures; in particular, the cumulative distribution function $\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-\frac{t^2}{2}}\, dt$ satisfies that $\tilde{\gamma}_n\left(\prod_{i=1}^n [a_i, b_i]\right) = \prod_{i=1}^n \left(\Phi(b_i) - \Phi(a_i)\right)$ for $b_i > a_i$. It follows that

$$|\psi(X)| = \tilde{\gamma}_n(X) \tag{4.93}$$

holds for any measurable $X \subset \mathbb{R}^n$ where $\psi : \mathbb{R}^n \to [0,1]^n$ is the map defined by $\psi(x_1, \ldots, x_n) = (\Phi(x_1), \ldots, \Phi(x_n))$. On the orther hand, $e^{-\frac{t^2}{2}} \leqslant 1$ for any $t \in \mathbb{R}$ in the definition of Φ yields that $|\Phi(r) - \Phi(s)| \leqslant \frac{1}{\sqrt{2\pi}} |r - s|$ for any $r, s \in \mathbb{R}$, and hence if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then

$$\left\|\psi(x) - \psi(y)\right\| = \sqrt{\sum_{i=1}^n \left(\Phi(x_i) - \Phi(y_i)\right)^2} \leqslant \sqrt{\frac{1}{2\pi} \sum_{i=1}^n (x_i - y_i)^2} = \frac{\|x - y\|}{\sqrt{2\pi}}. \tag{4.94}$$

Now let $t > 0$, and let measurable $A \subset [0,1]^n$ satisfy that $|A| \geqslant \frac{1}{2}$. We deduce from (4.94) that $\psi\left((\psi^{-1}A)^{\sqrt{2\pi}\, t}\right) \subset A^{(t)}$, thus (4.93) and Theorem 4.D.3 imply that

$$|A^{(t)}| \geqslant \tilde{\gamma}_n\left((\psi^{-1}A)^{\sqrt{2\pi}\, t}\right) > 1 - \frac{1}{2} \cdot e^{-\pi t^2},$$

proving Theorem 4.D.4. ∎

In turn, combining Proposition 4.D.1 with Theorem 4.D.2, Theorem 4.D.3 and Theorem 4.D.4 yields the following.

Remark 4.D.5 (Concentration of 1-Lipschitz functions). Let $t > 0$.

(i) Using the uniform probability measure $\mathbb{P}$ on S^n, if m is the median of a 1-Lipschitz function $f : S^n \to \mathbb{R}$, then

$$\mathbb{P}\left(\{|f - m| > t\}\right) \leqslant e^{-\frac{nt^2}{2}} \tag{4.95}$$

(ii) If m is the median of a 1-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to the standard Gaussian density $\tilde{\gamma}_n$, then

$$\tilde{\gamma}_n\left(\{|f - m| > t\}\right) \leqslant e^{-\frac{t^2}{2}}. \tag{4.96}$$

(iii) If m is the median of a 1-Lipschitz function $f : [0,1]^n \to \mathbb{R}$ with respect to the Lebesgue measure, then

$$\left|\{|f - m| > t\}\right| \leqslant e^{-\pi t^2}. \tag{4.97}$$

Finally we discuss how a Poincaré inequality like (4.67) or (4.71) implies measure concentration, even if the error term is of order e^{-ct}, not e^{-ct^2}. We recall that if μ is an absolutely continuous probability measure on $\mathbb{R}^n$, and $f \in L^2(\mu)$ is locally Lipschitz, then

$$\mathrm{Var}_\mu(f) = \int_{\mathbb{R}^n} f^2 \, d\mu - \left(\int_{\mathbb{R}^n} f \, d\mu\right)^2.$$

We say that μ satisfies a Poincaré inequality with spectral gap $\lambda > 0$ (cf. Remark 4.7.11) if for any locally Lipschitz function $f \in L^2(\mu)$, we have

$$\lambda \cdot \mathrm{Var}_\mu(f) \leqslant \int_{\mathbb{R}^n} \|Df\|^2 \, d\mu. \tag{4.98}$$

Theorem 4.D.6 (Gromov, V. Milman [340]). *If μ is an absolutely continuous probability measure on $\mathbb{R}^n$ satisfying the Poincaré inequality (4.98) with spectral gap $\lambda > 0$, and $f : \mathbb{R}^n \to \mathbb{R}$ is 1-Lipschitz, then*

$$\mu\left(\{|f - \mathbb{E}f| > t\}\right) \leqslant 4e^{-\sqrt{\lambda}\cdot t}. \tag{4.99}$$

Proof. First we may assume that f is bounded by replacing f by the function $f_M(x) = f(x)$ if $|f(x)| \leqslant M$, $f_M(x) = M$ if $f(x) \geqslant M$, and $f_M(x) = -M$ if $f(x) \leqslant -M$ for arbitrary large $M > 0$. Secondly, we may assume that $\mathbb{E}f = \int_{\mathbb{R}^n} f \, d\mu = 0$.

For $\theta > 0$, we consider $g_\theta = e^{\frac{\theta f}{2}}$, and hence the Poincaré inequality (4.98) yields that

$$\int_{\mathbb{R}^n} g_\theta^2 \, d\mu - \left(\int_{\mathbb{R}^n} g_\theta \, d\mu\right)^2 \leqslant \lambda^{-1} \int_{\mathbb{R}^n} \|Dg_\theta\|^2 \, d\mu. \tag{4.100}$$

Here the 1-Lipschitz condition on f implies that

$$\|Dg_\theta\| = \left\| \frac{\theta}{2} \cdot e^{\frac{\theta f}{2}} \cdot Df \right\| \leqslant \frac{\theta}{2} \cdot e^{\frac{\theta f}{2}},$$

thus we deduce from (4.100) that

$$\int_{\mathbb{R}^n} e^{\theta f}\, d\mu - \left(\int_{\mathbb{R}^n} e^{\frac{\theta f}{2}}\, d\mu \right)^2 \leqslant \frac{\theta^2}{4\lambda} \int_{\mathbb{R}^n} e^{\theta f}\, d\mu.$$

In particular, $S(\theta) = \int_{\mathbb{R}^n} e^{\theta f}\, d\mu$ satisfies that $S(\theta) \leqslant \left(1 - \frac{\theta^2}{4\lambda}\right)^{-1} S\left(\frac{\theta}{2}\right)^2$. Iterating the last formula starting from $\theta = \sqrt{\lambda}$ yields that

$$S(\sqrt{\lambda}) \leqslant S\left(\frac{\sqrt{\lambda}}{2^m}\right)^{2^m} \cdot \prod_{i=1}^{m} (1 - 4^{-i})^{-1}. \tag{4.101}$$

As there exists $M > 0$ such that $|f(x)| \leqslant M$ for $x \in \mathbb{R}^n$ and $e^t \leqslant 1 + t + t^2$ if $|t| \leqslant 1$, we deduce that if m is large, then for any $x \in \mathbb{R}^n$, we have

$$1 + \frac{\sqrt{\lambda}}{2^m} \cdot f(x) \leqslant \exp\left(\frac{\sqrt{\lambda}}{2^m} \cdot f(x)\right) \leqslant 1 + \frac{\sqrt{\lambda}}{2^m} \cdot f(x) + \frac{M^2 \lambda}{2^{2m}}.$$

It follows from $\int_{\mathbb{R}^n} f\, d\mu = 0$ that $1 \leqslant S\left(\frac{\sqrt{\lambda}}{2^m}\right) \leqslant 1 + \frac{M^2\lambda}{2^{2m}} < \exp\left(\frac{M^2\lambda}{2^{2m}}\right)$, and hence $\lim_{m\to\infty} S\left(\frac{\sqrt{\lambda}}{2^m}\right)^{2^m} = 1$. On the other hand, $(1-t)^{-1} \leqslant e^{t+t^2}$ for $t \in (0, \frac{1}{2})$, thus

$$\prod_{i=1}^{m} (1 - 4^{-i})^{-1} \leqslant \exp\left(\sum_{i=1}^{\infty} \left(\frac{1}{4^i} + \frac{1}{16^i} \right) \right) < 2;$$

therefore, letting m tend to infinity in (4.101) implies that $\int_{\mathbb{R}^n} e^{\sqrt{\lambda} f}\, d\mu = S(\sqrt{\lambda}) < 2$. It follows that

$$\mu(\{f > t\}) = \mu\left(\left\{ e^{\sqrt{\lambda} f} > e^{\sqrt{\lambda} t} \right\} \right) \leqslant e^{-\sqrt{\lambda} t} \int_{\mathbb{R}^n} e^{\sqrt{\lambda} f}\, d\mu < 2e^{-\sqrt{\lambda} t}.$$

Applying the same argument to $-f$ shows that $\mu(\{f < -t\}) < 2e^{-\sqrt{\lambda} t}$, completing the proof of Theorem 4.D.6. ∎

Combining Proposition 4.D.1 and (4.99) implies concentration of measure in the case a Poincaré inequality holds for a probability measure.

Corollary 4.D.7. *If μ is an absolutely continuous probability measure on $\mathbb{R}^n$ satisfying the Poincaré inequality (4.98) with spectral gap $\lambda > 0$, then*

$$\alpha_\mu(t) \leqslant 8e^{-\frac{\sqrt{\lambda}}{2} \cdot t}. \tag{4.102}$$

Therefore the Poincaré inequality (4.71) for isotropic log-concave measures yields the following estimate about the concentration of the measure.

Theorem 4.D.8. *If μ is an absolutely continuous isotropic log-concave probability measure on $\mathbb{R}^n$, then*

$$\alpha_\mu(t) \leqslant 8e^{-\frac{c}{\sqrt{\log n}}\cdot t}. \tag{4.103}$$

where $c \in (0,1)$ is an absolute constant.

Chapter 5

The Isoperimetric Inequality for sets of Finite Perimeter in $\mathbb{R}^n$ and the Sobolev inequality for BV functions

As we shall see in this chapter, the natural setup for the Isoperimetric Inequality in $\mathbb{R}^n$ and its stability is in the framework of sets of finite perimeter (that includes bounded sets of Lipschitz boundary). In addition, functions of bounded variations (BV functions) - that are the functional analogues of sets of finite perimeter - form the natural family for the Sobolev inequality, and in this family, unlike in the case of C_c^1 functions, the Sobolev inequality (see Theorem 4.2.1) does have extremizers; namely, multiples of characteristic functions of balls.

Given the technicalities in the subject, this chapter is mostly a survey. For an in-depth study of sets of Finite Perimeter and BV functions, see, for example, Ambrosio, Fusco, Pallara [25] and Maggi [557].

To motivate this section, let us describe an idea how to prove the Isoperimetric Inequality for the surface area $P(E)$ of a convex body $E \subset \mathbb{R}^n$ with C_+^2 boundary (see Section 8.6.1 for the detailed argument). We may assume that $|E| = |B^n|$, so that Caffarelli's Theorem 8.6.8 from [171] yields the existence of a C^1 diffeomorphism (Brenier map) $T : E \to B^n$ such that $D(T)(x)$ is a positive definite symmetric matrix and $\det DT(x) = 1$ for $x \in \operatorname{int} E$. It follows from the AM-GM inequality for the eigenvalues of DT that $\operatorname{div} T(x) \geqslant n(\det DT(x))^{\frac{1}{n}} = n$, with equality if and only if $DT(x) = I_n$. Thus, we deduce from the Divergence Theorem 2.A.1 and $\|T\| \leqslant 1$ that

$$n|B^n|^{\frac{1}{n}}|E|^{\frac{n-1}{n}} = \int_E n\,dx \leqslant \int_E \operatorname{div} T(x)dx = \int_{\partial E} \langle T(x), \nu_E(x)\rangle\, d\mathcal{H}^{n-1}(x) \leqslant P(E).$$

Also, if equality holds, then $\operatorname{div} T(x) = n$ for $x \in \operatorname{int} E$; therefore, $DT \equiv I_n$ and we conclude that $T(x) = x + z$ for a $z \in \mathbb{R}^n$, hence, $E = T^{-1}(B^n) = B^n - z$.

Now, this simple argument does not extend even to the case of convex bodies because typically the Brenier map does not extend smoothly up to the boundary. However, one can prove that T is a function of bounded variation (BV function), and this allows one to use a generalized version of the Divergence Theorem in order to repeat the proof above (see Section 5.2.1). Since the notions of BV functions and sets of finite perimeter are essentially equivalent, we develop the two theories in parallel.

5.1 Sets of Finite Perimeter and BV functions

In order to motivate the definition of sets of finite perimeter, we present an example of an open set of finite Lebesgue measure in $\mathbb{R}^n$ whose boundary has infinite Lebesgue measure.

Example 5.1.1. For a dense sequence $\{(x_k\}_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$, we consider $E = \bigcup_{k\in\mathbb{N}}(x_k + 2^{-k}\operatorname{int} B^n)$. Then

$$|E| \leqslant \sum_{k\in\mathbb{N}} |(x_k + 2^{-k}B^n)| = |B^n| \sum_{k\in\mathbb{N}} 2^{-nk} < \infty.$$

However, E is dense in $\mathbb{R}^n$, so $|\partial E| = |(\operatorname{cl} E)\backslash E| = \infty$.

For $A \subset \mathbb{R}^n$ open, we write $C^1_c(A;\mathbb{R}^n)$ to denote the space of C^1 vector fields $T : \mathbb{R}^n \to \mathbb{R}^n$ with compact support whose support is contained in A. We recall that $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure, see Section 1.D.

Idea to define perimeter: If $E \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} E$, then the Divergence Theorem 2.1.4 (known also as Gauss-Green theorem) yields that

$$\int_E \operatorname{div}(T)\, d\mathcal{H}^n = \int_{\partial E} \langle T, \nu_{\partial E}\rangle d\mathcal{H}^{n-1} \leqslant \mathcal{H}^{n-1}(\partial E)$$

for any vector field $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ such that $\|T\| \leqslant 1$. On the other hand, if T_0 is the vector field $T_0(sz) = \psi(s)\,\nu_E(z)$ for $z \in \partial E$ and $s \geqslant 0$ where $\psi : [0,\infty) \to [0,1]$ is C^1_c and satisfies $\psi(1) = 1$ and $\psi(0) = 0$, then $T_0 \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, $\|T_0\| \leqslant 1$, and

$$\int_E \operatorname{div}(T_0)\, d\mathcal{H}^n = \int_{\partial E} \langle T_0, \nu_{\partial E}\rangle\, d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial E);$$

therefore,

$$\mathcal{H}^{n-1}(\partial E) = \sup\left\{ \int_E \operatorname{div}(T)\, d\mathcal{H}^n \mid T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \|T\| \leqslant 1 \right\}.$$

Definition 5.1.2. A measurable set $E \subseteq \mathbb{R}^n$ is a set of *finite perimeter* if

$$P(E) := \sup\left\{ \int_E \operatorname{div}(T)\, d\mathcal{H}^n \mid T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \|T\| \leqslant 1 \right\} < \infty.$$

A measurable set $E \subseteq \mathbb{R}^n$ is *of locally finite perimeter* if, for every bounded open set $A \subset \mathbb{R}^n$, it holds

$$P(E;A) := \sup\left\{ \int_E \operatorname{div}(T)\, d\mathcal{H}^n \mid T \in C^1_c(A;\mathbb{R}^n),\ \|T\| \leqslant 1 \right\} < \infty.$$

Proposition 5.1.3. *A set $E \subset \mathbb{R}^n$ has locally finite perimeter if and only if there exists a $\mathbb{R}^n$-valued Radon measure $\mu_E = \nu_E \, d|\mu_E|$ on $\mathbb{R}^n$ such that $\|\nu_E\| = 1$ holds $|\mu_E|$ a.e. in $\mathbb{R}^n$ and*

$$\int_E \operatorname{div}(T) = \int_{\mathbb{R}^n} \langle T, d\mu_E \rangle = \int_E \langle T, \nu_E \rangle \, d|\mu_E|$$

for any $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$. Moreover, the set $E \subset \mathbb{R}^n$ has finite perimeter if and only if $|\mu_E|(\mathbb{R}^n) < \infty$.

Proof. Assume that E has locally finite perimeter, and consider the linear functional

$$L : C_c^1(\mathbb{R}^n; \mathbb{R}^n) \longrightarrow \mathbb{R}, \qquad T \longmapsto \int_E \operatorname{div}(T).$$

Given an open set A with compact closure, if $T \in C_c^1(A; \mathbb{R}^n)$ then

$$|L(T)| = \left| \int_E \operatorname{div}(T) \right| \leqslant \|T\|_\infty \left| \int_E \operatorname{div} \left(\frac{T}{\|T\|_\infty} \right) \right| \leqslant \|T\|_\infty P(E; A).$$

As $|L(T)| \leqslant \|T\|_\infty P(E; A)$ for any $T \in C_c^1(A; \mathbb{R}^n)$, it follows that for every open A with compact closure,

$$L : (C_c^1(A; \mathbb{R}^n), \| \cdot \|_\infty) \longrightarrow \mathbb{R}$$

is a bounded, densely defined, linear functional, thus it extends uniquely to a bounded functional on $C_c(A; \mathbb{R}^n)$ with respect to $\| \cdot \|_\infty$. Now the Riesz Representation Theorem 10.1.6 gives the existence of an $\mathbb{R}^n$ valued Radon measure $\mu_E = \nu_E d|\mu_E|$ such that $L(T) = \int_{\mathbb{R}^n} \langle T, \nu_E \rangle d|\mu_E|$ for every $T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$, $\nu_E : \mathbb{R}^n \to \mathbb{R}^n$ is $|\mu_E|$ measurable, and $\|\nu_E\| = 1$ $|\mu_E|$ a.e. in $\mathbb{R}^n$.

For the other direction, if $\mu_E = \nu_E d|\mu_E|$ exists, then given $T \in C_c^1(A; \mathbb{R}^n)$ with $\|T\| \leqslant 1$ we have

$$\left| \int_E \operatorname{div}(T) \right| = \left| \int_{\mathbb{R}^n} \langle T, \nu_E \rangle d|\mu_E| \right| = \left| \int_A \langle T, \nu_E \rangle d|\mu_E| \right| \leqslant |\mu_E|(A)$$

so $P(E; A) \leqslant |\mu_E|(A) < \infty$. ∎

Example 5.1.4. According to Definition 5.1.2, the union of finitely many sets (locally) of finite perimeter is (locally) of finite perimeter.

Example 5.1.5. If $E \subset \mathbb{R}^n$ is compact and has Lipschitz boundary (cf. Definition 4.1.4), then E is a set of finite perimeter according to the Divergence Theorem 2.8.1. In addition, $P(E) = \mathcal{H}^{n-1}(\partial E)$, $|\mu_E| = \mathcal{H}^{n-1} \llcorner \partial E$, and $\nu_E(x)$ is the unit exterior normal to ∂E at $\mathcal{H}^{n-1}$ a.e. $x \in \partial E$ again by the Divergence Theorem 2.8.1. If $A \subset \mathbb{R}^n$ is open bounded, then $P(E; A) = \mathcal{H}^{n-1}(A \cap \partial E)$.

Actually, if E is the finite union of compact sets with Lipschitz boundary, then E is still a set of finite perimeter by Example 5.1.4, but it is possible that $P(E) =$

$\mathcal{H}^{n-1}(\partial^* E) < \mathcal{H}^{n-1}(\partial E)$, for example, if $E = E_1 \cup E_2$ for compact sets $E_1, E_2 \subset \mathbb{R}^2$ with Lipschitz boundary that meet at a Cantor set in a segment with positive $\mathcal{H}^1$ measure (see Section 5.1.1, in particular, (5.6)).

Example 5.1.6. If $E \subset \mathbb{R}^n$ satisfies that $\mathcal{H}^{n-1}(K \cap \partial E) < \infty$ for any compact $K \subset \mathbb{R}^n$, then E is a set of locally finite perimeter. In particular, if $\mathcal{H}^{n-1}(\partial E) < \infty$, then E is a set of finite perimeter.

To show this, it is sufficient to verify the latter statement; therefore, we assume that $\mathcal{H}^{n-1}(\partial E) < \infty$ for some $E \subset \mathbb{R}^n$. According to 2.10.6 in Federer [260], a certain integral geometric measure $\mathcal{P}_1^{n-1}(\partial E)$ is also finite as $\mathcal{P}_1^{n-1}(\partial E) \leqslant C_n \mathcal{H}^{n-1}(\partial E)$ for some $C_n > 0$ depending on n, and hence 4.5.12 in Federer [260] says that E is a set of finite perimeter.

The following facts about sets of finite perimeter are contained in Ambrosio, Fusco, Pallara [25] and Maggi [557].

Remark 5.1.7. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter.

- $P(\lambda E) = \lambda^{n-1} P(E)$ for every $\lambda > 0$.
- $P(x + \Phi E) = P(E)$ for every $\Phi \in O(n)$ and $x \in \mathbb{R}^n$.
- $\mathbb{R}^n \backslash E$ is also a set of finite perimeter, $P(\mathbb{R}^n \backslash E) = P(E)$ and $\mu_{\mathbb{R}^n \backslash E} = -\mu_E$.
- $\operatorname{supp} \mu_E = \{x \in \mathbb{R}^n : 0 < |E \cap (x + rB^n)| < |rB^n| \text{ for any } r > 0\} \subset \partial E$.
- $P(E; A) = |\mu_E|(A)$ and $P(E) = |\mu_E|(\mathbb{R}^n)$ for any bounded open $A \subset \mathbb{R}^n$.
- For any measurable $F \subset \mathbb{R}^n$ with $|E \Delta F| = 0$ (here Δ stands for the symmetric difference), F is a set of finite perimeter with $\mu_E = \mu_F$, and hence $P(E) = P(F)$.
- There exists a Borel set $F \subset \mathbb{R}^n$ with $|E \Delta F| = 0$ such that $\operatorname{supp} \mu_E = \operatorname{supp} \mu_F = \partial F$.
- There exist compact $E_m \subset \mathbb{R}^n$ such that ∂E_m is a C^∞ manifold, $m \geqslant 0$,

$$\lim_{m \to \infty} P(E_m) = P(E) \quad \text{and} \quad \lim_{m \to \infty} |E \Delta E_m| = 0, \tag{5.1}$$

  and μ_{E_m} tends weakly* to μ_E.
- $\liminf_{\varrho \to 0^+} \frac{|E + \varrho B^n| - |E|}{\varrho} \geqslant P(E)$ holds in general for the outer Minkowski content, but if E is compact, $\operatorname{int} E \neq \varnothing$, and ∂E has Lipschitz boundary, then $P(E) = \mathcal{H}^{n-1}(\partial E) = \lim_{\varrho \to 0^+} \frac{|E + \varrho B^n| - |E|}{\varrho}$ (see also Ambrosio, Colesanti, Villa [23]).

As mentioned above, for compact subsets of $\mathbb{R}^n$ with Lipschitz boundary, the perimeter is the outer Minkowski content. Also, the latter satisfies the Isoperimetric inequality Theorem 4.1.6 by the Brunn-Minkowski inequality. As a consequence, approximating a general set of finite perimeter by smooth compact sets, thanks to (5.1) we deduce that the Isoperimetric inequality holds for any $E \subset \mathbb{R}^n$ of finite perimeter,

namely,

$$P(E) \geqslant n\omega_n^{1/n}|E|^{\frac{n-1}{n}}. \tag{5.2}$$

One of the main goals of this Chapter 5 is to characterize the case of equality in (5.2) (cf. Theorem 5.2.1). Before that, let us discuss some properties of sequences of sets finite perimeter based on Ambrosio, Fusco, Pallara [25] and Maggi [557].

Remark 5.1.8 (Sequences of sets of finite perimeter).

- The perimeter is lower semi-continuous; namely, if $E_m \subset \mathbb{R}^n$ has finite perimeter, $m \geqslant 0$, $\{P(E_m)\}$ is bounded, and there exists a measurable $F \subset \mathbb{R}^n$ such that $\lim_{m\to\infty} |F\Delta E_m| = 0$, then F has finite perimeter, μ_{E_m} tends weakly* to μ_F, and
$$\liminf_{m\to\infty} P(E_m) \geqslant P(F).$$
- "Local compactness of the family of sets of finite perimeter": If, for some $R > 0$, $E_m \subset RB^n$ has finite perimeter with $\sup_m\{P(E_m)\} < \infty$, then there exists a set $F \subset RB^n$ of finite perimeter and a subsequence $\{E_{m'}\} \subset \{E_m\}$ such that $\lim_{m'\to\infty} |F\Delta E_{m'}| = 0$ and $\mu_{E_{m'}}$ tends weakly* to μ_F.
More generally, if $E_m \subset \mathbb{R}^n$ has locally finite perimeter for $m \geqslant 0$, and $\sup_m\{P(E_m; RB^n)\} < \infty$ for any $R > 0$, then there exists a set $F \subset \mathbb{R}^n$ of locally finite perimeter and a subsequence $\{E_{m'}\} \subset \{E_m\}$ such that $\lim_{m'\to\infty} |(F\Delta E_{m'}) \cap B_R| = 0$ for any $R > 0$, and $\mu_{E_{m'}}$ tends weakly* to μ_F.

5.1.1 Reduced boundary and rectifiability of sets of locally finite perimeter

If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter, then it follows from the Lebesgue-Besicovitch theorem on differentiation of measures that for $|\mu_E|$-a.e. x, it holds

$$\lim_{r\to 0^+} \frac{\mu_E(x + rB^n)}{|\mu_E|(rB^n)} = \nu_E(x) \quad \text{and} \quad |\nu_E(x)| = 1. \tag{5.3}$$

Definition 5.1.9. Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. The set of points x such that (5.3) holds is called the *reduced boundary of* E and denoted by $\partial^* E$. Also, at points of the reduced boundary, ν_E is the *measure theoretic outer unit normal to* E.

One can prove that $\partial^* E \subset \partial E$. The following fundamental structure theorem shows that the reduced boundary is the "correct" measure notion of boundary.

Theorem 5.1.10 (De Giorgi Rectifiability Theorem, I). *If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter, then*

$$|\mu_E| = \mathcal{H}^{n-1}\llcorner_{\partial^* E}, \qquad \text{and} \qquad \mu_E = \nu_E\, d\mathcal{H}^{n-1}\llcorner_{\partial^* E}. \tag{5.4}$$

In particular

$$\int_E \operatorname{div} T = \int_{\partial^* E} \langle T, \nu_E\rangle d\mathcal{H}^{n-1} \tag{5.5}$$

for every $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$. *Moreover,* $\partial^* E = N \cup (\bigcup K_j)$ *where* $\mathcal{H}^{n-1}(N) = 0$, *the sets* K_j *are compact, and* $K_j \subseteq M_j$ *where* M_j *is a* C^1 *manifold of dimension* $n-1$ *and for every* $x \in K_j$, $\nu_E(x)$ *is normal to* $T_x M_j$.

Remarks. Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter.

- If $x \in \partial^* E$, then (see also (5.16))

$$\lim_{r\to 0^+} \frac{|E \cap (x + rB^n)|}{|rB^n|} = \frac{1}{2}. \tag{5.6}$$

- If $\varphi : \mathbb{R}^n \to [0,\infty)$ is Borel measurable, then

$$\int_{\mathbb{R}^n} \varphi \, d|\mu_E| = \int_{\partial^* E} \varphi \, d\mathcal{H}^{n-1}. \tag{5.7}$$

  In particular, if $E \subset \mathbb{R}^n$ is a set of finite perimeter, then

$$|\mu_E|(\mathbb{R}^n) = P(E) = \mathcal{H}^{n-1}(\partial^* E). \tag{5.8}$$

The importance of the reduced boundary is clarified by the following result (cf. [25, Theorem 3.59]). Here we use the L^1_{loc} convergence of sets, defined by setting $E_h \to E$ if, for every compact set C, we have $|C \cap (E_h \Delta E)| \to 0$.

Theorem 5.1.11 (De Giorgi Rectifiability Theorem, II). *If* $E \subset \mathbb{R}^n$ *is a set of locally finite perimeter and* $x \in \partial^* E$, *then*

$$\frac{(E-x)}{r} \to \{y \in \mathbb{R}^n : \langle \nu_E(x), y - x\rangle < 0\} \tag{5.9}$$

as $r \to 0^+$.

Formula (5.5) shows the validity of the divergence theorem on sets of finite perimeter for every vector field $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$. We shall need a refinement of this result (cf. Theorem 5.1.20), as discussed below. The first step now is to introduce the space of functions with bounded variation.

5.1.2 Functions of bounded variation (BV functions)

Writing $\int_E \operatorname{div}(T)\,dx = \int_{\mathbb{R}^n} \mathbf{1}_E \operatorname{div}(T)\,dx$ in (5.1.2) and replacing the indicator function of E by an arbitrary function f, we are naturally lead to the definition of BV functions.

Definition 5.1.12. A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is *BV* if

$$\sup\left\{ \int_{\mathbb{R}^n} f \operatorname{div}(T) d\mathcal{H}^n \mid T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \|T\| \leqslant 1 \right\} < \infty.$$

A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is *locally BV* if, for every bounded open set A, it holds

$$\sup\left\{\int_{\mathbb{R}^n} f\,\mathrm{div}(T)d\mathcal{H}^n \mid T \in C_c^1(A;\mathbb{R}^n),\ \|T\| \leqslant 1\right\} < \infty.$$

As for sets of finite perimeter, BV functions enjoy a series of structure theorems. First of all, being a (locally) BV function is equivalent to asking that the distributional derivative of f is a (locally) finite Radon measure.

Proposition 5.1.13. *A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is* locally BV *if and only if there exist n Radon measures $\{\mu_{i,f}\}_{1\leqslant i\leqslant n}$, such that*

$$\int_{\mathbb{R}^n} f\,\partial_i\varphi\,d\mathcal{H}^n = -\int \varphi\,d\mu_{i,f} \qquad \forall\,\varphi \in C_c^1(\mathbb{R}^n).$$

These measures are usually denoted by $\{D_i f\}_{1\leqslant i\leqslant n}$, and one considers the vector valued measure $Df = (D_1 f, \ldots, D_n f)$, so that the following identity holds:

$$\int_{\mathbb{R}^n} f\,\mathrm{div}(T)\,d\mathcal{H}^n = -\int_{\mathbb{R}^n}\langle T, d(Df)\rangle = -\sum_{i=1}^n \int_{\mathbb{R}^n} T^i\,d(D_i f)$$

for all $T = (T^1, \ldots, T^n) \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$.

Remark. If E is a set of locally finite perimeter, then $\mu_E = -D\mathbf{1}_E$.

Since Df is a (vector-valued) measure, we can define its total variation as

$$|Df|(E) := \sup\left\{\sum_{i\in\mathbb{N}}\left|\int_{E_i} d(Df)\right| : E_i \cap E_j = \varnothing,\ \bigcup_{i\in\mathbb{N}} E_i \subset E\right\}$$

for all Borel sets $E \subset \mathbb{R}^n$. BV functions and sets of finite perimeter are intrinsically related by the coarea formula [25, Theorem 3.40]: if $A \subset \mathbb{R}^n$ is open and $f \in BV(\mathbb{R}^n)$, then

$$\int_A |Df| = \int_{\mathbb{R}} P(\{f > t\}; A)dt = \int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial^*\{f > t\} \cap A)\,dt. \tag{5.10}$$

This is just a particular case of the coarea formula, which we state in the BV case below (see for example Grasmair [334]):

Theorem 5.1.14 (Coarea Formula). *If $f \in BV(\mathbb{R}^n)$ and $\psi : \mathbb{R}^n \to [0,\infty]$ is a Borel function, then*

$$\int_{\mathbb{R}^n} \psi\,d|Df| = \int_{\mathbb{R}}\int_{\partial^*\{f>t\}} \psi\,d\mathcal{H}^{n-1}\,dt. \tag{5.11}$$

Definition 5.1.15. The Sobolev space $W^{1,1}(\mathbb{R}^n)$ is the set of functions $f \in BV(\mathbb{R}^n)$ such that $|Df|$ is absolutely continuous with respect to the Lebesgue measure.

Example 5.1.16 (BV functions on an interval). For a function $f : [a,b] \to \mathbb{R}$, f is a BV function if and only if its total variation in the classical sense is finite; namely,

$$|Df|([a,b]) = \sup\left\{\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| : a = x_0 < x_1 < \ldots < x_k = b\right\} < \infty.$$

Note that if f is Lipschitz, then $|Df|([a,b]) = \int_a^b |f'|$ (actually, this also works for any $f \in W^{1,1}(\mathbb{R})$).

5.1.3 Anisotropic perimeter

All previous concepts can be generalized to the setting of anisotropic perimeter. More precisely, given a convex body $K \subset \mathbb{R}^n$ with $o \in \mathrm{int}K$, h_K its support function, and $\|\cdot\|_K$ its norm (see Section 1.9), we define the anisotropic perimeter as

$$P_K(E) := \sup\left\{\int_E \mathrm{div}(T)\, d\mathcal{H}^n \mid T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \|T\|_K \leqslant 1\right\},$$

and, for every open set A,

$$P_K(E;A) := \sup\left\{\int_E \mathrm{div}(T)\, d\mathcal{H}^n \mid T \in C_c^1(A;\mathbb{R}^n),\ \|T\|_K \leqslant 1\right\}.$$

Note that, since $c_1\|\cdot\| \leqslant \|\cdot\|_K \leqslant c_2\|\cdot\|$ for $c_2 \geqslant c_1 > 0$, the notion of sets of finite perimeter is independent of the choice of the norm.

Given a (vector-valued) measure μ, we can define its total variation with respect to the dual norm $\|\cdot\|_K$ as

$$\|-Df\|_{K^*}(E) := \sup\left\{\sum_{i\in\mathbb{N}} \left\|\int_{E_i} d(Df)\right\|_{K^*} : E_i \cap E_j = \varnothing,\ \bigcup_{i\in\mathbb{N}} E_i \subset E\right\}.$$

Then, (5.10) and (5.11) generalize as follows.

Theorem 5.1.17 (Anisotropic Coarea Formula). *If $f \in BV(\mathbb{R}^n)$, $\psi : \mathbb{R}^n \to [0,\infty]$ is a Borel function, $K \subset \mathbb{R}^n$ is a convex body with $o \in \mathrm{int}\,K$ and $A \subset \mathbb{R}^n$ open, then*

$$\int_A d\|-Df\|_{K^*} = \int_{\mathbb{R}} P_K(\{f > t\};A)\,dt, \tag{5.12}$$

$$\int_{\mathbb{R}^n} \psi\, d\|-Df\|_{K^*} = \int_{\mathbb{R}} \int_{\partial^*\{f>t\}} \psi\, h_K(\nu_{\{f>t\}})\, d\mathcal{H}^{n-1}\, dt. \tag{5.13}$$

We deduce from (5.12) and (5.13) applied to $\psi \equiv 1$ and $f = \mathbf{1}_E$ for a set of finite perimeter $E \subset \mathbb{R}^n$ the following simple representation of P_K:

Corollary 5.1.18. *If $E \subset \mathbb{R}^n$ is a set of finite perimeter and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$P_K(E) = \int_{\partial^* E} \|\nu_E\|_{K^*}\, d\mathcal{H}^{n-1} = \int_{\partial^* E} h_K(\nu_E)\, d\mathcal{H}^{n-1}. \tag{5.14}$$

5.1.4 A divergence theorem for BV vector fields on sets of finite perimeter

Our goal here it to generalize (5.5) to the case of vector fields $X \in BV(\mathbb{R}^n;\mathbb{R}^n)$.

For a Borel set $E \subset \mathbb{R}^n$ and $\ell \in [0,1]$, we denote by $E^{(\ell)}$ the set of points x of $\mathbb{R}^n$ having density ℓ with respect to E, i.e., $x \in E^{(\ell)}$ if

$$\lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \ell\,.$$

We use the notation $\partial_{1/2}E$ for $E^{(1/2)}$. A theorem by Federer [25, Theorem 3.61] relates the reduced boundary $\partial^* E$ to the set of points of density $1/2$, ensuring that, if E is a set of finite perimeter then these sets are $\mathcal{H}^{n-1}$-equivalent; namely,

$$\mathcal{H}^{n-1}(\mathbb{R}^n \backslash (E^{(1)} \cup E^{(0)} \cup \partial^* E)) = 0\,, \tag{5.15}$$

$$\mathcal{H}^{n-1}(\partial^* E \Delta \partial_{1/2} E) = 0\,. \tag{5.16}$$

Let now E and F be sets of finite perimeter. By [25, Proposition 3.38, Example 3.68, Example 3.97], $E \cap F$ is a set of finite perimeter and, if we let

$$J_{E,F} = \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = \nu_F(x)\}\,, \tag{5.17}$$

then, up to $\mathcal{H}^{n-1}$-null sets,

$$\partial^*(E \cap F) = J_{E,F} \cup [\partial^* E \cap F^{(1)}] \cup [\partial^* F \cap E^{(1)}]\,, \tag{5.18}$$

Moreover, at $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E \cap F$ we find

$$\nu_{E\cap F}(x) = \begin{cases} \nu_E(x)\,, & \text{if } x \in \partial^* E \cap F^{(1)}\,, \\ \nu_F(x)\,, & \text{if } x \in \partial^* F \cap E^{(1)}\,, \\ \nu_E(x) = \nu_F(x)\,, & \text{if } x \in J_{E,F}\,. \end{cases} \tag{5.19}$$

In the particular case that $F \subseteq E$, (5.18) and (5.19) reduce to

$$\partial^* F = [\partial^* F \cap \partial^* E] \cup [\partial^* F \cap E^{(1)}]\,, \tag{5.20}$$

$$\nu_F(x) = \nu_E(x)\,, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^* F \cap \partial^* E, \tag{5.21}$$

where (5.20) is valid up to $\mathcal{H}^{n-1}$-null sets. We shall also use the following lemma concerning the union of two sets of finite perimeter:

Lemma 5.1.19. *Let E and F be sets of finite perimeter with $|E \cap F| = 0$. Then*

$$\nu_{E\cup F}\, d\mathcal{H}^{n-1}\lfloor \partial^*(E\cup F) = \nu_E\, d\mathcal{H}^{n-1}\lfloor(\partial^*E\backslash\partial^*F) + \nu_F\, d\mathcal{H}^{n-1}\lfloor(\partial^*F\backslash\partial^*E)\,, \tag{5.22}$$

and $\nu_E(x) = -\nu_F(x)$ at $\mathcal{H}^{n-1}$-a.e. $x \in \partial^*E \cap \partial^*F$.

Proof. As $|E\cap F| = 0$, we have $1_{E\cup F} = 1_E + 1_F$ a.e., and in turn (5.4) yields that

$$\begin{aligned} \nu_{E\cup F}\, d\mathcal{H}^{n-1}\lfloor\partial^*(E\cup F) &= D1_{E\cup F} = D1_E + D1_F \\ &= \nu_E\, d\mathcal{H}^{n-1}\lfloor\partial^*E + \nu_F\, d\mathcal{H}^{n-1}\lfloor\partial^*F\,. \end{aligned} \tag{5.23}$$

Since $\partial_{1/2}E \cap \partial_{1/2}F \subseteq (E\cup F)^{(1)}$, we have $\mathcal{H}^{n-1}(\partial^*E \cup F) \cap \partial^*(E\cap\partial^*F) = 0$ by (5.16). In particular, (5.22) follows from (5.23). Moreover,

$$0 = \int_C (\nu_E + \nu_F)\, d\mathcal{H}^{n-1}\,, \quad \text{for every Borel set } C \subseteq \partial^*E\cap\partial^*F\,,$$

i.e. $\nu_E = -\nu_F$ at $\mathcal{H}^{n-1}$-a.e. point in $\partial^*E\cap\partial^*F$. ∎

Let us endow the space of $n\times n$ tensors $\mathbb{R}^{n\times n}$ with the metric $|L| = \sqrt{\operatorname{trace}(L^tL)}$. In particular, if $T \in BV_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ and DT is its $\mathbb{R}^{n\times n}$-valued distributional derivative (which is a measure), then we denote by $|DT|(C)$ the total variation of DT on the Borel set C defined with respect to this metric.

Since DT is a measure, we can decompose it into its absolutely continuous and singular parts with respect to the Lebesgue measure. More precisely, we denote by ∇T the density of DT with respect to Lebesgue measure, and by D_sT the corresponding singular part, so that $DT = \nabla T\, d\mathcal{H}^n + D_sT$. If $\operatorname{Div} T$ is the distributional divergence of T and if DT takes values in the set of $n\times n$-tensors that are symmetric and positive definite, then $\operatorname{Div} T$ is a non-negative Radon measure on $\mathbb{R}^n$, which is bounded above and below by the total variation of T: for every Borel set C in $\mathbb{R}^n$,

$$\frac{1}{\sqrt{n}}\int_C d(\operatorname{Div} T) \leqslant \int_C d|DT| \leqslant \int_C d(\operatorname{Div} T), \tag{5.24}$$

as a consequence of the inequality $n^{-1/2}\sum_{i=1}^n \ell_i \leqslant (\sum_{i=1}^n \ell_i^2)^{1/2} \leqslant \sum_{i=1}^n \ell_i$ whenever $\ell_i \geqslant 0$. Moreover, if we set $\operatorname{div} T(x) = \operatorname{trace}(\nabla T(x))$, then

$$\operatorname{Div} T = \operatorname{div} T\, dx + (\operatorname{Div} T)_s\,, \quad (\operatorname{Div} T)_s = \operatorname{trace}(D_sT) \geqslant |D_sT|\,, \tag{5.25}$$

As a consequence of (5.25), $\operatorname{Div} T - \operatorname{div} T\, d\mathcal{H}^n$ is a non-negative Radon measure.

Whenever $T \in BV(\mathbb{R}^n;\mathbb{R}^n)$ and E is a set of finite perimeter, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^*E$ there exists a vector $\operatorname{tr}_E(T)(x) \in \mathbb{R}^n$ such that

$$\lim_{r\to 0}\frac{1}{r^n}\int_{B_r(x)\cap\{y:(y-x)\cdot\nu_E(x)<0\}} |T(y) - \operatorname{tr}_E(T)(x)|dy = 0\,, \tag{5.26}$$

called the *inner trace of T on E*, see [25, Theorem 3.77]. Note that, as a byproduct of (5.9) we have in fact

$$\lim_{r\to 0}\frac{1}{r^n}\int_{B_r(x)\cap E}|T(y)-\mathrm{tr}_E(T)(x)|dy=0\,. \tag{5.27}$$

Moreover, as a consequence of [25, Example 3.97] (applied to the pair of functions T and 1_E) the Divergence Theorem holds true in the following form:

Theorem 5.1.20 (Divergence Theorem for BV vector fields). *If $T\in BV(\mathbb{R}^n;\mathbb{R}^n)$ and E is a set of finite perimeter, then*

$$\int_{E^{(1)}} d(\mathrm{Div}\,T)=\int_{\partial^* E}\langle \mathrm{tr}_E(T),\nu_E\rangle\, d\mathcal{H}^{n-1}. \tag{5.28}$$

5.2 Characterization of isoperimetric sets

5.2.1 The Anisotropic Isoperimetric inequality

In this section, we wish to characterize isoperimetric sets for the anisotropic isoperimetric inequality. This argument is contained in Figalli, Maggi, Pratelli [278] in a more general setting. The main ideas are presented in Section 8.6.1 in the technically much simpler case of convex bodies with C^2_+ boundary.

Theorem 5.2.1 (Anisotropic Isoperimetric Inequality with Equality). *Let E be a set of finite perimeter with $0<|E|<\infty$. Then $P_K(E)\geqslant n|K|^{1/n}|E|^{(n-1)/n}$, with equality if and only if $|E\Delta(x_0+rK)|=0$ for some $x_0\in\mathbb{R}^n$ and $r>0$.*

After various attempts, a satisfactory stability version of the Anisotropic Isoperimetric Theorem 5.2.1 was provided by Figalli, Maggi, Pratelli [278]. We recall that, for measurable sets $X,Y\subset\mathbb{R}^n$ with $|X|,|Y|>0$, if $\alpha=|X|^{\frac{-1}{n}}$ and $\beta=|Y|^{\frac{-1}{n}}$, then

$$A(X,Y)=\min\{|\alpha X\Delta(z+\beta Y)|:\ z\in\mathbb{R}^n\}\,.$$

Theorem 5.2.2 (Figalli, Maggi, Pratelli). *For $\theta_n=2^{-16}n^{-17}$, if $E\subset\mathbb{R}^n$ has finite perimeter and $|E|>0$, and $K\subset\mathbb{R}^n$ is a convex body with $o\in\mathrm{int}K$, then*

$$P_K(E)\geqslant n|K|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}\left[1+\theta_n\cdot A(K,E)^2\right]. \tag{5.29}$$

Remark. Here the exponent 2 of $A(K,E)^2$ is optimal, and θ_n can't be larger than $36n^{-2}$ (see Remark 8.6.6).

To prove Theorem 5.2.1, we shall need some preliminary results.

A set of finite perimeter E is said to be *indecomposable* if for every $F \subseteq E$ having finite perimeter and such that

$$\mathcal{H}^{n-1}(\partial^* E) = \mathcal{H}^{n-1}(\partial^* F) + \mathcal{H}^{n-1}(\partial^*(E\backslash F)), \tag{5.30}$$

we have that $\min\{|F|, |E\backslash F|\} = 0$. We shall need the following lemma.

Lemma 5.2.3. *Let E be an indecomposable set and let $f \in BV(\mathbb{R}^n)$. If $\int_{E^{(1)}} d|Df| = 0$, then there exists $c \in \mathbb{R}$ such that $f(x) = c$ for a.e. $x \in E$.*

Proof. Let $F_t = E \cap \{f > t\}$. As E is indecomposable, it suffices to show that (5.30) holds with $F = F_t$ for a.e. $t \in \mathbb{R}$. In fact it is enough to prove that

$$\mathcal{H}^{n-1}(\partial^* E) \geqslant \mathcal{H}^{n-1}(\partial^* F_t) + \mathcal{H}^{n-1}(\partial^*(E\backslash F_t)), \tag{5.31}$$

for a.e. $t \in \mathbb{R}$, as the converse inequality follows from the subadditivity of the distributional perimeter [25, Proposition 3.38 (d)]. To this end, we start by noticing that

$$\{f \leqslant t\}^{(1)} = \{f > t\}^{(0)}, \quad \partial_{1/2}\{f > t\} = \partial_{1/2}\{f \leqslant t\}, \tag{5.32}$$

$$\mathcal{H}^{n-1}(\partial^*\{f > t\}\Delta\partial^*\{f \leqslant t\}) = 0, \quad \mathcal{H}^{n-1}(J_{E,\{f>t\}} \cap J_{E,\{f\leqslant t\}}) = 0, \tag{5.33}$$

where (5.32) is trivially checked, and where (5.33) follows from (5.32), Lemma 5.1.19 and (5.16). We now come to the proof of (5.31).

Recall the notation $J_{E,F}$ introduced in (5.17). By (5.18), as $E\backslash F_t = E \cap \{f \leqslant t\}$, we have that, up to $\mathcal{H}^{n-1}$-null sets,

$$\partial^* F_t = J_{E,\{f>t\}} \cup [E^{(1)} \cap \partial^*\{f > t\}] \cup [\partial^* E \cap \{f > t\}^{(1)}], \tag{5.34}$$

$$\partial^*(E\backslash F_t) = J_{E,\{f\leqslant t\}} \cup [E^{(1)} \cap \partial^*\{f \leqslant t\}] \cup [\partial^* E \cap \{f \leqslant t\}^{(1)}]. \tag{5.35}$$

Hence, by the Coarea Formula (5.11) we get

$$0 = \int_{E^{(1)}} d|Df| = \int_{\mathbb{R}} \mathcal{H}^{n-1}(E^{(1)} \cap \partial^*\{f > t\})dt,$$

i.e. $\mathcal{H}^{n-1}(E^{(1)} \cap \partial^*\{f > t\}) = 0$ for a.e. $t \in \mathbb{R}$. By (5.33), we also have $\mathcal{H}^{n-1}(E^{(1)} \cap \partial^*\{f \leqslant t\}) = 0$ for a.e. $t \in \mathbb{R}$. Therefore, thanks to (5.32) we eventually deduce from (5.34) and (5.35), that

$$\begin{aligned}\mathcal{H}^{n-1}(\partial^* F_t) + \mathcal{H}^{n-1}(\partial^*(E\backslash F_t)) &= \mathcal{H}^{n-1}(J_{E,\{f>t\}}) + \mathcal{H}^{n-1}(J_{E,\{f\leqslant t\}}) \\ &\quad + \mathcal{H}^{n-1}(\partial^* E \cap [\{f > t\}^{(1)} \cup \{f > t\}^{(0)}]).\end{aligned} \tag{5.36}$$

Since (5.33) implies $\mathcal{H}^{n-1}(J_{E,\{f>t\}}) + \mathcal{H}^{n-1}(J_{E,\{f\leqslant t\}}) \leqslant \mathcal{H}^{n-1}(\partial^* E \cap \partial^*\{f > t\})$, (5.31) follows from (5.15) and (5.36). ∎

Before proving Theorem 5.2.1, we first give a rigorous proof of the anisotropic isoperimetric inequality itself for sets of finite perimeter using the Brenier map.

Theorem 5.2.4. *Whenever $|E| < \infty$, we have*

$$P_K(E) \geqslant n|K|^{1/n}|E|^{1/n'}.$$

Proof. By scaling, we can assume that $|E| = |K|$. The Brenier-McCann Theorem [159, 565] ensures the existence of a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that, if we set $T = \nabla\varphi$, then $T(x)$ belongs to K for a.e. $x \in \mathbb{R}^n$ and $T_\#(1_E(x)dx) = 1_K(y)dy$, i.e.

$$\int_K h(y)dy = \int_E h(T(x))dx\,, \tag{5.37}$$

for every Borel function $h : \mathbb{R}^n \to [0,\infty]$. As T is the gradient of convex function, its distributional derivative DT takes values in the set of symmetric and non-negative definite $n \times n$-tensors. Therefore $T \in BV(\mathbb{R}^n; K)$ (see e.g. [3, Proposition 5.1]) and (5.24) and (5.25) hold. Moreover, as a consequence of (5.37), $\det \nabla T(x) = 1$ for a.e. $x \in E$, see [566]. Since $\nabla T(x)$ is a positive semi-definite symmetric tensor for a.e. $x \in \mathbb{R}^n$, we can define measurable functions $\ell_k : \mathbb{R}^n \to [0,\infty)$ and $e_k : \mathbb{R}^n \to S^{n-1}$, $k = 1, ..., n$, such that

$$0 < \ell_k \leqslant \ell_{k+1}\,, \quad e_i \cdot e_j = \delta_{i,j}\,, \quad \nabla T = \sum_{k=1}^n \ell_k e_k \otimes e_k\,.$$

Then the arithmetic-geometric mean inequality implies that, for a.e. $x \in E$,

$$n = n(\det \nabla T(x))^{1/n} = n\left(\prod_{k=1}^n \ell_k(x)\right)^{1/n} \leqslant \sum_{k=1}^n \ell_k(x) = \mathrm{div} T(x)\,. \tag{5.38}$$

By (5.38), (5.25), and (5.28), we have

$$\begin{aligned} n|K|^{1/n}|E|^{1/n'} = n|E| &= \int_E n(\det \nabla T(x))^{1/n} dx \\ &\leqslant \int_E \mathrm{div}\, T(x)\, dx = \int_{E^{(1)}} \mathrm{div}\, T(x)\, dx \qquad (5.39) \\ &\leqslant \int_{E^{(1)}} d(\mathrm{Div}\, T) = \int_{\partial^* E} \langle \mathrm{tr}_E(T), \nu_E\rangle\, d\mathcal{H}^{n-1}. \qquad (5.40) \end{aligned}$$

Since T takes values in K, it follows from (5.26) that $\|\mathrm{tr}_E(T)(x)\|_K \leqslant 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$, and hence

$$\begin{aligned} n|K|^{1/n}|E|^{1/n'} &\leqslant \int_{\partial^* E} \|\mathrm{tr}_E(T)\|_K \|\nu_E\|_{K^*} d\mathcal{H}^{n-1} \\ &\leqslant \int_{\partial^* E} \|\nu_E\|_{K^*} d\mathcal{H}^{n-1} = P_K(E), \end{aligned} \tag{5.41}$$

which proves the anisotropic isoperimetric inequality. ∎

Proof of Theorem 5.2.1. Assume that $P(E) = n|K|^{1/n}|E|^{(n-1)/n}$. We first claim that E is indecomposable.

Indeed, let F be a set of finite perimeter contained in E and such that (5.30) holds true. The usual considerations based on (5.15), (5.16) and (5.18) serves to show that

$$\mathcal{H}^{n-1}(\partial^* F) = \mathcal{H}^{n-1}(\partial^* F \cap E^{(1)}) + \mathcal{H}^{n-1}(\partial^* E \cap \partial^* F)\,,$$
$$\mathcal{H}^{n-1}(\partial^*(E\backslash F)) = \mathcal{H}^{n-1}(\partial^* F \cap E^{(1)}) + \mathcal{H}^{n-1}(\partial^* E\backslash\partial^* F)\,.$$

Thus, by (5.30), we deduce $\mathcal{H}^{n-1}(\partial^* F \cap E^{(1)}) = 0$. In particular

$$\begin{aligned} P_K(F) + P_K(E\backslash F) &= \int_{\partial^* F\cap\partial^* E} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\partial^* F\cap E^{(1)}} \|\nu_F\|_* d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial^* E\backslash\partial^* F} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\partial^* F\cap E^{(1)}} \|-\nu_F\|_* d\mathcal{H}^{n-1} \\ &= \int_{\partial^* F\cap\partial^* E} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\partial^* E\backslash\partial^* F} \|\nu_E\|_* d\mathcal{H}^{n-1} = P_K(E). \end{aligned}$$

Hence, by the anisotropic isoperimetric inequality

$$\begin{aligned} P_K(E) &= P_K(F) + P_K(E\backslash F) \geqslant n|K|^{1/n}(|F|^{(n-1)/n} + |E\backslash F|^{(n-1)/n}) \\ &\geqslant n|K|^{1/n}(|F| + |E\backslash F|)^{(n-1)/n} = n|K|^{1/n}|E|^{(n-1)/n} = P_K(E)\,, \end{aligned}$$

and so, by strict concavity of the function $t \mapsto t^{(n-1)/n}$, $\min\{|F|, |E\backslash F|\} = 0$. Thus, E is indecomposable.

We now assume without loss of generality that $|E| = |K|$, and repeat the proof of Theorem 5.2.4. As $P(E) = n|K|^{1/n}|E|^{1/n'}$, we deduce in particular that the Brenier map T between E and K satisfies

$$0 = \int_E \left\{ \frac{\operatorname{div} T(x)}{n} - (\det \nabla T(x))^{1/n} \right\} dx + \int_{E^{(1)}} \frac{d(\operatorname{Div} T)_s}{n}\,.$$

In particular, since the last term vanishes, we deduce that $T \in W^{1,1}(\mathbb{R}^n; K)$. As $\det \nabla T(x) = 1$ a.e. on E, the equality $\frac{\operatorname{div} T(x)}{n} = (\det \nabla T(x))^{1/n}$ implies that $\nabla T(x) = I_n$ at a.e. $x \in E$, or equivalently $\nabla S = 0$ a.e. in E, with $S = T - x \in W^{1,1}(\mathbb{R}^n; \mathbb{R}^n)$. Thus, applying Lemma 5.2.3 to each component of the vector field S we deduce the existence of a vector $x_0 \in \mathbb{R}^n$ such that $T(x) = x - x_0$ for a.e. $x \in E^{(1)}$. As $T(x) \in K$ for a.e. $x \in E$, we deduce that $E^{(1)}$ is a subset of $x_0 + K$, and since $|K| = |E| = |E^{(1)}|$ we conclude the proof. ∎

5.2.2 Wulff theorem on minimizing surface energy

For a lower semicontimuous and bounded $\varrho : S^{n-1} \to (0, \infty)$ and $E \subset \mathbb{R}^n$ of finite perimeter, the associated surface energy of E is

$$\mathcal{E}_\varrho(\partial E) = \int_{\partial^* E} \varrho(\nu_E)\, d\mathcal{H}^{n-1}.$$

Extending the results of Section 4.4, we characterize sets minimizing the surface energy $\mathcal{E}_\varrho(\partial E)$ among sets of finite perimeter of given volume. The corresponding Wulff shape is

$$W_\varrho = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant \varrho(u)\ \forall u \in S^{n-1}\}. \tag{5.42}$$

Since $\inf \varrho > 0$ as ϱ is lower semi-continuous, W_ϱ is a convex body with $o \in \operatorname{int} W_\varrho$. The connection of the problem to crystallography is discussed in Section 4.4 and in the Comments to Chapter 4.

Theorem 5.2.5 (Wulff's inequality on minimizing surface energy). *Let $\varrho : S^{n-1} \to (0, \infty)$ be bounded and lower semi-continuous. If $E \subset \mathbb{R}^n$ is a set of finite perimeter with $0 < |E| < \infty$ and $|E| = |rW_\varrho|$ for $r > 0$, then*

$$\mathcal{E}_\varrho(\partial E) \geqslant \mathcal{E}_\varrho\left(\partial(rW_\varrho)\right) = n\,|W_\varrho|^{\frac{1}{n}} \cdot |E|^{\frac{n-1}{n}}. \tag{5.43}$$

Equality holds in (5.43) if and only if $|E\Delta(z + rW_\varrho)| = 0$ for some $z \in \mathbb{R}^n$.

In order to prove Theorem 5.2.5, we recall two facts. First, we deduce from the Anisotropic Isoperimetric inequality Theorem 5.2.1 and (5.14) that if K is a convex body with $o \in \operatorname{int} K$ and $E \subset \mathbb{R}^n$ is a set of finite perimeter with $0 < |E| < \infty$, then

$$\int_{\partial^* E} h_K(\nu_E)\, d\mathcal{H}^{n-1} \geqslant n|K|^{1/n}|E|^{(n-1)/n}, \tag{5.44}$$

with equality if and only if $|E\Delta(z + rK)| = 0$ for some $z \in \mathbb{R}^n$ and $r > 0$. Secondly, Lemma 4.4.3 says that if $\varrho : S^{n-1} \to \mathbb{R}_{>0}$ is bounded and lower semi-continuous, then

$$\int_{\partial W_\varrho} \varrho(\nu_{W_\varrho})\, d\mathcal{H}^{n-1} = \int_{\partial W_\varrho} h_{W_\varrho}(\nu_{W_\varrho})\, d\mathcal{H}^{n-1}. \tag{5.45}$$

Proof of Theorem 5.2.5. For $W = W_\varrho$, we have $\varrho \geqslant h_W$ by the definition of W_ϱ. Therefore, the Anisotropic Isoperimetric Inequality (5.44) and (5.45) yield

$$\begin{aligned}\mathcal{E}_\varrho(\partial E) &= \int_{\partial^* E} \varrho \circ \nu_E\, d\mathcal{H}^{n-1} \geqslant \int_{\partial^* E} h_W \circ \nu_E\, d\mathcal{H}^{n-1} \\ &\geqslant \int_{\partial(rW)} h_W \circ \nu_{rW}\, d\mathcal{H}^{n-1} = \int_{\partial(rW)} \varrho \circ \nu_{rW}\, d\mathcal{H}^{n-1} = \mathcal{E}_\varrho\left(\partial(rW)\right)\end{aligned}$$

where $\int_{\partial(rW)} h_W \circ \nu_{rW}\, d\mathcal{H}^{n-1} = n\,|W_\varrho|^{\frac{1}{n}} \cdot |E|^{\frac{n-1}{n}}$.

Equality in (5.43) yields equality in the Anisotropic Isoperimetric Inequality; therefore, $|E\Delta(z + rW_\varrho)| = 0$ for some $z \in \mathbb{R}^n$. ∎

The stability version Theorem 5.2.2 of the Anisotropic Isoperimetric Inequality due to by Figalli, Maggi, Pratelli [278] and the argument above yields the following stability version of Wulff's theorem:

Theorem 5.2.6 (Figalli, Maggi, Pratelli). *Let $\theta_n = 2^{-16}n^{-17}$, and $\varrho : S^{n-1} \to (0,\infty)$ be bounded and lower semi-continuous. If $E \subset \mathbb{R}^n$ has finite perimeter and $|E| > 0$, then*

$$\mathcal{E}_\varrho(\partial E) \geqslant n|W_\varrho|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}\left[1 + \theta_n \cdot A(W_\varrho, E)^2\right]. \tag{5.46}$$

Remark. Figalli, Zhang [281] proved an even stronger stability estimate in the crystalline case when W_ϱ is a polytope.

5.3 The anisotropic Sobolev inequality for BV functions

The isoperimetric inequality has a correspondent for *BV* functions: define the anisotropic total variation of a BV function f as

$$TV_K(f) := \int_{\mathbb{R}^n} \|-Df\|_{K^*}.$$

As a consequence of the Anisotropic Isoperimetric inequality Theorem 5.2.1 and of the Coarea formula (5.12), one can prove the following anisotropic Sobolev inequality:

Theorem 5.3.1 (Anisotropic Sobolev inequality for BV functions). *If $f \in L^1_{loc}(\mathbb{R}^n)$ vanishes at infinity; namely,*

$$|\{x \in \mathbb{R}^n : |f(x)| > t\}| < \infty\,, \quad \forall t > 0\,, \tag{5.47}$$

and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then

$$TV_K(f) \geqslant n|K|^{1/n}\|f\|_{n'}, \qquad n' = \frac{n}{n-1}, \tag{5.48}$$

with equality if and only if $f = a\mathbf{1}_{x+rK}$ a.e. on $\mathbb{R}^n$ for some $a, r > 0$ and $x \in \mathbb{R}^n$.

We show here a proof of this inequality, with a characterization of the equality case. One of the advantages of allowing functions with bounded variation is that some functions do satisfy the equality condition unlike in the case of the classical Sobolev inequality (cf. Theorem 4.2.1 and Theorem 4.3.5). We first need the following elementary result:

Lemma 5.3.2. *If $\alpha > 1$ and $F : [0,\infty) \to [0,\infty)$ is a decreasing function with $F(t) > 0$ for some $t > 0$, then*

$$\alpha \int_0^\infty t^{\alpha-1} F(t)^\alpha \, dt \leqslant \left(\int_0^\infty F(t)\, dt \right)^\alpha . \tag{5.49}$$

Equality holds if and only if there exists $c, T > 0$ such that $F(t) = c\mathbf{1}_{[0,T]}(t)$ for $t \neq T$.

Proof. Since F is decreasing, $tF(t) \leqslant \int_0^t F(s)\, ds$, which in turn yields that

$$\alpha \big(tF(t)\big)^{\alpha-1} F(t) \leqslant \alpha \left(\int_0^t F(s)\, ds \right)^{\alpha-1} F(t) = \frac{d}{dt} \left(\int_0^t F(s)\, ds \right)^\alpha .$$

Integrating the inequality above over $(0,\infty)$, yields the inequality (5.49).

Now, if equality holds in (5.49), then for a.e. $t \in (0,\infty)$, we have

$$\text{either} \quad tF(t) = \int_0^t F(s)\, ds \quad \text{or} \quad F(t) = 0.$$

Since F is decreasing, we deduce that $F(t) = c\mathbf{1}_{[0,T]}(t)$ for some $c, T \geqslant 0$ whenever $t \neq T$. ∎

We can now prove the Anisotropic Sobolev inequality (5.48), including the characterization of the equality case.

Proof of Theorem 5.3.1. We can assume that f is not identically zero.

Combining the coarea formula (5.12) (with $A = \mathbb{R}^n$) with Theorem 5.2.1 (applied to $\{|f| > t\}$ for a.e. $t \in (0,\infty)$) and Lemma 5.3.2 (with $F(t) = |\{|f| > t\}|^{1/n'}$ and $\alpha = n'$), we have

$$\begin{aligned} TV_K(f) &= \int_{-\infty}^\infty P_K(\{f > t\})\, dt = \int_0^\infty P_K(\{|f| > t\})\, dt \geqslant \int_0^\infty n|K|^{\frac{1}{n}} |\{|f| > t\}|^{\frac{1}{n'}}\, dt \\ &\geqslant n|K|^{\frac{1}{n}} \left(n' \int_0^\infty t^{n'-1} |\{|f| > t\}|\, dt \right)^{\frac{1}{n'}} = n|K|^{\frac{1}{n}} \|f\|_{n'}, \end{aligned}$$

where the last equality follow from $\{|f| > t\} = \{|f|^{n'} > t^{n'}\}$, the substitution $s = t^{n'}$, and the Layer Cake formula (cf. Lemma 4.2.3). This proves (5.48).

Now, if equality holds then Theorem 5.2.1 implies that, for a.e. t, up to sets of measure zero the sets $\{f > t\}$ are of the form $x_t + r_t K$ for some $x_t \in \mathbb{R}^n$ and $r_t > 0$. Secondly, Lemma 5.3.2 implies that the function $F(t) = |\{f > t\}|^{1/n'}$ is equal to $c1_{[0,T]}$ for some constants $c, T > 0$, which implies the existence of a radius $r > 0$ such that $r_t = r$ for $t \in [0,T]$ and $r_t = 0$ for $t > T$. Hence, up to sets of measure zero,

$$\{f > t\} = x_t + rK \quad \text{for } t \in [0,T], \qquad \{f > t\} = \varnothing \quad \text{for } t > T.$$

However, since the sets $\{f > t\}$ are contained one inside the other, the only possibility is that x_t is a constant x, thus $f = T\mathbf{1}_{x+rK}$ a.e. on $\mathbb{R}^n$. ∎

The stability version of the Sobolev inequality Theorem 5.3.1 of optimal order for functions of bounded variation is due to Figalli, Maggi, Pratelli [279].

We note that for $1 < p < n$ and $q > 1$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$, the (isotropic) L_p Sobolev inequality says that if $f \in W^{1,p}(\mathbb{R}^n)$; namely, $f \in L^1_{loc}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \|Df\|^p < \infty$, then

$$\|f\|_q \leqslant C_{n,p} \|Df\|_p \tag{5.50}$$

where the optimal $C_{n,p} > 0$ and the extremizers were determined by Talenti [695] and Aubin [45] (see D. Cordero-Erausquin, B. Nazaret, C. Villani [217] for a simplified argument using optimal transport), and the stability version where the error term has optimal order is due to Bianchi, Egnell [92] if $p = 2$, and to Figalli, Zhang [280] if $1 < p < n$.

5.4 Sets of finite weighted perimeter

In this section, we discuss the weighted version of the finite perimeter introduced in Section 5.1 mostly based on Ambrosio, Di Marino, Gigli [24] and Camfield [177]. We recall that if $A \subset \mathbb{R}^n$ is open, then $C^1_c(A;\mathbb{R}^n)$ denotes the space of C^1 vector fields $T : \mathbb{R}^n \to \mathbb{R}^n$ of compact support whose support is contained in A.

Definition 5.4.1 (Sets of finite weighted perimeter). Let $\Omega \subset \mathbb{R}^n$ be open, and let $\omega : \Omega \to (0,\infty)$ be continuous. The *weighted perimeter* of a measurable set $E \subset \Omega$ is

$$\mathrm{Per}_\omega(E) := \sup\left\{ \int_E \mathrm{div}(T)dx \mid T \in C^1_c(\Omega;\mathbb{R}^n),\ \|T\| \leqslant \omega \right\},$$

and E is a set of *finite weighted perimeter* if $\mathrm{Per}_\omega(E) < \infty$.

Remark 5.4.2. According to Camfield [177, Theorem 3.4], a measurable $E \subset \Omega$ satisfies $\mathrm{Per}_\omega(E) < \infty$ if and only if E is a set of locally finite perimeter in the "classical sense" as in Section 5.1, and $\int_{\Omega\cap\partial^* E} \omega\, d\mathcal{H}^{n-1} < 0$ where $\partial^* E \subset \partial E$ is the reduced boundary (cf. Section 5.1). In this case, combining (5.7) and Camfield [177], Theorem 5.3 applied to $f = \mathbf{1}_E$ leads to

$$\mathrm{Per}_\omega(E) = \int_{\Omega\cap\partial^* E} \omega\, d\mathcal{H}^{n-1}. \tag{5.51}$$

We will need the following technical statement that is a consequence of (5.51). If $E \subset \mathbb{R}^n$ is measurable such that $E \cap \Omega \neq \varnothing$ and $\mathrm{Per}_\omega(E \cap \Omega) < \infty$, then

$$\mathrm{Per}_\omega(E \cap \Omega) \leqslant \int_{\Omega\cap\partial E} \omega\, d\mathcal{H}^{n-1}. \tag{5.52}$$

Ambrosio, Di Marino, Gigli [24] compare the weighted perimeter to the lower weighted outer Minkowski content with respect to the absolutely continuous measure $d\theta = \omega\, d\mathcal{H}^{n-1}$; namely, if $E \subset \Omega$ is measurable set, then

$$\mathcal{M}_{\theta,-}(E) = \liminf_{\varrho\to 0^+} \frac{\theta(E^{(\varrho)} \cap \Omega) - \theta(E)}{\varrho} \tag{5.53}$$

where again $E^{(\varrho)} = E + \varrho B^n$, and prove (see the claim in the proof of Theorem 3.6 in [24]) that if $\mathrm{Per}_\omega(E) < \infty$, then

$$\mathcal{M}_{\theta,-}(E) \geqslant \mathrm{Per}_\omega E. \tag{5.54}$$

It also follows that finiteness of the weighted lower outer Minkowski content implies finite weighted perimeter.

In the rest of the section, we discuss some cases when equality holds in (5.54); namely, if E is the union of finitely many compact sets with Lipschitz boundary, or if the typical level set of a Lipschitz function (see Theorem 5.4.4 and Theorem 5.4.7). However, we start with Theorem 5.4.3 due to Ambrosio, Di Marino, Gigli [24] showing that we can always have equality (5.54) at least asymptotically.

Theorem 5.4.3 (Ambrosio, Di Marino, Gigli [24]). *Let $\Omega \subset \mathbb{R}^n$ be open, let $\omega : \Omega \to (0,\infty)$ be continuous, and let $d\theta = \omega\, d\mathcal{H}^n$ be the absolutely continuous measure on Ω. If $\mathrm{Per}_\omega E < \infty$ for a measurable $E \subset \Omega$ with $\mu(E) > 0$, then there exists a sequence of measurable subsets $E_k \subset \Omega$ such that $\lim_{k\to\infty} \theta(E_k \cap E) = 0$ and*

$$\lim_{k\to\infty} \mathcal{M}_{\theta,-}(E_k) = \mathrm{Per}_\omega E.$$

Remark. In particular, Theorem 5.4.3 applies if $\mathcal{M}_{\theta,-}(E) < \infty$.

Let us introduce a notation for the weighted Minkowski content that is used in the literature about the Cheeger constant (cf. (4.59)). Let $\Omega \subset \mathbb{R}^n$ be open, and let $p : \mathbb{R}^n \to [0,\infty)$ be a function such that

- $p|_\Omega$ is positive, bounded and continuous,
- $p(x) = 0$ if $x \notin \operatorname{cl}\Omega$.

For the absolutely continuous measure $d\mu = p\, d\mathcal{H}^{n-1}$ on $\mathbb{R}^n$, if $E \subset \mathbb{R}^n$ is measurable and satisfies that $\mu(E^{(r)}) < \infty$ for some $r > 0$, then we write

$$\mu_+(E) = \liminf_{\varrho\to 0^+} \frac{\mu(E^{(\varrho)}) - \mu(E)}{\varrho},$$

to denote the corresponding lower outer weighted Minkowski content, which is possibly infinity.

The argument for Theorem 5.4.4 is based on Ambrosio, Colesanti, Villa [23]. We recall that an $X \subset \mathbb{R}^n$ has Lipschitz boundary (cf. Definition 4.1.4) if X is the closure of its non-empty interior, and for any $x \in \partial X$, there exists $r > 0$ (depending on x, X) such that $\partial X \cap \operatorname{int}(x + rB^n)$ is the graph of a Lipschitz function. Such an X is a set of locally finite perimeter, and if X is compact, then X is of finite perimeter and $\mathcal{H}^{n-1}(\partial^* X) = \mathcal{H}^{n-1}(\partial X)$ (see Example 5.1.5 or [23]). In addition, the union E of finitely many compact sets with Lipschitz boundary is a set of finite perimeter by Example 5.1.4, and hence we can speak about its reduced boundary $\partial^* E \subset \partial E$ (cf. Section 5.1).

Theorem 5.4.4. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $p : \mathbb{R}^n \to [0, \infty)$ be a function such that $p|_\Omega$ is positive, bounded and continuous, and $p(x) = 0$ if $x \notin \operatorname{cl}\Omega$. If $E \subset \mathbb{R}^n$ is the union of finitely many compact sets with Lipschitz boundary and $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$, then the absolute continuos measure $d\mu = p\, d\mathcal{H}^{n-1}$ on $\mathbb{R}^n$ satisfies that*

$$\mu_+(E) = \lim_{\varrho \to 0^+} \frac{\int_{E(\varrho)} p - \int_E p}{\varrho} = \int_{\partial^* E} p\, d\mathcal{H}^{n-1}. \tag{5.55}$$

Remark. In particular, if $E \subset \mathbb{R}^n$ is a compact set with Lipschitz boundary satisfying that $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$, then

$$\mu_+(E) = \lim_{\varrho \to 0^+} \frac{\int_{E(\varrho)} p - \int_E p}{\varrho} = \int_{\partial E} p\, d\mathcal{H}^{n-1}. \tag{5.56}$$

Proof. As $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$, we may assume that $p(x) = 0$ if $x \in \partial\Omega$.

Since $\mathcal{H}^{n-1}(\partial E) < \infty$, there exists at most countable many $t > 0$ satisfying that $\mathcal{H}^{n-1}(\partial E \cap \partial\{p > t\}) > 0$. As $p|_\Omega$ is positive, continuous and bounded, it can approximated from below by an increasing sequence of step functions and from above by a decreasing sequence of step function where each step function is the linear combination of finitely many functions of the form $\mathbf{1}_{\{p>t\}}$ for $t \geqslant 0$ where we also assume that $\mathcal{H}^{n-1}(\partial E \cap \partial\{p > t\}) = 0$. Therefore, we conclude (5.55) from Proposition 2 in Ambrosio, Colesanti, Villa [23] stating that

$$\lim_{\varrho \to 0^+} \frac{\int_{E(\varrho)} \mathbf{1}_A - \int_E \mathbf{1}_A}{\varrho} = \int_{\partial^* E} \mathbf{1}_A\, d\mathcal{H}^{n-1}$$

for any open $A \subset \Omega$ such that $\mathcal{H}^{n-1}(\partial E \cap \partial A) = 0$. ∎

Next, we point out the consequence Corollary 5.4.6 of Theorem 5.4.4 stating that in order to prove certain isoperimetric type inequality for closed subsets with Lipschitz boundary with respect to an absolutely continuous probability measure on $\mathbb{R}^n$, it is sufficient to verify the inequality for compact subsets with Lipschitz boundary. For this, we need to show that we have some freedom to wiggle the direction that witnesses

that the boundary of an $E \subset \mathbb{R}^n$ near $x \in \partial E$ is Lipschitz. We recall if $v \in S^{n-1}$, then $\Pi_{v^\perp}$ is the orthogonal projection onto $v^\perp$, that is a linear transformation.

Lemma 5.4.5. *Let $X = \{x + f(x)u : x \in \Omega\}$ be the graph of a Lipschitz function $f : \Omega \to \mathbb{R}$ where $u \in S^{n-1}$ and $\Omega \subset u^\perp$ is relatively open. Under these conditions, there exists $\delta > 0$ (depending on the Lipschitz constant of f) such that if $\|v - u\| \leqslant \delta$ for $v \in S^{n-1}$, then $X = \{z + g(z)v : z \in \Pi_{v^\perp} X\}$ for a Lipschitz function $g : \Pi_{v^\perp} X \to \mathbb{R}$ where $\Pi_{v^\perp} X \subset v^\perp$ is relatively open.*

Remark. If $|f(x) - f(y)| \leqslant L\,\|x - y\|$ for $x, y \in \Omega$, then our argument gives $\delta = \frac{1}{4(1+L)}$, and $|g(\tilde{x}) - g(\tilde{y})| \leqslant 2(1 + L)\|\tilde{x} - \tilde{y}\|$ for $\tilde{x}, \tilde{y} \in \Pi_{v^\perp} X$.

Proof. Let $L > 0$ such that $|f(x) - f(y)| \leqslant L\,\|x - y\|$ for $x, y \in \Omega$, and let $v \in S^{n-1}$ such that

$$\|v - u\| \leqslant \frac{1}{4(1 + L)}, \tag{5.57}$$

and let $\varphi = \angle(u, v) \in (0, \frac{\pi}{2})$. The main observation is that if $x + f(x)u = \tilde{x} + g(\tilde{x})v$ for $\tilde{x} \in v^\perp$, then $g(\tilde{x}) = \langle x + f(x)u, v\rangle$.

For $x, y \in \Omega$, the triangle inequality yields that

$$\begin{aligned} |\langle x + f(x)u, v\rangle - \langle y + f(y)u, v\rangle| &\leqslant \|x - y\| + \langle u, v\rangle |f(x) - f(y)| \\ &\leqslant (1 + L)\|x - y\|. \end{aligned} \tag{5.58}$$

On the other hand, let $\tilde{x} = \Pi_{v^\perp}(x + f(x)u)$ and $\tilde{y} = \Pi_{v^\perp}(y + f(y)u)$, and hence the triangle inequality implies that

$$\begin{aligned} \|\tilde{x} - \tilde{y}\| &\geqslant \|\Pi_{v^\perp}(x - y)\| - |f(x) - f(y)| \cdot \|\Pi_{v^\perp} u\| \\ &\geqslant \|x - y\| \cos\varphi - L\|x - y\| \sin\varphi. \end{aligned}$$

Here $\sin\varphi \leqslant \max\{\frac{1}{4}, \frac{1}{4L}\}$ by the condition (5.57), thus $\cos\varphi > \frac{3}{4}$ and

$$\|\tilde{x} - \tilde{y}\| \geqslant \tfrac{1}{2}\,\|x - y\|. \tag{5.59}$$

We conclude that $\Pi_{v^\perp}|_X$ is injective, and hence there exists a function $g : \Pi_{v^\perp}|_X$ such that $x + f(x)u = \tilde{x} + g(\tilde{x})v$ for $x \in \Omega$ and $\tilde{x} = \Pi_{v^\perp}(x + f(x)u)$; moreover, the Open Mapping Theorem (cf. Hatcher [381]) yields that $\Pi_{v^\perp} X \subset v^\perp$ is relatively open being the image of a continuous and injective map $\Omega \to v^\perp$. Finally, combining (5.58) and (5.59) implies that $|g(\tilde{x}) - g(\tilde{y})| \leqslant 2(1 + L)\|\tilde{x} - \tilde{y}\|$ for $\tilde{x}, \tilde{y} \in \Pi_{v^\perp} X$. ∎

Corollary 5.4.6. *Let $\Omega \subset \mathbb{R}^n$ be open such that $\mathcal{H}^{n-1}(RB^n \cap \partial\Omega) < \infty$ for any $R > 0$, let $p : \mathbb{R}^n \to [0, \infty)$ be a function such that $p|_\Omega$ is positive, bounded and continuous, and $p(x) = 0$ if $x \notin \operatorname{cl}\Omega$, and let $\mu(\mathbb{R}^n) < \infty$ for the absolutely continuous measure $d\mu = p\,d\mathcal{H}^{n-1}$ on $\mathbb{R}^n$.*

If for some $c > 0$, $\mu_+(E) \geqslant c \cdot \mu(E)$ holds for any compact $E \subset \mathbb{R}^n$ with Lipschitz boundary and $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$, then

$$\int_{\partial E} p \, d\mathcal{H}^{n-1} \geqslant c \cdot \mu(E) \tag{5.60}$$

holds for any $E \subset \mathbb{R}^n$ with Lipschitz boundary and $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$.

Remarks. Readily, no need for the condition $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$ if $\Omega = \mathbb{R}^n$. Let us state some versions of Corollary 5.4.6 that follow from the same argument.

(i) If μ is a probability measure, then assuming that

$$\mu_+(E) \geqslant c \cdot \min\{\mu(E), 1 - \mu(E)\}$$

holds for any compact $E \subset \mathbb{R}^n$ with Lipschitz boundary and $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$, then

$$\int_{\partial E} p \, d\mathcal{H}^{n-1} \geqslant c \cdot \min\{\mu(E), 1 - \mu(E)\} \tag{5.61}$$

holds for any $E \subset \mathbb{R}^n$ with Lipschitz boundary and $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$.

(ii) If p is even, then assuming that $\mu_+(E) \geqslant c \cdot \mu(E)$ holds for any o-symmetric compact $E \subset \mathbb{R}^n$ with Lipschitz boundary and $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$, then (5.60) holds for any o-symmetric $E \subset \mathbb{R}^n$ with Lipschitz boundary and $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$.

Proof of Corollary 5.4.6. Let closed $E \subset \mathbb{R}^n$ have Lipschitz boundary and satisfy $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$. Since $\mathcal{H}^{n-1}(RB^n \cap \partial\Omega) < \infty$ for any $R > 0$, and integrating in polar coordinates (cf. (10.13)) yields that

$$\int_0^\infty \int_{\partial(rB^n)} p \, d\mathcal{H}^{n-1} \, dr = \int_{\mathbb{R}^n} p < \infty,$$

there exists a sequence $R_k > 0$ going to infinity with k such that $(R_k B^n) \cap \operatorname{int} E \neq \varnothing$ and $\mathcal{H}^{n-1}(\partial(R_k B^n) \cap \partial\Omega) = 0$ for each R_k, and

$$\lim_{k\to\infty} \int_{\partial(R_k B^n)} p \, d\mathcal{H}^{n-1} = 0. \tag{5.62}$$

Readily $\mathcal{H}^{n-1}(\partial E_k \cap \partial\Omega) = 0$ holds for each $E_k = (R_k B^n) \cap E$, and we claim that each E_k has Lipschitz boundary, and hence Theorem 5.4.4 implies that

$$\int_{\partial(E_k \cap R_k B^n)} p \, d\mathcal{H}^{n-1} = \mu_+(E_k \cap R_k B^n) \geqslant c \cdot \mu(E_k \cap R_k B^n). \tag{5.63}$$

To prove the claim, it is sufficient to check whether $\partial E_k \cap \partial(R_k B^n)$ is a graph of a Lipschitz function in a small neighbourhood of a $z \in \partial E_k \cap \partial(R_k B^n)$. According

to Lemma 5.4.5, there exist a $v \in S^{n-1}$ that is not tangent to $R_k B^n$ at z, a relatively open $\Omega \subset v^\perp$ with $\Pi_{v^\perp} z \in \Omega$, and and a Lipschitz function $f : \Omega \to \mathbb{R}$ such that $\{x + f(x)v : x \in \Omega\} \subset \partial E_k$ parametrizes a small neighbourhood of z on ∂E_k. As v is not tangent to $R_k B^n$, we may assume that $tv \in \operatorname{int}(E \cap R_k B^n)$ for small $t > 0$, and after possibly shrinking Ω, we may also assume that $\{x + g(x)v : x \in \Omega\} \subset \partial(R_k B^n)$ for a Lipschitz function $g : \Omega \to \mathbb{R}$ parametrizes a small neighbourhood of z on $\partial(R_k B^n)$. Now $\max\{f, g\}$ is also Lipschitz and parametrizes a small neighbourhood of z on $\partial(E_k \cap R_k B^n)$, proving that $E_k \cap R_k B^n$ has Lipschitz boundary.

Finally, we conclude from (5.62) and (5.63) that

$$\int_{\partial E} p \, d\mathcal{H}^{n-1} = \lim_{k\to\infty} \int_{\partial(E_k \cap R_k B^n)} p \, d\mathcal{H}^{n-1} \geqslant \lim_{k\to\infty} c \cdot \mu(E_k \cap R_k B^n) = c \cdot \mu(E).$$

■

In Theorem 5.4.7, we have to restrict the density function of the measure to be log-concave in order to ensure, for example, that the corresponding Ricci curvature is non-negative. For properties of a log-concave function $p : \mathbb{R}^n \to [0, \infty)$, see Section 10.9. In particular, $p = e^{-V}$ for a convex function $V : \mathbb{R}^n \to (-\infty, \infty]$, and if $0 < \mu(\mathbb{R}^n) < \infty$ for the log-concave measure $d\mu = p \, d\mathcal{H}^{n-1}$, then the common support of p and μ is convex, and p is positive and continuous on the non-empty interior of its support. The following theorem is based on the paper Ambrosio, Di Marino, Gigli [24].

Theorem 5.4.7. *If $d\mu = p \, d\mathcal{H}^n$ is a non-trivial finite measure on $\mathbb{R}^n$ for a log-concave density $p : \mathbb{R}^n \to [0, \infty)$, and $f \in L^1(\mathbb{R}^n, \mu)$ is Lipschitz and not a constant function on* supp p*, then*

$$\int_{\mathbb{R}} \mu_+ (\{f \geqslant t\}) \, dt = \int_{\mathbb{R}} \int_{f^{-1}(t)} p \, d\mathcal{H}^{n-1} \, dt, \tag{5.64}$$

and $\mu_+ (\{f \geqslant t\}) = \int_{f^{-1}(t)} p \, d\mathcal{H}^{n-1}$ for $\mathcal{H}^1$ a.e. t in the image of supp p *by f.*

Proof. Let Ω be the non-empty interior of the convex support of p where we may assume that $o \in \Omega$, and let $\omega = p|_\Omega$, that is positive and continuous on Ω. For the absolutely continuous measure $d\theta = \omega \, d\mathcal{H}^{n-1}$ on Ω, if $E \subset \mathbb{R}^n$ is measurable and $E \cap \Omega \neq \varnothing$, then $\mu(E) = \theta(E \cap \Omega)$, and hence (cf. (5.53))

$$\mu_+(E) = \mathcal{M}_{\theta,-}(E \cap \Omega). \tag{5.65}$$

We deduce from (5.54) that if $E \subset \mathbb{R}^n$ is measurable and $E \cap \Omega \neq \varnothing$, then

$$\mu_+(E) \geqslant \operatorname{Per}_\omega(E \cap \Omega). \tag{5.66}$$

Our first goal is to verify that equality holds in (5.66) for most level sets of a given Lipschitz function. For $k > 1$, let $\Omega_k = (1 - \frac{1}{k})\Omega \cap \operatorname{int}(kB^n)$, let $\omega_k = \omega \cdot \mathbf{1}_{\Omega_k}$, and

let $d\theta_k = \omega_k d\mathcal{H}^n$ on Ω. In addition, we write $d(\cdot,\cdot)$ to denote the Euclidean metric in $\mathbb{R}^n$.

We need the claim that the metric measure space (Ω_k, d, θ_k) is a doubling $\mathrm{RCD}(0,\infty)$ space in the terminology of Ambrosio, Di Marino, Gigli [24], where abusing the notation, here we write ω_k and θ_k to denote the restrictions of ω and θ to Ω_k. Now $\theta_k = e^{-V_k}$ for a convex $V_k : \Omega_k \to \mathbb{R}$ where the Hessian of V_k is almost everywhere positive semi-definite, thus the measure θ_k comes from a Riemannian metric on Ω_k of non-negative Ricci curvature, verifying that the measure θ_k represents a $\mathrm{RCD}(0,\infty)$ space. For doubling, as θ_k is bounded and bounded away from zero on Ω_k, it is sufficient to consider the case when θ_k is the Lebesgue measure on Ω_k. However, if $x \in \Omega_k$ and $\Omega_k \nsubseteq x + rB^n$ for $r > 0$, then

$$\frac{1}{2}x + \frac{1}{2}\Big((\Omega_k \cap (x + 2rB^n)) \setminus (x + rB^n) \Big) \subset \Omega_k \cap (x + rB^n),$$

and hence the metric measure space (Ω_k, d, θ_k) is also doubling besides being an $\mathrm{RCD}(0,\infty)$ space.

Combining the paragraph above Proposition 4.2 and the paragraph above Remark 4.4 in Ambrosio, Di Marino, Gigli [24], we deduce that the function $f \mapsto \int_{\Omega_k} \|Df\|\, d\theta_k$ of Lipschitz functions $f \in L^1(\Omega_k, \theta_k)$ is lower semi-continuous in the $L^1(\Omega_k, \theta_k)$ metric. As θ is the monotone increasing limit of θ_k (now θ_k is a measure on Ω), it follows that the function $f \mapsto \int_\Omega \|Df\|\, d\theta$ of Lipschitz functions $f \in L^1(\Omega, \theta)$ is lower semi-continuous in the $L^1(\Omega, \theta)$ metric. It follows from the paragraph above Proposition 4.2 in Ambrosio, Di Marino, Gigli [24] that the measure space (Ω, d, θ) satisfies property (4.2) in [24].

Now if $f : \mathbb{R}^n \to [0,\infty)$ is Lipschitz and satisfies that $f \in L^1(\mathbb{R}^n, \mu)$ and $\{f > 0\} \cap \Omega \neq \varnothing$, then (5.65) and Proposition 4.2 in Ambrosio, Di Marino, Gigli [24] yield that

$$\mu_+ (\{f \geqslant t\}) = \mathrm{Per}_\omega (\{f \geqslant t\} \cap \Omega) \tag{5.67}$$

for a.e. $t > 0$ in the image of Ω by f (compare (5.66)).

Next, combining (4.1) and (4.2) in Ambrosio, Di Marino, Gigli [24] leads to the formula

$$\int_0^\infty \mathrm{Per}_\omega (\{f \geqslant t\} \cap \Omega)\, dt = \int_\Omega \|Df\|\, d\theta = \int_{\mathbb{R}^n} \|Df\|\, d\mu < \infty, \tag{5.68}$$

where the Coarea formula (10.12) says that

$$\int_{\mathbb{R}^n} \|Df\|\, d\mu = \int_0^\infty \int_{f^{-1}(t)} p\, d\mathcal{H}^{n-1} dt. \tag{5.69}$$

Since $\mathcal{H}^{n-1}(kB^n \cap \partial\Omega) < \infty$ for $k \geqslant 1$ where the empty set has zero measure, and $f^{-1}(t) \cap f^{-1}(s) = \varnothing$ for $t \neq s$, we deduce that

$$\mathcal{H}^{n-1}\left(f^{-1}(t) \cap \partial\Omega\right) = 0 \tag{5.70}$$

for a.e. $t > 0$ in the image of Ω by f. Therefore, (5.70) and (5.52) yield that

$$\mathrm{Per}_\omega(\{f \geqslant t\} \cap \Omega) \leqslant \int_{\Omega \cap f^{-1}(t)} \omega \, d\mathcal{H}^{n-1} = \int_{f^{-1}(t)} p \, d\mathcal{H}^{n-1} \tag{5.71}$$

for a.e. $t > 0$ in the image of Ω by f. Combining (5.71), (5.68) and (5.69) implies that for a.e. $t > 0$ in the image of Ω by f, we have

$$\mathrm{Per}\,(\{f \geqslant t\}) = \int_{f^{-1}(t)} p \, d\mathcal{H}^{n-1},$$

and in turn (5.67) yields that

$$\mu_+\,(\{f \geqslant t\}) = \int_{f^{-1}(t)} p \, d\mathcal{H}^{n-1}, \tag{5.72}$$

verifying Theorem 5.4.7 if $f(x) \geqslant 0$ for $x \in \mathbb{R}^n$.

Finally, let $f \in L^1(\mathbb{R}^n, \mu)$ be Lipschitz. We may assume that f is not a constant function, and replacing f by $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$, we may also assume that $f(x) \geqslant 0$ for $x \in \mathbb{R}^n$, completing the proof of Theorem 5.4.7. ∎

5.5 Comments to Chapter 5

The material of this chapter is discussed in the monographs Federer [260], Ambrosio, Fusco, Pallara [25] and Maggi [557], and in the paper Figalli, Maggi, Pratelli [278].

Continuous functions of bounded variations were seemingly first considered by Camille Jordan in his studies about the convergence of Fourier series in 1881. General functions of bounded variation in several variables were treated by Lamberto Cesari around 1936, and the intimate connections between BV functions and sets of finite perimeter were established by Renato Caccioppoli and Ennio de Giorgi in the 1950s. Sets of finite perimeter are sometimes called Caccioppoli sets. De Giorgi [231] proved the classical isperimetric inequality for sets of finite perimeter *via* Steiner Symmetrization, also characterizing the equality case in 1958 (see also Talenti [696]). The Anisotropic Isoperimetric inequality for sets of finite perimeter was proved by Taylor [699] in 1978, and later Brothers, Morgan [162] and Fonseca, Müller [292] presented simplified arguments.

The notion of Wulff shape originates from the paper Wulff [721] related to Crystallography, and see Maggi [557], Brothers, Morgan [162] and Fonseca, Müller [292] for a dicussion of Wulff's theorem for sets of finite perimeter, and the papers Taylor [699], Miracle-Sole [592] and Figalli, Maggi [276] for the role of Wulff shapes within crystallography. Many examples of Wulff shapes and the relation to the underlying periodic and quasi-periodic structure are discussed in Böröczky, Schnell, Wills [148]. In particular, Wulff shapes are also successful models of certain quasi-crystals.

Chapter 6

Associated ellipsoids, Blaschke-Santaló inequality and the Reverse Isoperimetric Inequality

This chapter focuses on affine invariant properties of convex bodies in $\mathbb{R}^n$ centered around the study of some affine equivariant associated ellipsoids; namely, the maximal volume inscribed so-called John Ellipsoid, and the minimal volume circumscribed so called Löwner Ellipsoid are introduced in the first section, the Ellipsoid of Inertia in Section 6.4, and the so-called M-Ellipsoid in the closing Section 6.9. One of the main significance of these ellipsoids is that they provide reasonably good approximation of the corresponding convex body. We discuss various related inequalities, like the Blaschke-Santaló inequality, and the reverse forms of the Isoperimetric, the Blaschke-Santaló and the Brunn-Minkowski inequalities, where the associated ellipsoids have significant roles in the arguments. In addition, an important topic of this Chapter is the study of convex bodies in a kind of "isotropic position" when one of the associated ellipsoids is a centered Euclidean ball, and also inequalities related to the Slicing Conjecture posed by Bourgain and Keith Ball in the mid 1980's and proved recently by Klartag, Lehec [470].

6.1 John and Löwner ellipsoid

We recall that an ellipsoid E is of the form $E = \Phi B^n + v$ for $\Phi \in \mathrm{GL}(n)$ and $v \in \mathbb{R}^n$ where v is the center of E. Actually, there exists a positive definite symmetric matrix A such that $E = AB^n + v$ (see Section 6.A). In this section, we introduce the inscribed John and the circumscribed Löwner ellipsoid that approximate best in terms of volume difference. We only provide the arguments in the case of the John ellipsoid because they are analogous but slightly more technical in the case of the Löwner ellipsoid. The following statement is proved in Proposition 6.1.5 and Proposition 6.1.7.

Theorem 6.1.1 (John). *If $K \subset \mathbb{R}^n$ is a convex body, then there exists a unique "John" ellipsoid E of maximal volume contained in K, and writing x_0 to denote the center of E, we have*

$$E \subset K \subset x_0 + n(E - x_0).$$

If K is origin symmetric ($K = -K$), then $x_0 = o$, and $E \subset K \subset \sqrt{n}\, E$.

Remark. ΦE is the John ellipsoid of ΦK for any $\Phi \in \mathrm{GL}(n)$.

For $u \in S^{n-1}$, $u \otimes u = u \cdot u^t$ is a rank one positive semi-definite $n \times n$ symmetric matrix with $\mathrm{tr}(u \otimes u) = 1$. We postpone the proof of John's condition Theorem 6.1.2 characterizing the maximal volume inscribed ellipsoid by the contact points to Section 6.A because the argument is somewhat lengthy even if natural.

Theorem 6.1.2 (John). *B^n is the John ellipsoid of $K \subset \mathbb{R}^n$ if and only if $B^n \subset K$ and there exist $u_1, \dots, u_k \in S^{n-1} \cap \partial K$ and $c_1, \dots, c_k > 0$, $k \leqslant \frac{n(n+3)}{2}$, such that*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n, \tag{6.1}$$

$$\sum_{i=1}^{k} c_i u_i = o \tag{6.2}$$

where I_n denotes the $n \times n$ identity matrix.

If K is origin symmetric ($K = -K$), then (6.1) is sufficient.

Remarks.

- (6.1) yields that $\langle x, y \rangle = \sum_{i=1}^{k} c_i \langle x, u_i \rangle \langle y, u_i \rangle$ for $x, y \in \mathbb{R}^n$, and hence the discrete measure μ on S^{n-1} concentrated on $\{u_1, \dots, u_k\}$ with $\mu(u_i) = c_i$ is called isotropic.
- $\sum_{i=1}^{k} c_i = n$ follows by comparing traces in (6.1).
- $\langle x, u_i \rangle \leqslant 1$ for $x \in K$ and $i = 1, \dots, k$ as K and B^n share the same supporting hyperplanes at $u_1, \dots, u_k$.

Next we consider the minimum volume circumscribed so-called Löwner ellipsoid.

Theorem 6.1.3 (John, Löwner). *If $K \subset \mathbb{R}^n$ convex body, then there exists a unique "Löwner" ellipsoid E of minimal volume containing K, and writing x_0 to denote the center of E, we have*

$$x_0 + \frac{1}{n}(E - x_0) \subset K \subset E.$$

If K is origin symmetric ($K = -K$), then $x_0 = o$ and $\frac{1}{\sqrt{n}} \cdot E \subset K \subset E$.

Remark. ΦE is the Löwner ellipsoid of ΦK for any $\Phi \in \mathrm{GL}(n)$.

Theorem 6.1.4 (John). *B^n is the Löwner ellipsoid containing a convex body $K \subset \mathbb{R}^n$ if and only if $K \subset B^n$ and there exist $u_1, \dots, u_k \in S^{n-1} \cap \partial K$ and $c_1, \dots, c_k > 0$,*

$k \leqslant \frac{n(n+3)}{2}$, *such that*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n, \tag{6.3}$$

$$\sum_{i=1}^{k} c_i u_i = o. \tag{6.4}$$

If K is origin symmetric ($K = -K$), then (6.3) is sufficient.

In the remaining of this section, we discuss Theorem 6.1.1. We prove directly the uniqueness of the John ellipsoid (see Proposition 6.1.5), and a consequence of it. After that we show how the conditions (6.1) and (6.2) yield that John's ellipsoid is a good approximation of the convex body (see Proposition 6.1.7).

Proposition 6.1.5. *For any convex body $K \subset \mathbb{R}^n$, there exists a unique ellipsoid of maximal volume contained in K.*

Proof. The arguments is indirect, we suppose that there exist two ellipsoids of maximal volume $A\,B^n + x \neq B\,B^n + y$ contained in K for positive definite matrices A, B and $x, y \in \mathbb{R}^n$, and hence $\det A = \det B$. We may assume that $y = -x$ and $B = I_n$.

Now $(\frac{1}{2}\,A + \frac{1}{2}\,I_n)B^n \subset K$ as K convex and $y = -x$ where $\det(\frac{1}{2}\,A + \frac{1}{2}\,I_n) > 1$ unless $A = I_n$ according to (3.5.2), and hence $A = I_n$ and $x \neq o$. We deduce that $E \subset K$ for the ellipsoid $E = \Phi B^n$ for the $\Phi \in \mathrm{GL}(n)$ where $\Phi x = (1 + \frac{1}{2}\|x\|)x$ and $\Phi w = w$ for $w \in x^{\perp}$. Since $|E| = (1 + \frac{1}{2}\|x\|)|B^n|$, $E \subset K$ contradicts the maximality of the volume of the ellipsoid $A\,B^n + x$. ∎

Theorem 6.1.6 is a nice application of the uniqueness of John's ellipsoid to characterize the maximal compact subgroups of $\mathrm{GL}(n)$.

Theorem 6.1.6. *For any compact subgroup $G \subset \mathrm{GL}(n)$, there exists a scalar product on $\mathbb{R}^n$ that is invariant under G. In addition, the maximal compact subgroups of $\mathrm{GL}(n)$ are the conjugates of $O(n)$.*

Proof. Let $v_1, \ldots, v_n$ be any basis of $\mathbb{R}^n$, and let

$$K = \mathrm{conv}\{\pm \Phi v_i : \Phi \in G \text{ and } i = 1, \ldots, n\}.$$

As G is compact, K is an o-symmetric convex body invariant under G, and hence the John ellipsoid E of K is also invariant under G. We may choose the scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$ in a way such that E is the unit ball of the corresponding Euclidean norm, thus $G \subset \Phi\, O(n)\, \Phi^{-1}$ where $E = \Phi\, B^n$ for some suitable $\Phi \in \mathrm{GL}(n)$. ∎

Based on the conditions (6.1) and (6.2) for the John ellipsoid, we prove how well the John ellipsoid approximates the convex body using a simple argument due to Keith Ball.

Proposition 6.1.7 (John). *For convex body $K \subset \mathbb{R}^n$, if x_0 is the center of the unique John ellipsoid E of maximal volume contained in K, then*

$$E \subset K \subset x_0 + n(E - x_0).$$

If, in addition, K is o-symmetric, then $x_0 = o$, and $E \subset K \subset \sqrt{n}\, E$.

Proof. We assume that B^n is the John ellipsoid, and use the notation of Theorem 6.1.2.

If $K = -K$ and $x \in K$, then the Remarks after Theorem 6.1.2 yield that

$$\|x\|^2 = \sum_{i=1}^{k} c_i \langle u_i, x\rangle^2 \leqslant \sum_{i=1}^{k} c_i = n.$$

For a general convex body K, we need to prove that $\|x\| \leqslant n$ for $x \in K$. Let $\|x\| = r$, and hence $-r \leqslant \langle x, u_i\rangle \leqslant 1$ for $i = 1, \dots, k$ by the Remarks after Theorem 6.1.2. It also follows using these Remarks and (6.2) that

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{k} c_i (1 - \langle x, u_i\rangle)(r + \langle x, u_i\rangle) \\ &= r \sum_{i=1}^{k} c_i + (1 - r) \sum_{i=1}^{k} c_i \langle x, u_i\rangle - \sum_{i=1}^{k} c_i \langle x, u_i\rangle^2 \\ &= rn - \|x\|^2 = \|x\|(n - \|x\|), \end{aligned}$$

and hence $\|x\| \leqslant n$. ∎

6.2 Some characterizations of ellipsoids

In this section, we discuss some characterizations of ellipsoids based on some properties of sections (where the first two are due to Brunn [163, 164] around 1889). These properties are useful for example when discussing the equality cases of inequalities when Steiner symmetrization is applied during the argument (see, for example, Theorem 8.C.3 about the Affine Isoperimetric inequality).

Theorem 6.2.1 (Brunn). *A convex body $K \subset \mathbb{R}^n$, $n \geqslant 2$, is an ellipsoid if and only if for any $u \in S^{n-1}$, the midpoints of the secants of K parallel to u are contained in a hyperplane.*

Proof. We observe that the property described in Theorem 6.2.1 is invariant under affine transformations, and hence any ellipsoid satisfies this property. On the other hand, if K satisfies this property, then we may assume that B^n is the maximum volume John ellipsoid in K.

For a $u \in S^{n-1}$, let H be the hyperplane containing the midpoints of the secants of K parallel to u. and let Φ be the affine transformation that fixes each point of H, and maps $x + u$ into $x - u$ for $x \in H$. Then Φ keeps volume, and $\Phi K = K$ as the midpoints of the secants of K parallel to u lie in H. Thus $\Phi B^n \subset K$ has the same volume as B^n, and hence the uniqueness of the John ellipsoid (cf. Proposition 6.1.5) yields that $\Phi B^n = B^n$. It follows that $H = u^\perp$, and in turn that K is symmetric through $v^\perp$ for all $v \in S^{n-1}$. Therefore, $\|x\| S^{n-1} \subset K$ for any $x \in K$, thus K is a centered ball. ∎

We do not provide the involved argument for the other two statements. We recall that a compact convex set C is centrally symmetric if there exists a $z \in Z$ satisfying $-(C - z) = C - z$ (and naturally, the empty set is centrally symmetric). The following statement is used, for example, in Section 6.5 when characterizing the equality case of the Blaschke-Santaló inequality (6.31) in the case of o-symmetric convex bodies.

Theorem 6.2.2 (Brunn). *Given $2 \leqslant m < n$, a convex body $K \subset \mathbb{R}^n$ is an ellipsoid if and only if the intersection with any affine m-space is empty or centrally symmetric.*

We could not end this section without stating the following beautiful strengthening of Theorem 6.2.2 due to Larman [494]:

Theorem 6.2.3 (False Center Theorem). *Given $2 \leqslant m < n$, a point $p \in \mathbb{R}^n$ and a convex body $K \subset \mathbb{R}^n$, if the intersection of K with any affine m-space containing p is centrally symmetric, then either K is symmetric through p, or K is an ellipsoid.*

6.3 Reverse isoperimetric inequality

For small $\varepsilon \in (0, 1)$ and the box $X_\varepsilon = [-\varepsilon^{-(n-1)}, \varepsilon^{-(n-1)}] \times [-\varepsilon, \varepsilon]^{n-1}$, we have $|X_\varepsilon| = 2^n$ but $P(X_\varepsilon) > 1/\varepsilon$ (the area of a "long" facet); therefore, the isoperimetric quotient $S(X_\varepsilon)^n/|X_\varepsilon|^{n-1}$ can be arbitrary large in general. The "Reverse Isoperimetric inequality" in this section says that each convex body has a linear image whose isoperimetric quotient is at most as bad as of a regular simplex, and hence "simplices have the worst isoperimetric quotient" up to linear transforms (cf. Theorem 6.3.1). For origin symmetric convex bodies, "cubes have the worst isoperimetric quotient" up to linear transforms (cf. Theorem 6.3.2).

Let Δ^n denote the regular simplex circumscribed around B^n, and hence each facet touches B^n.

Theorem 6.3.1 (Reverse Isoperimetric Inequality, Keith Ball [54]). *For any convex body K in $\mathbb{R}^n$, there exists $\Phi \in \mathrm{GL}(n)$ such that*

$$\frac{S(\Phi K)^n}{|\Phi K|^{n-1}} \leqslant \frac{S(\Delta^n)^n}{|\Delta^n|^{n-1}} = \frac{n^{3n/2}(n+1)^{(n+1)/2}}{n!},$$

where strict inequality can be attained if and only if K is not a simplex.

We note that a *parallelepiped* is the linear image of a cube, and consider the centered cube $W^n = [-1,1]^n$ of edge length 2.

Theorem 6.3.2 (Reverse Isoperimetric Inequality in the o-symmetric case, Keith Ball [53]). *For any o-symmetric convex body K in $\mathbb{R}^n$, there exists $\Phi \in \mathrm{GL}(n)$ such that*

$$\frac{S(\Phi K)^n}{|\Phi K|^{n-1}} \leqslant \frac{S(W^n)^n}{|W^n|^{n-1}} = 2^n n^n,$$

where strict inequality can be attained if and only if K is not a parallelepiped.

Stability versions of the Reverse Isoperimetric Inequality Theorem 6.3.1 and Theorem 6.3.2 are provided by Böröczky, Hug [135] and Böröczky, Fodor, Hug [128], respectively. To measure the volume difference between the shapes of the convex bodies $K, C \subset \mathbb{R}^n$ up to linear maps, we consider the centered convex bodies (cf. Section 1.11) $\widetilde{K} = |K|^{\frac{-1}{n}}(K - \sigma_K)$ and $\widetilde{C} = |C|^{\frac{-1}{n}}(C - \sigma_C)$ of volume one, and define

$$\Theta(K,C) = \min\left\{\left|\widetilde{K}\Delta(\Phi\widetilde{C})\right| : \Phi \in \mathrm{GL}(n) \text{ and } \det\Phi = \pm 1\right\}.$$

Then Böröczky, Hug [135] prove for any convex body K in $\mathbb{R}^n$ that there exists $\Phi \in \mathrm{GL}(n)$ such that

$$\frac{S(\Phi K)^n}{|\Phi K|^{n-1}} \leqslant \left(1 - n^{-cn}\Theta(K,\Delta^n)^4\right)\frac{S(\Delta^n)^n}{|\Delta^n|^{n-1}}, \tag{6.5}$$

and if K is o-symmetric, then Böröczky, Fodor, Hug [128] verify that

$$\frac{S(\Phi K)^n}{|\Phi K|^{n-1}} \leqslant \left(1 - n^{-cn}\Theta(K,W^n)^3\right)\frac{S(W^n)^n}{|W^n|^{n-1}}, \tag{6.6}$$

where $c \in (0,1)$ is an absolute constant. We note that the optimal exponent of $\Theta(K,\Delta^n)$ in (6.5) and of $\Theta(K,W^n)$ in (6.6) is most probably 1.

To start the preparation for the proof of the Reverse Isoperimetric Inequality Theorem 6.3.1 and Theorem 6.3.2, we note that a polytope P is circumscribed around B^n if each facet of P touches B^n.

Lemma 6.3.3. *If $rB^n \subset K$ for a convex body K in $\mathbb{R}^n$ and $r > 0$, then $P(K) \leqslant \frac{n}{r}|K|$, and equality holds if K is a polytope circumscribed around rB^n.*

Proof. The inequality $P(K) \leqslant \frac{n}{r}|K|$ follows from

$$P(K) = \lim_{\varrho\to 0^+}\frac{|K + \varrho B^n| - |K|}{\varrho} \leqslant \lim_{\varrho\to 0^+}\frac{|K + \frac{\varrho}{r}K| - |K|}{\varrho} = \frac{n}{r}|K|.$$

If K is a polytope circumscribed around rB^n, then considering the bounded "cones" with apex o and of height r over the facets (cf. (1.26)) shows that $|K| = \frac{r}{n} P(K)$ in this case. ∎

The proof of the Reverse Isoperimetric inequality both in the o-symmetric and non-symmetric cases is based on the rank one Brascamp-Lieb inequality Theorem 6.3.4 in harmonic analysis. We sketch a proof using optimal transport in Section 6.B.

Theorem 6.3.4 (Brascamp-Lieb, Keith Ball). *If $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ satisfy*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n, \tag{6.7}$$

and $f_1, \dots, f_k \in L^1(\mathbb{R})$ are non-negative, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x, u_i \rangle)^{c_i} \, dx \leqslant \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i \right)^{c_i}. \tag{6.8}$$

Remarks.

(i) If $n = 1$, then the Brascamp-Lieb inequality (6.8) is the Hölder inequality (10.5).

(ii) Inequality (6.8) is optimal, and we provide two types of examples for equality:

- If $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ satisfy (6.7), and $f_i(t) = e^{-\pi t^2}$ for $i = 1, \dots, k$, then each $\int_{\mathbb{R}} f_i = 1$, and

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x, u_i \rangle)^{c_i} \, dx = \int_{\mathbb{R}^n} e^{-\pi \sum_{i=1}^{k} c_i \langle x, u_i \rangle^2} \, dx = \int_{\mathbb{R}^n} e^{-\pi \langle x, x \rangle^2} \, dx = 1.$$

- If $u_1, \dots, u_n$ is an orthonormal basis and $c_1 = \dots = c_k = 1$, and hence (6.7) holds, and $f_1, \dots, f_n \in L^1(\mathbb{R})$ any functions, then the Fubini Theorem yields

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} f_i(\langle x, u_i \rangle)^{c_i} \, dx = \prod_{i=1}^{n} \left(\int_{\mathbb{R}} f_i \right)^{c_i}.$$

More precisely, Theorem 6.3.4 is the so-called Geometric form - discovered by Keith Ball - of the rank one Brascamp-Lieb inequality (see the Comments Section 6.10), which, as Keith Ball realized it, matches nicely the form of John's theorem as in Theorem 6.1.2.

Equality in Theorem 6.3.4 has been characterized by Barthe [67]. It is more involved; therefore, we only quote the special case that we need.

Theorem 6.3.5 (Barthe). *Let $\int_{\mathbb{R}} f_i > 0$ for $i = 1, \dots, k$, such that none of the f_is is Gaussian in Theorem 6.3.4, and equality holds in (6.8). Then there exists an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ such that $\{u_1, \dots, u_k\} \subset \{\pm e_1, \dots, \pm e_n\}$ and $\sum_{u_i \in \mathbb{R} e_p} c_i = 1$ for each e_p, and if $u_i = -u_j$, then $f_i(t) = \lambda_{ij} f_j(-t)$ for $\lambda_{ij} > 0$.*

It is a natural question how well an inscribed ellipsoid can approximate a convex body in terms of volume. This question was answered by Keith Ball [53, 54], see Theorem 6.3.6 for the origin symmetric case, and Theorem 6.3.7 in general. We note that Böröczky, Hug [135] and Böröczky, Fodor, Hug [128] prove stability versions - that are similar to (6.5) and (6.6) - of the volume ratio estimates Theorem 6.3.7 and Theorem 6.3.6, respectively.

Theorem 6.3.6 (Volume Ratio in the origin symmetric case, Keith Ball [53]). *For any o-symmetric convex body K in $\mathbb{R}^n$, the maximal volume John ellipsoid $E \subset K$ satisfies*

$$\frac{|K|}{|E|} \leqslant \frac{|W^n|}{|B^n|} = \frac{2^n}{\omega_n},$$

where strict inequality is attained unless K is a parallelepiped.

Proof. We may assume after a linear transformation that $E = B^n$. According to John's Theorem 6.1.2, there exists symmetric set $u_1, \dots, u_{2k} \in S^{n-1} \cap \partial K$ and $c_1, \dots, c_{2k} > 0$ with $u_{k+i} = -u_i$ and $c_{k+i} = c_i$, $i = 1, \dots, k$, such that

$$\sum_{i=1}^{2k} c_i u_i \otimes u_i = \mathrm{I}_n.$$

For $i = 1, \dots, 2k$ and $t \in \mathbb{R}$, let $f_i = \mathbf{1}_{[-1,1]}$. Now $K \subset P$ for the polytope $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant 1, i = 1, \dots, 2k\}$ according to the Remarks after John's Theorem 6.1.2 where $\mathbf{1}_P(x) = \prod_{i=1}^{2k} f_i(\langle x, u_i \rangle) = \prod_{i=1}^{2k} f_i(\langle x, u_i \rangle)^{c_i}$. It follows from the Brascamp-Lieb inequality (6.8) and $\sum_{i=1}^{2k} c_i = n$ that

$$|K| \leqslant |P| = \int_{\mathbb{R}^n} \prod_{i=1}^{2k} f_i(\langle x, u_i \rangle)^{c_i} \, dx \leqslant \prod_{i=1}^{2k} \left(\int_{\mathbb{R}} f_i \right)^{c_i} = 2^{\sum_{i=1}^{2k} c_i} = 2^n = |W^n|.$$

If $|K| = |W^n|$, then $|K| = |P|$, and Theorem 6.3.5 yields that $k = n$ and $u_1, \dots, u_n$ is an orthonormal basis of $\mathbb{R}^n$; therefore, K is a cube. ∎

Concerning the volume ratio of general convex bodies, we only sketch the argument because it involves a somewhat technical calculation.

Theorem 6.3.7 (Volume Ratio, Keith Ball [54]). *For any convex body K in $\mathbb{R}^n$, the maximal volume John ellipsoid $E \subset K$ satisfies*

$$\frac{|K|}{|E|} \leqslant \frac{|\Delta^n|}{|B^n|} = \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!\omega_n},$$

where strict inequality is attained unless K is a simplex.

Sketch of the proof of Theorem 6.3.7. We may assume that B^n is the John ellipsoid of K, and let $c_1, \dots, c_k > 0$ be the coefficients and $u_1, \dots, u_k \in S^{n-1} \cap \partial K$ be the contact points satifying (6.1) and (6.2) in John's Theorem 6.1.2; namely,

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n \quad \text{and} \quad \sum_{i=1}^{k} c_i u_i = o. \tag{6.9}$$

Again, $K \subset P$ for the polytope $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant 1, i = 1, \dots, k\}$ according to the Remarks after John's Theorem 6.1.2. The main idea is to lift $u_1, \dots, u_k$ to $\mathbb{R}^{n+1}$, and employ the Brascamp-Lieb inequality in $\mathbb{R}^{n+1}$. In particular, $\mathbb{R}^n$ is identified with $w^\perp$ for a fixed $w \in S^n \subset \mathbb{R}^{n+1}$, and let $\tilde{u}_i = -\sqrt{\frac{n}{n+1}} \cdot u_i + \sqrt{\frac{1}{n+1}} \cdot w$ and $\tilde{c}_i = \frac{n+1}{n} \cdot c_i$ for $i = 1, \dots, k$. Therefore, $\sum_{i=1}^{k} \tilde{c}_i \tilde{u}_i \otimes \tilde{u}_i = \mathrm{I}_{n+1}$ follows from (6.9). For $i = 1, \dots, k$, we consider the probability density

$$f_i(t) = \begin{cases} e^{-t} & \text{if } t \geqslant 0; \\ 0 & \text{if } t < 0 \end{cases}$$

on $\mathbb{R}$ where some not too complicated calculations show that

$$\int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} f_i(\langle x, \tilde{u}_i \rangle)^{\tilde{c}_i} = \frac{|P|}{|\Delta^n|}.$$

We conclude from the Brascamp-Lieb inequality (6.8) that $|K| \geqslant |P| \geqslant |\Delta^n|$.

If $|K| = |\Delta^n|$, then $K = P$ and equality holds in the Brascamp-Lieb inequality. Therefore, Theorem 6.3.5 provides an orthonormal basis $e_1, \dots, e_{n+1}$ of $\mathbb{R}^{n+1}$ such that $\{\tilde{u}_1, \dots, \tilde{u}_k\} \subset \{\pm e_1, \dots, \pm e_{n+1}\}$. Since $\langle w, \tilde{u}_i \rangle = \sqrt{\frac{1}{n+1}}$ for $i = 1, \dots, k$, we conclude that $k = n + 1$ and $\tilde{u}_1, \dots, \tilde{u}_{n+1}$ is an an orthonormal basis of $\mathbb{R}^{n+1}$, and hence P is congruent to Δ^n. ∎

Proof of the Reverse Isoperimetric Inequality Theorem 6.3.1 *and Theorem* 6.3.2*:* After applying an affine transformation, we may assume that the John ellipsoid of K is B^n both in Theorem 6.3.1 and Theorem 6.3.2.

For Theorem 6.3.1, Theorem 6.3.7 yields that $|K| \leqslant |\Delta^n|$, thus we deduce from Lemma 6.3.3 that

$$\frac{P(K)^n}{|K|^{n-1}} \leqslant \frac{n^n |K|^n}{|K|^{n-1}} = n^n |K| \leqslant n^n |\Delta^n| = \frac{S(\Delta^n)^n}{|\Delta^n|^{n-1}}.$$

If equality holds in Theorem 6.3.1, then the equality case of Theorem 6.3.7 yields that K is congruent to Δ^n.

For Theorem 6.3.2, we use the same argument, only with Theorem 6.3.6 in place of Theorem 6.3.7.

Finally, Petty's condition in Theorem 6.C.1, together with the symmetries of the regular simplex and the cube ensures that both the regular simplex and the cube have minimal surface area among linear images of the same volume. ∎

6.4 Ellipsoid of inertia, Isotropic position of a convex body

6.4.1 Legendre ellipsoid, Isotropic constant

We have already seen how useful some notions of physics are in studies, like the notions of centroid and Wulff shape in Section 1.11 and Section 4.4. Now we consider the so-called ellipsoid of inertia or Legendre ellipsoid that has the same centroid and the same inertia as the corresponding convex body.

For the basic properties of the centroid σ_K of a convex body $K \subset \mathbb{R}^n$, see Section 1.11. In particular, K is centered if the centroid satisfies $\sigma_K = o$.

Definition 6.4.1. Let $K \subset \mathbb{R}^n$ be a centered convex body.

Matrix M_K of inertia: $\langle M_K u, v\rangle = \int_K \langle u, x\rangle\langle v, x\rangle \, dx$ for $u, v \in \mathbb{R}^n$ for the positive definite symmetric $n \times n$ matrix M_K; or in other words, $\langle M_K u, u\rangle = \int_K \langle u, x\rangle^2 \, dx$ for $u \in S^{n-1}$.

Ellipsoid of inertia or *Legendre ellipsoid:* E_K centered ellipsoid such that if $u \in \mathbb{R}^n$, then $\int_{E_K} \langle u, x\rangle^2 \, dx = \int_K \langle u, x\rangle^2 \, dx$.

Remarks.

(i) $E_{\Phi K} = \Phi E_K$ and $M_{\Phi K} = |\det \Phi| \cdot \Phi M_K \Phi^t$ for $\Phi \in \mathrm{GL}(n)$, and hence the existence of E_K can be seen by first transforming M_K into a diagonal matrix.

(ii) If $|K| = 1$ and $d\mu = \mathbf{1}_K d\mathcal{H}^n$, then $M_K = \mathrm{Cov}(\mu)$ is the covariance matrix for the log-concave probability measure μ (cf. Section 4.7).

In this section, we focus on convex bodies in quasi-isotropic position; namely, when the ellipsoid of inertia is a ball.

Definition 6.4.2 (Isotropic position). Let $K \subset \mathbb{R}^n$ be a centered convex body.

Isotropic constant: $L_K = (\det M_K)^{\frac{1}{2n}} / |K|^{\frac{n+2}{2n}}$.

Quasi-isotropic position: K is in quasi-isotropic position if E_K is ball; or equivalently, if $M_K = L_K^2 |K|^{\frac{n+2}{n}} I_n$, which is turn equivalent to saying that

$$\int_K \langle u, x\rangle\langle v, x\rangle \, dx = L_K^2 |K|^{\frac{n+2}{n}} \langle u, v\rangle \quad \text{for } u, v \in \mathbb{R}^n.$$

Isotropic position: K is in isotropic position if $|K| = 1$ and K is in quasi-isotropic position, and hence $M_K = L_K^2 I_n$ and

$$\int_K \langle u, x\rangle^2 \, dx = L_K^2 \quad \text{for } u \in S^{n-1}.$$

Remarks.

(i) $L_{\Phi K} = L_K$ for $\Phi \in \mathrm{GL}(n)$ and centered convex body K.

(ii) It follows from the Remarks after Definition 6.4.1 that every centered convex body K has a linear image that is in isotropic position.

It might be confusing but in this book, we are considering two different types of notions of isotropy. The point is that if $K \subset \mathbb{R}^n$ is centered convex body in isotropic position, and hence $|K| = 1$, then $d\mu = \mathbf{1}_K d\mathcal{H}^n$ is a log-concave probability measure with mean zero, but μ is typically not an isotropic log-concave measure in the sense of Definition 4.7.12 as $\mathrm{Cov}(\mu) = L_K^2 I_n$.

Lemma 6.4.3. *If $K \subset \mathbb{R}^n$ is a centered convex body in isotropic position, and hence $|K| = 1$, then the log-concave probability measure $d\mu = |\widetilde{K}|^{-1} \mathbf{1}_{\widetilde{K}} d\mathcal{H}^n$ has zero mean and is isotropic for $\widetilde{K} = L_K^{-1} K$.*

Proof. On the one hand, $\mathrm{Cov}(\mu) = \lambda I_n$ for $\lambda > 0$ because K is in isotropic position. On the other hand,

$$\mathrm{tr}\,\mathrm{Cov}(\mu) = |\widetilde{K}|^{-1} \int_{\widetilde{K}} \|x\|^2 \, dx = L_K^{-2} \int_K \|x\|^2 \, dx = L_K^{-2} \,\mathrm{tr}\, M_K = n,$$

and hence $\mathrm{Cov}(\mu) = I_n$. ∎

In order to verify the nice characterization Lemma 6.4.5 of the isotropic constant and quasi-isotropic position, we need the following simple consequence of the AM-GM inequality applied to the eigenvalues of a positive definite matrix.

Lemma 6.4.4. *If A is an $n \times n$ positive definite matrix, then $\mathrm{tr}\, A \geqslant n(\det A)^{\frac{1}{n}}$, with equality if and only if $A = \lambda I_n$ for $\lambda > 0$.*

Lemma 6.4.5. *If $K \subset \mathbb{R}^n$ is a centered convex body, then*

$$nL_K^2 |K|^{\frac{n+2}{n}} = \min_{\Phi \in \mathrm{SL}(n)} \int_{\Phi K} \|x\|^2 \, dx = \min_{\Phi \in \mathrm{SL}(n)} \int_K \|\Phi x\|^2 \, dx,$$

and $nL_K^2 |K|^{\frac{n+2}{n}} = \int_K \|x\|^2 \, dx = \int_{E_K} \|x\|^2 \, dx$ if and only if K is in quasi-isotropic position.

Proof. We may assume that $|K| = 1$ and K is in isotropic position, and hence $M_K = L_K^2 I_n$. For any $\Phi \in \mathrm{SL}(n)$, given orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$,

$$\int_{\Phi K} \|x\|^2 \, dx = \sum_{i=1}^{n} \int_{\Phi K} \langle e_i, x\rangle^2 \, dx = \operatorname{tr} M_{\Phi K} = \operatorname{tr}\left(\Phi M_K \Phi^t\right) = L_K^2 \operatorname{tr}\left(\Phi\Phi^t\right) \geqslant n L_K^2$$

by Lemma 6.4.4 applied to $A = \Phi\Phi^t$. If $\int_{\Phi K} \|x\|^2 \, dx = n L_K^2$, then the equality case of Lemma 6.4.4 yields that $\Phi\Phi^t = I_n$, and hence $\Phi \in SO(n)$. ∎

Now, we show that the isotropic constant L_K is minimized by ellipsoids, and hence it is at least a positive absolute constant for any centered convex body K in any dimension.

Proposition 6.4.6. *If $K \subset \mathbb{R}^n$ is a centered convex body, then $L_K \geqslant L_{B^n} > \frac{1}{\sqrt{2e\pi}}$ where $L_K = L_{B^n}$ if and only if K is an ellipsoid.*

Remark. $L_{B^n} \sim \frac{1}{\sqrt{2e\pi}}$ as n tends to infinity as $\Gamma(t+1) \sim \left(\frac{t}{e}\right)^t \sqrt{2\pi t}$.

Proof. We may assume that $|K| = |B^n|$ and K is in quasi-isotropic position. It follows that $|K \backslash B^n| = |B^n \backslash K|$ and $\|x\| > 1 \geqslant \|y\|$ holds for $x \in K \backslash B^n$ and $y \in B^n \backslash K$; therefore, Lemma 6.4.5 yields

$$n L_K^2 |B^n|^{\frac{n+2}{n}} = \int_K \|x\|^2 \, dx \geqslant \int_{B^n} \|x\|^2 \, dx = n L_{B^n}^2 |B^n|^{\frac{n+2}{n}},$$

with equality if and only if $K = B^n$.

To estimate L_{B^n}, we use polar coordinates (cf. (10.13)), and that $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ where $\Gamma(t+1) \geqslant \left(\frac{t}{e}\right)^t \sqrt{2\pi t}$ for $t \geqslant 1$. It follows that

$$L_{B^n}^2 = \frac{\frac{1}{n}\int_{B^n} \|x\|^2 \, dx}{\omega_n^{\frac{n+2}{n}}} = \frac{\frac{1}{n+2}\,\omega_n}{\omega_n^{\frac{n+2}{n}}} = \frac{\Gamma\left(\frac{n}{2}+1\right)^{\frac{2}{n}}}{(n+2)\pi} > \frac{n}{(n+2)2\pi e} \cdot (\pi n)^{\frac{1}{n}} > \frac{1}{2\pi e}.$$

∎

A nice geometric consequence of Proposition 6.4.6 is that the volume of the ellipsoid of inertia is almost at least the volume of the corresponding convex body.

Corollary 6.4.7. *If $K \subset \mathbb{R}^n$ is a centered convex body, then $|E_K| \geqslant |K|$, with equality if and only if K is an ellipsoid.*

Proof. We may assume that $E_K = B^n$, and hence Lemma 6.4.5 yields that

$$n L_K^2 |K|^{\frac{n+2}{n}} = \int_K \|x\|^2 \, dx = \int_{B^n} \|x\|^2 \, dx = n L_{B^n}^2 |B^n|^{\frac{n+2}{n}}.$$

Since $L_K \geqslant L_{B^n}$ by Proposition 6.4.6, we deduce $|K| \leqslant |B^n|$, together with the characterization of equality. ∎

We note that the proof of Proposition 6.4.6 actually yields the following statement.

Lemma 6.4.8. *If $X \subset \mathbb{R}^n$ is a bounded measurable set with $|X| > 0$, then*

$$\int_X \|x\|^2 \, dx \geqslant \frac{n\omega_n^{-\frac{2}{n}}}{n+2} \cdot |X|^{\frac{n+2}{n}},$$

where equality holds if and only if $|X \Delta B| = 0$ for a centered ball B.

The fundamental paper Kannan, Lovász, Simonovits [454], stating the KLS Conjecture 4.7.13, also proved that the ellipsoid of inertia approximates just as well the corresponding centered convex body (cf. (6.10)) as any ellipsoid can do.

Proposition 6.4.9 (Kannan, Lovász, Simonovits). *If a convex body $K \subset \mathbb{R}^n$ is in isotropic position, then*

$$L_K^{\frac{-2}{n+2}} \sqrt{\frac{n+2}{n}} \, B^n \subset K \subset L_K^{\frac{-2}{n+2}} \sqrt{(n+2)n} \, B^n.$$

Remark. In particular, if $K \subset \mathbb{R}^n$ is a centered convex body and $\alpha_n = \sqrt{\frac{n+2}{n}} \cdot \left(\frac{\omega_n}{n+2}\right)^{\frac{1}{n+2}}$ where $\alpha_n \sim \sqrt{2e\pi/n}$ as n tends to infinity, then

$$\alpha_n \cdot E_K \subset K \subset n\alpha_n \cdot E_K. \tag{6.10}$$

Another interesting well-known property of the isotropic constant is that it is linked to the second moment of the volume of a random simplex whose one vertex is the origin (cf. V. Milman, Pajor [582] and Klartag, V. Milman [473]).

Proposition 6.4.10. *For a centered convex body $K \subset \mathbb{R}^n$, we have*

$$L_K^{2n} |K|^{n+2} = \frac{1}{n!} \int_K \dots \int_K \left| \det[x_1, \dots, x_n] \right|^2 dx_1 \dots dx_n.$$

6.4.2 The Slicing Theorem

For the "slices" of a convex body (intersections by hyperplanes), Fradelizi [293] prove the following estimates.

Lemma 6.4.11. *Let $K \subset \mathbb{R}^n$ be a centered convex body with $|K| = 1$.*

(i) $\mathcal{H}^{n-1}(u^\perp \cap K) \geqslant \frac{1}{\sqrt{12}\sqrt{\|M_K u\|}}$ for any $u \in S^{n-1}$ and

$$\max_{t \in \mathbb{R}} \mathcal{H}^{n-1}((tu + u^\perp) \cap K) \leqslant \|M_K u\|^{-\frac{1}{2}};$$

(ii) There exists $u \in S^{n-1}$ with $\mathcal{H}^{n-1}(u^\perp \cap K) \geqslant \frac{1}{\sqrt{12} L_K}$;

(iii) If K is in isotropic position, then $\mathcal{H}^{n-1}(u^\perp \cap K) \geqslant \frac{1}{\sqrt{12}L_K}$ for any $u \in S^{n-1}$, and $\max_{t\in\mathbb{R}} \mathcal{H}^{n-1}((tu + u^\perp) \cap K) \leqslant L_K^{-1}$.

Proof. Apply (4.72), (4.76) and (4.77) to $d\mu = \mathbf{1}_K d\mathcal{H}^n$ to get (i). Since $\det M_K = L_K^{2n}$ (cf. Definition 6.4.2), (iii) directly follows from (i), and for (ii), we can choose u to correspond to the minimal eigenvalue of M_K (that is at most L_K^2). ∎

Independently, Bourgain [151] and Keith Ball [52] posed the following fundamental so-called Slicing Conjecture in 1986 that has been the driving force behind an incredible progress within the field "Asymptotic Geometric Analysis", and has been finally resolved by Klartag, Lehec [470] in 2024.

Theorem 6.4.12 (Slicing Theorem). *For an absolute constant $c > 0$, if $K \subset \mathbb{R}^n$, $n \geqslant 2$, is a convex body with $|K| = 1$, then there exists a hyperplane H with*

$$\mathcal{H}^{n-1}(H \cap K) \geqslant c.$$

Remark. It is equivalent saying that there exists an absolute constant $C > 1$ such that

$$\frac{1}{\sqrt{2e\pi}} < L_K < C \tag{6.11}$$

for any $n \geqslant 2$ and centered convex body $K \subset \mathbb{R}^n$ (cf. Proposition 6.4.6 and Lemma 6.4.11).

The importance of the Slicing Theorem is exhibited by the fact how many more equivalent formulations it has (see for example Klartag, V. Milman [473] and Brazitikos, Giannopoulos, Valettas, Vritsiou [157]). Here we list just some of them.

Remark 6.4.13 (Consequences of the Slicing Theorem). There exists an absolute constant $c > 1$ such that if $K \subset \mathbb{R}^n$ is a centered convex body with $|K| = 1$, then the following properties hold.

- For the Legendre ellipsoid (ellipsoid of inertia) E_K, we have

  $$|E_K \cap K| \geqslant c^{-1} \text{ and } |E_K| \leqslant c^n,$$

  and if $\widetilde{E}_K$ is the rescaled copy of E_K with $|\widetilde{E}_K| = |K| = 1$, then $\widetilde{E}_K$ is an M-ellipsoid (cf. Section 6.9); namely,

  $$|K \cap \widetilde{E}_K| \geqslant c^{-n} \text{ and } |K^* \cap \widetilde{E}_K^*| \geqslant c^{-n}|\widetilde{E}_K^*| \geqslant c^{-n}|K^*|.$$

- If K is in isotropic position, then for any $u \in S^{n-1}$, we have (cf. (6.11) and Lemma 6.4.11)

  $$c^{-1} \leqslant \mathcal{H}^{n-1}(u^\perp \cap K) \leqslant c.$$

- The second moment of the determinant of n independently chosen random vectors in K according to the uniform distribution is between $c^{-2n}n^{-n/2}$ and $c^{2n}n^{-n/2}$ (cf. Proposition 6.4.10).
- The expected volume of the convex hull of $n+1$ independently chosen random points in K according to the uniform distribution is between $c^{-n}n^{-n/2}$ and $c^{n}n^{-n/2}$.
- Combining Paouris [612] and (6.11) yields that if the convex body K is in isotropic position and $t \geqslant 1$, then

$$\left|\{x \in K : \|x\| \geqslant ct\sqrt{n}\}\right| \leqslant e^{-\sqrt{n}\,t}.$$

Here if $|r_n B^n| = 1$ for $r_n > 0$, then $r_n \sim \sqrt{n}/\sqrt{2e\pi}$ for large n according to the Stirling formula (cf. (10.1)).

A strengthened version of Theorem 6.4.12 is the Strong Slicing Conjecture, see, for example, Klartag, Lehec [470] and Fradelizi, Sola [304].

Conjecture 6.4.14 (Strong Slicing Conjecture). *If $\Delta^n \subset \mathbb{R}^n$ is a centered simplex and $K \subset \mathbb{R}^n$ is a centered convex body, then*

$$L_K \leqslant L_{\Delta^n} = \frac{(n!)^{\frac{1}{n}}}{(n+1)^{\frac{n+1}{2n}}\sqrt{n+2}}, \tag{6.12}$$

$$L_K \leqslant L_{[-1,1]^n} = \frac{1}{\sqrt{12}} \quad \text{if } K \text{ is o-symmetric}. \tag{6.13}$$

We note that the Strong Slicing Conjecture (6.12) for non-symmetric convex bodies yields the Mahler conjecture Conjecture 6.6.2 for the polars of non-symmetric convex bodies (see Klartag [467]); namely, if $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then

$$|K| \cdot |K^*| \geqslant |\Delta^n| \cdot |(\Delta^n)^*|.$$

On the other hand, the Strong Slicing Conjecture for o-symmetric bodies (6.13) implies the more than a century old Minkowski conjecture about lattices (see Autissier [47] and Magazinov [556]). We recall that a lattice $L \subset \mathbb{R}^n$ is of the form $\Phi\mathbb{Z}^n$ for a $\Phi \in \mathrm{GL}(n)$ and $\det L = |\det \Phi|$. Now the Minkowski conjecture - that is open for $n \geqslant 11$ (see Aranov [34]), - states that if $L \subset \mathbb{R}^n$ is a lattice with $\det L = 1$, and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then there exists a $y = (y_1, \ldots, y_n) \in L$ such that

$$\prod_{i=1}^{n} |x_i - y_i| \leqslant 2^{-n}.$$

6.4.3 Cheeger constant for a convex body and a Poincaré-type inequality

Let $K \subset \mathbb{R}^n$ be a convex body. We prove the essentially optimal Poincaré-type inequalities (6.18) and (6.24) due to Kolesnikov, E. Milman [480] involving the Cheeger

constant assigned to the uniform log-concave probability measure $\mu = \mu_K$ associated to the convex body K; namely, $d\mu = |K|^{-1}\mathbf{1}_K\, d\mathcal{H}^n$. Abusing the notation introduced in Section 4.7, we set $C_{\mathrm{Che}}(K) = C_{\mathrm{Che}}(\mu) > 0$ for the Cheeger constant; namely, $C_{\mathrm{Che}}(K) > 0$ is minimal such that for every $X \subset \operatorname{int} K$ locally of finite perimeter, we have (see (4.62) for the notion of Cheeger constant, and Section 5.1.1 for the notion of reduced boundary of a set of locally finite perimeter)

$$C_{\mathrm{Che}}(K) \cdot \mathcal{H}^{n-1}\left((\partial^* X) \cap \operatorname{int} K\right) \geqslant \min\{|X|, |K| - |X|\}. \tag{6.14}$$

It follows from the definition that $C_{\mathrm{Che}}(K)$ is translation invariant, and if $\lambda > 0$, then

$$C_{\mathrm{Che}}(\lambda K) = \lambda\, C_{\mathrm{Che}}(K). \tag{6.15}$$

Remark 6.4.15. (i) To determine $C_{\mathrm{Che}}(K)$, it is enough to consider the case when $|X| = \frac{1}{2}|K|$ in (6.14) (see E. Milman [577] and Sternbergand, Zumbrun [693]).

(ii) According to (4.63), we have

$$C_{\mathrm{Che}}(B^n) = \frac{\omega_n}{2\omega_{n-1}} < \sqrt{\frac{\pi}{2n}}. \tag{6.16}$$

(iii) If $|K| = 1$ and K is centered, then the minimal eigenvalue of M_K is at most L_K^2 by $\det M_K = L_K^{2n}$ (cf. Definition 6.4.2), and hence (4.76) and Proposition 6.4.6 yield that

$$C_{\mathrm{Che}}(K) \geqslant \frac{\sqrt{2}}{e}\, L_K > \frac{1}{\sqrt{e^3\pi}}.$$

For a convex body $K \subset \mathbb{R}^n$ and a Lipschitz function $f : K \to \mathbb{R}$, we consider the median m_f with respect to $\mu = \mu_K$; namely,

$$|\{f > m_f\}| \leqslant |K|/2 \quad \text{and} \quad |\{f < m_f\}| \leqslant |K|/2.$$

In particular, we deduce from Theorem 4.7.8 that

$$\int_K |f - m_f|\, d\mathcal{H}^n \leqslant C_{\mathrm{Che}}(K) \cdot \int_K \|Df\|\, d\mathcal{H}^n. \tag{6.17}$$

We recall that $\partial' K$ denotes the family of regular points of ∂K; namely, the family of $\mathcal{H}^{n-1}$ a.e. points $x \in \partial K$ where there exists a unique exterior unit normal $\nu_K(x)$ (see Section 1.5). Now we consider a version of (6.17) involving ∂K.

Lemma 6.4.16 (Kolesnikov-Milman). *If $rB^n \subset K \subset RB^n$ holds for a convex body $K \subset \mathbb{R}^n$ and $R \geqslant r > 0$, and $f : K \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial K} |f - m_f|\, d\mathcal{H}^{n-1} \leqslant \frac{nC_{\mathrm{Che}}(K) + R}{r} \cdot \int_K \|Df\|\, d\mathcal{H}^n. \tag{6.18}$$

Remark. Figalli, Maggi, Pratelli [277] proved a version of (6.18) with explicit factor $\frac{\sqrt{2}\,nR}{\log 2 \cdot r}$ instead of $\frac{nC_{\mathrm{Che}}(K)+R}{r}$. The advantage of (6.18) is that K has an affine image $\widetilde{K}$ such that $C_{\mathrm{Che}}(\widetilde{K}) \leqslant c(\log n)^{\frac{1}{2}}$ and $\tilde{r}B^n \subset \widetilde{E} \subset \tilde{R}B^n$ with $\tilde{r} \geqslant c'(\log n)^{-\frac{1}{2}}$ and $\tilde{R} \leqslant n\tilde{r}$ for absolute constants $c, c' > 0$ (see (6.21) and (6.22)).

Proof. Since $\langle x, \nu_K(x)\rangle = h_K(\nu_K(x)) \geqslant r$ for $x \in \partial' K$, we deduce by applying the Divergence Theorem 2.1.4 to the Lipschitz function $x \mapsto |f(x) - m_f| \cdot x$ and by integration by parts that

$$\begin{aligned}\int_{\partial K} |f - m_f| \, d\mathcal{H}^{n-1} &\leqslant \frac{1}{r}\int_{\partial K} |f(x) - m_f| \langle x, \nu_K(x)\rangle \, d\mathcal{H}^{n-1}(x)\\ &= \frac{1}{r}\int_K \mathrm{div}\,(|f(x) - m_f| \cdot x)\, dx\\ &= \frac{1}{r}\int_K n\,|f(x) - m_f|\, dx + \frac{1}{r}\int_K \langle D|f(x) - m_f|, x\rangle\, dx.\end{aligned}$$

Here the first term is at most $nC_{\mathrm{Che}}(K)/r$ by (6.17). For the second term, we observe that $D|f(x) - m_f| = 0$ if $f(x) - m_f = 0$ and $f(x) - m_f$ is differentiable at $x \in \mathrm{int}K$. On the other hand, if $f(x) - m_f \neq 0$ and $f(x) - m_f$ is differentiable at an $x \in \mathrm{int}K$, then $\|D|f(x) - m_f|\| = \|Df(x)\|$. Since $\|x\| \leqslant R$ for $x \in K$, we conclude (6.18). ∎

We deduce from (6.16), (6.18) and $n\sqrt{\frac{\pi}{2n}} + 1 < 2\sqrt{n}$ for $n \geqslant 2$ the following inequality for balls:

Corollary 6.4.17. *If $f : B^n \to \mathbb{R}$ is Lipschitz, then*

$$\int_{S^{n-1}} |f - m_f| \, d\mathcal{H}^{n-1} \leqslant 2\sqrt{n} \cdot \int_{B^n} \|Df\| \, d\mathcal{H}^n. \tag{6.19}$$

Remark. The estimate in (6.19) is close to be sharp, as there exists a Lipschitz function $f : B^n \to \mathbb{R}$ such that

$$\int_{S^{n-1}} |f - m_f| \, d\mathcal{H}^{n-1} \geqslant \sqrt{n} \cdot \int_{B^n} \|Df\| \, d\mathcal{H}^n. \tag{6.20}$$

To verify (6.20), fix $u \in S^{n-1}$, and consider the Lipschitz function $f_\varrho : B^n \to \mathbb{R}$ for small $\varrho > 0$ such that if $x \in B^n$, then $f_\varrho(x) = 1$ if $\langle x, u\rangle \geqslant \varrho$, $f_\varrho(x) - 1$ if $\langle x, u\rangle \leqslant -\varrho$, and $f_\varrho(x) = \frac{\langle x,u\rangle}{\varrho}$ if $|\langle x, u\rangle| \leqslant \varrho$. It follows that $m_{f_\varrho} = 0$. Since $\frac{n\omega_n}{2\omega_{n-1}} > \sqrt{n}\sqrt{\frac{\pi n}{2(n+1)}} > \sqrt{n}$ according to (10.3), we conclude (6.20) for $f = f_\varrho$ and small enough $\varrho > 0$.

The optimal factor in (6.19) is most probably $\frac{n\omega_n}{2\omega_{n-1}}$. Our argument yields the factor $\sqrt{\frac{\pi}{2}}\sqrt{n} + 1$ in (6.19), and $\sqrt{\frac{\pi}{2}}\sqrt{n} - 1$ for large n in (6.20).

According Lemma 6.4.3, for any convex body $K \subset \mathbb{R}^n$, there exists a $\Phi \in \mathrm{GL}(n)$ such that the (log-concave) uniform probability measure $\tilde{\mu}$ on $\widetilde{K} = \Phi(K - \sigma_K)$ is

isotropic; namely, $d\tilde{\mu} = |\widetilde{K}|^{-1}\mathbf{1}_{\widetilde{K}}\, d\mathcal{H}^n$, the mean satisfies $\int_{\mathbb{R}^n} x\, d\tilde{\mu}(x) = o$ and $\operatorname{cov}\tilde{\mu} = I_n$ (see also Section 4.7). In this case, Klartag [468] proves (cf. Theorem 4.7.14) that

$$C_{\mathrm{Che}}(\widetilde{K}) = C_{\mathrm{Che}}(\tilde{\mu}) \leqslant c(\log n)^{\frac{1}{2}} \tag{6.21}$$

for an absolute constant $c > 0$. In addition, we deduce from combining Lemma 6.4.3 and Proposition 6.4.9 about the Kannan, Lovász, Simonovits ellipsoid that

$$\tilde{r}B^n \subset \widetilde{K} \subset \tilde{R}B^n \text{ where } \tilde{r} \geqslant L_K^{-\frac{n+4}{n+2}} \text{ and } \tilde{r} \leqslant \tilde{R} \leqslant n\tilde{r}. \tag{6.22}$$

Here the Slicing Theorem 6.4.12 due to Klartag, Lehec [470] implies that

$$\tilde{r} \geqslant c' \tag{6.23}$$

for some absolute constant $c' > 0$. The main result of this section is the following estimate.

Proposition 6.4.18 (Kolesnikov-Milman). *If $K \subset \mathbb{R}^n$ is a centered convex body in quasi-isotropic position, and $f : K \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial K} |f - m_f|\, d\mathcal{H}^{n-1} \leqslant cn\sqrt{\log n} \cdot \int_K \|Df\|\, d\mathcal{H}^n \tag{6.24}$$

for an absolute constant $c > 0$.

Remark. Assuming the KLS conjecture (see Section 4.7), the argument below yields (6.24) without the logarithmic factor, which would be the optimal estimate (see Example 6.4.19).

Proof. It follows from the discussion above that we may assume that the log-concave uniform probability measure μ on K with $d\mu = |K|^{-1}\mathbf{1}_K\, d\mathcal{H}^n$ is isotropic. In this case, combining Lemma 6.4.16, (6.21), (6.22) and (6.23) yields (6.24). ∎

Example 6.4.19. If $K \subset \mathbb{R}^n$ is the regular simplex circumscribed around B^n (and hence K is a centered convex body in quasi-isotropic position), then there exists a Lipschitz function $f : K \to \mathbb{R}$ such that

$$\int_{\partial K} |f - m_f|\, d\mathcal{H}^{n-1} \geqslant \frac{n}{2} \cdot \int_K \|Df\|\, d\mathcal{H}^n. \tag{6.25}$$

We fix an exterior unit normal u to a facet F of K, and let $t = 1 - (n+1)(1 - 2^{\frac{-1}{n}}) > 0$, and hence $|\{x \in K : \langle x, u\rangle \leqslant t\}| = \frac{1}{2}|K|$. For small $\varrho > 0$, we consider the Lipschitz function $f_\varrho : K \to \mathbb{R}$ such that if $x \in K$, then

$$f_\varrho(x) = \begin{cases} 1 & \text{if } \langle x, u\rangle \geqslant t + \varrho; \\ \frac{\langle x,u\rangle - t}{\varrho} & \text{if } t - \varrho \leqslant \langle x, u\rangle \leqslant t + \varrho; \\ -1 & \text{if } \langle x, u\rangle \leqslant t - \varrho. \end{cases}$$

It follows that $m_{f_\varrho} = 0$, the set $\{x \in K : \langle x, u\rangle = t\}$ has $\mathcal{H}^{n-1}$-measure $2^{\frac{1-n}{n}}\mathcal{H}^{n-1}(F)$ and $\|Df_\varrho(x)\| = \varrho^{-1}$ if $x \in \operatorname{int} K$ and $t - \varrho < \langle x, u\rangle < t + \varrho$; therefore,

$$\lim_{\varrho\to 0^+} \int_{\partial K} \left|f_\varrho - m_{f_\varrho}\right| \, d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial K) = (n+1)\mathcal{H}^{n-1}(F);$$

$$\lim_{\varrho\to 0^+} \int_K \|Df_\varrho\| \, d\mathcal{H}^n = 2 \cdot 2^{\frac{1-n}{n}}\mathcal{H}^{n-1}(F) = 2^{\frac{1}{n}}\,\mathcal{H}^{n-1}(F).$$

Finally, we improve on (6.24) if the convex body K is close to be a ball.

Proposition 6.4.20. *If $K \subset \mathbb{R}^n$ is a convex body with $|K| = |B^n|$ and $|K\Delta B^n| \leqslant (4n)^{-2n}|B^n|$, and $f : K \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial K} |f - m_f| \, d\mathcal{H}^{n-1} \leqslant 8\sqrt{n} \cdot \int_K \|Df\| \, d\mathcal{H}^n. \tag{6.26}$$

Remark. The estimate in (6.26) is essentially optimal according to (6.20).

Proof. We may assume that $K \neq B^n$, and observe that $|K\backslash B^n| = |B^n\backslash K| \leqslant (4n)^{-2n}|B^n|$. Let $r \in (0,1)$ be maximal with the property that $rB^n \subset K$; therefore, $B^n\backslash K$ contains a circular cone having height $1 - r$ and radius $\sqrt{1-r}$ of the base. We deduce that $|B^n\backslash K| \geqslant \frac{\omega_{n-1}}{n}(1-r)^{\frac{n+1}{2}}$, and hence $r \geqslant 1 - \frac{1}{4n}$, which in turn yields that $K \subset 2B^n$.

According to Lemma 6.4.16, Proposition 6.4.20 follows from the estimate

$$C_{\mathrm{Che}}(K) \leqslant \frac{4}{\sqrt{n}}. \tag{6.27}$$

To estimate $C_{\mathrm{Che}}(K)$, it is sufficient to consider any closed set $X \subset K$ with $|X| = \frac{1}{2}|B^n|$ and locally Lipschitz boundary according to Remark 6.4.15. For $X_0 = X \cap rB^n$, $\frac{1}{4}|rB^n| < |X_0| < \frac{3}{4}|rB^n|$ follows from $|Y\backslash(rB^n)| \leqslant |K\backslash(rB^n)| = (1-r^n)|B^n|$ for $Y = X$ and $Y = K\backslash X$; therefore, the estimate $C_{\mathrm{Che}}(rB^n) < r\sqrt{\frac{\pi}{2n}}$ (cf. (6.15) and (6.16)) and the inequality (6.14) yield that

$$\mathcal{H}^{n-1}(\partial X \cap \operatorname{int} K) \geqslant \mathcal{H}^{n-1}(\partial X_0 \cap \operatorname{int}(rB^n)) \geqslant r^{-1}\sqrt{\frac{2n}{\pi}} \cdot |rB^n| > \frac{\sqrt{n}}{2}\,|B^n|.$$

In turn, we conclude (6.27). ∎

Various additional aspects of the interrelation among Poincarè type inequalities, the Cheeger constant and the Slicing Theorem is discussed by Klartag, Lehec [469].

Remark 6.4.21 (Isoperimetric problem in an open bounded convex set). Given a convex body $K \subset \mathbb{R}^n$ with C^2 boundary and $0 < t < |K|$, the theory of minimal surfaces asks for the subset $X \subset \operatorname{int} K$ of finite perimeter minimizing $\mathcal{H}^{n-1}\left((\partial X) \cap \operatorname{int} K\right)$ under the condition $|X| = t$. According to classical results, see the surveys Ros [634] and Cozzi, Figalli [225], there exists a minimizing $X \subset \operatorname{int} K$. Moreover; except for a

relatively closed singular set of Hausdorff dimension at most $n-8$, $(\partial X) \cap \operatorname{int} K$ is a smooth embedded hypersurface with constant mean curvature meeting ∂K orthogonally. For example, when $K = B^n$, and the minimizing X is the intersection with a ball if $t < \omega_n/2$, and is a half ball (intersection with a half-space) if $t = \omega_n/2$, which facts lead to (6.16) (see (4.63) for more details).

6.5 The Blaschke-Santaló inequality for the polar body

This section discusses the fundamental Blaschke-Santaló inequality (6.31) for a convex body and its polar. The main reasons why this linearly invariant inequality is discussed in this chapter about associated ellipsoids are that extremal bodies are centered ellipsoids, and the characterization of the equality case is greatly simplified by using the ellipsoids of inertia from Section 6.4 (see Section 6.E for the argument).

We recall from Section 1.9 the definition of the polar body. If $K \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int} K$, then

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for } y \in K\}.$$

As we have seen in Section 1.9, K^* convex body with $o \in \operatorname{int} K^*$, and it satisfies $(K^*)^* = K$, $\varrho_{K^*}(u) = \frac{1}{h_K(u)}$ for $u \in S^{n-1}$, and $(\Phi K)^* = \Phi^{-t} K^*$ for $\Phi \in \mathrm{GL}(n)$. In particular, $|K| \cdot |K^*|$ is called the volume product, which is invariant under linear transformations as $|\Phi K| \cdot |(\Phi K)^*| = |K| \cdot |K^*|$ for $\Phi \in \mathrm{GL}(n)$.

Lemma 6.5.1 (Santaló point). *For a convex body $K \subset \mathbb{R}^n$, there exists a unique so-called Santaló point $\aleph_K \in \operatorname{int} K$ that minimizes the strictly convex function $\xi \mapsto |(K-\xi)^*|$ for $\xi \in \operatorname{int} K$, and satisfies*

$$\int_{S^{n-1}} \frac{u}{(h_K(u) - \langle \aleph_K, u\rangle)^{n+1}}\, du = 0. \tag{6.28}$$

In particular, $\aleph_K = o$ if and only if $\sigma_{K^*} = o$ for the centroid of the polar.

Remarks.

- The Santaló point is affine equivariant, and hence it is the origin if K is o-symmetric.
- For a hyperplane $H \subset \mathbb{R}^n$ with a normal vector $v \in S^{n-1}$ and satisfying $H \cap \operatorname{int} K \neq \varnothing$, the same argument yields the existence of a unique $\aleph_{K,H} \in H \cap \operatorname{int} K$ minimizing $\xi \mapsto |(K-\xi)^*|$ for $\xi \in H \cap \operatorname{int} K$, where $z = \aleph_{K,H}$ for $z \in H \cap \operatorname{int} K$ if and only if

$$\int_{(K-z)^*} x\, dx \in \mathbb{R} v. \tag{6.29}$$

Proof. Since $h_{K-\xi}(u) = h_K(u) - \langle \xi, u\rangle$ for $\xi \in \operatorname{int} K$, Lemma 1.11.6 yields

$$g_K(\xi) = |(K-\xi)^*| = \frac{1}{n}\int_{S^{n-1}} \varrho_{(K-\xi)^*}(u)^n \, du = \frac{1}{n}\int_{S^{n-1}} (h_K(u) - \langle \xi, u\rangle)^{-n} \, du.$$

We claim that if $\xi_m \in \operatorname{int} K$ tends to a $z_0 \in \partial K$, then

$$\lim_{m\to\infty} g_K(\xi_m) = \infty. \tag{6.30}$$

For $D = \operatorname{diam} K$ and an exterior unit normal u_0 at z_0, we consider the half-ball $K_0 = \{x \in z_0 + DB^n : \langle x, u_0\rangle \leqslant \langle z_0, u_0\rangle\} \supset K$, and hence it is sufficient to prove (6.30) for $g_{K_0} \leqslant g_K$ instead of g_K. For the hemisphere $S_+ = \{x \in S^{n-1} : \langle x, u_0\rangle > 0\}$ and $B^{n-1} = B^n \cap u_0^\perp$, we have $h_{K_0}(u) - \langle z_0, u\rangle = D\|(u-u_0)|u_0^\perp\|$ for $u \in S_+$, thus (cf. (2.1))

$$\int_{S_+} (h_{K_0}(u) - \langle z_0, u\rangle)^{-n} \, du \geqslant D^{-n} \int_{B^{n-1}} \|y\|^{-n} \, dy = \infty.$$

We conclude (6.30) for g_{K_0} instead of g_K, and in turn for g_K, as well.

Since g_K is strictly convex (as $t \mapsto t^{-n}$ strictly convex), (6.30) yields the existence of a a unique $\aleph_K \in \operatorname{int} K$ minimizing g_K, and hence (6.28) follows from $Dg_K(\aleph_K) = o$. Finally, if $\aleph_K = o$, then (6.28) is equivalent to $\sigma_{K^*} = o$ by $\varrho_{K^*} = 1/h_K$ and Lemma 1.11.6. ∎

We note that (6.30) in the proof of Lemma 6.5.1 is not trivial, as it does not hold if $K = K_0$ and $g_K(\xi)$ is replaced by $\int_{S^{n-1}} (h_K(u) - \langle \xi, u\rangle)^{-\alpha} \, du$ for an $\alpha < n-1$.

The main statement of this section is the following inequality proved by Blaschke [97] in dimensions $n = 2, 3$ around 1920, and by Santaló [642] in any dimensions in 1949. The equality case was clarified by Petty [619] in 1985.

Theorem 6.5.2 (Blaschke-Santaló inequality). *If the centroid or the Santaló point of a convex body $K \subset \mathbb{R}^n$ is the origin, then*

$$|K| \cdot |K^*| \leqslant |B^n|^2, \tag{6.31}$$

with equality if and only if K is a centered ellipsoid.

As $\varrho_{K^*} = h_K^{-1}$ according to (1.21), and $|K^*| = \frac{1}{n}\int_{S^{n-1}} \varrho_{K^*}^n \, d\mathcal{H}^{n-1}$ follows from using polar coordinates (cf. Lemma 1.11.6), a useful equivalent form of the Blaschke-Santaló inequality (6.31) is that if $K \subset \mathbb{R}^n$ is a centered convex body, then

$$\int_{S^{n-1}} h_K^{-n} \, d\mathcal{H}^{n-1} \leqslant \frac{n|B^n|^2}{|K|}. \tag{6.32}$$

We prove the Blaschke-Santaló inequality Theorem 6.5.2 for general convex bodies in Section 6.E. Here we only provide the simple argument in Keith Ball's PhD thesis [52] in the o-symmetric case.

Let us recall the definition of the Steiner symmetrization. For $u \in S^{n-1}$, and convex body $K \subset \mathbb{R}^n$, the Steiner symmetrial $\Theta_{u^\perp} K$ of K is (see Section 1.10)

$$\Theta_{u^\perp} K = \left\{ x + \frac{t-s}{2} \cdot u : x \in K | H \,\&\, x + tu \in K \,\&\, x + su \in K \right\},$$

which readily satisfies $|\Theta_{u^\perp} K| = |K|$. We also need the Brunn-Minkowski inequality (1.37) yielding that

$$|\tfrac{1}{2} C + \tfrac{1}{2} (-C)| \geqslant |C| \tag{6.33}$$

for any convex body $C \subset \mathbb{R}^n$ with equality if and only if C and $-C$ are translates; or equivalently, if and only if C is centrally symmetric.

Proposition 6.5.3 (Keith Ball [52]). *If $K \subset \mathbb{R}^n$ o-symmetric convex body and $u \in S^{n-1}$, then $|(\Theta_{u^\perp} K)^*| \geqslant |K^*|$. where $|(\Theta_{u^\perp} K)^*| = |K^*|$ implies that any section of K^* by a hyperplane parallel to $u^\perp$ is centrally symmetric.*

Proof. Let $\widetilde{K} = \Theta_{u^\perp} K$, and for o-symmetric convex body C and $t \in \mathbb{R}$, let

$$C_t^* = \{x \in u^\perp : x + tu \in C^*\}.$$

The core observation is the claim that if $t \in \mathbb{R}$, then

$$\frac{1}{2}(K_t^* + K_{-t}^*) \subset \widetilde{K}_t^*, \tag{6.34}$$

which is equivalent to saying that if $x + tu \in K^*$, $y - tu \in K^*$ and $z + su \in \widetilde{K}$ for $x, y, z \in u^\perp$ and $s \in \mathbb{R}$, then

$$\left\langle \frac{x+y}{2} + tu, z + su \right\rangle \leqslant 1. \tag{6.35}$$

We have $s = \frac{s_1 - s_2}{2}$ where $z + s_1 u \in K$, $z + s_2 u \in K$ by the definition of Steiner symmetrization, and hence $\langle x + tu, z + s_1 u \rangle \leqslant 1$ and $\langle y - tu, z + s_2 u \rangle \leqslant 1$; or in other words, $\langle x, z \rangle + t s_1 \leqslant 1$ and $\langle y, z \rangle - t s_2 \leqslant 1$. We deduce (6.35), and in turn (6.34).

Now $K = -K$ yields $K^* = -K^*$, and hence $K_{-t}^* = -K_t^*$ for $t \in \mathbb{R}$. It follows from (6.93) and Brunn-Minkowski inequality (6.33) that if $t \in \mathbb{R}$, then using $\mathcal{H} = \mathcal{H}^{n-1}$ to denote the $(n-1)$-dimensional Hausdorff measure, we have

$$\mathcal{H}(\widetilde{K}_t^*) \geqslant \left(\frac{1}{2} \mathcal{H}(K_t^*)^{\frac{1}{n-1}} + \frac{1}{2} \mathcal{H}(K_{-t}^*)^{\frac{1}{n-1}} \right)^{n-1} = \mathcal{H}(K_t^*). \tag{6.36}$$

We conclude from the Fubini theorem that $|(\Theta_{u^\perp} K)^*| = |\widetilde{K}^*| \geqslant |K^*|$.

If $|(\Theta_{u^\perp} K)^*| = |K^*|$, then equality in (6.36) and the equality condition in the Brunn-Minkowski inequality (6.33) imply that K_t^* is centrally symmetric for $t \in \mathbb{R}$. ∎

Proof the Blaschke-Santaló inequality Theorem 6.5.2 for o-symmetric bodies: We may assume that $|K| = |B^n|$ for an o-symmetric convex body K. Since iterated Steiner symmetrisations applied to K may lead to a centered ball of the same volume according to Theorem 1.10.7, Proposition 6.5.3 yields that $|K^*| \leqslant |(B^n)^*| = |B^n|$.

If $|K^*| = |(B^n)^*|$, then we deduce from Proposition 6.5.3 that any hyperplane section parallel to $u^\perp$ of K^* is centrally symmetric for any $u \in S^{n-1}$, and hence Brunn's Theorem 6.2.2 yields that K^* is a centered ellipsoid, which in turn implies that $K = (K^*)^*$ is a centered ellipsoid. ∎

The proof of Brunn's Theorem 6.2.2 characterizing ellipsoids - that is used above in order to characterize equality in the Blaschke-Santaló inequality Theorem 6.5.2, - is rather technical, and we do not provide it in this book. Instead, we present a simpler argument to characterize equality in the Blaschke-Santaló inequality using the ellipsoid of inertia in Section 6.E.

6.6 Reverse Blaschke-Santaló inequality and Mahler's conjecture

We recall that according to the Blaschke-Santaló inequality (6.31), if the centroid or the Santaló point of a convex body $K \subset \mathbb{R}^n$ is the origin, then $|K| \cdot |K^*| \leqslant |B^n|^2$. For a reverse inequality for any convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, let $\aleph_K \in \operatorname{int} K$ be the Santaló point (cf. Lemma 6.5.1), and hence the orgin is the centroid of $(K - \aleph_K)^*$. We deduce the existence of a KLS-ellipsoid $E \subset (K - \aleph_K)^*$ such that $(K - \aleph_K)^* \subset nE$ (cf. (6.10)), and hence $\frac{1}{n} E^* \subset K - \aleph_K$ and

$$|K| \cdot |K^*| \geqslant |K - \aleph_K| \cdot |(K - \aleph_K)^*| \geqslant n^{-n} |E^*| \cdot |E| = n^{-n} |B^n|^2 \tag{6.37}$$

by the linear invariance of the volume product (cf. Proposition 1.9.2). In this section, improvements of (6.37) are discussed, mostly without the rather involved proofs.

Any inequality of the form $|K| \cdot |K^*| \geqslant c^n |B^n|^2$, where $c \in (0, 1)$ is an absolute constant, holding for any convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ (or at least for any o-symmetric K) is called a Reverse Blaschke-Santaló inequality. The first such inequality is due to Bourgain, V. Milman [153] in 1987, and the best estimate is the following statement proved by G. Kuperberg [490] (see the arxiv version for the case of non-symmetric convex bodies).

Theorem 6.6.1 (Reverse Blaschke-Santaló inequality). *If $K \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int} K$, then*

$$|K| \cdot |K^*| > 4^{-n} |B^n|^2, \tag{6.38}$$

and even $|K| \cdot |K^*| > \frac{\pi^n}{n!} > 2^{-n} |B^n|^2$ provided $K = -K$.

The exact minimum of the volume product in $\mathbb{R}^n$ is a classical conjecture attributed to Mahler. More precisely, Mahler [560] from 1939 only stated Conjecture 6.6.2 in the

o-symmetric case. But since Mahler did settle the 2-dimensional case concerning the extremality of simplices in his 1938 paper [559], he must have been aware of the possibility that centered simplices are extremal in higher dimensional spaces, as well.

Conjecture 6.6.2 (Mahler, 1939). *Let $K \subset \mathbb{R}^n$ be convex body with $o \in \mathrm{int}K$.*

- $|K| \cdot |K^*| \geqslant \frac{(n+1)^{n+1}}{(n!)^2}$, *with equality if and only if K is a centered simplex.*
- *If, in addition, $K = -K$, then $|K| \cdot |K^*| \geqslant \frac{4^n}{n!}$, with equality if and only if K is a Hanner polytope (see below, for example, $K = [-1,1]^n$ is a Hanner polytope).*

Remark 6.6.3 (Hanner polytopes). A Hanner polytope is an o-symmetric convex polytope defined by induction on $n \geqslant 1$. If $n = 1$, then any o-symmetric segment is a Hanner polytope. If L and L' are complementary linear subspaces of $\mathbb{R}^n$, $n \geqslant 2$, and $P \subset L$ and $P' \subset L'$ are Hanner polytopes, then $P + P'$ and $\mathrm{conv}\{P, P'\}$ are Hanner polytopes in $\mathbb{R}^n$. For example, the cube $W_n = [-1,1]^n$, and its polar, the cross-polytope $C_n = \mathrm{conv}\{\pm e_1, \ldots, \pm e_n\}$ are Hanner polytopes, and if $K \subset \mathbb{R}^n$ is a Hanner polytope, then K^* is also a Hanner polytope and $|K| \cdot |K^*| = |W_n| \cdot |W_n^*|$. However, if $n = 4$, then a cylinder over the octahedron is a Hanner polytope that is different from the linear images of W_n or C_n.

The Mahler Conjecture is still open after more than 80 years of intensive research.

Remark 6.6.4 (Some known cases of the Mahler conjecture).

- $n = 2$ (Mahler [559] in 1938).
- $n = 3$ and $K = -K$ (Iriyeh, Shibata [434] in 2020, see Fradelizi, Hubard, Meyer, Roldán-Pensado, Zvavitch [300] for a simpler argument).
- K unconditional (Saint-Raymond [643] in 1980, see Theorem 6.6.5 for the elegant argument due to Meyer [574]).
  Equality holds in this case if and only if K is a Hanner polytope.
- K zonoid (cf. Example 1.6.3) (Reisner [631] in 1985). Equality holds in this case if and only if K is a centered parallelopiped (linear image of $[-1,1]^n$).
- K has n independent hyperplane symmetries (Barthe, Fradelizi [71] in 2013).

We remark that further progress was achieved by the paper by Reisner, Schütt and Werner [632], which attempts to understand the boundary structutre of a minimizer of the volume product. According to Klartag [467], the Strong Slicing Conjecture (6.12) for non-symmetric convex bodies would yield the Mahler conjecture Conjecture 6.6.2 for non-symmetric convex bodies.

Finally, we present the simple argument due to Meyer [574] verifying the Mahler conjecture for unconditional convex bodies, originally proved by Saint-Raymond [643]. We recall that given an orthonomal basis of $\mathbb{R}^n$, a convex body $K \subset \mathbb{R}^n$ is unconditional if $(x_1, \ldots, x_n) \in K$ implies that $(\pm x_1, \ldots, \pm x_n) \in K$.

Theorem 6.6.5 (Mahler Conjecture for unconditional bodies). *If $K \subset \mathbb{R}^n$ is an unconditional convex body, then $|K| \cdot |K^*| \geqslant \frac{4^n}{n!}$.*

Proof. Induction on $n \geqslant 1$ where the case $n = 1$ trivial.

Writing $e_1, \ldots, e_n$ to denote the orthonomal basis of $\mathbb{R}^n$, $n \geqslant 2$, let $C = K \cap [0,\infty)^n$ and $C^\circ = K^* \cap [0,\infty)^n$, and for $i = 1, \ldots, n$, let $C_i = K \cap [0,\infty)^n \cap e_i^\perp$ and $C_i^\circ = K^* \cap [0,\infty)^n \cap e_i^\perp$. It follows that K can be dissected into 2^n congruent copies of C, while $K \cap e_i^\perp$ can be dissected into 2^{n-1} copies of C_i, and

$$C^\circ = \{x \in [0,\infty)^n : \langle x, y\rangle \leqslant 1 \ \text{ for } y \in C\},$$
$$C_i^\circ = \left\{x \in [0,\infty)^n \cap e_i^\perp : \langle x, y\rangle \leqslant 1 \ \text{ for } y \in C_i\right\}.$$

In particular, Theorem 6.6.5 is equivalent with the claim

$$|C| \cdot |C^\circ| \geqslant \frac{1}{n!}, \tag{6.39}$$

and we deduce from the induction hypothesis that if $i = 1, \ldots, n$ and $\mathcal{H} = \mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, then

$$\mathcal{H}(C_i) \cdot \mathcal{H}(C_i^\circ) \geqslant \frac{1}{(n-1)!}. \tag{6.40}$$

For $x = (x_1, \ldots, x_n) \in (0,\infty)^n$, the interiors of the sets $\mathrm{conv}\{x, C_i\}$ are pairwise disjoint for $i = 1, \ldots, n$, thus

$$|C| \geqslant \frac{1}{n} \sum_{i=1}^n x_i \mathcal{H}(C_i) \ \text{ for } x = (x_1, \ldots, x_n) \in C, \tag{6.41}$$

$$|C^\circ| \geqslant \frac{1}{n} \sum_{i=1}^n y_i \mathcal{H}(C_i^\circ) \ \text{ for } y = (y_1, \ldots, y_n) \in C^\circ. \tag{6.42}$$

We conclude from (6.41) that $p = \left(\frac{\mathcal{H}(C_1)}{n|C|} \ldots, \frac{\mathcal{H}(C_n)}{n|C|}\right) \in C^o$.

Applying (6.42) to p, and using (6.40) imply that

$$|C| \cdot |C^\circ| \geqslant |C| \cdot \frac{1}{n} \sum_{i=1}^n \frac{\mathcal{H}(C_i)}{n|C|} \cdot \mathcal{H}(C_i^\circ) \geqslant \frac{1}{n} \sum_{i=1}^n \frac{1}{n!} = \frac{1}{n!},$$

which in turn yields (6.39). ∎

6.7 Functional Santaló inequality and reverse form

In this section, we review the functional versions of the Blaschke-Santaló inequality (6.31) and its Reverse form (6.38). Since most of the arguments are rather involved, we

only provide the simple proof of the functional Santaló inequality for even functions due to Keith Ball. First we define what is meant by the polar of a non-negative function (see Section 1.A for the properties of the Legendre transform).

Definition 6.7.1. Let $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ and $f : \mathbb{R}^n \to [0, \infty)$ be non-trivial measurable functions ($\varphi(x) < \infty$ and $f(y) > 0$ for some $x, y \in \mathbb{R}^n$).

Legendre transform (convex conjugate): $\mathcal{L}(\varphi)(x) = \sup_{y\in\mathbb{R}^n} \{\langle x, y\rangle - \varphi(y)\}$;

Polar: $f^\circ(x) = \inf_{y\in\mathbb{R}^n} \left\{ \frac{e^{-\langle x,y\rangle}}{f(y)} \right\}$, and hence if $f = e^{-\varphi}$, then $f^\circ = e^{-\mathcal{L}(\varphi)}$.

Remarks.

- $\mathcal{L}(\varphi)$ is convex and lower semi-continuous (see the Remark after Definition 1.A.5), and hence f° is log-concave and upper semi-continuous.
- If φ is convex and lower semi-continuous, then $\mathcal{L}\mathcal{L}(\varphi) = \varphi$ (cf. Lemma 1.A.7), and hence if f is log-concave and upper semi-continuous, then $(f^\circ)^\circ = f$.
- If $x, y \in \mathbb{R}^n$, then
$$f(x) \cdot f^\circ(y) \leqslant e^{-\langle x,y\rangle}. \tag{6.43}$$

Example 6.7.2. Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \text{int}K$.

(i) $(\mathbf{1}_K)^\circ(x) = e^{-\|x\|_{K^*}}$;

(ii) $\left(e^{-\frac{1}{2}\|x\|_K^2}\right)^\circ = e^{-\frac{1}{2}\|x\|_{K^*}^2}$, and hence $\left(e^{-\frac{1}{2}\|x\|^2}\right)^\circ = e^{-\frac{1}{2}\|x\|^2}$;

(iii) if $g(x) = af(\Phi x)$ for $\Phi \in \mathrm{GL}(n)$ and $a > 0$, then $g^\circ(x) = \frac{1}{a} f^\circ(\Phi^{-t}x)$.

For the exponential expressions occurring in Example 6.7.2, we have the following integral formulas. We recall (cf. Section 1.9) that if $K \subset \mathbb{R}^n$ is a convex body with $o \in \text{int}\, K$ and $x \in \mathbb{R}^n$, then

$$\|x\|_K = \min\{t \geqslant 0 : x \in tK\}.$$

Lemma 6.7.3. *If $K \subset \mathbb{R}^n$ is a convex body with $o \in \text{int}\, K$, then*

(i) $\displaystyle\int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x\|_K^2}\, dx = \frac{(2\pi)^{\frac{n}{2}}|K|}{|B^n|}$;

(ii) $\displaystyle\int_{\mathbb{R}^n} e^{-\|x\|_K}\, dx = |K|$.

Proof. (i) and (ii) follow from the formula $\int_{\mathbb{R}^n} \psi(\|x\|_K)\, dx = n|K| \int_0^\infty \psi(r) r^{n-1}\, dr$ for continuous $\psi : [0, \infty) \to [0, \infty)$, which holds as $\frac{\partial}{\partial r} \int_{rK} \psi(\|x\|_K)\, dx = nr^{n-1}|K|\psi(r)$. ∎

Artstein-Avidan, Klartag, V. Milman [40] proved the functional version (6.44) of the Blaschke-Santaló inequality, together with the characterization of equality, extending Keith Ball's earlier result about even functions in his PhD thesis [52]. Here we only

provide Keith Ball's simple argument in the even case, which started off the quest for functional versions of inequalities on convex bodies.

Theorem 6.7.4 (Functional Santaló). *If* $0 < \int_{\mathbb{R}^n} f < \infty$ *and* $\int_{\mathbb{R}^n} xf(x)\,dx = o$ *for measurable* $f : \mathbb{R}^n \to [0, \infty)$, *then*

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \leqslant (2\pi)^n. \tag{6.44}$$

Equality holds if and only if $f(x) = ae^{-\frac{1}{2}\|\Phi x\|^2}$ *for* $a > 0$ *and* $\Phi \in \mathrm{GL}(n)$.

Proof of the functional Santaló inequality (6.44) *if* f *is even:* The sets $X_r = \{f > e^{-r}\}$ and $K_s = \{f^\circ > e^{-s}\}$ are o-symmetric and K_s is convex, and if $x \in X_r$ and $y \in K_s$, then $e^{-r}e^{-s} \leqslant e^{-\langle x,y\rangle}$ by (6.43), and hence $\langle x, y\rangle \leqslant r + s$. It follows by the definition of the polar of a convex body that $X_r \subset (r+s)(K_s)^*$, thus $|X_r| \cdot |K_s| \leqslant (r+s)^n \omega_n^2$ by the Blaschke-Santaló inequlity (6.31).

We now apply the Prékopa-Leindler inequality (3.5) to the functions $\varphi(r) = e^{-r}|\{f > e^{-r}\}|$, $\psi(s) = e^{-s}|\{f^\circ > e^{-s}\}|$ and $h(t) = \omega_n e^{-t}(2t)^{\frac{n}{2}}$ for $t \geqslant 0$ and $h(t) = 0$ if $t < 0$. These functions satisfy $\sqrt{\varphi(r)\psi(s)} \leqslant h(\frac{r+s}{2})$, and conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ &= \int_{\mathbb{R}} e^{-r}|\{f > e^{-r}\}|\,dr \cdot \int_{\mathbb{R}} e^{-s}|\{f^\circ > e^{-s}\}|\,ds \\ &\leqslant \left(\int_0^\infty \omega_n e^{-t}(2t)^{\frac{n}{2}}\,dt\right)^2 = \left(\frac{(2\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\int_0^\infty e^{-s}s^{\frac{n}{2}}\,ds\right)^2 = (2\pi)^n.\end{aligned}$$

■

Unsurprisingly, the Blaschke-Santaló inequality (6.31) does play a significant role in the proof in the functional Santaló inequality (6.44), and Example 6.7.2 (i) and Lemma 6.7.3 (i) show that in turn, the functional Santaló inequality implies the Blaschke-Santaló inequality for convex bodies. We note that Lehec [502, 503] provided direct proofs of the functional Santaló inequality (6.44) without the characterization of equality not using (and hence yielding) the Blaschke-Santaló inequality for convex bodies. For a reversed inequality, Berndtsson [82] proved a Reverse Functional Santaló inequality for even log-concave functions using complex analysis that was inspired by G. Kuperberg's Reverse Blaschke-Santaló inequality (6.38), even if Berndtsson's does not actually use G. Kuperberg's method or result in [490]. We note that in a Reverse Functional Santaló inequality, we have to assume that the function is log-concave.

Theorem 6.7.5 (Berndtsson's Even Reverse Functional Santaló inequality). *If* $f : \mathbb{R}^n \to [0, \infty)$ *is even and log-concave with* $0 < \int_{\mathbb{R}^n} f < \infty$, *then*

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \geqslant \pi^n. \tag{6.45}$$

Fradelizi [294] extended (6.45) to possibly non-even log-concave functions with a slightly smaller factor. More precisely, Fradelizi [294] provided the strategy for how to obtain the factor $(c/2)^n$ in the Reverse Functional Santaló inequality for any log-concave function if the factor c^n, $c > 0$ absolute constant, is known in the even case in (6.45) in place of π^n.

Theorem 6.7.6 (Reverse Functional Santaló inequality (Berndtsson, Fradelizi)). *If $f : \mathbb{R}^n \to [0, \infty)$ is log-concave with $0 < \int_{\mathbb{R}^n} f < \infty$, then*

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \geqslant \left(\frac{\pi}{2}\right)^n. \tag{6.46}$$

Fradelizi, Meyer [298] proposed the following conjecture about the optimal constant in the functional Santaló inequality:

Conjecture 6.7.7 (Functional Mahler conjecture). *If $f : \mathbb{R}^n \to [0, \infty)$ is log-concave with $0 < \int_{\mathbb{R}^n} f < \infty$, then*

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \geqslant e^n \tag{6.47}$$

where equality holds if $f(x_1, \ldots, x_n) = e^{-\sum_{i=1}^n x_i} \mathbf{1}_{[-1,\infty)^n}(x_1, \ldots, x_n)$.

If, in addition, f is even, then

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \geqslant 4^n \tag{6.48}$$

where equality holds if $f(x_1, \ldots, x_n) = e^{-\sum_{i=1}^n |x_i|}$.

Remark. Fradelizi, Meyer [298] proved (6.47) if $n = 1$, Fradelizi, Nakhle [302] established (6.48) for $n = 1, 2$, and Fradelizi, Meyer [297, 298] verified (6.48) for unconditional functions where Fradelizi, Gordon, Meyer, Reisner [299] even managed to characterize the equality in (6.48) for unconditional functions (which case is actually related to Hanner polytopes).

6.8 Volume approximation by polytopes

The reason why we discuss volume approximation by polytopes in this chapter is that the Blaschke-Santaló inequality (6.31) and the Reverse Blaschke-Santaló inequality (6.38) are used to verify that for an ellipsoid, volume approximation by inscribed polytopes of given (not too high) number of vertices is essentially equivalent to volume approximation by circumscribed polytopes of the same number of facets (cf. Theorem 6.8.3). In turn, Theorem 6.D.1 due to Sas [651] on volume approximation in the plane can be used, for example, to settle the equality case in the Blaschke-Santaló

inequality for non-symmetric convex bodies in any dimension (see the argument characterizing equality in Theorem 6.5.2 in Section 6.E).

One of our main results is Theorem 6.8.1 due to Macbeath [555] claiming that ellipsoids are the worst approximable convex bodies by inscribed polytopes in the sense of volume. Here the key tool is Steiner symmetrization, and hence we recall that for $u \in S^{n-1}$ and convex body $K \subset \mathbb{R}^n$, the Steiner symmetrial $\Theta_{u^\perp} K$ of K is

$$\Theta_{u^\perp} K = \left\{ x + \frac{t-s}{2} \cdot u : x \in K|u^\perp \ \& \ x + tu \in K \ \& \ x + su \in K \right\}.$$

The properties of Steiner symmetrization that we need here are that $\Theta_{u^\perp} K$ is also a convex body with $|\Theta_{u^\perp} K| = |K|$ (cf. Proposition 1.10.3), and there is a sequence of iterated Steiner symmetrizations whose results tend to a ball (cf. Theorem 1.10.7).

Theorem 6.8.1 (Macbeath). *If $C \subset \mathbb{R}^n$ is a convex body with $|C| = |rB^n|$ for $r > 0$ and $k \geqslant n+1$, then for any polytope $P \subset rB^n$ with at most k vertices, there exists a polytope $Q \subset C$ with at most k vertices such that $|Q| \geqslant |P|$.*

Remark. In particular, ellipsoids are worst approximable by inscribed polytopes of given number of vertices, but it is known only if in the planar case that ellipsoids are the only extremal bodies (see the elegant argument presented in Theorem 6.D.1 due to Sas [651]). The fact for triangles in $\mathbb{R}^2$ and for tetrahedra in $\mathbb{R}^3$, the ellipses or ellipsoids, respectively are the only extremizers have been verified earlier by Gross [342] and Blaschke [97], Section 72.

Proof. Since Steiner symmetrization preserves volume, and there is a sequence of iterated Steiner symmetrizations such that the image tends to a ball (see Theorem 1.10.7), it is enough to prove the following statement.

Claim. If $C \subset \mathbb{R}^n$ convex body and $u \in S^{n-1}$, then for any polytope $P \subset \Theta_{u^\perp} C$ with at most k vertices, there exists a polytope $Q \subset C$ with at most k vertices such that

$$|Q| \geqslant |P|. \tag{6.49}$$

In order to find the suitable Q in (6.49), let $x_i + \frac{1}{2}(t_i - s_i)u$, $i = 1, \dots, m$, $m \leqslant k$ be the vertices of P for $x_i \in C|u^\perp$, $t_i, s_i \in \mathbb{R}$ with $x_i + t_i u, x_i + s_i u \in C$. For $Q^+ = \operatorname{conv}\{x_i + t_i u : i = 1, \dots, m\}$ and $Q^- = \operatorname{conv}\{x_i + s_i u : i = 1, \dots, m\}$, we claim that

$$\frac{\mathcal{H}^1(\ell \cap Q^+) + \mathcal{H}^1(\ell \cap Q^-)}{2} \geqslant \mathcal{H}^1(\ell \cap P) \ \text{ for any } z \in P|u^\perp \text{ and } \ell = z + \mathbb{R}u. \tag{6.50}$$

To prove (6.50), let $\ell \cap P = \operatorname{conv}\{p, q\}$ for $p, q \in \Theta_{u^\perp} C$ with $p - q = \mathcal{H}^1(\ell \cap P) \cdot u$. We deduce the existence of $\alpha_i, \beta_i \geqslant 0$ for $i = 1, \dots, m$ such that $p = z + \frac{1}{2} \sum_{i=1}^m \alpha_i (t_i -$

$s_i)u$, $q = z + \frac{1}{2}\sum_{i=1}^m \beta_i(t_i - s_i)u$ and $\sum_{i=1}^m \alpha_i = \sum_{i=1}^m \beta_i = 1$, and hence

$$\mathcal{H}^1(\ell \cap P) = \frac{1}{2}\sum_{i=1}^m (\alpha_i - \beta_i)(t_i - s_i);$$

$$\mathcal{H}^1(\ell \cap Q^+) \geqslant \sum_{i=1}^m (\alpha_i - \beta_i)t_i \quad \text{as } z + \sum_{i=1}^m \alpha_i t_i u,\ z + \sum_{i=1}^m \beta_i t_i u \in Q^+;$$

$$\mathcal{H}^1(\ell \cap Q^-) \geqslant \sum_{i=1}^m (\beta_i - \alpha_i)s_i \quad \text{as } z + \sum_{i=1}^m \alpha_i s_i u,\ z + \sum_{i=1}^m \beta_i s_i u \in Q^-.$$

Combining these three estimates yields (6.50).

It follows from (6.50) that $\frac{1}{2}(|Q^+| + |Q^-|) \geqslant |P|$; therefore, we may choose either $Q = Q^+$ or $Q = Q^-$, that satisfies (6.49), and in turn proves Theorem 6.8.1. ∎

We deduce from Theorem 6.8.1 that upper bounds in the case of volume approximation of balls by inscribed ellipsoids lead to upper bounds in the case of volume approximation of any convex body. Therefore, the rest of the section is dedicated to volume approximation of Euclidean balls. We start with an elementary estimate due to G. Elekes [245] that shows that "Computing the volume is difficult"; namely, polytopes with at most polynomially many (in n) vertices provide bad approximation of the ball.

Theorem 6.8.2 (Elekes). *If $P \subset B^n$ a polytope with at most k vertices, then $\frac{|P|}{|B^n|} \leqslant \frac{k}{2^n}$.*

Remark. As Bárány, Füredi [63] explains, the simple estimate of Theorem 6.8.2 is a quite reasonable one if k is around $2^{n/2}$ (see the improved estimate (6.51) below if k is subexponential in n).

Proof. Let $v_1, \ldots, v_k$ be the vertices of P. For any $x \in P \setminus \{o\}$, there exists a v_i contained in the halfspace having x both as a boundary point and as an inward normal, and hence $\langle v_i, x\rangle \geqslant \langle x, x\rangle$, which in turn yields the inequalities $\|x - \frac{1}{2}v_i\|^2 \leqslant \frac{1}{4}\langle v_i, v_i\rangle \leqslant \frac{1}{4}$. We deduce that $P \subset \bigcup_{i=1}^k (\frac{1}{2}v_i + \frac{1}{2}B^n)$, thus $|P| \leqslant \frac{k}{2^n}|B^n|$. ∎

The following statement provides the true order of volume approximation of a ball if at most exponentially many vertices (in the inscribed case) or at most exponentially many facets (in the circumscribed case) are used.

Theorem 6.8.3. *For an absolute constant $c > 1$ and $2n \leqslant k \leqslant 2^n$, if $P_k \subset B^n$ polytope of maximal volume with k vertices and $P_{(k)} \supset B^n$ polytope of minimal volume with k*

facets, then

$$c^{-1} \cdot \sqrt{\frac{\log \frac{k}{n}}{n}} \leqslant \sqrt[n]{\frac{|P_k|}{|B^n|}} \leqslant c \cdot \sqrt{\frac{\log \frac{k}{n}}{n}}; \tag{6.51}$$

$$c^{-1} \cdot \sqrt{\frac{n}{\log \frac{k}{n}}} \leqslant \sqrt[n]{\frac{|P_{(k)}|}{|B^n|}} \leqslant c \cdot \sqrt{\frac{n}{\log \frac{k}{n}}}. \tag{6.52}$$

Remarks.

(i) As we may asssume allowing $2k$ vertices/facets that the polytopes are o-symmetric, the lower bound in (6.51) and the upper bound in (6.52) are equivalent according to the Blaschke-Santaló inequality (6.31) and the Reverse Blaschke-Santaló inequality (6.38), and similarly, the upper bound in (6.51) and the lower bound in (6.52) are equivalent.

(ii) Comparing (6.51) and (2.47) shows that if $2n \leqslant k \leqslant 2^n$, then for a polytope $P_k \subset B^n$ with k vertices, maximizing the inradius or the volume are equivalent.

(iii) The upper bound in (6.51) is due to Bárány, Füredi [63], and the equivalent lower bound in (6.52) are proved by Carl, Pajor [179] and Gluskin [326], all three papers are from 1988. The argument of Bárány, Füredi [63] is retold in Section 6 of Böröczky, Wintsche [150], while the argument of Gluskin [326] is presented in the survey Ball [57] (see Galicer, Litvak, Merzbacher, Pinasco [323] for a recent approach).

(iv) The lower bound in (6.51) follows from Theorem 2.7.7 due to Böröczky, Wintsche [150]. For the equivalent upper bound in (6.52), Assaf Naor gave the following construction (cf. Keith Ball, Böröczky, Naor [60]): Assuming n is large, we may assume that $k > 16n$, and we choose integers $m \geqslant 2$ such that $\frac{k}{4n} \leqslant \frac{2^m}{m} \leqslant \frac{k}{2n}$ (and hence $m < n$), $r \geqslant 1$ maximal such that $rm \leqslant n$, and $0 \leqslant \ell \leqslant m-1$ such that $rm + \ell = n$. The approximating polytope $\widetilde{P}_{(k)}$ is the direct sum of r copies (if $\ell = 0$) or $r+1$ copies (if $\ell > 0$) of centered regular cross-polytopes of inradius 1 in pairwise orthogonal linear subspaces, where we use r cross-polytopes of dimension m, and if $\ell > 0$, then we also use an extra cross-polytope of dimension ℓ. Thus the number of facets of $\widetilde{P}_{(k)}$ is $r2^m$ if $\ell = 0$, or $r2^m + 2^\ell$ if $\ell > 0$, and hence lies between $k/8$ and k. According to the bound $m! > (m/e)^m$ by the Stirling formula (10.2), the volume of the m-dimensional regular cross-polytopes $\widetilde{C}_m$ of inradius 1 can be estimated by

$$\sqrt[m]{\mathcal{H}^m(\widetilde{C}_m)} = \sqrt[m]{\frac{\sqrt{m}^m \cdot 2^m}{m!}} < \frac{2e}{\sqrt{m}}$$

thus elementary calculations show that $\widetilde{P}_{(k)}$ satisfies the upper bound in (6.52).

(v) If $2n \leqslant k \leqslant 2^n$, then (6.51) and (6.52) are applied in the isomorphic reverse isoperimetric inequality Theorem 6.8.4 for polytopes with at most k vertices, and later in the isomorphic isoperimetric inequality Theorem 7.7.9 in Section 7.7 for polytopes with at most k facets in the form

$$c^{-1} \cdot n^{-1} \sqrt{\log \frac{k}{n}} \leqslant |P_k|^{\frac{1}{n}} \leqslant c \cdot n^{-1} \sqrt{\log \frac{k}{n}}; \tag{6.53}$$

$$c^{-1} \left(\log \frac{k}{n} \right)^{\frac{-1}{2}} \leqslant |P_{(k)}|^{\frac{1}{n}} \leqslant c \left(\log \frac{k}{n} \right)^{\frac{-1}{2}}. \tag{6.54}$$

For large k, it follows from Theorem 2.7.3 and Theorem 6.8.1 that if $k \geqslant 2^n$ and $C \subset \mathbb{R}^n$ is a convex body, then there exists a polytope $P_k \subset C$ with at most k vertices such that

$$\frac{|C \backslash P_k|}{|C|} \leqslant \frac{c \cdot n}{k^{\frac{2}{n-1}}} \tag{6.55}$$

where $c > 1$ is an absolute constant. This estimate is optimal even considering the factor because Prochno, Schütt, Werner [628] proved the existence absolute constants $a, b > 0$ such that if $P_k \subset B^n$ polytope of maximal volume with at most k vertices, then

$$\frac{|B^n \backslash P_k|}{|B^n|} \geqslant \frac{a \cdot n}{k^{\frac{2}{n-1}}} \quad \text{provided } k \geqslant b \cdot n^{\frac{n-1}{2}}. \tag{6.56}$$

If ∂C has C^2 boundary and $P_k \subset C$ is a polytope of maximal volume with at most k vertices and $P_{(k)} \supset C$ is a polytope of minimal volume with at most k facets, then the limits $\lim_{k\to\infty} k^{\frac{2}{n-1}} |C \backslash P_k|$ and $\lim_{k\to\infty} k^{\frac{2}{n-1}} |P_{(k)} \backslash C|$ exist and are positive (see Section 8.10 and Böröczky [117]).

Concerning both random and best approximation by polytopes in terms of volume difference, the earlier history is discussed by Gruber [347,349], and more recent developments are reviewed by Prochno, Schütt, Werner [628].

A related problem is the Reverse Isoperimetric inequality Theorem 6.3.2 for origin symmetric convex bodies, stating that if $K \subset \mathbb{R}^n$ is an o-symmetric convex body, then there exists $\Phi \in \mathrm{GL}(n)$ such that the isoperimetric quotient satisfies

$$\frac{P(\Phi K)}{|\Phi K|^{\frac{n-1}{n}}} \leqslant 2n, \tag{6.57}$$

with equality for cubes. However, cubes have rather high number; namely, 2^n vertices. Next, we present an improvement due to Keith Ball, Böröczky, Naor [60] on the bound of (6.57) if n is large and an o-symmetric polytope has significantly fewer than 2^n vertices.

Theorem 6.8.4. *If $Q \subset \mathbb{R}^n$ is an o-symmetric polytope with at most k vertices where $2n \leqslant k \leqslant 2^n$, then there exists a $\Phi \in \mathrm{GL}(n)$ such that*

$$\frac{P(\Phi Q)}{|\Phi Q|^{\frac{n-1}{n}}} \leqslant c\sqrt{n} \cdot \sqrt{\log \frac{k}{n}} \tag{6.58}$$

for an absolute constant $c > 0$.

Proof. According to John's Theorem 6.1.1, there exists a $\Phi \in \mathrm{GL}(n)$ such that

$$\tfrac{1}{\sqrt{n}} B^n \subset \Phi Q \subset B^n.$$

We apply first Lemma 6.3.3, and then (6.53) to conclude that

$$\frac{P(\Phi Q)}{|\Phi Q|^{\frac{n-1}{n}}} \leqslant n\sqrt{n}|\Phi Q|^{\frac{1}{n}} \leqslant c\sqrt{n} \cdot \sqrt{\log \frac{k}{n}}.$$

■

6.9 The M-ellipsoid, the Covering number and the Reverse Brunn-Minkowski inequality

As introduced by V. Milman, Pajor [583], for a centered convex body $K \subset \mathbb{R}^n$, an "M-ellipsoid" E is just any ellipsoid satisfying either of the properties (6.59), (6.60) or Theorem 6.9.4 up to a factor c^n where c is some absolute constant where the three properties are equivalent according to V. Milman, Pajor [583]. The main goal of this section to show how the mere existence of the M-ellipsoid leads to the Reverse Brunn-Minkowski inequality Theorem 6.9.5 (see Artstein-Avidan, Giannopoulos, V. Milman [38], Chapter 8 for a throughout discussion of the topic).

Given a centered convex body $K \subset \mathbb{R}^n$ of diameter D, the volume of its intersection with an ellipsoid of inradius r is at most $2rD^{n-1}\omega_{n-1}$, and hence the Blaschke Selection Theorem 1.7.4 yields the existence of an o-symmetric ellipsoid $E \subset \mathbb{R}^n$ with $|E| = |K|$ maximizing $|K \cap E|$ where $|K^*| \leqslant |E^*|$ by the Blaschke-Santaló inequality (6.31). Now V. Milman, Pajor [583] prove that this E has the following property.

Theorem 6.9.1 (Existence of an M-ellipsoid). *For a centered convex body $K \subset \mathbb{R}^n$, if E is an o-symmetric ellipsoid with $|E| = |K|$ maximizing $|K \cap E|$, then*

$$|K \cap E| \geqslant c^n|K| \ \text{ and } \ |K^* \cap E^*| \geqslant c^n|E^*| \geqslant c^n|K^*| \tag{6.59}$$

for an absolute constant $c \in (0, 1)$.

Remarks. Let $K \subset \mathbb{R}^n$ be a centered convex body.

- Any ellipsoid E satisfying (6.59) for an absolute constant $c \in (0,1)$ is called an M-ellipsoid for K, and hence the notion depends on the choice of the absolute constant c. If $\Phi \in \mathrm{GL}(n)$, then ΦE is an M-ellipsoid of ΦK.
- According to Klartag, Lehec [470], if E is the rescaled copy of the Legendre ellipsoid E_K (see Section 6.4.1) with $|E| = |K|$, then E is an M-ellipsoid.
- As $|K \cap E| \cdot |K + E| < 4^n |K| \cdot |E|$ according to (1.33), the M ellipsoid E satisfies

$$|K + E| \leqslant C^n |K| \text{ and } |K^* + E^*| \leqslant C^n |K^*| \tag{6.60}$$

  for the absolute constant $C = 4/c$ where c comes from (6.59).
- Assuming that $|K \cap E| \geqslant \tilde{c}^n |K|$ for $\tilde{c} > 0$ and an o-symmetric ellipsoid $E \subset \mathbb{R}^n$ with $|E| = |K|$, G. Kuperberg's Reverse Blaschke-Santaló inequality Theorem 6.6.1 and $|K \cap E| \cdot |K + E| < 4^n |K| \cdot |E|$ (cf. (1.33)) yield

$$|K^* \cap E^*| = \left|(\mathrm{conv}\{K, E\})^*\right| \geqslant \left|(K + E)^*\right| \geqslant \frac{4^{-n}|E| \cdot |E^*|}{|K + E|} \geqslant \left(\frac{\tilde{c}}{16}\right)^n |E^*|.$$

  Therefore, the crucial part of (6.59) in the definition of the M ellipsoid is the lower bound for $|K \cap E|$. We note that most arguments establishing the existence of the M-ellipsoid use the Reverse Blaschke Santaló inequality, see, for example, Brazitikos, Giannopoulos, Valettas, Vritsiou [157].

Besides using the intersections as in (6.59), or Minkowski sums as in (6.60), an M ellipsoid can be defined *via* covering numbers as Theorem 6.9.4 shows.

Definition 6.9.2 (Covering number). For convex bodies $K, L \subset \mathbb{R}^n$, the covering number $N(K, L)$ is the minimal $N \geqslant 1$ such that $K \subset (x_1 + L) \cup \ldots \cup (x_N + L)$ for some $x_1, \ldots, x_N \in \mathbb{R}^n$.

Remark. $N(K, L) = N(\Phi K, \Phi L)$ for $\Phi \in \mathrm{GL}(n)$.

For a convex body $L \subset \mathbb{R}^n$, Erdős, Rogers [248] construct a covering of $\mathbb{R}^n$ by translates of L such that any point of $\mathbb{R}^n$ is covered at most $4n \ln n$ times. If a translate $x + L$ intersects a convex body $K \subset \mathbb{R}^n$, then $x \in K - L$, and hence $x + L \subset K - L + L$; therefore, we deduce the following estimate for the covering number.

Theorem 6.9.3 (Rogers bound on the covering number). *For convex bodies* $K, L \subset \mathbb{R}^n$,

$$N(K, L) \leqslant 4n \log n \cdot \frac{|K + L - L|}{|L|}. \tag{6.61}$$

Remark. The bound (6.61) is close to be optimal in general, as if $L = B^n$ and $nB^n \subset K$, then $N(K, B^n) \geqslant cn \cdot \frac{|K+2B^n|}{|B^n|}$ for an absolute constant $c > 0$ as any covering of $\mathbb{R}^n$ by unit balls has density at least $c_0 n$ for an absolute constant $c_0 > 0$ according to Coxeter,

Few, Rogers [223] (see also Böröczky [118], Theorems 8.2.1 and 9.5.2).

The following estimate is a consequence of (6.60) and (6.61):

Theorem 6.9.4. *If $K \subset \mathbb{R}^n$ is convex body and E is an M-ellipsoid, then $N(K,E) \leqslant c^n$ for an absolute constant $c > 0$.*

Remark. Assuming $\sigma_K = o$, also $N(K^*, E^*) \leqslant c^n$.

We recall that according to the Brunn-Minkowski inequality, if $K, L \subset \mathbb{R}^n$ are convex bodies, then $|K+L|^{\frac{1}{n}} \geqslant |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}$. The following reverse form of the Brunn-Minkowski inequality, indicated by V. Milman, Pajor [583], is a generalization of V. Milman's Reverse Brunn-Minkowski inequality for o-symmetric convex bodies in [581], and was seemingly first stated in this form only much later by Brazitikos, Giannopoulos, Valettas, Vritsiou [157].

Theorem 6.9.5 (Reverse Brunn-Minkowski inequality). *There exists an absolute constant $C > 1$ with the following property: For convex bodies $K, L \subset \mathbb{R}^n$, one finds an $\Phi \in \mathrm{SL}(n)$ such that if $\alpha, \beta > 0$, then*

$$|\alpha \Phi K + \beta L|^{\frac{1}{n}} \leqslant C(\alpha |K|^{\frac{1}{n}} + \beta |L|^{\frac{1}{n}}).$$

Remark. Assuming that $\sigma_K = \sigma_L = o$, we also have

$$|\alpha (\Phi K)^* + \beta L^*|^{\frac{1}{n}} \leqslant C(\alpha |K^*|^{\frac{1}{n}} + \beta |L^*|^{\frac{1}{n}}).$$

Proof. We may assume that $\sigma_K = \sigma_L = o$, and $|K| = |rB^n|$ and $|L| = |\varrho B^n|$ for $r, \varrho > 0$.

Theorem 6.9.4 yields the existence of $\Omega, \Psi \in \mathrm{SL}(n)$ and an absolute constant $C_0 > 1$ such that $N(\Omega K, rB^n) \leqslant C_0^n$ and $N(\Psi L, \varrho B^n) \leqslant C_0^n$, and hence $\alpha \Omega K \subset X_K + \alpha r B^n$ and $\beta \Psi L \subset X_L + \beta \varrho B^n$ where $\#X_K, \#X_L \leqslant C_0^n$. In turn, we deduce that $\Omega K + \Psi L \subset X_K + X_L + (\alpha r + \beta \varrho) B^n$ and $\#(X_K + X_L) \leqslant \#X_K \cdot \#X_L \leqslant C_0^{2n}$, thus

$$|\alpha \Omega K + \beta \Psi L| \leqslant C_0^{2n} \cdot |(\alpha r + \beta \varrho) B^n| = C_0^{2n} \left(\alpha |K|^{\frac{1}{n}} + \beta |L|^{\frac{1}{n}} \right)^n.$$

Therefore, we can choose $C = C_0^2$ and $\Phi = \Omega^{-1} \Phi$. ∎

We note that the M-ellipsoids also have significant role in the resolution of the Slicing Conjecture by Klartag, Lehec [470] (see Theorem 6.4.12).

6.10 Comments to Chapter 6

The subject of Chapter 6 is discussed in depth by Artstein-Avidan, Giannopoulos, V. Milman [38], and see also the survey Ball [56] for various aspects of this topic.

For a convex body $K \subset \mathbb{R}^n$, the properties of the inscribed ellipsoid of maximal volume ("John ellipsoid") and circumscribed ellipsoid of minimal volume ("Löwner ellipsoid") were essentially established by John [451] in 1948, and put it into the right context by Keith Ball [55], who also established that "John's conditions" are sufficient for the extremality of the unit ball (see also Gruber, Schuster [350] for a streamlined argument). According to Busemann, Löwner was independently aware of many properties of the extremal ellipsoids in the middle of the 20th century, hence the naming (see Henk [384] on the history of the problem). Note that around 1908, Voronoi [713] established an analogous property of the convex hull of the minimal vectors of an extremal lattice to John's conditions on a ball being the Löwner ellipsoid (see perfect forms in Schürmann [669]).

Hug, Schneider [431] prove that for any convex body $K \subset \mathbb{R}^n$, the ratio of the volumes of the Löwner ellipsoid over the John ellipsoid of K is at most n^n, and equality occurs if and only if K is a simplex. Hug, Schneider [431] even verify a stability version of this statement.

There are numerous widely used characterizations of ellipsoids like the one in Brunn's [164] Theorem 6.2.1. For a comprehensive surveys about characterizations of ellipsoids among convex bodies, see Petty [618] and Soltan [686].

The Reverse Isoperimetric inequality (cf. Theorems 6.3.1 and 6.3.2) is due to Keith Ball [54]. We note that Livshyts [529] proves an analogous statement for non-negative functions on $\mathbb{R}^n$ whose level sets are convex.

The proof of the Reverse Isoperimetric inequality is based on the Geometric form (6.8) of the rank one Brascamp-Lieb inequality due to Keith Ball [54], and the general Brascamp-Lieb inequality (see (6.62) below) was proved by Brascamp, Lieb [154] (see Bennett, Carbery, Christ, Tao [78] for a comprehensive study of the Brascamp-Lieb inequality). Equality in the Geometric the rank one Brascamp-Lieb inequality (6.8) was clarified by Barthe [67], and in the general Brascamp-Lieb inequality (6.62) by Valdimarsson [708].

The general form of the Brascamp-Lieb inequality (6.62) (see Barthe [67] for an elegant proof using optimal transport) is as follows. Let $B_i : \mathbb{R}^n \to H_i$ be surjective linear maps where H_i is n_i-dimensional Euclidean space, $n_i \geqslant 1$, for $i = 1, \ldots, k$, and let $c_1, \ldots, c_k > 0$ satisfy $\sum_{i=1}^k c_i n_i = n$. For non-negative $f_i \in L^1(H_i)$, we have

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(B_i x)^{c_i}\, dx \leqslant C \prod_{i=1}^k \left(\int_{H_i} f_i \right)^{c_i} \tag{6.62}$$

where C is determined by choosing centered Gaussians $f_i(x) = e^{-\langle A_i x, x\rangle}$, A_i positive definite. The Brascamp-Lieb inequality (6.62) generalizes for example Young's convolution inequality and the Hölder inequality. Other nice proofs of the Brascamp-Lieb inequality (6.62) are provided by Barthe, Cordero-Erausquin [70] based on the heat equation, and by Lehec [504] using a probabilistic argument.

The so-called Geometric form of the Brascamp-Lieb inequality (6.62) is when H_i is a linear subspace of $\mathbb{R}^n$, $B_i = \Pi_{H_i}$ and $\sum_{i=1}^k c_i \Pi_{H_i} = I_n$. In this case, the optimal factor in (6.62) is $C = 1$, as it was verified by Ball [54] in the rank one case (each $n_i = 1$), and by Barthe [67] in general.

Naor [598] states the following Isomorphic Reverse Isoperimetry Conjecture:
For any o-symmetric convex body $K \subset \mathbb{R}^n$, there exists $\varphi \in \mathrm{SL}(n)$ and an o-symmetric convex body $C \subset \Phi K$ such that $|C| \geqslant c^{-n}|K|$ and $P(C) \leqslant c\sqrt{n}|C|^{\frac{n-1}{n}}$ for an absolute constant $c > 1$.
In particular, the Isomorphic Reverse Isoperimetry Conjecture says that while $C \subset \Phi K$ has essentially the same volume as ΦK, and the isoperimetric ratio for C is essentially the same as for a ball where $P(B^n)/|B^n|^{\frac{n-1}{n}} \sim \sqrt{2\pi e}\sqrt{n}$ as n tends to infinity (cf. (10.1)). For example, the isoperimetric ratio of a cube is much larger for large n as $P([-1,1]^n)/|[-1,1]^n|^{\frac{n-1}{n}} = 2n$.

Many properties of a convex body related to some kind of isotropic position can be found in Artstein-Avidan, Giannopoulos, V. Milman [38, 39] and Brazitikos, Giannopoulos, Valettas, Vritsiou [157] where the Slicing Conjecture is also discussed in detail (see Klartag [468] for more recent developments). We note that Kannan, Lovász, Simonovits [454] uses another normalization to define isotropicity, they say that a convex body K is in isotropic position if and only if $\sigma_K = o$ and $M_K = I_n$.

For a brief history of the Slicing Conjecture (there exists an absolute constant c such that $L_K \leqslant c$ for any convex body $K \subset \mathbb{R}^n$), it was posed independently by Bourgain [151] and in the PhD thesis Ball [52] in 1986, and Bourgain [151] proved $L_K \leqslant c\, n^{\frac{1}{4}} \log n$ in [152] in 1991. In spite of serious efforts, for 30 years, the only improvement on the bound was $L_K \leqslant c\, n^{\frac{1}{4}}$ by Paouris [611] and Klartag [465] in 2006. A breakthrough $L_K \leqslant n^{o(1)}$ has been achieved by Yuansi Chen [196] in 2021, whose estimate was eventually improved to the resolution of the Slicing Conjecture by Klartag, Lehec [470] (see Theorem 6.4.12).

The Blaschke-Santaló inequality (6.31) in all dimensions about the maximum of the volume product of a centered convex body is due Santaló [642] in 1949, whose proof is based on relating the inequality to the Affine Isoperimetric Inequality Theorem 8.9.10 (see Section 6.F). The characterization of the equality case in the Blaschke-Santaló inequality (6.31) was achieved by [642] for convex bodies with C^2_+ boundary, and by Petty [619] in general (only in 1985). A direct proof of the Blaschke-Santaló inequality *via* Steiner symmetrization in the o-symmetric case was provided in Ball's PhD thesis [52], whose argument was extended to all convex bodies by Meyer, Pajor [575], who also characterized the equality case (see Section 6.E). Further arguments verifying the Blaschke-Santaló inequality are provided by Andrews [26, Theorem 7.3] (using geometric flows), and in the o-symmetric case, by Bianchi, Kelly [96] (using Fourier analysis). Stability versions of the Blaschke-Santaló inequality are due to Böröczky [119] for $n \geqslant 3$, Ivaki [436] for $n = 2$, and Ball, Böröczky [59] for $n \geqslant 2$.

Lehec [502,503] provided simple proofs of the functional Santaló inequality (6.44) without the characterization of equality not using (and hence yielding) the Blaschke-Santaló inequality for convex bodies. Fradelizi, Meyer [296] proved a generalized version of the functional Santaló inequality, which was further generalized to more than two even functions by Nakamura, Tsuji [595] extending the work of Kolesnikov, Werner [481] and Kalantzopoulos, Saroglou [453]. Stability versions of the functional Santaló inequality (6.44), and in general, of the inequalities proved by Fradelizi, Meyer [296] are provided by Barthe, Böröczky, Fradelizi [69].

Fradelizi, Gozlan, Sadovsky, Zugmeyer [295] verified a version of the Blaschke-Santaló inequality for not necessarily centered convex bodies. Actually, they even considered any compact set $K \subset \mathbb{R}^n$ such that $|K| > 0$ and $o \in \operatorname{int} \operatorname{conv} K$. For such a set, the centroid $\sigma_K = \frac{1}{|K|} \int_K x \, dx$ and the polar body $K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \, \forall y \in K\}$ make sense where K^* is a convex body containing the origin in its interior. Writing San_{K^*} to denote the Santaló point of K^*, [295] proves that

$$|K| \cdot |K^*| \leqslant |B^n|^2 \cdot (1 - \langle \operatorname{San}_{K^*}, \sigma_K \rangle)^{n+1} . \tag{6.63}$$

If K is convex and $\sigma_K = o$, then (6.63) is just the Blaschke-Santaló inequality (6.31). In addition, if K is convex, then [295] verifies that $\langle \operatorname{San}_{K^*}, \sigma_K \rangle \leqslant 0$ in (6.63). We note that Fradelizi, Gozlan, Sadovsky, Zugmeyer [295] also generalize Talagrand's inequality in optimal transport based on the functional Blaschke-Santaló inequality.

The Mahler conjecture in $\mathbb{R}^n$; namely, that the volume product of is minimized by centered simplices among all convex bodies and by cubes among all o-symmetric convex bodies was stated by Mahler [560] in 1939. It has been verified in the plane earlier by Mahler himself in [559], but it had been open in general in any dimension $n \geqslant 3$ until the Mahler conjecture was proved by Iriyeh, Shibata [434] in 2020 at least for o-symmetric convex bodies in $\mathbb{R}^3$ (and the argument is simplified by Fradelizi, Hubard, Meyer, Roldán-Pensado, Zvavitch [300]). In addition, Mahler's conjecture has been verified in various special cases like

- for zonoids by Reisner [631] (see Gordon, Meyer, Reisner [331] for a simpler proof), and Böröczky, Hug [134] provided stability version;
- for unconditional convex bodies by Saint-Raymond [643] (see Meyer [574] or Theorem 6.6.5 for a simple proof), and Kim, Zvavitch [461] provided a stability version;
- for convex bodies with n independent hyperplane symmetries (by rank n action of a Coxeter group) by Barthe, Fradelizi [71].

Some other cases when the Mahler conjecture is known include sections and projections of Hanner polytopes (see Karasev [455] using symplectic techniques), and convex bodies in $\mathbb{R}^3$ invariant under the rotational symmetries of a centered regular tetrahedron (see Iriyeh, Shibata [435]).

Kim, Reisner [460] proved that centered simplices are local minimums for the volume product, providing even a stability estimate, and Kim [459] proved the analogous results for the Hanner polytopes among o-symmetric convex bodies (extending earlier work by Nazarov, Petrov, Ryabogin, Zvavitch [604]).

Concerning the order of the minimum of the volume product, the ground breaking work Bourgain, V. Milman [153] proved the Reverse Blaschke-Santaló inequality $|K| \cdot |K^*| \geqslant c^n V(B^n)^2$ for any o-symmetric convex body $K \subset \mathbb{R}^n$ and an absolute constant $c \in (0,1)$ (see Giannopoulos, Paouris, Vritsiou [321] for a simpler proof). Explicit estimates for the value of c was obtained, for example, by Nazarov [603], but the best estimate Theorem 6.6.1 with $c = \frac{1}{4}$ is due to G. Kuperberg [490] (see the arxiv version for the case of non-symmetric convex bodies).

The functional Santaló inequality was proved by Keith Ball [52] for even functions in his PhD thesis, and was extended to centered functions with positive integral by Artstein-Avidan, Klartag, V. Milman [40], also characterizing equality. Versions of the functional Santaló inequality were provided by Fradelizi, Meyer [296] and Lehec [502, 503]. The reverse functional Santaló inequality for log-concave functions with a lower bound c^n for an unknown absolute constant $c > 0$ is due to Klartag, V. Milman [472], and simpler arguments have been provided by Fradelizi, Meyer [297] and Giannopoulos, Paouris, Vritsiou [321]. The beautiful argument by Berndtsson [82] uses complex analysis to verify the lower bound π^n in the case even log-concave functions, and is inspired by ideas in G. Kuperberg [490]. For the case of any log-concave functions, our bound $(\pi/2)^n$ might have not appeared in print, and is based on Fradelizi's [294] strategy to obtain the lower bound $(c/2)^n$ in the Reverse Functional Santaló inequality if the lower bound c^n, $c > 0$ absolute constant, is known in the even case.

Fradelizi, Sola [304] discuss a functional version of the Strong Slicing Conjecture, and show that the Functional Mahler conjecture (6.47) would follow from it. In addition, Fradelizi, Nakhle [303] states the Functional Mahler conjecture for s-concave functions on $\mathbb{R}^n$ and $s > -\frac{1}{n}$, and proves it in dimensions $n = 1, 2$.

For o-symmetric convex bodies, an M-ellipsoid was defined in V. Milman [581], where the Reverse Brunn-Minkowski was established. The definition of the M-ellipsoid and the Reverse Brunn-Minkowski was extended to the non-symmetric case by V. Milman, Pajor [583] (see Giannopoulos, Paouris, Vritsiou [321] for a nice survey). The M-ellipsoid is not unique by its definition, and how close two M-ellipsoids are is dicussed by V. Milman, Pajor [584].

For a survey about volume approximation of the unit ball with polytopes of low complexity, see for example Keith Ball [57], Section 6 of Böröczky, Wintsche [150], and Galicer, Litvak, Merzbacher, Pinasco [323]. Randomized algorithms proved to be effective in estimating the volume of a polytope of few vertices within a convex body (see Lee, Vempala [501]).

According to the Dvoretzky theorem, any convex body K in $\mathbb{R}^n$ has a section that is essentially a ball. This line of research was initiated by Groethendick, whose question

was answered by Dvoretzky and later by V. Milman, and finally Gordon [330] proved the most precise statement: For $\varepsilon \in (0,1)$ and $n \geqslant C$, if $2 \leqslant m \leqslant c\varepsilon^2 \log n$ and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then there exist $r > 0$ and a linear subspace L of dimension m such that

$$rB^n \cap L \subset K \cap L \subset (1+\varepsilon)(rB^n \cap L) \tag{6.64}$$

where $c \in (0,1)$ and $C > 1$ are absolute constants. Actually, if $\varepsilon \in (0,1)$ is fixed, $m \leqslant f(\varepsilon, n)$ and $K \subset \mathbb{R}^n$ is an o-symmetric convex body, then the section by a random linear m-plane L is almost spherical in the sense of (6.64) with high probability where r depends on L, K, ε and $f(\varepsilon, n)$ tends to infinity with n (see Mendelson [573]).

6.A Supplement: The John condition on the inscribed maximal volume ellipsoid

To prove the classical properties in Theorem 6.A.2 of John's maximal volume ellipsoid contained in a given convex body, we use various properties of matrices, which properties are summarized below.

Remark 6.A.1 (Some properties of Positive Semi-definite Matrices).

(i) $u \otimes u = uu^t$ for $u \in \mathbb{R}^n \backslash \{o\}$ is a rank one symmetric $n \times n$ matrix, $\operatorname{tr} u \otimes u = \|u\|^2$.

(ii) Writing I_n to denote the $n \times n$ identity matrix, if A is any $n \times n$ matrix, then

$$\frac{d}{dt} \det(I_n + tA)\Big|_{t=0} = \operatorname{tr} A.$$

(iii) $E \subset \mathbb{R}^n$ is an ellipsoid if and only if $E = \Phi\, B^n + v$ for $\Phi \in \mathrm{GL}(n)$ and $v \in \mathbb{R}^n$. According to the Left Polar Decomposition, $\Phi = AQ$ for a positive definite symmetric matrix A and an orthogonal matrix Q where $A^2 = \Phi\Phi^t$, and hence $E = AB^n + v$.
Note that if $u_1, \ldots, u_n$ is an orthonormal basis representing the principal directions of E, then $Au_i = a_i u_i$ where $a_i > 0$ is the principle axis corresponding to u_i.

It follows from Remark 6.A.1 (iii) that the space of o-symmetric ellipsoids can be identified with the space of positive definite matrices.

Theorem 6.A.2 (John). *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) There exists a unique so-called John ellipsoid $E \subset K$ of maximal volume.

(ii) B^n *is the John ellipsoid of* $K \subset \mathbb{R}^n$ *if and only if* $B^n \subset K$ *and there exist* $c_1, \dots, c_k > 0$ *and* $u_1, \dots, u_k \in S^{n-1} \cap \partial K$, $k \leqslant \frac{n(n+3)}{2}$, *such that*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = I_n, \tag{6.65}$$

$$\sum_{i=1}^{k} c_i u_i = o. \tag{6.66}$$

In addition, if K *is* o*-symmetric, then* (6.65) *is sufficient.*

Remarks.

(a) The uniqueness yields that ΦE is the John ellipsoid of ΦK for $\Phi \in \mathrm{GL}(n)$, for example, E is o-symmetric if K is o-symmetric.

(b) $\sum_{i=1}^{k} c_i = n$ by equating traces in (6.65).

(c) $\|x\|^2 = \sum_{i=1}^{k} c_i \langle u_i, x\rangle^2$ for $x \in \mathbb{R}^n$ by (6.65).

(d) $\langle x, u_i\rangle \leqslant 1$ for $x \in K$ and $i = 1, \dots, k$ as K and B^n share the same supporting hyperplanes at $u_1, \dots, u_k$.

We recall (3.5.2), that says that if A, B are symmetric positive definite $n \times n$ matrices and $\lambda \in (0, 1)$, then

$$\det((1-\lambda)A + \lambda B) \geqslant (\det A)^{1-\lambda} (\det B)^{\lambda} \tag{6.67}$$

where equality holds if and only if $A = B$.

Proof of Theorem 6.A.2.

Step 1 There exists a maximal volume ellipsoid $E \subset K$.

Fix an ellipsoid $\widetilde{E} \subset K$. The family of pairs (A, v) where A is a positive definite matrix and $v \in \mathbb{R}^n$ with $A\,B^n + v \subset K$ and $|A\,B^n + v| \geqslant |\widetilde{E}|$ is compact, as the largest eigenvalue of A is at most $\operatorname{diam} K$.

Step 2 There is a unique ellipsoid of maximal volume contained in K.

Indirect. We suppose that B^n is an ellipsoid of maximal volume contained in K, and $B^n \neq A\,B^n + v \subset K$ for a positive definite matrix A and $v \in \mathbb{R}^n$ with $\det A = 1$. It follows that $(\frac{1}{2} I + \frac{1}{2} A)B^n + \frac{1}{2} v \subset K$ as K convex where $\det(\frac{1}{2} I_n + \frac{1}{2} A) > 1$ unless $A = I_n$ according to (6.67); therefore, $A = I_n$ and $v \neq o$. We deduce that $E \subset K$ for the ellipsoid $E = \Phi B^n + \frac{1}{2} v$ where $\Phi v = (1 + \frac{1}{2}\|v\|)v$ and $\Phi w = w$ for $w \in v^{\perp}$, and hence $\det \Phi = 1 + \frac{1}{2}\|v\| > 1$. This contradiction verifies the uniqueness of the John ellipsoid.

Step 3 If $K = -K$ and B^n is the John ellipsoid, then (6.65) and $k \leqslant d = \frac{n(n+1)}{2}$ hold.

We identify the vector space of $n \times n$ matrices with $\mathbb{R}^{n^2}$, and for any $n \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, we define their scalar product to be $\langle A, B\rangle = \sum_{i,j=1,\dots,n} a_{ij} b_{ij}$.

Let $V \subset \mathbb{R}^{n^2}$ be the linear subspace of symmetric $n \times n$ matrices with $d = \dim V = \frac{n(n+1)}{2}$; namely, $V = \{A = [a_{ij}] : a_{ij} = a_{ji}\ \forall i \neq j\}$. In particular, V can be identified with R^d.

For the compact set $U = \partial K \cap S^{n-1}$, we consider the compact set $\widetilde{U} = \{u \otimes u : u \in U\} \subset V$. Since $\langle I_n, u \otimes u\rangle = \operatorname{tr} u \otimes u = 1$ for any $u \in U$, Lemma 1.3.9 yields that $C = \operatorname{pos} \widetilde{U} \subset V$ is a closed convex cone. As every $u \times u \in \widetilde{U}$ is positive semi-definite, each $M \in C$ is positive semi-definite, as well, and hence $C \cap (-C) = \{o_V\}$.

The proof of (6.65) is indirect; namely, we suppose that $I_n \notin C$, and seek a contradiction. Lemma 1.3.5 implies the existence of an $A \in V$ with

$$\langle A, I_n\rangle > 0 \text{ and } \langle A, u \otimes u\rangle < 0 \ \text{ for } u \in U, \tag{6.68}$$

while $\operatorname{tr} A > 0$ and Remark 6.A.1 (ii) yield

$$\det(I_n + tA) > 1 \ \text{ for small } t > 0. \tag{6.69}$$

It follows from (6.68) that there exists a $\delta > 0$ and open neighbourhood $\mathcal{N} \subset S^{n-1}$ of U such that $\langle Au, u\rangle = \langle A, u \otimes u\rangle < -\delta$ for $u \in \mathcal{N}$, and hence if $t > 0$ is small, then $(I_n + tA)u \in B^n$ for $u \in \mathcal{N}$. As compact set $S^{n-1}\backslash\mathcal{N}$ lies in $\operatorname{int} K$, we deduce that $(I_n + tA)B^n \subset K$ for small $t > 0$, which is a contradiction (yielding (6.65)) by (6.69).

The bound $k \leqslant d$ follows from Lemma 1.3.8.

Step 4 If (6.65) holds for an o-symmetric convex K with $B^n \subset K$, then B^n is the John ellipsoid.

Let $U = \partial K \cap S^{n-1}$. Our argument for Step 5 is indirect, thus we suppose that $(I_n + A)B^n \subset K$ for symmetric $n \times n$ matrix A with $\det(I_n + A) > 1$. It follows that $\langle (I_n + A)u, u\rangle \leqslant 1$ for $u \in U$, and $f'(t) > 0$ for the function $f(t) = \log\det(I_n + tA)$ as f is concave by (6.67) and $f(1) > f(0)$. Thus $\langle A, u \otimes u\rangle = \langle Au, u\rangle \leqslant 0$ holds for $u \in U$ by $\langle (I_n + A)u, u\rangle \leqslant 1$ and $\langle u, u\rangle \leqslant 1$ on the one hand, while Remark 6.A.1 (ii) and $f'(t) > 0$ yield that $\langle A, I_n\rangle = \operatorname{tr} A = f'(t) > 0$ on the other hand. We deduce that $I_n \notin \operatorname{pos}\{u \otimes u : u \in U\} \subset \mathbb{R}^d$, that contradicts (6.65).

Step 5 If $K \subset \mathbb{R}^n$ is a convex body and B^n is the John ellipsoid, then (6.65) and (6.66) hold.

We consider the space $\mathbb{R}^{n^2} \oplus \mathbb{R}^n$ of pairs (A, v) where A is an $n \times n$ matrix and $v \in \mathbb{R}^n$, and for two such pairs (A, v) and (B, w), their scalar product is $\langle (A, v), (B, w)\rangle = \langle A, B\rangle + \langle v, w\rangle$. Let $W \subset \mathbb{R}^{n^2} \oplus \mathbb{R}^n$ be the linear subspace of pairs of the form (A, v) where A is a symmetric $n \times n$ matrix and $v \in \mathbb{R}^n$, and let $d_0 = \dim W = \frac{n(n+3)}{2}$.

For $U = \partial K \cap S^{n-1}$, the set $\widetilde{U}_0 = \{(u \otimes u, u) : u \in U\} \subset W$ compact. Since

$$\langle (I_n, o), (u \otimes u, u)\rangle = \operatorname{tr} u \otimes u = 1$$

for any $u \in U$, Lemma 1.3.9 yields that $C_0 = \operatorname{pos} \widetilde{U} \subset W$ is a closed convex cone. We have $C_0 \cap (-C_0) = o_W$, because if $\sum_{i=1}^d \lambda_i (u_i \otimes u_i, u_i) = -\sum_{i=1}^d \tilde{\lambda}_i (\tilde{u}_i \otimes \tilde{u}_i, \tilde{u}_i)$ for $\lambda_i, \tilde{\lambda}_i \geqslant 0$ and $u_i, \tilde{u}_i \in U$, then $u \otimes u$ being positive semi-definite for $u \in U$ yields that each $\lambda_i = \tilde{\lambda}_i = 0$.

The proof of the properties (6.65) and (6.66) is indirect, we suppose that $(I_n, o) \notin C_0$, and seek a contradiction. Lemma 1.3.5 implies the existence of $(A, v) \in W$ for symmetric matrix A and $v \in \mathbb{R}^n$ with $\langle (A, v), (I_n, o) \rangle > 0$ and $\langle (A, v), (u \otimes u, u) \rangle < 0$ for $u \in U$, and hence Remark 6.A.1 (ii) yields

$$\det(I_n + tA) > 1 \text{ for small } t > 0 \text{ and } \operatorname{tr} A = \langle (A, v), (I_n, o) \rangle > 0; \tag{6.70}$$

$$\langle Au + v, u \rangle = \langle (A, v), (u \otimes u, u) \rangle < 0 \text{ for } u \in U. \tag{6.71}$$

We deduce from (6.71) that there exists $\delta > 0$ and open neighbourhood $\mathcal{N} \subset S^{n-1}$ of U such that $\langle Au + v, u \rangle < -\delta$ for $u \in \mathcal{N}$, and hence if $t > 0$ is small, then $(I_n + tA)u + tv \in B^n$ for $u \in \mathcal{N}$. Therefore, $(I_n + tA)B^n + tv \subset K$ holds for small $t > 0$, which contradicts (6.70).

Step 6 If (6.65) and (6.66) hold for a convex body $K \subset \mathbb{R}^n$ with $B^n \subset K$, then B^n is the John ellipsoid.

This statement can be proved using the argument in Step 4. ∎

6.B Supplement: The rank one Geometric Brascamp-Lieb inequality

In this section, we sketch Barthe's argument using optimal transport and presented in [67] of the form (6.73) of the Brascampl-Lieb inequality due to Keith Ball .

Theorem 6.B.1. *If $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ satisfy*

$$\sum_{i=1}^k c_i u_i \otimes u_i = I_n, \tag{6.72}$$

and $f_1, \dots, f_k \in L^1(\mathbb{R})$ are non-negative, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(\langle x, u_i \rangle)^{c_i} \, dx \leqslant \prod_{i=1}^k \left(\int_{\mathbb{R}} f_i \right)^{c_i}. \tag{6.73}$$

Remark. For any $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ satisfying (6.7), we have equality if we choose each f_i to be the same Gaussian probability density; for example, if $f_i(t) = e^{-\pi t^2}$ for $i = 1, \dots, k$.

In the rest of the section, we assume that $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ satisfy (6.72), thus for any $z \in \mathbb{R}^n$, we have

$$z = \sum_{i=1}^{k} c_i \langle u_i, z \rangle u_i \quad \text{and} \quad \|z\|^2 = \sum_{i=1}^{k} c_i \langle u_i, z \rangle^2.$$

Before proving the Brascampl-Lieb inequality, we verify two auxiliary estimates due to Keith Ball [52].

Lemma 6.B.2. *If $t_1, \dots, t_k > 0$, then*

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i \right) \geqslant \prod_{i=1}^{k} t_i^{c_i}.$$

Proof. Let $v_i = \sqrt{c_i}\, u_i$ for $i = 1, \dots, k$, and hence $\sum_{i=1}^{k} t_i c_i u_i \otimes u_i = \sum_{i=1}^{k} t_i v_i \otimes v_i$, $\langle v_i, v_i \rangle = c_i$ for $i = 1, \dots, k$ and $\sum_{i=1}^{k} v_i \otimes v_i = I_n$. In this argument, J is always an element of the family Θ_n^k of all n element subsets of $\{1, \dots, k\}$, and for $J = \{i_1, \dots, i_n\} \in \Theta_n^k$, we define

$$d_J = \det[v_{i_1}, \dots, v_{i_n}]^2 \text{ and } t_J = t_{i_1} \cdots t_{i_n}.$$

Applying the Cauchy-Binet formula to the $n \times k$ matrix $U = [\sqrt{t_1}\, v_1, \dots, \sqrt{t_k}\, v_k]$ yields

$$\det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i \right) = \det\left(U U^{\top} \right) = \sum_{J \in \Theta_n^k} t_J d_J, \tag{6.74}$$

and hence $\sum_{J \in \Theta_n^k} d_J = \det\left(\sum_{i=1}^{k} v_i \otimes v_i \right) = 1$ follows from (6.74) with $t_i = 1$. Therefore, applying the AM-GM inequality in (6.74) leads to

$$\det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i \right) \geqslant \prod_{J \in \Theta_n^k} t_J^{d_J}. \tag{6.75}$$

For a fixed $i \in \{1, \dots, k\}$, the factor t_i occurs exactly $\sum_{J,\, i \in J} d_J$ time in $\prod_J t_J^{d_J}$. Moreover, (6.74) applied to the vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$ implies

$$\begin{aligned} \sum_{J,\, i \in J} d_I = \sum_{J} d_J - \sum_{J,\, i \notin J} d_J &= 1 - \det\left(\sum_{j \neq i} v_j \otimes v_j \right) \\ &= 1 - \det\left(I_n - v_i \otimes v_i \right) = \langle v_i, v_i \rangle. \end{aligned}$$

Substituting this into (6.75) yields the lemma. ∎

Lemma 6.B.3. *If $z = \sum_{i=1}^{k} c_i \theta_i u_i$ for $\theta_1, \dots, \theta_k \in \mathbb{R}$, then*

$$\|z\|^2 \leqslant \sum_{i=1}^{k} c_i \theta_i^2. \tag{6.76}$$

Proof. The condition $z = \sum_{i=1}^k c_i\theta_i u_i$ and the Cauchy-Schwarz inequality yield that

$$\|z\|^2 = \left\langle z, \sum_{i=1}^k c_i\theta_i u_i \right\rangle = \sum_{i=1}^k c_i\theta_i\langle z, u_i\rangle \leqslant \sqrt{\sum_{i=1}^k c_i\theta_i^2}\sqrt{\sum_{i=1}^k c_i\langle u_i, z\rangle^2},$$

and hence the lemma follows from $\|z\|^2 = \sum_{i=1}^k c_i\langle u_i, z\rangle^2$. ∎

Following Barthe [67], we prove the Brascamp-Lieb inequality (6.73) via optimal transport based on the Gaussian density $g(t) = e^{-\pi t^2}$. Approximating f_i first by piecewise linear functions with compact support, and then in turn these functions by continuous functions, we may assume that each f_i is a positive continuous probabiity density function. For $i = 1, \ldots, k$, the transport map $T_i : \mathbb{R} \to \mathbb{R}$ is defined by

$$\int_{-\infty}^{t} f_i(s)\, ds = \int_{-\infty}^{T_i(t)} g(s)\, ds.$$

It follows that each T_i is differentiable, monotone inscreasing and

$$f_i(t) = g(T_i(t)) \cdot T_i'(t) \text{ holds for } t \in \mathbb{R}. \tag{6.77}$$

Now we consider the differentiable map $\Theta : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Theta(x) := \sum_{i=1}^k c_i T_i(\langle u_i, x\rangle)\, u_i, \qquad x \in \mathbb{R}^n,$$

which satisfies

$$D\Theta(x) = \sum_{i=1}^k c_i T_i'(\langle u_i, x\rangle)\, u_i \otimes u_i.$$

We claim that Θ is injective; namely,

$$\Theta(x_2) \neq \Theta(x_1) \text{ for } x_2 \neq x_1.$$

To prove the claim, we observe that if $v \in \mathbb{R}^n\backslash\{o\}$ and $x \in \mathbb{R}^n$, then

$$\langle v, D\Theta(x)v\rangle = v^t \left(\sum_{i=1}^k c_i T_i'(\langle u_i, x\rangle)\, u_i u_i^t\right) v = \sum_{i=1}^k c_i T_i'(\langle u_i, x\rangle)\, \langle u_i, v\rangle^2 > 0,$$

thus $D\Theta$ is positive definite. We deduce that $\varphi'(t) > 0$ for $t \in [0, 1]$ for the function $\varphi(t) = \langle x_2 - x_1, \Theta((1-t)x_1 + tx_2)\rangle$, and hence $\langle x_2 - x_1, \Theta(x_2)\rangle = \varphi(1) > \varphi(0) = \langle x_2 - x_1, \Theta(x_1)\rangle$.

Proof of the Brascampl-Lieb inequality (6.73). Using first $f_i(t) = g(T_i(t)) \cdot T_i'(t)$ for $g(t) = e^{-\pi t^2}$, then Lemmas 6.B.2 and 6.B.3, and finally the injectivity of Θ, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(\langle u_i, x\rangle)^{c_i}\, dx &= \int_{\mathbb{R}^n} \left(\prod_{i=1}^k g(T_i(\langle u_i, x\rangle))^{c_i}\right) \left(\prod_{i=1}^k T_i'(\langle u_i, x\rangle)^{c_i}\right) dx \\
&\leqslant \int_{\mathbb{R}^n} \left(\prod_{i=1}^k e^{-\pi c_i T_i(\langle u_i, x\rangle)^2}\right) \det\left(\sum_{i=1}^k c_i T_i'(\langle u_i, x\rangle)\, u_i \otimes u_i\right) dx \\
&\leqslant \int_{\mathbb{R}^n} e^{-\pi \|\Theta(x)\|^2} \det\left(D\Theta(x)\right) dx \\
&\leqslant \int_{\mathbb{R}^n} e^{-\pi \|y\|^2}\, dy = 1.
\end{aligned}$$

■

6.C Supplement: Minimal surface area position - isotropic surface area measure

If $X = \mathbb{R}^n$ or $X = S^{n-1}$, we say that a measure on X is quasi-isotropic (sometimes simply called isotropic) if the integral of the positive semi-definite rank one $n \times n$ matrix $u \mapsto u \otimes u$ is a multiple of I_n. In this section, we prove Petty's isotopicity condition in Petty [615] on when a convex body is in a minimal surface area position (see also Giannopoulos, Papadimitrakis [322], while the paper Giannopoulos, V. Milman [320] considers an extension for the mean projections defined in Theorem 7.1.2).

Theorem 6.C.1 (Petty). *For any convex body $K \subset \mathbb{R}^n$, $P(\Phi K)$ for $\Phi \in \mathrm{SL}(\mathbb{R}, n)$ attains its minimum at some $K_0 = \Phi_0 K$ for some $\Phi_0 \in \mathrm{SL}(\mathbb{R}, n)$ where K_0 is unique up to orthogonal transformations, and is characterized by the property that*

$$\int_{S^{n-1}} u \otimes u\, dS_{K_0} = \frac{P(K_0)}{n} \cdot I_n. \tag{6.78}$$

In order to prepare for the proof of Theorem 6.C.1, first we verify a statement showing that given the volume of a convex body, the surface area is large if the body is elongated.

Lemma 6.C.2. *If $K \subset \mathbb{R}^n$ is a convex body, then*

$$P(K) \geqslant \alpha_n (\operatorname{diam} K)^{\frac{1}{n-1}} |K|^{\frac{n-2}{n-1}} \text{ for } \alpha_n > 0 \text{ depending on } n.$$

Proof. If $n = 2$, then $P(K) \geqslant 2\operatorname{diam} K$ as K contains a segment of length $\operatorname{diam} K$. If $n \geqslant 3$, then acccording to John's theorem Theorem 6.1.1, we may assume that $E \subset$

$K \subset nE$ for a centered ellipsoid E. Let $a_1 \geqslant \ldots \geqslant a_n > 0$ be the semi axes of E, and hence E contains an $(n-1)$-dimensional ellipsoid of surface area $2\omega_{n-1}a_1 \ldots a_{n-1}$, and K satisfies $\operatorname{diam} K \leqslant 2a_1$ and $|K| \leqslant n^n\omega_n a_1 \ldots a_n$. Since $(a_2 \ldots a_{n-1})^{\frac{1}{n-2}} \geqslant a_n$, we deduce that

$$P(K) \geqslant 2\omega_{n-1}a_1 \ldots a_{n-1} \geqslant 2\omega_{n-1}a_1^{\frac{1}{n-1}}(a_1 \ldots a_n)^{\frac{n-2}{n-1}} \geqslant \alpha_n \cdot (\operatorname{diam} K)^{\frac{1}{n-1}}|K|^{\frac{n-2}{n-1}}$$

for $\alpha_n = 2\omega_{n-1}(2n)^{-\frac{1}{n-1}}(n^n\omega_n)^{-\frac{n-2}{n-1}}$. ∎

Next we need a formula for the surface area of a linear image.

Lemma 6.C.3. *If $K \subset \mathbb{R}^n$ is a convex body and $\Phi \in \mathrm{SL}(n)$, then*

$$P(\Phi^{-t}K) = \int_{S^{n-1}} \|\Phi(u)\|\, dS_K(u).$$

Proof. Let $\widetilde{\Phi} : S^{n-1} \to S^{n-1}$ be defined by $\widetilde{\Phi}(u) = \frac{\Phi(u)}{\|\Phi(u)\|}$. As $V_{\Phi^{-t}K} = \widetilde{\Phi}_* V_K$ by the linear equivariance Proposition 2.6.15 of the cone volume measure $dV_K = \frac{1}{n}\, h_K\, dS_K$, and $h_{\Phi^{-t}K}(\widetilde{\Phi}_* u) = h_K(u)/\|\Phi(u)\|$ by Lemma 2.6.14, we deduce that

$$P(\Phi^{-t}K) = n\int_{S^{n-1}} \frac{1}{h_{\Phi^{-t}K}}\, dV_{\Phi^{-t}K} = n\int_{S^{n-1}} \frac{\|\Phi(u)\|}{h_K(u)}\, dV_K(u) = \int_{S^{n-1}} \|\Phi(u)\|\, dS_K(u).$$

∎

We also need an algebraic condition when the surface area measure is quasi-isotropic.

Lemma 6.C.4. *For any convex body $K \subset \mathbb{R}^n$,*

$$\int_{S^{n-1}} u \otimes u\, dS_K = \frac{P(K)}{n} \cdot I_n \tag{6.79}$$

if and only if any real $n \times n$ matrix Ψ satisfies

$$\int_{S^{n-1}} \langle u, \Psi u\rangle\, dS_K(u) = \frac{\operatorname{tr}\Psi}{n} \cdot P(K). \tag{6.80}$$

Proof. We fix an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. If (6.80) holds, then choosing Ψ in (6.81) in all possible ways such that one entry is 1 and the rest is zero, we deduce that

$$\int_{S^{n-1}} \langle u, e_i\rangle\langle u, e_j\rangle\, dS_K(u) = \begin{cases} 0 & \text{if } i \neq j \\ \frac{P(K_0)}{n} & \text{if } i = j, \end{cases}$$

which in turn yields (6.79)

On the other hand, if (6.79) holds, then the previous argument yields that (6.80) holds whenever one entry of Ψ is 1 and the rest is zero. Any $n \times n$ matrix is the linear combination of such matrices; therefore, we conclude (6.80) . ∎

Proof of Theorem 6.C.1. The existence of a $\Phi_0 \in \mathrm{SL}(\mathbb{R}, n)$ such that $P(\Phi_0 K) \leqslant P(\Phi K)$ for any $\Phi \in \mathrm{SL}(\mathbb{R}, n)$ follows from Lemma 6.C.2. Let $K_0 = \Phi_0 K$. To prove (6.78), we consider a real $n \times n$ matrix Ψ, small $\varepsilon > 0$ and $\Omega = (I_n + \varepsilon \Psi)/\det(I_n + \varepsilon \Psi)^{\frac{1}{n}} \in \mathrm{SL}(\mathbb{R}, n)$, and hence $P(\Omega^{-t} K_0) \geqslant P(K_0)$ and Lemma 6.C.3 yield that

$$\int_{S^{n-1}} \|u + \varepsilon \Psi u\| \, dS_{K_0}(u) \geqslant \det(I_n + \varepsilon \Psi)^{\frac{1}{n}} P(K_0).$$

Letting $\varepsilon > 0$ tend to zero, we deduce that

$$\int_{S^{n-1}} \langle u, \Psi u \rangle \, dS_{K_0}(u) \geqslant \frac{\operatorname{tr} \Psi}{n} \cdot P(K_0), \tag{6.81}$$

and as (6.81) holds for $-\Psi$, as well, it follows that

$$\int_{S^{n-1}} \langle u, \Psi u \rangle \, dS_{K_0}(u) = \frac{\operatorname{tr} \Psi}{n} \cdot P(K_0). \tag{6.82}$$

We conclude (6.78) from (6.82) and Lemma 6.C.4.

Next we show that if S_K is quasi-isotropic; namely, it satisfies (6.78), then K is in minimal surface area position, and such position is unique up to orthogonal transformations. In order to verify $P(\Phi^{-t} K) \geqslant P(K)$ for $\Phi \in \mathrm{SL}(\mathbb{R}, n)$, using the polar decomposition, we write $\Phi = U\Omega$ for $U \in O(n)$ and positive semi-definite symmetric Ω with $\det \Omega = 1$, and hence Lemma 6.C.3, Lemma 6.C.4 and the the AM-GM inequality applied to the eigenvalues of Ω (cf. Lemma 3.5.2) yield that

$$\begin{aligned} P(\Phi^{-t} K) = P(U^{-t} \Omega^{-t} K) = P(\Omega^{-t} K) = \int_{S^{n-1}} \|\Omega u\| \, dS_K(u) \\ \geqslant \int_{S^{n-1}} \langle u, \Omega u \rangle \, dS_K(u) = \frac{\operatorname{tr} \Omega}{n} \cdot P(K) \geqslant (\det \Omega)^{\frac{1}{n}} \cdot P(K) = P(K). \end{aligned} \tag{6.83}$$

If $P(\Phi^{-t} K) = P(K)$, then $\frac{\operatorname{tr} \Omega}{n} = (\det \Omega)^{\frac{1}{n}}$ in (6.83), thus the equality conditions in the AM-GM inequality (or in Lemma 3.5.2) imply that all eigenvalues of Ω are the same, and hence $\Omega = I_n$. Therefore, $\Phi^{-t} = U^{-t} \in O(n)$. ∎

6.D Supplement: Maximal area of inscribed polygons and the extremality of ellipses

We have seen in Macbeath' Theorem 6.8.1 that ellipsoids are the worst approximable convex bodies in terms of volume approximation by inscribed polytopes of given number of vertices. In this section, we provide the elegant argument due to Sas [651] verifying that ellipses are the only extremizers in the plane.

Theorem 6.D.1 (Sas). *If $k \geqslant 3$ and P_k is a polygon with at most k vertices of maximal area contained in a convex body $C \subset \mathbb{R}^2$, then*

$$|P_k| \geqslant |C| \cdot \frac{k}{2\pi} \sin \frac{2\pi}{k},$$

with equality if and only if C is an ellipsoid.

Proof. We may assume that $(-1,0), (1,0) \in \partial C$ are the endpoints of a diameter of C, and hence $(x, f(x)), (x, -g(x)) \in \partial C$ for the concave functions $f, g : [-1,1] \to [0,\infty)$ where we may assume that $f(0) > 0$. It follows that

$$\begin{array}{rcll} f(x), g(x) & \leqslant & 2\sqrt{1-x} & \text{for } x \in [0,1]; \\ f(x), g(x) & \leqslant & 2\sqrt{x+1} & \text{for } x \in [-1,0]. \end{array} \tag{6.84}$$

Parameterize ∂C as $(x(t), y(t)) = (\cos t, \varphi(t) \sin t)$ for $t \in \mathbb{R}$ where $\varphi(t)$ is 2π-periodic, is continuous on $(0,\pi)$ and $(\pi, 2\pi)$, and $\varphi \leqslant 2$ on $[0, 2\pi]$ by (6.84), and hence $y(t) = f(\cos t)$ if $t \in [0,\pi]$ and $y(t) = -g(\cos t)$ if $t \in [\pi, 2\pi]$.

For $t \in \mathbb{R}$, let $Q_t = \operatorname{conv}\{p_i(t_i)\}_{i=0,\dots,k-1}$ where $p_i(t) = (x(t_i), y(t_i))$ and $t_i = t + \frac{i2\pi}{k}$, $i \in \mathbb{Z}$, therefore,

$$\begin{aligned} |Q_t| &= \frac{1}{2} \sum_{i=0}^{k-1} x(t_i) y(t_{i+1}) - x(t_{i+1}) y(t_i) = \frac{1}{2} \sum_{i=0}^{k-1} \varphi(t_i) \sin t_i (\cos t_{i+1} - \cos t_{i-1}) \\ &= \sin \frac{2\pi}{k} \sum_{i=0}^{k-1} \varphi\left(t + \frac{i2\pi}{k}\right) \sin^2\left(t + \frac{i2\pi}{k}\right). \end{aligned}$$

Using substitution $x = \cos t$, the mean value of $|Q_t|$ is

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |Q_t| \, dt &= \frac{k}{2\pi} \sin \frac{2\pi}{k} \int_0^{2\pi} \varphi(t) \sin^2 t \, dt = \frac{k}{2\pi} \sin \frac{2\pi}{k} \left(\int_1^{-1} -f(x) \, dx + \int_{-1}^{1} g(x) \, dx \right) \\ &= \frac{k}{2\pi} \sin \frac{2\pi}{k} \cdot |C|, \end{aligned}$$

and hence there exists a $t \in [0, 2\pi]$ with $|Q_t| \geqslant \frac{k}{2\pi} \sin \frac{2\pi}{k} \cdot |C|$.

Equality in Theorem 6.D.1 yields that $|Q_t| = |P_k|$ holds for $t \in [0, 2\pi]$, thus for any $t \in [0, 2\pi]$, the line through $p_0(t)$ and parallel to $p_1(t) - p_{-1}(t) \neq o$ is a supporting line to C. As $p_1(t) - p_{-1}(t)$ is continuous and C is convex, it follows that ∂C is C^1, and hence $y(t)$ differentiable on $(0,\pi)$ and on $(\pi, 2\pi)$, and satisfies

$$y'(t) = \frac{y(t + \frac{2\pi}{k}) - y(t - \frac{2\pi}{k})}{2 \sin \frac{2\pi}{k}}. \tag{6.85}$$

We deduce from (6.85) that $\lim_{t\to 0} y'(t) = y(\frac{2\pi}{k}) - y(-\frac{2\pi}{k})$ and $\lim_{t\to\pi} y'(t) = y(\pi + \frac{2\pi}{k}) - y(\pi - \frac{2\pi}{k})$; therefore, $y(t)$ is continuous, 2π periodic and piecewise smooth, which in turn yields that it has a Fourier series

$$y(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \text{ and } y'(t) = \sum_{n=1}^{\infty}(nb_n \cos nt - na_n \sin nt).$$

It follows from (6.85) that

$$\sum_{n=1}^{\infty}(nb_n \cos nt - na_n \sin nt) = \sum_{n=1}^{\infty} \frac{\sin \frac{n2\pi}{k}}{\sin \frac{2\pi}{k}}(b_n \cos nt - a_n \sin nt).$$

Comparing coefficients leads to $a_n = b_n = 0$ for $n \geqslant 2$; or in other words, $y(t) = a_0 + a_1 \cos t + b_1 \sin t$. It follows that ∂C is parametrized as $(\cos t, a_0 + a_1 \cos t + b_1 \sin t)$, and hence C is an ellipse. ∎

6.E Supplement: The Blaschke-Santaló inequality for general convex bodies

The proof of the Blaschke-Santaló inequality Theorem 6.5.2 presented in this section goes back to Keith Ball's PhD thesis [52] in the o-symmetric case, and the to Meyer, Pajor [575] in the general case. First, we show that the Brunn-Minkowski type inequality in Theorem 3.6.4 for the coordinatewise product directly yields the Blaschke Santaló inequality for unconditional convex bodies. Let $K, C \subset \mathbb{R}^n$ be unconditional convex bodies; namely, if $(x_1, \ldots, x_n) \in K$, then $(\pm x_1, \ldots, \pm x_n) \in K$. According to Theorem 3.6.4, we have

$$\left|K^{\frac{1}{2}} \cdot C^{\frac{1}{2}}\right|^2 \geqslant |K| \cdot |C|, \tag{6.86}$$

where

$$K^{\frac{1}{2}} \cdot C^{\frac{1}{2}} = \left\{ \left(\pm\sqrt{|x_1| \cdot |y_1|}, \ldots, \pm\sqrt{|x_n| \cdot |y_n|}\right) : (x_1, \ldots, x_n) \in K \right.$$
$$\left. \& \ (y_1, \ldots, y_n) \in C \right\}.$$

In addition, equality holds in (6.86) if and only if $C = \Phi K$ for a positive definite diagonal matrix Φ, and hence $K^{\frac{1}{2}} \cdot C^{\frac{1}{2}} = \Phi^{\frac{1}{2}} K$. The argument proving Proposition 6.E.1 is taken from Keith Ball's PhD thesis [52].

Proposition 6.E.1 (Blaschke-Santaló inequality for unconditional convex bodies). *If $K \subset \mathbb{R}^n$ is an unconditional convex body, then*

$$|K| \cdot |K^*| \leqslant |B^n|^2, \tag{6.87}$$

with equality if and only if K is an ellipsoid.

Proof. As K is unconditional, and $\langle x, y\rangle \leqslant 1$ holds for $x \in K$ and $y \in K^*$, we deduce that $K^{\frac{1}{2}} \cdot (K^*)^{\frac{1}{2}} \subset B^n$. Therefore, (6.86) yields the Blaschke-Santaló inequality (6.87).

Now, assume that equality holds in (6.87); in particular, we have $K^{\frac{1}{2}} \cdot (K^*)^{\frac{1}{2}} = B^n$. It follows from the equality conditions in (6.87) that there exists a positive definite diagonal matrix $\Psi \in \mathrm{GL}(n)$ such that $\Psi K = B^n$, and hence K is an ellipsoid. ∎

To prove the Blaschke-Santaló inequality Theorem 6.5.2 for general convex bodies, we need the following "harmonic" version of Prékopa-Leindler inequality due to Keith Ball [52]. We recall that the harmonic mean of $x, y > 0$ is $\left(\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)\right)^{-1} = \frac{2xy}{x+y}$.

Proposition 6.E.2 (Keith Ball). *Let $f, g, h : [0, \infty) \to [0, \infty)$ be continuous and positive on some bounded intervals whose left endpoint is 0, and be zero outside of these intervals, and let they satisfy*

$$h\left(\frac{2xy}{x+y}\right) \geqslant f(x)^{\frac{y}{x+y}} g(y)^{\frac{x}{x+y}} \tag{6.88}$$

for any $x, y > 0$. Then

$$\frac{1}{\int_0^\infty h} \leqslant \frac{1}{2}\left(\frac{1}{\int_0^\infty f} + \frac{1}{\int_0^\infty g}\right). \tag{6.89}$$

Proof. We set $F = \int_0^\infty f > 0$, $G = \int_0^\infty g > 0$ and $H = \int_0^\infty h > 0$, and hence there exists some strictly increasing differentiable functions $x : [0, 1) \to [0, \infty)$ and $y : [0, 1) \to [0, \infty)$ with $x(0) = y(0) = 0$ such that if $t \in (0, 1)$, then

$$F \cdot t = \int_0^{x(t)} f(s)\, ds$$
$$G \cdot t = \int_0^{y(t)} f(s)\, ds,$$

and hence

$$F = f(x(t)) \cdot x'(t) \text{ and } G = g(y(t)) \cdot y'(t). \tag{6.90}$$

Using the substitution $r = r(t) = \frac{2x(t)y(t)}{x(t)+y(t)}$ for $t \in (0, 1)$ where $r(t)$ is strictly increasing being the harmonic mean of $x(t)$ and $y(t)$, we deduce from (6.90), and then from (6.88)

that

$$H = \int_0^\infty h(r)\,dr \geqslant 2\int_0^1 h\left(\frac{2xy}{x+y}\right)\frac{(x'y+xy')(x+y)-xy(x'+y')}{(x+y)^2}\,dt$$
$$=2\int_0^1 h\left(\frac{2xy}{x+y}\right)\frac{\frac{Fy^2}{f(x)}+\frac{Gx^2}{g(y)}}{(x+y)^2}\,dt$$
$$\geqslant 2\int_0^1 f(x)^{\frac{y}{x+y}}g(y)^{\frac{x}{x+y}}\cdot\frac{\frac{y}{x+y}\cdot\frac{Fy}{f(x)}+\frac{x}{x+y}\cdot\frac{Gx}{g(y)}}{x+y}\,dt \tag{6.91}$$

where x and y are the increasing functions of t. Now, the AM-GM inequality yields that if $a, b > 0$, $t \in (0,1)$, $x = x(t)$ and $y = y(t)$, then

$$\frac{y}{x+y}\cdot a + \frac{y}{x+y}\cdot b \geqslant a^{\frac{y}{x+y}}b^{\frac{x}{x+y}},$$

which is applied to (6.91) first with $a = \frac{Fy}{f(x)}$ and $b = \frac{Gx}{g(y)}$, and then with $a = \frac{1}{Fy}$ and $b = \frac{1}{Gx}$, implies that

$$H \geqslant 2\int_0^1 (Fy)^{\frac{y}{x+y}}(Gx)^{\frac{x}{x+y}}\frac{1}{x+y}\,dt \geqslant 2\int_0^1\left(\frac{1}{F}+\frac{1}{G}\right)^{-1}dt = \frac{2}{\frac{1}{F}+\frac{1}{G}}.$$

■

Let us recall the definition of the Steiner symmetrization. For $u \in S^{n-1}$, and convex body $K \subset \mathbb{R}^n$, the Steiner symmetrial $\Theta_{u^\perp}K$ of K is (see Section 1.10)

$$\Theta_{u^\perp}K = \left\{x + \frac{t-s}{2}\cdot u : x \in K | u^\perp \;\&\; x+tu \in K \;\&\; x+su \in K\right\},$$

which readily satisfies $|\Theta_{u^\perp}K| = |K|$. We also need the Brunn-Minkowski inequality in the form (cf. Corollary 1.10.10 (i))

$$|(1-\lambda)K + \lambda C| \geqslant |K|^{1-\lambda}|C|^\lambda \tag{6.92}$$

for convex bodies $K, C \subset \mathbb{R}^n$ and $\lambda \in (0,1)$. For the following statement, proved by Keith Ball [52] if K is o-symmetric, and by Meyer, Pajor [575] in general, we say that a hyperplane $H \subset \mathbb{R}^n$ bounding a half-space H^+ is a halving hyperplane for a convex body $K \subset \mathbb{R}^n$ if $|K \cap H^+| = |K \backslash H^+|$.

Proposition 6.E.3. *For a $u \in S^{n-1}$, and a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, if $u^\perp$ is a halving hyperplane for K^*, then $|(\Theta_{u^\perp}K)^*| \geqslant |K^*|$.*

Remark. Thus, if K is o-symmetric, then $|(\Theta_{u^\perp}K)^*| \geqslant |K^*|$ for any $u \in S^{n-1}$.

Proof. Let $\widetilde{K} = \Theta_{u^\perp} K$, and for any convex body C with $o \in \operatorname{int} C$ and $t \in \mathbb{R}$, let

$$C_t^* = \{x \in u^\perp : x + tu \in C^*\}.$$

The core observation is the claim that if K_t^* and K_{-s}^* are non-empty for $t, s > 0$, then

$$\frac{s}{t+s} K_t^* + \frac{t}{t+s} K_{-s}^* \subset \widetilde{K}_{\frac{2ts}{t+s}}^*, \tag{6.93}$$

which is equivalent to saying that if $x + tu \in K^*$, $y - su \in K^*$ and $z + ru \in \widetilde{K}$ for $x, y, z \in u^\perp$ and $r \in \mathbb{R}$, and $w = \frac{s}{t+s} x + \frac{t}{t+s} y$, then

$$\langle w, z \rangle + \frac{2ts}{t+s} \cdot r \leqslant 1. \tag{6.94}$$

We have $r = \frac{r_1 - r_2}{2}$ where $z + r_1 u \in K$ and $z + r_2 u \in K$ by the definition of Steiner symmetrization, and hence $\langle x, z \rangle + t r_1 \leqslant 1$ and $\langle y, z \rangle - s r_2 \leqslant 1$ by the definition of K^*. Taking the linear combination of the last two inequalities with coefficients $\frac{s}{t+s}$ and $\frac{s}{t+s}$ yields (6.94), and in turn (6.93).

Now, (6.93) and the Brunn-Minkowski inequality (6.92) imply that the condition $h(\frac{2ts}{t+s}) \geqslant f(t)^{\frac{s}{t+s}} g(s)^{\frac{t}{t+s}}$ in Proposition 6.E.2 hold for $t, s > 0$ where

$$f(r) = \mathcal{H}^{n-1}(K_r^*),\ g(r) = \mathcal{H}^{n-1}(K_{-r}^*) \text{ and } h(r) = \mathcal{H}^{n-1}(\widetilde{K}_r^*)$$

for $r > 0$. Therefore, as $u^\perp$ is a halving hyperplane for K^*, Proposition 6.E.2 yields that

$$\frac{2}{|\widetilde{K}^*|} = \frac{1}{\int_0^\infty h} \leqslant \frac{1}{2} \left(\frac{1}{\int_0^\infty f} + \frac{1}{\int_0^\infty g} \right) = \frac{1}{2} \left(\frac{2}{|K^*|} + \frac{2}{|K^*|} \right) = \frac{2}{|K^*|}.$$

■

Corollary 6.E.4. *Let $C \subset \mathbb{R}^n$ be a convex body such that o is the Santaló point, let H be a halving hyperplane for C, and let $z = \aleph_{C,H} \in H \cap \operatorname{int} C$ minimize $\xi \mapsto |(C - \xi)^*|$ for $\xi \in H \cap \operatorname{int} C$ (cf. Lemma 6.5.1). Then the convex body $\widetilde{C} \subset \mathbb{R}^n$ defined by*

$$\widetilde{C}^* = \Theta_{H-z} \left((C - z)^* \right)$$

is symmetric through $H - z$, its Santaló point is the origin, and satisfies

$$|C| \cdot |C^*| \leqslant |\widetilde{C}| \cdot |\widetilde{C}^*| \tag{6.95}$$

where equality in (6.95) yields that $o \in H$ and $z = o$.

Proof. We deduce (6.95) from applying Proposition 6.E.3 to $K = (C - z)^*$ and $u^\perp = H - z$ where $u \in S^{n-1}$, and noting that $|C^*| \leqslant |(C - z)^*|$ because o is the Santaló point of C. As $z = \aleph_{C,H}$, the Remark after Lemma 6.5.1 yields that $\int_K x\,dx \in \mathbb{R}u$. Since Fubini's theorem implies that $\int_{\Theta_{u^\perp} K} x\,dx \in \mathbb{R}u$, as well, and $\Theta_{u^\perp} K$ is symmetric through $u^\perp$, we conclude that the origin is the centroid of $\Theta_{u^\perp} K$, and hence the Santaló point of $\widetilde{C}$.

Finally, the argument above actually proved $|C| \cdot |(C - z)^*| \leqslant |\widetilde{C}| \cdot |\widetilde{C}^*|$, thus equality in (6.95) yields that z is the Santaló point of C (cf. Lemma 6.5.1). ∎

Proof the Blaschke-Santaló inequality Theorem 6.5.2 without the equality case. Let us fix an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$, and let H_1 be the halving hyperplane for K orthogonal to e_1. Let $K_1 = \widetilde{C}$ where $\widetilde{C}$ is constructed from $C = K$ using the halving hyperplane $H = H_1$ as done in Corollary 6.E.4. It follows that C_1 is symmetric through $e_1^\perp$, its Santaló point is the origin, and satisfies

$$|K| \cdot |K^*| \leqslant |K_1| \cdot |K_1^*|. \tag{6.96}$$

Next, we construct convex bodies K_i, $i = 1, \dots, n$, by induction on i where K_i is symmetric through $e_1^\perp, \dots, e_i^\perp$, its Santaló point is the origin, and satisfies $|K| \cdot |K^*| \leqslant |K_i| \cdot |K_i^*|$. If K_{i-1} has been constructed for some $i = 2, \dots, n$, then we define $K_i = \widetilde{C}$ where $\widetilde{C}$ is constructed from $C = K_{i-1}$ using the halving hyperplane $H = H_i$ of C_{i-1} parallel to $e_i^\perp$ as done in Corollary 6.E.4. In particular,

$$|K| \cdot |K^*| \leqslant |K_{i-1}| \cdot |K_{i-1}^*| \leqslant |K_i| \cdot |K_i^*|. \tag{6.97}$$

Since K_{i-1} is symmetric through $e_1^\perp, \dots, e_{i-1}^\perp$, that are orthogonal to $e_i^\perp$, we deduce that K_i is symmetric through $e_1^\perp, \dots, e_i^\perp$.

Finally, as K_n is unconditional, we conclude from Proposition 6.E.1 that $|K| \cdot |K^*| \leqslant |K_n| \cdot |K_n^*| \leqslant |B^n|^2$. ∎

In order to characterize the case of equality in the Blaschke-Santaló inequality, we use the notion of isotropic convex bodies. According to Section 6.4, for any centered convex body $K \subset \mathbb{R}^n$ ($\sigma_K = o$), there exists $\Phi \in \mathrm{SL}(n)$ satisfying that ΦK is in quasi-isotropic position; namely, there exists $L_K > 0$ such that

$$\int_{\Phi K} \langle u, x\rangle^2\,dx = L_K^2 |K|^{\frac{n+2}{n}} \quad \text{for any } u \in S^{n-1}. \tag{6.98}$$

Here $L_{\Phi K} = L_K$ for $\Phi \in \mathrm{GL}(n)$, and Proposition 6.4.6 says that

$$L_K \geqslant L_{B^n}. \tag{6.99}$$

We note that a centered ellipsoid is in a quasi-isotropic position if and only if it is a Euclidean ball.

Proof for the equality case of the Blaschke-Santaló inequality Theorem 6.5.2. We may assume that K is centered ($\sigma_K = o$), $|K| = |B^n|$ and K is in a quasi-isotropic position (cf. (6.98)). Indirectly, we suppose that K maximizes the volume product and it is not an ellipsoid; or equivalently,

$$|K| = |K^*| = |B^n| \quad \text{and} \tag{6.100}$$

$$K \neq B^n, \tag{6.101}$$

and seek a contradiction.

Let us remark that according to Lemma 6.5.1, Corollary 6.E.4 and the already proved Blaschke-Santaló inequality (6.31), if $C \subset \mathbb{R}^n$ is a centered convex body with $|C| \cdot |C^*| = |B^n|^2$, then for any $u \in S^{n-1}$, $u^\perp$ is a halving hyperplane for C^*, and hence

$$|(\Theta_{u^\perp} C)^*| = |C^*|. \tag{6.102}$$

Let r be maximal with the property $rB^n \subset K$, and hence $r < 1$ as K is not a ball and $|K| = |B^n|$, and let $v \in S^{n-1}$ such that $rv \in \partial K$, thus $h_K(v) = r$. Now the Schwartz rounding $\widetilde{K} = \Theta_{\mathbb{R}v} K$ can be obtained as the limit of sequential Steiner symmetrizations from K using hyperplanes containing v according to Lemma 1.10.14; therefore, (6.102) yields that $\widetilde{K}$ is a convex body with axial rotational symmetry around $\mathbb{R}v$ satisfying the properties that

$$|\widetilde{K}| = |\widetilde{K}^*| = |B^n|, \tag{6.103}$$

$$h_{\widetilde{K}}(v) = r, \tag{6.104}$$

$$\sigma_{\widetilde{K}} = o. \tag{6.105}$$

We claim that

$$\widetilde{K} \text{ is not an ellipsoid.} \tag{6.106}$$

We suppose that $\widetilde{K}$ is an ellipsoid, and seek a contradiction. Then $h_K(-v) = r$, and for any $t \in [-r, r]$, we have

$$\mathcal{H}^{n-1}(K \cap (te_n + e_n^\perp)) = \mathcal{H}^{n-1}(K_{n-1} \cap (te_n + e_n^\perp)) = \frac{1}{r}\left(1 - \frac{t^2}{r^2}\right)^{\frac{n-1}{2}} \omega_{n-1},$$

and hence (6.98), (6.99), $|K| = |B^n|$ and K being in quasi-isotropic position imply that

$$\begin{aligned} L_{B^n}^2 |B^n|^{\frac{n+2}{n}} \leqslant L_K^2 |K|^{\frac{n+2}{n}} &= \int_K \langle v, x\rangle^2 \, dx = \int_{-r}^{r} \frac{t^2}{r}\left(1 - \frac{t^2}{r^2}\right)^{\frac{n-1}{2}} \omega_{n-1} \, dt \\ &= r^2 \int_{-1}^{1} s^2 \left(1 - s^2\right)^{\frac{n-1}{2}} \omega_{n-1} \, ds = r^2 \int_{B^n} \langle v, x\rangle^2 \, dx = r^2 L_{B^n}^2 |B^n|^{\frac{n+2}{n}}, \end{aligned}$$

which absurd by $r < 1$. In turn, we conclude (6.106).

Finally, we choose a linear 2-subspace $L \subset \mathbb{R}^n$ with $v \in L$, and hence $\widetilde{K} \cap L$ is not an ellipse by (6.106). It follows from Sas's Theorem 6.D.1 that there exists $w_1, w_2, w_3, w_4 \in \partial \widetilde{K} \cap L$ such that $\mathcal{H}^2(Q) > \frac{2}{\pi}\, \mathcal{H}^2(\widetilde{K} \cap L)$ for the quadrilateral $Q = \operatorname{conv}\{w_1, w_2, w_3, w_4\}$ where $\frac{2}{\pi}\, \mathcal{H}^2(\widetilde{K} \cap L)$ is the maximal area of a quadrilateral contained in an ellipse of area $\mathcal{H}^2(\widetilde{K} \cap L)$. Now $\bar{K} = \Theta_{u_2^\perp} \Theta_{u_1^\perp} \widetilde{K}$ where $u_1 = (w_1 - w_3)/\|w_1 - w_3\|$ and $u_2 \in S^{n-1} \cap L \cap u_1^\perp$ satisfies that $\bar{K}$ is unconditional for an orthonormal basis $u_1, u_2, \dots, u_n$ of $\mathbb{R}^n$ where $u_i \in L^\perp$ for $i > 2$, and $\bar{Q} \subset \bar{K}$ for the quadrilateral $\bar{Q} = \Theta_{u_2^\perp} \Theta_{u_1^\perp} Q$ with $\mathcal{H}^2(\bar{Q}) = \mathcal{H}^2(Q)$. Since $\bar{K} \cap L$ contains the quadrilateral $\bar{Q}$ of area $\mathcal{H}^2(\bar{Q}) > \frac{2}{\pi}\, \mathcal{H}^2(\bar{K} \cap L) = \frac{2}{\pi}\, \mathcal{H}^2(\widetilde{K} \cap L)$, Sas' Theorem 6.D.1 yields that $\bar{K} \cap L$, and in turn $\bar{K}$ is not an ellipsoid. However, $\bar{K}$ is unconditional, and $|\bar{K}| = |\bar{K}^*| = |B^n|$ follows from (6.102) and (6.103), which facts then contradict the equality case in Proposition 6.E.1. In particular, equality in the Blaschke-Santaló inequality (6.31) implies that K is a centered ellipsoid. ∎

Remark 6.E.5. Using the hemispherical transform (originally defined by Funk [309]), one can avoid the last step using Sas' theorem in the argument above. For $u \in S^{n-1}$, let us consider the half-space $H_u^+ = \{x \in \mathbb{R}^n : \langle x, u \rangle \geqslant 0\}$. If $g : S^{n-1} \to \mathbb{R}$ is a continuous function, then its hemispherical transform on S^{n-1} is defined by

$$\mathcal{F}g(u) = \int_{H_u^+ \cap S^{n-1}} g \, d\mathcal{H}^{n-1}.$$

A basic theorem in integral geometry (see Groemer [339], Proposition 3.4.11) is that

$$\text{if } \mathcal{F}g \text{ is an even function on } S^{n-1}, \text{ then } g \text{ is even, as well.} \tag{6.107}$$

Now, let $M \subset \mathbb{R}^n$ be a convex body whose Santaló point is the origin, and satisfies $|M| \cdot |M^*| = |B^n|^2$, and hence Corollary 6.E.4 and the Blaschke-Santaló inequality (6.31) yields that $u^\perp$ is a halving hyperplane for M^* for any $u \in S^{n-1}$. For $g = \varrho_M^n$ and $u \in S^{n-1}$, the formula

$$\mathcal{F}g(u) = n \left| H_u^+ \cap M \right|$$

holds according to Lemma 2.6.6. Since $u^\perp$ is a halving hyperplane for M for any $u \in S^{n-1}$, $\mathcal{F}g$ is even, and hence g, and in turn ϱ_M is even by (6.107). Therefore, the C in (6.102), and in turn the $\widetilde{K}$ in (6.106) are o-symmetric. We deduce that $\widetilde{K}$ is unconditional, and hence Proposition 6.E.1 leads to a contradiction.

6.F Supplement: From the Affine Isoperimetric to the Blaschke-Santaló inequality via Minkowski's inequality and the Minkowski problem

In this section, we discuss the intimate connection between the Blaschke-Santaló inequality (6.31) and the Affine Isoperimetric Inequality in Section 8.9. For a convex body $K \subset \mathbb{R}^n$, the Gaussian curvature $\kappa_K(x) \geqslant 0$ exists at $\mathcal{H}^{n-1}$ a.e. $x \in \partial K$, and the curvature function $f_K(u)$ exists at $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$ (see Section 8.1). Actually, if $\kappa_K(x) > 0$ at an $x \in \partial K$, then $f_K(\nu_K(x)) = 1/\kappa_K(x)$. Now the affine surface area of K is defined as (see Section 8.9)

$$\Omega(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}}\, du.$$

According to Theorem 8.9.10, the affine surface area is equi-affine invariant ($\Omega(\Phi K + z) = \Omega(K)$ for $z \in \mathbb{R}^n$ and $\Phi \in \mathrm{GL}(n)$ with $\det \Phi = \pm 1$), is upper semi-continuous ($\Omega(K) \geqslant \limsup_{m\to\infty} \Omega(K_m)$ if $\lim_{m\to\infty} K_m = K$), and satisfies the Affine Isoperimetric inequality saying that

$$\Omega(K) \leqslant n\omega_n^{\frac{2}{n+1}} |K|^{\frac{n-1}{n+1}}, \tag{6.108}$$

with equality if and only if K is an ellipsoid.

In Section 8.9 (following Lutwak [551]), we verify the Affine Isoperimetric inequality including the characterization of the case of equality based on the Blaschke-Santaló inequality Theorem 6.5.2. In this section, we sketch the argument of Santaló [642] proving first the Affine Isoperimetric inequality directly using Steiner symmetrization in a simple manner, and then applying his condition (6.109) on the Santaló point, the Minkowski inequality (6.110) and the Minkowski Theorem (6.111) about the characterization of surface area measure to prove the Blaschke-Santaló inequality (6.112).

Let $K \subset \mathbb{R}^n$ be a convex body. Let us recall that the (unique) Santaló point $\aleph_K \in \operatorname{int} K$ minimizes the integral $\xi \mapsto |(K-\xi)^*|$ for $\xi \in \operatorname{int} K$, and (6.28) yields that $\aleph_K = o$ if and only if

$$\int_{S^{n-1}} \frac{u}{h_K(u)^{n+1}}\, du = o. \tag{6.109}$$

The Minkowski inequality (2.31) says that if $C \subset \mathbb{R}^n$ in another convex body, then

$$\int_{S^{n-1}} h_K\, dS_C \geqslant n\, |K|^{\frac{1}{n}} |C|^{\frac{n-1}{n}}, \tag{6.110}$$

and according to Minkowski's Theorem 9.2.3, for any positive continuous function g on S^{n-1}, there exists a convex body C such that $dS_C(u) = g(u)\, du$ and $f_C = g$ (cf. Section 8.2) if and only if

$$\int_{S^{n-1}} u \cdot g(u)\, du = o. \tag{6.111}$$

We also note that for any convex body C with $o \in \operatorname{int} C$, we have $|C| = \frac{1}{n}\int_{S^{n-1}} \varrho_K(u)^n \, du$ and $\varrho_{C^*} = 1/h_C$ (cf. Lemma 1.11.6 and (1.21)).

Now, let us sketch how Santaló [642] proved the Blaschke-Santaló inequality (6.31); namely, if the Santaló point of K is $\aleph_K = o$, then

$$|K| \cdot |K^*| \leqslant |B^n|^2. \tag{6.112}$$

First, Santaló [642] verified the Affine Isoperimetric inequality (6.108) directly using Steiner symmetrization (cf. Theorem 8.C.2 and Theorem 8.C.3). Then he observed that the condition (6.109) on the Santaló point being the origin and the Minkowksi Theorem (6.111) yield the existence of a convex body C such that $dS_C(u) = f_C(u)\, du$ and $f_C = h_K^{-n-1}$. Next, applying the Minkowski inequality (6.110), and after that the Affine Isoperimetric inequality (6.108) yield that

$$\begin{aligned} n|K^*| = \int_{S^{n-1}} h_K(u)^{-n} \, du = \int_{S^{n-1}} h_K \, dS_C \geqslant n\,|K|^{\frac{1}{n}} |C|^{\frac{n-1}{n}} \\ \geqslant n\,|K|^{\frac{1}{n}} n^{-\frac{n+1}{n}} \, |B^n|^{-\frac{2}{n}} \Omega(C)^{\frac{n+1}{n}}. \end{aligned} \tag{6.113}$$

Since

$$\Omega(C) = \int_{S^{n-1}} f_C(u)^{\frac{n}{n+1}} \, du = \int_{S^{n-1}} h_K(u)^{-n} \, du = n|K^*|,$$

(6.113) implies the Blaschke-Santaló inequality (6.112).

Chapter 7

Steiner's formula and Mixed Volumes

This chapter discusses one of the most striking features of convex geometry - that the volume of a non-negative linear combination of given compact convex sets is polynomial in the coefficients as it was proved by Steiner in a special case around 1840, and by Minkowski in general around 1900. In turn, many of the coefficients of these polynomials - the so-called mixed volumes - have deep geometric meaning (for example, volume, surface area, mean projections, combinatorial quantities related to poset extensions, intersection numbers of algebraic hypersurfaces, etc), and satisfy some fundamental inequalities, like Minkowski's inequality, and its generalization, the Aleksandrov-Fenchel Inequality. At the end, we consider some topics related to the theory of mixed volumes, like Aleksandrov's variational formula for the Wulff shape, the L_p-Brunn-Minkowski inequality for $p > 1$, and the isoperimetric inequality for polytopes.

7.1 Steiner formula, Intrinsic volumes and Mean projections

For a compact convex set $K \subset \mathbb{R}^n$ (which we always assume to be non-empty in this book), its parallel domain of radius $r > 0$ (the set of points of distance at most r from K) is $K + rB^n$. One of the early results of convex geometry is that - as Steiner [691] observed already around 1840 - the volume of $K + rB^n$ is a polynomial in r and the coefficients have natural geometric meaning. To prove this statement, we need some properties of cones and polytopes discussed in Section 1.4. We recall that a polyhedral cone C is the intersection of finitely many half-spaces that have the orgin on their boundary; in particular, $\lambda x \in C$ for any $x \in C$ and $\lambda \geqslant 0$. Assuming that $\dim C = m$, the normalized angle of C is (cf. (1.8))

$$\beta(C) = \frac{\mathcal{H}^m(C \cap B^n)}{\mathcal{H}^m(B^m)} = \frac{\mathcal{H}^{m-1}(C \cap S^{n-1})}{\mathcal{H}^{m-1}(S^{n-1} \cap \mathrm{lin} C)} = \int_C e^{-\pi \|x\|^2} \, d\mathcal{H}^m(x).$$

In addition, a polytope $P \subset \mathbb{R}^n$ is the convex hull of finitely many vertices, which has finitely many faces (intersection with a supporting hyperplane). If F is a face of P, or $\dim P < n$ and $F = P$, and we set $\dim F = d$, then Lemma 1.4.10 assigns a polyhedral cone $N_P(F)$ (the so-called normal cone) of dimension $n - d$ to F in a way such that $z \in (\mathrm{relint}\, F) + N_P(F)$ if and only if $\Pi_P(z) \in \mathrm{relint}\, F$ for the closest point $\Pi_P(z)$ of P to z, and $N_P(F)$ satisfies

$$\left| \left(N_P(F) \cap \varrho\, B^n \right) + \mathrm{relint}\, F \right| = \varrho^{n-d} \omega_{n-d} \cdot \beta\left(N_P(F) \right) \mathcal{H}^d(F). \tag{7.1}$$

If $\dim P = n$, then it is practical to set $N_P(P) = \{o\}$. In particular, writing $\sqcup$ to denote disjoint union, Lemma 1.4.10 says that if $\varrho > 0$, then

$$P + \varrho B^n = \big((N_P(P) \cap \varrho B^n) + \operatorname{relint} P\big) \sqcup \bigsqcup_{F \text{ face of } P} \big((N_P(F) \cap \varrho B^n) + \operatorname{relint} F\big) \tag{7.2}$$

where $\omega_d = \mathcal{H}^d(B^d)$. Since orthogonal projection is a linear operation, we have the following.

Lemma 7.1.1. *If $X, Y \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^n$ is a linear subspace, then*

$$(X + Y)|L = (X|L) + (Y|L).$$

The main result of this section is the following formula for the parallel domain.

Theorem 7.1.2 (Steiner formula). *If $K \subset \mathbb{R}^n$ is a convex compact set and $r \geqslant 0$, then there exists unique $V_i(K) \geqslant 0$ called intrinsic i-volume for $i = 0, \dots, n$ such that*

$$|K + rB^n| = \sum_{i=0}^{n} V_i(K)\omega_{n-i} r^{n-i}. \tag{7.3}$$

In addition, these intrinsic i-volumes satisfy the following properties:

(i) Isometry Invariance: *$V_i(\Phi K + z) = V_i(K)$ for $\Phi \in O(n)$, $z \in \mathbb{R}^n$ and $i = 0, \dots, n$;*

(ii) $V_n(K) = |K|$, $V_{n-1}(K) = \frac{1}{2} P(K)$, $V_0(K) = 1$;

(iii) $V_i(K) > 0$ if $\dim K \geqslant i$, and $V_i(K) = 0$ if $\dim K < i$;

(iv) Dimension Invariance: *If $K \subset L$ for a linear d-subspace $L \subset \mathbb{R}^n$, then $V_i(K)$ with respect to L and $\mathbb{R}^n$ coincide. In particular, $V_i(K) = \mathcal{H}^i(K)$ if $\dim K = i$;*

(v) $V_i(K)$ is continuous in K for $i = 0, \dots, n$;

(vi) If P is a polytope, $\dim P \geqslant i$ and $\mathcal{F}^i(P)$ denotes the family of i-dimensional faces of P, then

$$V_i(P) = \sum_{F \in \mathcal{F}^i(P)} \mathcal{H}^i(F) \cdot \beta(N_P(F)); \tag{7.4}$$

(vii) $V_i(B^n) = \frac{\binom{n}{i}\omega_n}{\omega_{n-i}}$ for $i = 0, \dots, n$;

(viii) Mean projections (Kubota formula): *If $i = 1, \dots, n-1$, $\mathcal{L}^{i,n}$ is the space of linear i-planes (Grassmanian), and $\varrho_{i,n}$ is the Haar probability measure on $\mathcal{L}^{i,n}$ (invariant under $O(n)$), then*

$$V_i(K) = \frac{\binom{n}{i}\omega_n}{\omega_i \omega_{n-i}} \cdot \int_{\mathcal{L}^{i,n}} \mathcal{H}^i(K|L)\, d\varrho_{i,n}(L); \tag{7.5}$$

(ix) $V_i(K) \leqslant V_i(C)$ if $K \subset C$ for compact convex sets $K, C \subset \mathbb{R}^n$, and even $V_i(K) < V_i(C)$ if moreover $\dim C \geqslant i$ and $K \neq C$.

Remark. (Mean curvatures and mean width)

Mean curvature: If ∂K is C^2_+ and $i = 1, \dots, n-1$, then (see (8.32) in Theorem 8.3.4)

$$V_i(K) = \frac{1}{(n-i)\omega_{n-i}} \int_{\partial K} \sigma_{n-1-i}(\kappa_1(x), \dots, \kappa_{n-1}(x))\, dx$$

where $\sigma_{n-1-i}(\kappa_1(x), \dots, \kappa_{n-1}(x))$ is the $(n-1-i)th$ symmetric function of the principal curvatures at an $x \in \partial K$.

Mean width: If $K \subset \mathbb{R}^n$ is compact convex, then the width in the direction $u \in S^{n-1}$ is $\mathcal{H}^1(K|(\mathbb{R}u)) = h_K(u) + h_K(-u)$, and hence the Kubota formula (7.5) yields

$$V_1(K) = \frac{1}{2\omega_{n-1}} \int_{S^{n-1}} h_K(u) + h_K(-u)\, du = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} h_K\, d\mathcal{H}^{n-1}. \quad (7.6)$$

In particular, if $P = \operatorname{conv}\{v_1, \dots, v_m\} \subset \mathbb{R}^n$ is a polytope (for example, $v_1, \dots, v_m$ are the vertices), then

$$V_1(P) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \max_{i=1,\dots,k} \langle v_i, u \rangle\, du. \quad (7.7)$$

Proof of Theorem 7.1.2. Before verifying (7.3), we observe that if $\mathcal{C}$ is any family of compact convex sets such that the Steiner formula (7.3) holds for any $K \in \mathcal{C}$, then using Vandermonde matrices shows that there exist $\alpha_{n,i,j} \in \mathbb{R}$ depending on $n, i, j \in \mathbb{N}$ with $V_i(K) = \sum_{j=0}^n \alpha_{n,i,j} |K + jB^n|$ for any $K \in \mathcal{C}$ and $i = 0, \dots, n$; therefore, $V_i(K)$ is unique and it is a continuous function of $K \in \mathcal{C}$ for fixed $i = 0, \dots, n$ (see Lemma 1.7.5 for the continuity of the volume).

First we prove (7.3) and (i)-(vi) for a polytope $P \subset \mathbb{R}^n$. For $K = P$, the Steiner formula (7.3) with coefficients as in (7.4) follows from (7.1) and (7.2), proving (i)-(vi) where we note that for $m = \dim P$, if $m < n$, then $\beta(N_P(P)) = 1$, and if $m = n$ and $F_1, \dots, F_k$ are the facets, then $\beta(N_P(F_i)) = \frac{1}{2}$.

If $P \subset RB^n$ for a polytope P and $R > 0$, then $|P + B^n| \leqslant (R+1)^n |B^n|$, and hence the Steiner formula (7.3) yields that $V_i(P) \leqslant \frac{(R+1)^n \omega_n}{\omega_{n-i}}$ for $i = 1, \dots, n$. We deduce *via* approximating by polytopes (cf. (1.17)) and the continuity of the volume (cf. Lemma 1.7.5) that (7.3) and the properties (i)-(v) hold for any compact convex set $K \subset \mathbb{R}^n$ where in the case of (iii), we only know non-strict inequality $V_i(K) \geqslant 0$ if $\dim K \geqslant i$ at the moment.

Next, $V_i(B^n) = \frac{\binom{n}{i}\omega_n}{\omega_{n-i}}$ for $i = 0, \dots, n$ in (vii) follows from (7.3).

Kubota formula (7.5) in (viii) is proved by induction on $n \geqslant 2$. If $n = 2$, or $n \geqslant 3$ and $i = n-1$, then (7.5) is just the Cauchy formula (2.9); therefore, we may assume that $i \leqslant n-2$. Derivating the Steiner formula (7.3) with respect to r (cf. (2.8)) shows that

$$P(K + rB^n) = \sum_{i=0}^{n-1} (n-i) V_i(K) \omega_{n-i} r^{n-1-i}.$$

We deduce from the Cauchy formula (2.9) and Lemma 7.1.1 and by induction that

$$\begin{aligned}\sum_{i=0}^{n-1}(n-i)V_i(K)\omega_{n-i}r^{n-1-i} &= P(K+rB^n)\\ &= \frac{1}{\omega_{n-1}}\int_{S^{n-1}}\mathcal{H}^{n-1}((K+rB^n)|u^\perp)\,du\\ &= \frac{1}{\omega_{n-1}}\int_{S^{n-1}}\sum_{i=0}^{n-1}V_i(K|u^\perp)\omega_{n-1-i}r^{n-1-i}\,du.\end{aligned}$$

Therefore, equating coefficients of r^{n-1-i} for $i = 1,\dots,n-2$ leads to the formula

$$V_i(K) = \frac{\omega_{n-1-i}}{(n-i)\omega_{n-i}\omega_{n-1}}\cdot\int_{S^{n-1}}V_i(K|u^\perp)\,du.$$

Applying the Kubota formula in each $u^\perp$, $u\in S^{n-1}$ by induction, and using the uniqueness of the $O(n)$ invariant Haar probability measure on $\mathcal{L}^{i,n}$ yields that there exists $c_{n,i}>0$ depending on n and i with

$$V_i(K) = c_{n,i}\cdot\int_{\mathcal{L}^{i,n}}\mathcal{H}^i(K|L)\,d\varrho_{i,n}(L). \tag{7.8}$$

Substituting $K = B^n$ in (7.8) implies that $c_{n,i} = \frac{\binom{n}{i}\omega_n}{\omega_i\omega_{n-i}}$. Finally, the monotonicity property (ix), and in turn, the property that $V_i(K)>0$ if $\dim K\geqslant i$ in (iii) follow from the Kubota formula (7.5). ∎

7.2 Isoperimetric inequality for Mean projections

According to the Isoperimetric inequality for convex bodies (see Section 2.4), given the volume, the surface area is minimized by the ball. Here we extend this property to any mean projection; namely, to any intrinsic volume $V_i(K)$ for $i = 1,\dots,n-1$. This result is also a consequence of the version (7.31) of the Aleksandrov-Fenchel inequality where the equality case is characterized by Alesker, Dar, V. Milman [16].

Theorem 7.2.1 (Isoperimetric inequality for Mean projections). *If $K\subset\mathbb{R}^n$, $n\geqslant 2$ is a convex body with $|K| = |rB^n|$ for $r>0$ and $i = 1,\dots,n-1$, then $V_i(K)\geqslant V_i(rB^n)$ with equality if and only if K is a ball. In particular,*

$$V_i(K)\geqslant c_{i,n}|K|^{\frac{i}{n}}$$

for $c_{i,n} = \binom{n}{i}\omega_n^{\frac{n-i}{n}}/\omega_{n-i}$ with equality if and only if K is a ball.

The main method to verify Theorem 7.2.1 is Steiner symmetrization originally designed to prove the Isoperimetric inequality, and applied by Hadwiger [377] in our context. We recall that if $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then the Steiner symmetrial $\Theta_{u^\perp} K$ (cf. Definition 1.10.1) of K is

$$\Theta_{u^\perp} K = \left\{ x + \frac{t-s}{2} \cdot u : x \in K|u^\perp \,\&\, x + tu \in K \,\&\, x + su \in K \right\}.$$

Proposition 7.2.2. *If $K \subset \mathbb{R}^n$ is a convex body, $n \geqslant 2$, $i = 1, \dots, n-1$ and $u \in S^{n-1}$, then*

$$V_i(\Theta_{u^\perp} K) \leqslant V_i(K), \tag{7.9}$$

with equality if and only if K has a hyperplane of symmetry parallel to $u^\perp$.

Proof. Let $K_0 = \Theta_{u^\perp} K$ be the Steiner symmetrial and let

$$\widetilde{K} = \xi_{u^\perp} K = \left\{ x - tu : x \in K|u^\perp \,\&\, x + tu \in K \right\}$$

be the reflected image over $u^\perp$.

According to the Kubota formula (7.5), to prove (7.9), it is enough to verify that if $L \in \mathcal{L}^{i,n}$ such that $L \not\subset u^\perp$, then

$$V_i(K_0|L) \leqslant \frac{1}{2}\Big(V_i(K|L) + V_i(\widetilde{K}|L) \Big). \tag{7.10}$$

Writing $L_0 = L \cap u^\perp$ and $X' = X|L$ for any $X \subset \mathbb{R}^n$, $L \not\subset u^\perp$ yields that $u' \neq o$. Therefore, (7.10) follows by the Fubini theorem if for any line $\ell = z + \mathbb{R}u'$, $z \in K_0|L_0 = K|L_0$, we have

$$V_1(\ell \cap K_0') \leqslant \frac{1}{2} \left(V_1(\ell \cap K') + V_1(\ell \cap \widetilde{K}') \right). \tag{7.11}$$

To prove (7.11), we consider $v = u'/\|u'\|$, and observe that there exist $p_1, p_2 \in \partial K_0$ with $p_1', p_2' \in \ell$ and $V_1(\ell \cap K_0') = \langle v, p_2' - p_1' \rangle$. Let $p_j = a_j + \frac{1}{2}(t_j - s_j)u$ for $a_j + t_j u, a_j + s_j u \in K$ where $a_j \in K|u^\perp$, $t_j, s_j \in \mathbb{R}$, $j = 1, 2$, and hence $a_1', a_2' \in \ell$ and

$$\begin{aligned}
&V_1(\ell \cap K_0') = \left\langle v, a_2' - a_1' + \tfrac{1}{2}(t_2 - s_2 - t_1 + s_1)u' \right\rangle; \\
&V_1(\ell \cap K') = \max\{\langle v, x - y \rangle : x, y \in \ell \cap K'\} \geqslant \langle v, a_2' + t_2 u' - (a_1' + t_1 u') \rangle; \\
&V_1(\ell \cap \widetilde{K}') = \max\{\langle v, x - y \rangle : x, y \in \ell \cap \widetilde{K}'\} \geqslant \langle v, a_2' - s_2 u' - (a_1' - s_1 u') \rangle.
\end{aligned}$$

These last estimates yield the claim (7.11), which then implies (7.10), and in turn (7.9).

Equality in (7.9) yields that for any $L \in \mathcal{L}^{i,n}$ such that $L \not\subset u^\perp$, and for any line $\ell \subset L$ orthogonal to $L_0 = L \cap u^\perp$ and intersecting $K|L$ (and hence $u \in \ell + L^\perp$), we have (using the notation as above)

- $a_2 + t_2 u, a_1 + t_1 u \in \operatorname{relbd}(K \cap (\ell + L^\perp))$, and a tangent hyperplane in $\ell + L^\perp$ at $a_2 + t_2 u$ and at $a_1 + t_1 u$ is parallel to $L^\perp$;
- $a_2 - s_2 u, a_1 - s_1 u \in \operatorname{relbd}(\widetilde{K} \cap (\ell + L^\perp))$, and a tangent hyperplane in $\ell + L^\perp$ at $a_2 - s_2 u$ and at $a_1 - s_1 u$ is parallel to $L^\perp$.

We deduce for any fixed affine $(n-i+1)$-subspace A that is parallel to u and intersects $\operatorname{int} K$ the following properties: If $L \in \mathcal{L}^{i,n}$, $L \not\subset u^\perp$, satisfies that $L^\perp$ is parallel to A, and for the points $a + tu, a + su \in \operatorname{relbd}(K \cap A)$ with $t > s$ and $a \in K|L$ we have that a tangent hyperplane in A at $a + tu$ is parallel to $L^\perp$, then a tangent hyperplane in A at $a + su$ is parallel to the reflected image $\xi_{u^\perp} L^\perp$. It follows that if $f : (K \cap A)|u^\perp \to \mathbb{R}$ is the convex function and $g : (K \cap A)|u^\perp \to \mathbb{R}$ is the concave function satisfying that

$$K \cap A = \left\{ x + t \cdot u : x \in (K \cap A)|u^\perp \ \&\ f(x) \leqslant t \leqslant g(x) \right\},$$

then $-Df(x) = Dg(x)$ for $\mathcal{H}^i$ a.e $x \in (K \cap A)|u^\perp$. Thus $f(x) + g(x) = c$ for $x \in (K \cap A)|u^\perp$ where $c \in \mathbb{R}$ is a a constant, and hence $g(x) - \frac{c}{2} = \frac{c}{2} - f(x)$ for $x \in (K \cap A)|u^\perp$. We conclude that $A \cap K$ is symmetric through the hyperplane $u^\perp + \frac{c}{2} u$ (where $c \in \mathbb{R}$ depends on A).

Fix $a \in (\operatorname{int} K)|u^\perp$, and let $a + tu, a + su \in \partial K$ for $t > s$. The previous argument yields that $A \cap K$ is symmetric through the hyperplane $u^\perp + \frac{t+s}{2} u$ for any $(n-i+1)$-subspace A with $a + tu, a + su \in A$; therefore, K is symmetric through the hyperplane $u^\perp + \frac{t+s}{2} u$. ∎

Proof of Theorem 7.2.1. Proposition 7.2.2 and the use of iterated Steiner symmetrizations leading to a ball (cf. Lemma 1.10.4) yield (7.9).

Now, let $V_i(K) = V_i(rB^n)$. It follows from Proposition 7.2.2 that for any $u \in S^{n-1}$, there is a hyperplane $H_u \subset \mathbb{R}^n$ parallel to $u^\perp$ such that K is symmetric through H_u, and hence H_u contains the centroid σ_K by affine invariance (cf. Lemma 1.11.2). In turn, we deduce that $K - \sigma_K = rB^n$. ∎

7.3 Mixed volumes

We have already seen in the case of the Steiner formula (7.3) that a non-negative linear combination of a compact convex set and a ball is a polynomial of degree n in the coefficents. Following Minkowski's ideas in [590, 591], we extend this result to non-negative linear combination of any number of compact convex sets. For the notion of the surface area measure S_K on S^{n-1} of a compact convex set $K \subset \mathbb{R}^n$ introduced in Section 2.5, the property we need is that if $K \subset \mathbb{R}^n$ convex body and $C \subset \mathbb{R}^n$ is a compact convex set, then Proposition 2.5.9 yields

$$\lim_{r \to 0^+} \frac{|K + rC| - |K|}{r} = \int_{S^{n-1}} h_C \, dS_K = \int_{\partial' K} h_C \circ \nu_K \, d\mathcal{H}^{n-1}, \tag{7.12}$$

where the first equality holds also if K is a compact convex set.

Theorem 7.3.1 (Mixed volumes). *If $K_1, \dots, K_m, C_1, \dots, C_l \subset \mathbb{R}^n$ are compact convex sets, then*

$$\left| \sum_{i=1}^{m} \lambda_i K_i \right| = \sum_{\substack{\alpha_1+\dots+\alpha_m=n \\ \alpha_1,\dots,\alpha_m \in \mathbb{N}}} \frac{n!}{\alpha_1! \cdot \ldots \cdot \alpha_m!} \cdot V(K_1, \alpha_1; \dots; K_m, \alpha_m) \cdot \prod_{i=1}^{m} \lambda_i^{\alpha_i} \tag{7.13}$$

is a homogeneous polynomial of degree n in $\lambda_1, \dots, \lambda_m \geqslant 0$.

(i) $V(K_1, \alpha_1; \dots; K_m, \alpha_m) \geqslant 0$, it is uniquely determined, has the natural symmetry, and depends only on the positive α_i. We set $V(K_1, \dots, K_n) = V(K_1, 1; \dots; K_n, 1)$.

(ii) $\sum_{j=1}^{l} \varrho_j V(K_1, \dots, K_{n-1}, C_j) = V\left(K_1, \dots, K_{n-1}, \sum_{j=1}^{l} \varrho_j C_j\right)$ where $\varrho_j \geqslant 0$ and $j = 1, \dots, l$.

(iii) $n! V(K_1, \dots, K_n) = \sum_{i=1}^{n} (-1)^{n-i} \sum_{1 \leqslant j_1 < \dots < j_i \leqslant n} |K_{j_1} + \dots + K_{j_i}|$; or in other words, $n! V(K_1, \dots, K_n) = \left| \sum_{i=1}^{n} K_i \right| - \sum_{j=1}^{n} \left| \sum_{i \neq j} K_i \right| + \dots$.

(iv) $V(K, \dots, K) = |K|$.

(v) $V(K_1 + z_1, \dots, K_n + z_n) = V(K_1, \dots, K_n)$ for $z_i \in \mathbb{R}^n$, $i = 1, \dots, n$.

(vi) $V(\Phi K_1, \dots, \Phi K_n) = |\det \Phi| \cdot V(K_1, \dots, K_n)$ for $\Phi \in \mathrm{GL}(n)$.

(vii) Setting $V(K_1, n-i; K_2, i) = V(K_1, K_2; i)$, we have

$$|\lambda_1 K_1 + \lambda_2 K_2| = \sum_{i=0}^{n} \binom{n}{i} V(K_1, K_2; i) \lambda_1^{n-i} \lambda_2^{i}. \tag{7.14}$$

(viii) $V(K_1, \dots, K_n)$ is continuous in $K_1, \dots, K_n$.

(ix) If $K \subset \mathbb{R}^n$ is a convex body and $C \subset \mathbb{R}^n$ is a compact convex set, then

$$V(K, C; 1) = \frac{1}{n} \int_{S^{n-1}} h_C \, dS_K = \frac{1}{n} \int_{\partial' K} h_C \circ \nu_K \, d\mathcal{H}^{n-1} \tag{7.15}$$

where the first equality holds also if K is a compact convex set.

(x) $V(C_1, \dots, C_n) \leqslant V(K_1, \dots, K_n)$ if $C_i \subset K_i$.

Before the proof of Theorem 7.3.1, let us show that some funcdamental geometric quantities, like intrinsic volumes and the area of an orthonal projection, occur as mixed volumes.

Example 7.3.2.

Intrinsic volumes: For a compact convex set $K \in \mathcal{K}^n$, comparing the Steiner formula (7.3) to (7.14) shows that

$$P(K) = nV(K, B^n; 1); \tag{7.16}$$

$$V_i(K) = \frac{\binom{n}{i}}{\omega_{n-i}} V(B^n, K; i) \quad \text{for } i = 0, \dots, n. \tag{7.17}$$

We also note that the intrinsic volumes are proportional to the mean projections as they satisfy the Kubota formula (cf. (7.5)).

Orthogonal projection: For a compact convex set $K \in \mathcal{K}^n$, $x \in \mathbb{R}^n \backslash \{o\}$ and segment $s = \operatorname{conv}\{0, x\}$, (7.14) and calculating $|K + \lambda s|$ using integration over $x^\perp$ and the Fubini theorem yield that

$$nV(K, s; 1) = \|x\| \cdot \mathcal{H}^{n-1}(K|x^\perp) = \mathcal{H}^1(s) \cdot \mathcal{H}^{n-1}(K|x^\perp). \tag{7.18}$$

Let us review various properties of polytopes that we need to prove Theorem 7.3.1. For an n-polytope $P \subset R^n$ and $u \in S^{n-1}$, u is an exterior normal to the face $F_P(u) = \{x \in P : \langle x, u \rangle = h_P(u)\}$ of P (see Section 1.4 for properties of polytopes). We observe that if $P = \sum_{i=1}^m \lambda_i P_i$ for polytopes $P_1, \dots, P_m \subset \mathbb{R}^n$ and $\lambda_1, \dots, \lambda_m \geqslant 0$, then

$$F_P(u) = \sum_{i=1}^m \lambda_i F_{P_i}(u). \tag{7.19}$$

In addition, if $F_1, , \dots, F_k$ are the facets of P with exterior unit vectors $u_1, \dots, u_k$, then (2.4) says

$$|P| = \frac{1}{n} \sum_{i=1}^k h_P(u_i)\, |F_i|. \tag{7.20}$$

We also need the fact that the (discrete) support of the surface area measure of positive linear combination of polytopes does not depend on the coefficients. This property follows from Lemma 1.6.10 about the normal cones of a Minkowski sum of polytopes, but we also provide a direct argument.

Lemma 7.3.3. *If $P_1, \dots, P_n \subset \mathbb{R}^n$ are n-dimensional polytopes and $\lambda_1, \dots, \lambda_m > 0$, then* $\operatorname{supp} S_{\sum_{i=1}^m P_i} = \operatorname{supp} S_{\sum_{i=1}^m \lambda_i P_i}$.

Proof. If $0 < a \leqslant \lambda_1, \dots, \lambda_m \leqslant b$ and $u \in S^{n-1}$, then

$$a^{n-1}\mathcal{H}^{n-1}\left(\sum_{i=1}^m F_{P_i}(u)\right) \leqslant \mathcal{H}^{n-1}\left(\sum_{i=1}^m \lambda_i F_{P_i}(u)\right) \leqslant b^{n-1}\mathcal{H}^{n-1}\left(\sum_{i=1}^m F_{P_i}(u)\right).$$

■

Proof of Theorem 7.3.1. After the existence of mixed volumes have been established, property (v) about the individually translating the convex bodies follows from the translation invariance of volume, and property (vi) about applying simultaneous linear transformation again follows from the corresponding property of volume.

First, we consider the case of n-dimensional polytopes, and then verify the general case of compact convex sets *via* polytopal approximation.

Case 1 $K_i = P_i$ and $C_j = Q_j$ are n-dimensional polytopes.

We may assume $o \in \operatorname{int} P_i, \operatorname{int} Q_j$, and hence $h_{P_i}(u) > 0$, $h_{Q_j}(u) > 0$ for $u \in S^{n-1}$.

We prove Case 1 by induction on $n \geqslant 1$ where the case $n = 1$ is trivial as $V(s, 1) = \mathcal{H}^1(s)$ for a segment s. If $n \geqslant 2$, $u \in S^{n-1}$, $\alpha_1, \dots, \alpha_m \in \mathbb{N}$ with $\sum_{i=1}^m \alpha_i = n - 1$ and $F_i \subset u^\perp + t_i u$ are compact convex for $t_i \in \mathbb{R}$, $i = 1, \dots, m$, then let $v(F_1, \alpha_1; \dots; F_m, \alpha_m)$ be the $(n-1)$-dimensional mixed-volume.

For $P = \sum_{i=1}^m \lambda_i P_i$, $\lambda_i > 0$, and $\mathcal{U} = \operatorname{supp} S_{P_1 + \dots + P_m}$, (7.19), (7.20) and the induction hypothesis imply

$$\begin{aligned} |P| =& \frac{1}{n} \sum_{u \in \mathcal{U}} h_P(u) F_P(u) = \frac{1}{n} \sum_{u \in \mathcal{U}} \left(\sum_{u \in \mathcal{U}} \lambda_i h_{P_i}(u) \right) \cdot \mathcal{H}^{n-1} \left(\sum_{i=1}^m \lambda_i F_{P_i}(u) \right) \\ =& \frac{1}{n} \sum_{u \in \mathcal{U}} \left(\sum_{u \in \mathcal{U}} \lambda_i h_{P_i}(u) \right) \times \\ & \times \left(\sum_{\substack{\alpha_1 + \dots + \alpha_m = n-1 \\ \alpha_1, \dots, \alpha_m \in \mathbb{N}}} \frac{(n-1)!}{\alpha_1! \cdot \ldots \cdot \alpha_m!} \cdot v(F_{P_1}(u), \alpha_1; \dots; F_{P_m}, \alpha_m) \cdot \prod_{i=1}^m \lambda_i^{\alpha_i} \right). \end{aligned} \tag{7.21}$$

We conclude (7.13) with coefficients $V(P_1, \alpha_1; \dots; P_m, \alpha_m) \geqslant 0$ where the mixed volume $V(P_1, \alpha_1; \dots; P_m, \alpha_m)$ is symmetric in its variables (P_i, α_i), and if $P_{m-1} = P_m$, then $V(P_1, \alpha_1; \dots; P_{m-1}, \alpha_{m-1}; P_m, \alpha_m) = V(P_1, \alpha_1; \dots; P_{m-1}, \alpha_{m-1} + \alpha_m)$. Uniqueness in (i) of $V(P_1, \alpha_1; \dots; P_m, \alpha_m)$ follows from taking mixed partial derivatives at $\lambda_i = 0$ in (7.13). In particular, we deduce the properties (i) to (vii). As the volume and the Minkowski sum are continuous, (iii) yields the continuity of the mixed volume in (viii), and the representation (7.15) of the mixed volume using the surface area measure in (ix) follows from (7.14) in (vii) and (7.12).

Finally, to prove the monotonicity property (x), we deduce from (ii) that if $P = \sum_{i=1}^{n-1} \lambda_i P_i$ for $\lambda_i \geqslant 0$ and n-polytopes P_i, Q, $i = 1, \dots, n-1$, then

$$V(P, \dots, P, Q) = \sum_{\substack{\alpha_1 + \dots + \alpha_m = n-1 \\ \alpha_1, \dots, \alpha_m \in \mathbb{N}}} \frac{(n-1)!}{\alpha_1! \cdot \ldots \cdot \alpha_m!} \cdot V(P_1, \alpha_1; \dots; P_{n-1}, \alpha_{n-1}; Q, 1) \cdot \prod_{i=1}^m \lambda_i^{\alpha_i},$$

and hence for $\mathcal{U} = \operatorname{supp} S_{P_1+\ldots+P_{n-1}}$, we use (7.15) in (ix) and induction to deduce that

$$V(P_1,\ldots,P_{n-1},Q) = \frac{1}{n}\sum_{u\in\mathcal{U}} h_Q(u)\cdot v(F_{P_1}(u),\ldots,F_{P_{n-1}}(u)). \tag{7.22}$$

Since for n-polytopes $Q_1 \subset Q_2$, we have $h_{Q_1} \leqslant h_{Q_2}$, the formula (7.22) implies that $V(P_1,\ldots,P_{n-1},Q_1) \leqslant V(P_1,\ldots,P_{n-1},Q_2)$, finally conluding (x).

Case 2 $K_1,\ldots,K_m,C_1,\ldots,C_l \subset \mathbb{R}^n$ are general compact convex sets.

We choose $a > 0$ such that $K_1,\ldots,K_m,C_1,\ldots,C_l \subset \operatorname{int} W$ for $W = [-a,a]^n$, and by polytopal approximation (cf. (1.18)), we can choose n-dimensional polytopes $P_1^{(k)},\ldots,P_m^{(k)},Q_1^{(k)},\ldots,Q_l^{(k)} \subset W$ such that $P_i^{(k)} \to K_i$ and $Q_j^{(k)} \to C_i$, and hence (iv) and (x) yield that each mixed volume built from these polytopes is at most $|W|$. Except for the uniqueness in (i), we deduce (7.13) and the properties (i) to (vii) for any compact convex sets by polytopal approximation and by the properties of mixed volumes of polytopes proved in Case 1. Uniqueness in (i) of $V(K_1,\alpha_1;\ldots;K_m,\alpha_m)$ follows from taking mixed partial derivatives at $\lambda_i = 0$ in (7.13). Again, as the volume and the Minkowski sum are continuous, (iii) yields the continuity of the mixed volume in (viii), and (7.15) in (ix) follows from (7.14) in (vii) and (7.12). Finally, for the monotonicity property $V(C_1,\ldots,C_n) \leqslant V(K_1,\ldots,K_n)$ if $C_i \subset K_i$ in (x), then we may assume that $K_i \subset \operatorname{int} P_i^{(k)}$ (cf. (1.18)) and $Q_i^{(k)} \subset P_i^{(k)}$ for each i and k. ∎

Remark 7.3.4 (Mixed area measure for n-polytopes). The formula (7.22) in the argument above and approximating a compact convex set C by n-dimensional polytopes yields that if $P_1,\ldots,P_{n-1} \subset \mathbb{R}^n$ are n-polytopes, then

$$V(P_1,\ldots,P_{n-1},C) = \frac{1}{n}\sum_{u\in\operatorname{supp} S_{P_1+\ldots+P_{n-1}}} h_C(u)\cdot v(F_{P_1}(u),\ldots,F_{P_{n-1}}(u)). \tag{7.23}$$

Therefore, there exists a discrete measure $S_{P_1,\ldots,P_{n-1}}$ on S^{n-1} with $\operatorname{supp} S_{P_1,\ldots,P_{n-1}} \subset \operatorname{supp} S_{P_1+\ldots+P_{n-1}}$ such that for any convex compact set $C \subset \mathbb{R}^n$, we have

$$V(P_1,\ldots,P_{n-1},C) = \frac{1}{n}\int_{S^{n-1}} h_C\, dS_{P_1,\ldots,P_{n-1}}. \tag{7.24}$$

We will show the existence of mixed area measure $S_{K_1,\ldots,K_{n-1}}$ on S^{n-1} for any compact convex sets $K_1,\ldots,K_{n-1} \subset \mathbb{R}^n$ in Theorem 8.3.5.

We have seen in Theorem 7.3.1 (x) that the mixed volumes are monotone increasing in each of their variables. Now we exhibit an example showing that this monotonicity may not be strict.

Example 7.3.5 (Monotonicity of mixed volumes may not be strict). We observe that $B^n \subset W^n = [-1,1]^n$, $B^n \neq W^n$, but for any exterior unit normal u to a facet of W^n, we have $h_{W^n}(u) = 1 = h_{B^n}(u)$; therefore, (7.15) in Theorem 7.3.1 (ix) yields that $V(W^n, B^n; 1) = V(W^n, W^n; 1)$.

Proposition 7.3.6. *For convex compact $K_1, \dots, K_n \subset \mathbb{R}^n$, $V(K_1, \dots, K_n) > 0$ if and only $\exists x_i, y_i \in K_i$ such that $x_1 - y_1, \dots, x_n - y_n$ are independent.*

Proof. If $x_1 - y, \dots, x_n - y_n$ are independent, then for $s_i = [x_i, y_i]$, we have

$$V(K_1, \dots, K_n) \geqslant V(s_1, \dots, s_n) = |\det[x_1 - y_1, \dots, x_n - y_n]|/n! > 0.$$

If there exist no suitable $x_i, y_i \in K_i$, then after possibly translating and reindexing, there exist $1 \leqslant m \leqslant n$ and linear $(m-1)$-plane L such that $K_1, \dots, K_m \subset L$. Thus there exist compact convex sets $C \subset L$ and $M \subset L^{\perp}$ such that $K_i \subset C$ if $i \leqslant m$ and $K_j \subset C + M$ if $j > m$. We deduce that

$$V(K_1, \dots, K_n) \leqslant V(C, m; C + M, n - m) = \sum_{j=0}^{n-m} \binom{n-m}{j} V(C, m + j; M, n - m - j) = 0$$

as $|\alpha C + \beta M| = \alpha^{m-1}\beta^{n-m+1}\mathcal{H}^{m-1}(C)\mathcal{H}^{n-m+1}(M)$ for $\alpha, \beta > 0$ shows that the only positive mixed volume involving C and M is $V(C, m-1; M, n-m+1)$. ∎

7.4 The Minkowski inequality and the Aleksandrov-Fenchel inequality

The Minkowski inequality (7.26) is an equivalent form of the Brunn-Minkowski inequality below (cf. Remark 7.4.4), and the Aleksandrov-Fenchel inequality (7.30) is a far reaching generalization. We recall that if $K, C \subset \mathbb{R}^n$ are convex bodies and $\alpha, \beta > 0$, then the Brunn-Minkowski inequality Theorem 1.12.3 says that

$$|\alpha K + \beta C|^{\frac{1}{n}} \geqslant \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}} \tag{7.25}$$

with equality if and only if $K = \gamma C + z$ for $\gamma > 0$ and $z \in \mathbb{R}^n$; namely, K and C are homothetic. The following statement was also included in Lemma 1.12.2 about equivalent forms of the Brunn-Minkowski inequality, but we provide the simple argument for the reader's convenience:

Lemma 7.4.1. *If $K, C \in \mathcal{K}^n$ are compact and convex, then*

$$f(\lambda) = |(1 - \lambda)K + \lambda C|^{\frac{1}{n}}$$

is a concave function of $\lambda \in [0, 1]$.

If $K + C$ is a convex body, then f is linear if and only if K and C are homothetic convex bodies.

Proof. For $t \in [0,1]$, let $M_t = (1-t)K + tC$, and hence $M_{\frac{1}{2}t+\frac{1}{2}s} = \frac{1}{2}M_t + \frac{1}{2}M_s$ because K and C are convex; therefore, the Brunn-Minkowski inequality yields that f is concave.

If $K + C$ convex body and f is linear, then $0 < f(\frac{1}{2}) = \frac{1}{2}f(0) + \frac{1}{2}f(1)$ as f is concave, thus K and C are homothetic convex bodies by the equality case of the Brunn-Minkowski inequality. ∎

Theorem 7.4.2 (Minkowski inequality). *If $K, C \subset \mathbb{R}^n$ are convex bodies, then*

$$V(K,C;1)^n \geqslant |K|^{n-1}|C| \tag{7.26}$$

where equality holds if and only if K and C are homothetic.

Proof. We may assume that $|K| = |C|$. Since $f(\lambda) = |(1-\lambda)K + \lambda C|^{\frac{1}{n}}$ is concave where $f(\lambda) = \big(|K|(1-\lambda)^n + nV(K,C,1)(1-\lambda)^{n-1}\lambda + g(\lambda)\big)^{\frac{1}{n}}$, and the degree of λ in each term in the polynomial $g(\lambda)$ is at least two, the relation $f(0) = f(1) = |K|$ yields

$$0 \leqslant f'(0) = \frac{|K|^{\frac{-(n-1)}{n}}}{n} \cdot (-n|K| + nV(K,C;1));$$

therefore, $V(K,C;1) \geqslant |K|$.

Equality in the Minkowski inequality yields that $f'(0) = 0$, and hence f, being concave, is linear. Therefore, K and C are translates by Lemma 7.4.1. ∎

Using the representation (7.15) of certain mixed volumes *via* the integration over some surface area measure, we deduce the following useful forms of the Minkowski inequality (see also Theorem 2.5.10).

Corollary 7.4.3 (Minkowski inequality with surface area measure).

(i) If $K, C \subset \mathbb{R}^n$ are convex bodies, then

$$\int_{S^{n-1}} h_C \, dS_K \geqslant n|K|^{\frac{n-1}{n}}|C|^{\frac{1}{n}} \tag{7.27}$$

where equality holds if and only if K and C are homothetic.

(ii) If, in addition, $|K| = |C|$, then

$$\int_{S^{n-1}} h_C \, dS_K \geqslant \int_{S^{n-1}} h_K \, dS_K \tag{7.28}$$

where equality holds if and only if K and C are translates.

Remark 7.4.4 (Equivalence of Minkowski's and the Brunn-Minkowski inequalities). The argument in Theorem 7.4.2 shows that the Brunn-Minkowski inequality (7.25) yields the Minkowski inequality (7.26). On the other hand, assuming the Minkowski

inequality (7.26) for any convex bodies in $\mathbb{R}^n$, we deduce by the linearity of the mixed volumes that for any convex bodies $K, C \subset \mathbb{R}^n$, we have

$$|K+C| = V(K+C,K;1) + V(K+C,C;1) \geqslant |K+C|^{\frac{n-1}{n}}|K|^{\frac{1}{n}} + |K+C|^{\frac{n-1}{n}}|C|^{\frac{1}{n}},$$

verifying the Brunn-Minkowksi inequality (7.25).

The Minkowski inequality directly yields, for example, the Isoperimetric and Urysohn inequalities (see Theorem 7.2.1). For $K \subset \mathbb{R}^n$ convex body with $|K| = |rB^n|$, $r > 0$, (7.27) implies the Isoperimetric inequality

$$P(K) = \int_{S^{n-1}} h_{B^n}\, dS_K \geqslant n|B^n|^{\frac{1}{n}}|K|^{\frac{n-1}{n}} = P(rB^n)$$

with equality if and only if K is a ball. Similarly, for the Urysohn Inequality, (7.26) yields

$$V_1(K) = \frac{n}{\omega_{n-1}} V(B^n,K;1) \geqslant \frac{n}{\omega_{n-1}} |B^n|^{\frac{n-1}{n}}|K|^{\frac{1}{n}} = V_1(rB^n)$$

with equality if and only if K is a ball.

Remark 7.4.5 (Anisotropic perimeter). Given a convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int} C$, if $K \subset \mathbb{R}^n$ is compact convex, $n \geqslant 2$, then the Anisotropic Perimeter of K is

$$P_C(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho C| - |K|}{\varrho} = nV(K,C;1),$$

and hence $K \mapsto P_C(K)$ is continuous and monotone increasing in K. If K is a convex body, then (cf. also (2.18))

$$P_C(K) = \int_{\partial K} h_C(\nu_K(x))\, d\mathcal{H}^{n-1}(x) = \int_{\partial K} \|\nu_K(x)\|_*\, d\mathcal{H}^{n-1}(x).$$

In turn, the Minkowski inequality (7.26) is equivalent to the Anisotropic Isoperimetric inequality Theorem 2.4.4 stating that

$$P_C(K) \geqslant n|C|^{\frac{1}{n}}|K|^{\frac{n-1}{n}},$$

with equality if and only if C and K are homothetic.

We have proved the Minkowski inequality from the Brunn-Minkowski inequality (7.25) using the first derivative of an associated concave function. Using that the second derivative of a concave function is non-positive leads to the second inequality of Minkowski [590, 591].

Theorem 7.4.6 (Minkowski's second inequality). *If $K, C \subset \mathbb{R}^n$ are convex bodies, then*

$$V(K,\dots,K,C)^2 \geqslant |K| \cdot V(K,\dots,K,C,C). \tag{7.29}$$

Proof. For $g(\lambda) = |K + \lambda C| = \sum_{i=0}^{n} \binom{n}{i} V(K, C; i)\lambda^i$, the function $\lambda \mapsto g(\lambda)^{\frac{1}{n}}$ is concave for $\lambda \geqslant 0$ by the Brunn-Minkowski inequality (7.25), and hence

$$0 \geqslant \frac{\partial^2}{\partial \lambda^2} g(\lambda)^{\frac{1}{n}}\Big|_{\lambda=0} = \tfrac{1}{n} g"(0) g(0)^{\frac{1}{n}-1} - \tfrac{n-1}{n^2} (g'(0))^2 g(0)^{\frac{1}{n}-2}$$
$$= (n-1)|K|^{\frac{1}{n}-2} \left(V(K, C; 2)|K| - V(K, C; 1)^2 \right).$$

■

A far reaching generalization of Minkowski's first inequality (7.26) and second inequality (7.29) is the Aleksandrov-Fenchel inequality (7.30) below (proved actually only by Aleksandrov [5, 7, 11], see the Comments in Section 7.8). In this monograph, we provide two arguments, one using strongly isomorphic polytopes in Section 7.A, and one using the theory of elliptic operators in Section 8.5.2, both are based on Aleksandrov's original ideas as developed further by Shenfeld, van Handel [678].

Theorem 7.4.7 (Aleksandrov-Fenchel Inequality). *If $n \geqslant 3$ and $K_1, K_2, C_3, \ldots, C_n$ are compact convex sets in $\mathbb{R}^n$, then*

$$V(K_1, K_2, C_3, \ldots, C_n)^2 \geqslant V(K_1, K_1, C_3, \ldots, C_n) V(K_2, K_2, C_3, \ldots, C_n). \quad (7.30)$$

Unlike in the case of the Minkowski inequality, equality may occur in the Aleksandrov-Fenchel inequality (7.30) even if the bodies are not pairwise homotopic, and the cases of equality are not fully understood.

Remark 7.4.8 (Known equality cases of the Aleksandrov-Fenchel inequality (7.30)).

- K_1, K_2, C_i are convex bodies with C^1 boundary (see Schneider [658], equality if and only if the bodies pairwise homothetic, and even a stability estimate is proved in this case by Schneider [659]).
- K_1, K_2, C_i are full dimensional zonoids (see Schneider [657] for the result, and Example 1.6.3 for the notion of a zonoid). This case was generalized to "polyoids" by Hug, Reichert [429].
- K_1, K_2, C_i are polytopes (see Shenfeld, van Handel [680]). This case is especially important because the Aleksandrov-Fenchel inequality for polytopes appears in Algebraic Geometry and in Combinatorics (see the Comments Section 7.8).
- Hug, Reichert [429] provided a common generalization of the results above in the case of zonoids and polytopes.
- $C_3 = \ldots = C_n$ (Minkowski's second inequality (7.4.6)) where the characterization of the equality case had been open for 100 years (see Shenfeld, van Handel [679]).

Let us state some consequences of the Aleksandrov-Fenchel inequality (7.30). "Algebraic manipulation" starting with (7.30) leads to the following version.

Corollary 7.4.9 (Aleksandrov-Fenchel Inequality - general form).
If $2 \leqslant m \leqslant n-1$ *and* $K_1, \dots, K_m, C_{m+1}, \dots, C_n \subset \mathbb{R}^n$ *are compact convex sets, then*

$$V(K_1, \dots, K_m, C_{m+1}, \dots, C_n)^m \geqslant \prod_{i=1}^{m} V(K_i, \dots, K_i, C_{m+1}, \dots, C_n).$$

If $m = n$, then Corollary 7.4.9 reads as

$$V(K_1, \dots, K_n)^n \geqslant \prod_{i=1}^{n} V(K_i). \tag{7.31}$$

Alesker, Dar, V. Milman [16] give a simpler proof of (7.31) using optimal transport, and even manage to verify that equality holds in (7.31) for convex bodies $K_1, \dots, K_n$ if and only if $K_1, \dots, K_n$ are pairwise homothetic. The Isoperimetric inequality

$$V(B^n, K; i)^n \geqslant |B^n|^{n-i} |K|^i$$

for the mean i-dimensional projections of a convex body K (cf. Theorem 7.2.1) directly follows from the consequence (7.31) of the Aleksandrov-Fenchel inequality including the characterization of equality.

A useful consequence of Aleksandrov-Fenchel inequality (7.30) in combinatorics is that certain sequence of mixed volumes is log-concave, and hence unimodal (see, for example, Stanley's inequality (7.60) about posets).

Remark 7.4.10 ("Log-concavity" of some sequences of mixed volumes). For $i, j \geqslant 1$ and $i + j \leqslant n$, and compact, convex sets $K_1, K_2, C_{i+j+1}, \dots, C_n \subset \mathbb{R}^n$, if $C = C_{i+j+1}, \dots, C_n$ (there is no C if $i + j = n$), then

$$V(K_1, i; K_2, j; C)^2 \geqslant V(K_1, i-1; K_2, j+1; C) V(K_1, i+1; K_2, j-1; C). \tag{7.32}$$

7.5 Aleksandrov's lemma for Wulff shapes

The main goal of this section is to present Aleksandrov's result that the surface area measure is the first variation of volume in the sense of Theorem 7.5.2 below. We note that for a closed $\Omega \subset S^{n-1}$ and $\varphi : \Omega \to \mathbb{R}$, if $K = \{x \in \mathbb{R}^n : \langle x, u \rangle \leqslant \varphi(u) \text{ for } x \in \Omega\}$ is a convex body, then Lemma 2.5.6 yields that Ω is not contained in any closed hemisphere and

$$\{\nu_K(x) : x \in \partial' K\} \subset \operatorname{supp} S_K \subset \Omega. \tag{7.33}$$

Here we say that φ is lower semi-continuous if $\lim_{m\to\infty} x_m = x$ for $x_m, x \in \Omega$ yields that $\liminf_{m\to\infty} \varphi(x_m) \geqslant \varphi(x)$, and hence $\inf_{u\in\Omega} \varphi > 0$ in this case. For Wulff shapes - that have been introduced in Section 4.4 - the following statement follows from Lemma 4.4.3 where the integrals in (7.34) make sense by (7.33).

Lemma 7.5.1 (Aleksandrov). *Let $\Omega \subset S^{n-1}$ be closed and not contained in any closed hemisphere, and let $\varphi : \Omega \to (0, \infty)$ be bounded and lower semi-continuous. Then the Wulff shape*

$$W = \{x \in \mathbb{R}^n : \langle x, v \rangle \leqslant \varphi(v) \text{ for } v \in \Omega\}$$

is a convex body with $o \in \operatorname{int} W$, $h_W(\nu_W(x)) = \varphi(\nu_W(x))$ for $x \in \partial' W$ and

$$\int_{S^{n-1}} \varphi \, dS_W = \int_{S^{n-1}} h_W \, dS_W = n|W|. \tag{7.34}$$

Remark. The definition of the surface area measure S_W yields (cf. (2.21)) that (7.34) is equivalent with

$$\left(\int_{S^{n-1}} \varphi \, dS_W = \right) \int_{\partial' W} \varphi(\nu_W) \, d\mathcal{H}^{n-1} = \int_{\partial' W} h_W(\nu_W) \, d\mathcal{H}^{n-1}. \tag{7.35}$$

The typical cases in Aleksandrov's Theorem 7.5.2 - essentially due to Aleksandrov [11], and in this form to Schneider [661], Section 7.5 - are when Ω is discrete (and hence the Wulff shapes are polytopes) or $\Omega = S^{n-1}$.

Theorem 7.5.2 (Aleksandrov Lemma for Wulff shapes). *Let $\Omega \subset S^{n-1}$ be a closed set not contained in a closed hemisphere, let $\varphi : \Omega \times (-t_0, t_0) \to (0, \infty)$, $t_0 > 0$, be continuous such that*

$$\lim_{t \to 0} \frac{\varphi(u,t) - \varphi(u,0)}{t} = \partial_t \varphi(u,0)$$

exists uniformly in $u \in \Omega$ where $\partial_t \varphi(\cdot, 0)$ is continuous on Ω. For $t \in (-t_0, t_0)$ and $z_0 \in \mathbb{R}^n$, the Wulff shape

$$K_t = \{x \in \mathbb{R}^n : \langle x - z_0, u \rangle \leqslant \varphi(u,t) \text{ for } u \in S^{n-1}\}$$

satisfies that

$$\lim_{t \to 0} \frac{|K_t| - |K_0|}{t} = \int_{S^{n-1}} \partial_t \varphi(u,0) \, dS_{K_0}(u).$$

Remarks.

- If $\varphi : \Omega \times [0, t_0) \to (0, \infty)$, then

$$\lim_{t \to 0^+} \frac{|K_t| - |K_0|}{t} = \int_{S^{n-1}} \partial_t \varphi(u,0) \, dS_{K_0}(u). \tag{7.36}$$

- If $\varphi(u,t) = h_{K_0}(u) + tg(u)$ for some continuous $g : S^{n-1} \to \mathbb{R}$ and $t \in (-t_0, t_0)$, then

$$\lim_{t \to 0} \frac{|K_t| - |K_0|}{t} = \int_{S^{n-1}} g \, dS_{K_0}. \tag{7.37}$$

Proof of Theorem 7.5.2. We may assume that $z_0 = o$. For any $\varepsilon \in (0,1)$, if $|t|$ is small, then $(1-\varepsilon)\varphi(u,0) < \varphi(u,t) < (1+\varepsilon)\varphi(u,0)$ holds for any $u \in \Omega$ because $\frac{\varphi(u,t)-\varphi(u,0)}{t}$ tends uniformly in $u \in \Omega$ to $\partial_t \varphi(u,0)$ as $t \to 0$. We deduce that $\lim_{t\to 0} K_t = K_0$, and hence S_{K_t} tends to S_{K_0} weakly according to Proposition 2.6.13. Using again the uniform convergence of $\frac{\varphi(u,t)-\varphi(u,0)}{t}$, it follows that

$$\lim_{t\to 0} \int_{S^{n-1}} \frac{\varphi(u,t)-\varphi(u,0)}{t}\, dS_{K_t}(u) = \int_{S^{n-1}} \partial_t \varphi(u,0)\, dS_{K_0}(u) =: J. \tag{7.38}$$

On the one hand, using first (7.38), then (7.34) for K_t and the estimate $\varphi(u,0) \geqslant h_{K_0}(u)$ for $u \in S^{n-1}$, and later the Minkowski inequality and $K_t \to K_0$, we deduce that

$$\begin{aligned}
J &= \lim_{t\to 0} \frac{1}{t} \left(\int_{S^{n-1}} \varphi(u,t)\, dS_{K_t}(u) - \int_{S^{n-1}} \varphi(u,0)\, dS_{K_t}(u) \right) \\
&\leqslant \liminf_{t\to 0} \frac{1}{t} \left(\int_{S^{n-1}} h_{K_t}(u)\, dS_{K_t}(u) - \int_{S^{n-1}} h_{K_0}(u)\, dS_{K_t}(u) \right) \\
&= \liminf_{t\to 0} \frac{n}{t} \left(|K_t| - V(K_t; K_0, 1) \right) = \liminf_{t\to 0} \frac{n}{t} \cdot \frac{|K_t|^n - V(K_t; K_0, 1)^n}{\sum_{i=0}^{n-1} |K_t|^i V(K_t; K_0, 1)^{n-1-i}} \\
&\leqslant \liminf_{t\to 0} \frac{n}{t} \cdot \frac{|K_t|^n - |K_t|^{n-1}|K_0|}{\sum_{i=0}^{n-1} |K_t|^i V(K_t; K_0, 1)^{n-1-i}} = \liminf_{t\to 0} \frac{|K_t| - |K_0|}{t}.
\end{aligned}$$

On the other hand, using first (7.38), then (7.34) for K_0 and $\varphi(u,t) \geqslant h_{K_t}(u)$ and later the Minkowski inequality and $K_t \to K_0$, we deduce that

$$\begin{aligned}
J &= \lim_{t\to 0} \frac{1}{t} \left(\int_{S^{n-1}} \varphi(u,t)\, dS_{K_0}(u) - \int_{S^{n-1}} \varphi(u,0)\, dS_{K_0}(u) \right) \\
&\geqslant \limsup_{t\to 0^+} \frac{1}{t} \left(\int_{S^{n-1}} h_{K_t}(u)\, dS_{K_0}(u) - \int_{S^{n-1}} h_{K_0}(u)\, dS_{K_0}(u) \right) \\
&= \limsup_{t\to 0} \frac{n}{t} \left(V(K_0; K_t, 1) - |K_0| \right) = \limsup_{t\to 0} \frac{n}{t} \cdot \frac{V(K_0; K_t, 1)^n - |K_0|^n}{\sum_{i=0}^{n-1} V(K_0; K_t, 1)^{n-1-i} |K_t|^i} \\
&\geqslant \limsup_{t\to 0} \frac{n}{t} \cdot \frac{|K_0|^{n-1}|K_t| - |K_0|^n}{\sum_{i=0}^{n-1} V(K_0; K_t, 1)^{n-1-i} |K_t|^i} = \limsup_{t\to 0} \frac{|K_t| - |K_0|}{t}.
\end{aligned}$$

In turn, we conclude the Aleksandrov Lemma Theorem 7.5.2. ∎

7.6 The L_p-Brunn-Minkowski inequality for $p \geqslant 1$

In 1962, Firey [284] generalized the Minkowski addition of convex body - when support functions are added - into L_p addition for $p \geqslant 1$, and verified the corresponding

Brunn-Minkowski-type inequality. We recall that if $K, C \subset \mathbb{R}^n$ are convex bodies, $\lambda \in (0,1)$, and $\alpha, \beta > 0$, then the Brunn-Minkowski inequality (cf. Lemma 1.12.2) says that

$$\begin{aligned} |\alpha K + \beta C|^{\frac{1}{n}} &\geqslant \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}}, \text{ equality if and only if } K, C \text{ are homothetic;} \\ |(1-\lambda)\, K + \lambda\, C| &\geqslant |K|^{1-\lambda} |C|^{\lambda}, \text{ equality if and only if } K, C \text{ are translates.} \end{aligned} \tag{7.39}$$

In order to study the notion of L_p addition for $p > 1$, we recall some basic inequalities. As t^p is a strictly convex function of $t \geqslant 0$ (since the second derivative is positive), if $t, s \geqslant 0$ and $\lambda \in (0,1)$, then

$$((1-\lambda)t + \lambda\, s)^p \leqslant (1-\lambda)t^p + \lambda\, s^p \text{ with equality if and only if } t = s. \tag{7.40}$$

In addition, we need the Minkowski inequality (cf. (10.10)) for integrals in the form that if $p > 1$, $a_i, b_i \geqslant 0$ and $\mu_i > 0$ for $i = 1, \ldots, k$, then

$$\left(\sum_{i=1}^{k} \mu_i (a_i + b_i)^p \right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{k} \mu_i a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{k} \mu_i b_i^p \right)^{\frac{1}{p}} \tag{7.41}$$

with equality if and only if there exists $\lambda > 0$ such that $a_i = \lambda b_i$, $i = 1, \ldots, k$.

Lemma 7.6.1 (L_p linear combination of convex bodies, $p > 1$). *Let $p \geqslant 1$. For convex bodies $K, C \subset \mathbb{R}^n$ with $o \in K, C$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta > 0$, there exists a convex body $\alpha \cdot K +_p \beta \cdot C$ containing the origin whose support function is*

$$h_{\alpha \cdot K +_p \beta \cdot C} = \left(\alpha\, h_K^p + \beta\, h_C^p \right)^{\frac{1}{p}}. \tag{7.42}$$

In particular,

$$\alpha \cdot K +_p \beta \cdot C = \left\{ x \in \mathbb{R}^n : \langle x, u \rangle^p \leqslant \alpha\, h_K(u)^p + \beta\, h_C(u)^p,\ u \in S^{n-1} \right\}. \tag{7.43}$$

Remarks.

- If $o \in \operatorname{int} K$ and $o \in \operatorname{int} C$, then $o \in \operatorname{int}(\alpha \cdot K +_p \beta \cdot C)$.
- The case $p = 1$ is the classical Minkowski combination; namely, $\alpha \cdot K +_1 \beta \cdot C = \alpha\, K + \beta\, C$.

Proof. We may assume that $\alpha, \beta > 0$. Now $h_{\alpha K +_p \beta C}$ defined as in (7.42) is readily 1-homogeneous, continuous, and it is also convex as for $z = \frac{1}{2} x + \frac{1}{2} y$, $x, y \in \mathbb{R}^n$, applying the convexity of h_K, h_C and the Minkowski inequality (7.41) with $\mu_1 = \alpha$, $\mu_2 = \beta$, etc, yields

$$\begin{aligned} (\alpha\, h_K(z)^p + \beta\, h_C(z)^p)^{\frac{1}{p}} &\leqslant \left(\alpha \cdot \left(\tfrac{1}{2} h_K(x) + \tfrac{1}{2} h_K(y) \right)^p + \beta \cdot \left(\tfrac{1}{2} h_C(x) + \tfrac{1}{2} h_C(y) \right)^p \right)^{\frac{1}{p}} \\ &\leqslant \frac{1}{2} (\alpha\, h_K(x)^p + \beta\, h_C(x)^p)^{\frac{1}{p}} + \frac{1}{2} (\alpha\, h_K(y)^p + \beta\, h_C(y)^p)^{\frac{1}{p}}. \end{aligned}$$

Therefore, the non-negative function $\left(\alpha\, h_K^p + \beta\, h_C^p\right)^{\frac{1}{p}}$ is the support function of a compact convex set $M \subset \mathbb{R}^n$ with $o \in M$ (cf. Proposition 1.6.9). Since K and C being convex bodies yields $h_K(u) + h_K(-u) > 0$ and $h_C(u) + h_C(-u) > 0$ for any $u \in S^{n-1}$, we have $h_M(u) + h_M(-u) > 0$, and hence $M = \alpha \cdot K +_p \beta \cdot C$ is a convex body. ∎

We observe that the L_p linear combination is equivariant with respect to linear transformations.

Lemma 7.6.2. *If $p \geqslant 1$, $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in K, C$, $\alpha, \beta \geqslant 0$ with $\alpha + \beta > 0$, and $\Phi \in \mathrm{GL}(n)$, then*

$$\alpha \cdot (\Phi K) +_p \beta \cdot (\Phi C) = \Phi(\alpha \cdot K +_p \beta \cdot C). \tag{7.44}$$

Proof. Since $\alpha \cdot K +_p \beta \cdot C = \{x \in \mathbb{R}^n : \langle x, u\rangle^p \leqslant \alpha\, h_K(u)^p + \beta\, h_C(u)^p,\ u \in \mathbb{R}^n\}$ according to (7.43), we deduce (7.44) from the fact that $h_{\Phi M}(u) = h_M(\Phi^t u)$ (cf. Lemma 1.6.2) holds for any compact convex set $M \subset \mathbb{R}^n$ and $u \in \mathbb{R}^n$. ∎

Firey [284] proved the following simple, but very useful variant of the Brunn-Minkowski inequality:

Theorem 7.6.3 (Firey's L_p-Brunn-Minkowski inequality, $p > 1$). *Let $K, C \subset \mathbb{R}^n$ be convex bodies with $o \in K, C$, and let $p > 1$.*

(i) $|\alpha \cdot K +_p \beta \cdot C|^{\frac{p}{n}} \geqslant \alpha\, |K|^{\frac{p}{n}} + \beta\, |C|^{\frac{p}{n}}$ for $\alpha, \beta > 0$ with equality if and only if K and C are dilates.

(ii) $|(1-\lambda) \cdot K +_p \lambda \cdot C| \geqslant |K|^{1-\lambda} |C|^{\lambda}$ for $\lambda \in (0,1)$ with equality if and only if $K = C$.

Proof. For (ii), (7.40) yields that $h_{(1-\lambda)\cdot K +_p \lambda \cdot C}(u) \geqslant (1-\lambda) h_K(u) + \lambda\, h_C(u)$ for any $u \in S^{n-1}$ with equality if and only if $h_K(u) = h_C(u)$. Therefore, the Brunn-Minkowski inequality (7.39) implies

$$|(1-\lambda) \cdot K +_p \lambda \cdot C| \geqslant |(1-\lambda)\, K + \lambda\, C| \geqslant |K|^{1-\lambda} |C|^{\lambda} \tag{7.45}$$

where equality yields that $K = C$ by the equality case in (7.40).

Let us turn to (i). We set $\alpha_0 = |K|^{\frac{-1}{n}}$, $\beta_0 = |C|^{\frac{-1}{n}}$, $K = \alpha_0^{-1} K$ and $C = \beta_0^{-1} C$, and hence $|K_0| = |C_0| = 1$. For $\lambda = \frac{\beta \beta_0^p}{\alpha \alpha_0^p + \beta \beta_0^p}$, we deduce that

$$h_{\alpha \cdot K +_p \beta \cdot C} = (\alpha \alpha_0^p + \beta \beta_0^p)^{\frac{1}{p}} \left((1-\lambda) h_{K_0}^p + \lambda\, h_{C_0}^p\right)^{\frac{1}{p}}.$$

Since $|(1-\lambda) \cdot K_0 +_p \lambda \cdot C_0| \geqslant 1$ by (ii), we conclude the inequality in (i). If equality holds in (i), then $K_0 = C_0$ by $|(1-\lambda) \cdot K_0 +_p \lambda \cdot C_0| = 1$, and hence K and C are dilates. ∎

Similarly to the case of Brunn-Minkowski inequality (cf. Lemma 1.12.2), we deduce the following equivalent formulation of the L_p-Brunn-Minkowski inequality:

Corollary 7.6.4. *If $p > 1$ and $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in K, C$, then $\lambda \mapsto |(1-\lambda)\cdot K +_p \lambda \cdot C|^{\frac{p}{n}}$ is a concave function of $\lambda \in [0,1]$, which is a linear function if and only if K and C are dilates.*

Proof. We set $M_t = (1-t)\cdot K +_p t \cdot C$ for $t \in [0,1]$. We deduce from (7.40) that $\frac{1}{2}\cdot M_t + p\,\frac{1}{2}\cdot M_s \subset M_{\frac{1}{2}t+\frac{1}{2}s}$; therefore, Theorem 7.6.3 yields Corollary 7.6.4. ∎

Next, we discuss the L_p analogue of the Minkowski inequality due to Lutwak [550]. The classical Minkowski inequality says that if $K, C \subset \mathbb{R}^n$ are convex bodies, then

$$\int_{S^{n-1}} h_C \, dS_K \geqslant n|K|^{\frac{n-1}{n}}|C|^{\frac{1}{n}} \tag{7.46}$$

where equality holds if and only if K and C are homothetic. For the L_p analogue, we need the notion of cone volume measure (cf. Definition 2.6.1); namely, V_K is the Borel measure on S^{n-1} satisfying $dV_K = \frac{1}{n} h_K \, dS_K$ for a convex body $K \subset \mathbb{R}^n$ with $o \in K$, and hence $V_K(S^{n-1}) = |K|$. As S_K is a finite measure, we observe that $V_K(\{h_K = 0\}) = 0$ even in the case when $o \in \partial K$ where

$$\{h_K = 0\} = \{u \in S^{n-1} : h_K(u) = 0\} = N_K(o) \cap S^{n-1}$$

for the exterior normal cone $N_K(o)$ at $o \in \partial K$. On the other hand, possibly $S_K(\{h_K = 0\}) = S_K(N_K(o) \cap S^{n-1}) > 0$ if $o \in \partial K$; for example, when K is an n-polytope.

We also need the following form of Jensen's inequality (cf. (10.6)): If $p > 1$, μ is a Borel probability measure on the topological space X, and $f \in L^1(X, \mu)$ is non-negative, then

$$\int_X f \, d\mu \leqslant \left(\int_X f^p \, d\mu \right)^{\frac{1}{p}} \tag{7.47}$$

with equality if and only if f is μ a.e. constant.

Theorem 7.6.5 (Lutwak's L_p-Minkowski inequality, $p > 1$). *If $p > 1$, $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in K, C$, and either $o \in \operatorname{int} K$, or $o \in \partial K$ and $S_K(\{h_K = 0\}) = 0$, then*

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K \geqslant |K|^{\frac{n-p}{n}}|C|^{\frac{p}{n}} \tag{7.48}$$

with equality if and only if K and C are dilates.

Remarks.

- In order (7.48) to hold if $o \in \partial K$, then a condition like $S_K(\{h_K = 0\}) = 0$ is necessary (cf. Example 7.6.6). In addition, the left hand side of (7.48) might be infinity if $o \in \partial K$ even assuming that $S_K(\{h_K = 0\}) = 0$.

- It follows via Aleksandrov's Lemma 7.5.2 for the Wulff shapes that (7.48) holds for all $K, C \in \mathcal{K}_{(o)}$ if and only if the function $\lambda \mapsto |(1-\lambda)\cdot K +_p \lambda \cdot C|^{\frac{p}{n}}$ of $\lambda \in [0,1]$ is concave for all $K, C \in \mathcal{K}_{(o)}$, which is in turn equaivalent to the L_p-Brunn-Minkowski inequality Theorem 7.6.3 where $\mathcal{K}_{(o)}$ is the family of convex bodies $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$.

Proof. We may assume that $|K| = |C| = 1$. We deduce from the Jensen inequality (7.47), the fact that $S_K(N_K(o) \cap S^{n-1}) = 0$ if $o \in \partial K$, and from the Minkowski inequality (7.46) that

$$\left(\int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K\right)^{\frac{1}{p}} \geqslant \int_{S^{n-1}} \frac{h_C}{h_K}\, dV_K = \frac{1}{n}\int_{S^{n-1}} h_C\, dS_K \geqslant 1. \tag{7.49}$$

If equality holds in (7.49), then K and C are translates (by the equality case in the Minkowski inequality (7.46)), and there exists $t > 0$ such that $h_C = t\, h_K$ (by the equality case of the Jensen inequality (7.47)); therefore, $t = 1$ and $h_C = h_K$. ∎

Example 7.6.6. The L_p-Minkowski inequality (7.48) may not hold if $o \in \partial K$ and $S_K(N_K(o) \cap S^{n-1}) > 0$. Let $K \subset \mathbb{R}^n$ be a regular simplex of volume 1 such that the exterior unit normals are $u_0, \dots, u_n \in S^{n-1}$, the centroid of the facet with exterior unit normal u_0 is o, and hence the opposite vertex is $-t_0 u_0$ for some $t_0 > 0$. Let $C = K + s u_0$ for $s \in (0, t_0)$, and hence there exists $r \in (0,1)$ such that $h_C(u_i) = r h_K(u_i)$ for $i = 1, \dots, n$. Since $\operatorname{supp} V_K = \{u_1, \dots, u_n\}$ and $h_C(u_0) > 0$, it follows that

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K = \left(\int_{S^{n-1}} \frac{h_C}{h_K}\, dV_K\right)^p < \left(\frac{1}{n}\int_{S^{n-1}} h_C\, dS_K\right)^p = 1 = |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}}.$$

Lutwak [550] extended the notion of surface area measure to the L_p case (see also Section 9.3 and Section 9.4):

Definition 7.6.7 (L_p-surface area measure for $p > 1$). For $p > 1$, if $K \subset \mathbb{R}^n$ is a convex body with either $o \in \operatorname{int} K$, or $o \in \partial K$ and $S_K(\{h_K = 0\}) = 0$, then its L_p-surface area measure on S^{n-1} is

$$dS_{K,p} = h_K^{1-p}\, dS_K = n h_K^{-p}\, dV_K.$$

Remark. $S_{K,p}$ is a Borel measure. Possibly $S_{K,p}(S^{n-1}) = \infty$ if $o \in \partial K$, but even in that case, $S_{K,p}(X) < \infty$ for any compact $X \subset \{h_K > 0\} = S^{n-1} \backslash N_K(o)$.

Lutwak [550] considered the left hand side of (7.48) as the L_p-mixed volume; namely, if $p \geqslant 1$, $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in C$, and either $o \in \operatorname{int} K$, or $o \in \partial K$ and $S_K(\{h_K = 0\}) = 0$, then

$$V_p(K, C) = \int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K = \frac{1}{n}\int_{S^{n-1}} h_C^p h_K^{1-p}\, dS_K = \frac{1}{n}\int_{S^{n-1}} h_C^p\, dS_{K,p}. \tag{7.50}$$

In particular, $S_{K,1} = S_K$ and $V_1(K,C) = V(K,C;1)$.

Let us summarize the main properties of these notions; for example, $V_p(K,C)$ is invariant under $\mathrm{SL}(n)$.

Proposition 7.6.8. *Let $p > 1$, $K, C \subset \mathbb{R}^n$ be convex bodies with $o \in C$, and either $o \in \operatorname{int} K$, or $o \in \partial K$ and $S_K(\{h_K = 0\}) = 0$.*

(i) $V_p(K,K) = |K|$.

(ii) $V_p(K,C) = \frac{1}{n}\int_{S^{n-1}} h_C^p \, dS_{K,p} \geqslant |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}}$ *with equality if and only if K and C are dilates.*

(iii) If $\Phi \in \mathrm{GL}(n)$, then

$$V_p(\Phi K, \Phi C) = |\det \Phi| \cdot V_p(K,C). \tag{7.51}$$

(iv) If, in addition, $p \neq n$ and either $o \in \operatorname{int} C$, or $o \in \partial C$ and $S_C(\{h_C = 0\}) = 0$, then $S_{K,p} = S_{C,p}$ implies $K = C$.

Proof. (i) follows directly from the definition, and (ii) is just (7.48).

For (iii), we may assume that $|\det \Phi| = 1$ as $V_p(\lambda K, \lambda C) = \lambda^n V_p(K,C)$ for $\lambda > 0$. Now, (7.51) is a consequence of Proposition 2.6.15 and Lemma 2.6.14 as

$$\begin{aligned}\int_{S^{n-1}} \frac{h_{\Phi C}^p}{h_{\Phi K}^p} \, dV_{\Phi K} &= \int_{S^{n-1}} \frac{h_{\Phi C}\,(\Phi^{-t}u/\|\Phi^{-t}u\|)^p}{h_{\Phi K}\,(\Phi^{-t}u/\|\Phi^{-t}u\|)^p} \, dV_K(u) \\ &= \int_{S^{n-1}} \frac{h_{\Phi C}\,(\Phi^{-t}u)^p}{h_{\Phi K}\,(\Phi^{-t}u)^p} \, dV_K(u) = \int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K.\end{aligned}$$

For (iv), we deduce from (7.50), $S_{K,p} = S_{C,p}$ and the L_p-Minkowski inequality in (ii) that

$$|K| = \int_{S^{n-1}} \frac{h_K^p}{h_K^p} \, dV_K = \frac{1}{n}\int_{S^{n-1}} h_K^p \, dS_{K,p} = \frac{1}{n}\int_{S^{n-1}} h_K^p \, dS_{C,p} \geqslant |C|^{\frac{n-p}{n}} |K|^{\frac{p}{n}}; \tag{7.52}$$

therefore, $|K|^{\frac{n-p}{n}} \geqslant |C|^{\frac{n-p}{n}}$. Reversing the role of K and C in (2.32) implies $|C|^{\frac{n-p}{n}} \geqslant |K|^{\frac{n-p}{n}}$, and hence $|C| = |K|$ by $p \neq n$. In turn, equality in (7.52) implies equality in (ii), which fact combined with $|C| = |K|$ yields that $K = C$. ∎

Finally, the form (7.36) of the Aleksandrov lemma and (7.50) yield the following variational characterization of the L_p-surface area due to Lutwak [550].

Lemma 7.6.9 (Lutwak). *If $p > 1$, $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in C$ and $o \in \operatorname{int} K$, then*

$$\lim_{t \to 0^+} \frac{|K +_p t \cdot C|^{\frac{p}{n}} - |K|^{\frac{p}{n}}}{t} = nV_p(K,C) = \int_{S^{n-1}} h_C^p \, dS_{K,p}.$$

How natural notion is the L_p-surface area for $p > 1$ is exhibited by its appearance in the characterization of $\mathrm{SL}(n)$ equivariant valuations on polytopes valued at rank p tensors due to Haberl, Parapatits [375] (see Section 8.D for a survey on valuations on convex bodies).

7.7 Isoperimetric type problems for polytopes

The reason why the isoperimetric problem for polytopes is discussed in this chapter is that the fundamental properties of the mixed volumes yield a theorem due to Lindelöf [520] from 1869; namely, when minimizing the isoperimetric quotient $P(Q)/|Q|^{\frac{n-1}{n}}$ of a polytope $Q \subset \mathbb{R}^n$ of given exterior normals to the facets, we may assume that the polytope is circumscribed around a ball. Here the polytope $Q \subset \mathbb{R}^n$ is circumscribed around B^n if $B^n \subset Q$ and any facet of Q touches B^n, and hence in this case, Q satisfies

$$\frac{P(Q)}{|Q|^{\frac{n-1}{n}}} = n|Q|^{\frac{1}{n}}. \tag{7.53}$$

Theorem 7.7.1 (Lindelöf). *If $Q, Q_0 \subset \mathbb{R}^n$ are n-polytopes such that Q_0 is circumscribed around B^n and the exterior normal to any facet of Q is an exterior normal to some facet of Q_0, then*

$$\frac{P(Q)}{|Q|^{\frac{n-1}{n}}} \geqslant \frac{P(Q_0)}{|Q_0|^{\frac{n-1}{n}}},$$

with equality if and only if Q and Q_0 are homothetic.

Proof. Let $Q_0 = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant 1\}$ be the polytope whose facets touch B^n at $u_1, \dots, u_k$. If Q is any polytope whose unit facet unit normals are among $u_1, \dots, u_k$, then $h_{Q_0}(u) = 1$ for any unit exterior normal u at a facet of Q. It follows form the form (7.27) of the Minkowski inequality that $P(Q) = \int_{S^{n-1}} h_{Q_0}\, dS_Q \geqslant n|Q|^{\frac{n-1}{n}}|Q_0|^{\frac{1}{n}}$; therefore, $P(Q)/|Q|^{\frac{n-1}{n}} \geqslant n|Q_0|^{\frac{1}{n}} = P(Q_0)/|Q_0|^{\frac{n-1}{n}}$ (cf. (7.53)). If equality holds, then equality holds in the Minkowski inequality, and hence Q and Q_0 are homothetic. ∎

In $\mathbb{R}^2$, Weierstrass (cf. [714]) verified that regular k-gons have optimal isoperimetric quotient among k-gons at the end of the 19th century (see Theorem 4.C.1 for the spherical and hyperbolic versions).

Theorem 7.7.2 (Weierstrass). *For $k \geqslant 3$, among convex k-gons of given area in $\mathbb{R}^2$, the regular k-gons are the ones having minimal perimeter.*

Proof. Lindelöf's Theorem 7.7.1 yields that we may restrict our attention to k-gons circumscribed around B^2. According to (7.53), it is sufficient to prove that the area

of a k-gon Q circumscribed around B^2 is minimized if and only if Q is a regular k-gon. Let $\alpha_1, \dots, \alpha_k$ be the angles of Q, and hence $\frac{\alpha_i}{2} \in (0, \frac{\pi}{2})$ for $i = 1, \dots, k$ and $\sum_{i=1}^{k} \frac{\alpha_i}{2} = \frac{(k-2)\pi}{2}$. Since Q is circumscribed around B^2, and $t \mapsto \cot t$ is strictly convex on $(0, \frac{\pi}{2})$, we deduce that

$$|Q| = \sum_{i=1}^{k} \cot \frac{\alpha_i}{2} \geqslant k \cdot \cot \frac{(k-2)\pi}{2k},$$

with equality if and only if Q is a regular k-gon. ∎

According to Lindelöf's Theorem 7.7.1, among polytopes of given number of facets, the ones minimizing isoperimetric ratio $P(Q)/|Q|^{\frac{n-1}{n+1}}$ are circumscribed around some Euclidean ball. In the plane, even a dual statement holds according to Proposition 7.7.3, as it was observed by Steiner around 1840. On the other hand, in higher dimensions, among polytopes of given number of vertices, the vertices of an extremal one minimizing the isoperimetric quotient may not lie on the boundary of a ball (see Böröczky, Böröczky [121] in the case of polytopes with 5 vertices in $\mathbb{R}^3$).

We recall that a k-gon is cyclic if there exists a circle containing all vertices.

Proposition 7.7.3 (Steiner). *For $k \geqslant 4$, among k-gons of given side lengths, there are ones maximizing the area, and they are cyclic.*

Proof. As the diameter of such k-gons are bounded, the Blaschke Selection Theorem 1.7.4 yields the existence of a suitable "generalized k-gon" Q_0 of maximal area, where "generalized" means that some of the "sides" of given length might be aligned. We verify that if $|Q_0|$ is a local maximum in this class, then Q_0 is cyclic. As Q_0 is cyclic if and only if any four consecutive vertices of Q_0 are contained in some circle, we may assume that $k = 4$.

Let a_1, a_2, a_3, a_4 be the (generalized) sides of the generalized quadrilateral Q_0 in this order, and we may assume that if two sides are aligned, then they are a_1 and a_2. Let θ_0 be the angle of Q_0 enclosed by a_1 and a_2, and hence possible $\theta_0 = \pi$, but the other three angles of Q_0 are in $(0, \pi)$. We deform Q_0 by varying the angle θ of the sides a_1 and a_2, and write η to denote the angle of the varying quadrilateral enclosed by a_3 and a_4, with the initial condition $\eta = \eta_0$ when $\theta = \theta_0$. Expressing the diagonal separating θ and η using the Law of Cosines yields that

$$a_1^2 + a_2^2 - 2a_1a_2 \cos\theta = a_3^2 + a_4^2 - 2a_3a_4 \cos\eta,$$

and hence $a_1a_2 \sin\theta = a_3a_4 \sin\eta \cdot \frac{d\eta}{d\theta}$. Thus the area $f(\theta) = \frac{1}{2}a_1a_2 \sin\theta + \frac{1}{2}a_3a_4 \sin\eta$ of the varying quadrilateral satisfies

$$f'(\theta) = \frac{a_1a_2}{2} \cdot \cos\theta + \frac{a_3a_4}{2} \cdot \cos\eta \cdot \frac{d\eta}{d\theta} = \frac{a_1a_2}{2} \cdot \cos\theta + \frac{a_1a_2}{2} \cdot \sin\theta \cdot \frac{\cos\eta}{\sin\eta}.$$

If $\theta_0 = \pi$, then $\theta < \theta_0$ and $f'(\theta_0) = \frac{-1}{2} a_1 a_2 < 0$, which contradicts the local maximality of $|Q_0|$. Therefore, $\theta_0 < \pi$ and $f'(\theta_0) = 0$, which in turn yields that $\tan \theta_0 = -\tan \eta_0$. We deduce that $\theta_0 + \eta_0 = \pi$, thus Q_0 is a cyclic quadrilateral. ∎

In the rest of the section, we discuss the isoperimetric inequality for polytopes in higher dimension, also in terms of the mean projections. The following estimate, together with the Blaschke Selection Theorem 1.7.4, ensures the existence of extremal bodies.

Lemma 7.7.4. *For a convex body $K \subset \mathbb{R}^n$ and $1 \leqslant i \leqslant n-1$, $n \geqslant 2$, we have*

$$V_i(K) \geqslant \gamma \cdot |K|^{\frac{i-1}{n-1}} (\operatorname{diam} K)^{\frac{n-i}{n-1}} \quad \text{for } \gamma > 0 \text{ depending only on } n, i.$$

Proof. If $i = 1$, then $V_1(K) \geqslant \operatorname{diam} K$ as K contains a segment of length $\operatorname{diam} K$.

If $i \geqslant 2$, then the Aleksandrov-Fenchel inequality Corollary 7.4.9 yields that

$$V(B^n, K; i)^{n-1} = V(B^n, n-i; K, i-1; K, 1)^{n-1} \geqslant V(B^n, K; 1)^{n-i} |K|^{i-1},$$

where (7.17) yields that $V(B^n, K; i) = \binom{n}{i}^{-1} \omega_{n-i} V_i(K)$ and $V(B^n, K; 1) = \frac{\omega_{n-1}}{n} V_1(K) \geqslant \frac{\omega_{n-1}}{n} \operatorname{diam} K$. ∎

The isoperimetric-type inequalities for simplices in any dimensions were proved by Steiner [692] already in 1842 in the case of the perimeter, and much later by Hadwiger [377] in the middle of the 20th century in the case of other mean projections.

Theorem 7.7.5 (Isoperimetric Inequality for simplices). *For $i = 1, \dots, n-1$, $n \geqslant 2$, if C is a simplex and T is a regular simplex with $|C| = |T|$, then*

$$V_i(C) \geqslant V_i(T),$$

with equality if and only if C is a regular simplex.

Proof. There exists a simplex C_0 with $|C_0| = |T|$ and minimizing $V_i(C_0)$ according to Lemma 7.7.4 and Blaschke Selection Theorem 1.7.4, so all we need to prove is that C_0 is a regular simplex. We suppose that C_0 has two edges of different lengths, and seek a contradiction. It follows that C_0 has two intersecting edges of different length, thus $C_0 = \operatorname{conv}\{v_0, \dots, v_n\}$ where $\|v_2 - v_0\| \neq \|v_2 - v_0\|$.

For $u = (v_1 - v_0)/\|v_1 - v_0\| \in S^{n-1}$, we have

$$C_0 = \{x + tu : x \in C_0 | u^\perp \text{ and } g(x) \leqslant t \leqslant f(x)\}$$

where f, g are linear functions on $u^\perp$. We deduce from Lemma 1.10.12 that the Steiner symmetrial $\Theta_{u^\perp} C_0$ (cf. Definition 1.10.1) is also a simplex of diameter at most D (cf. Proposition 1.10.3 (v)), and $V_i(\Theta_{u^\perp} C_0) < V_i(C_0)$ by Proposition 7.2.2 as $\|v_2 - v_0\| \neq \|v_2 - v_0\|$ yields that there exists no hyperplane of symmetry for C_0 parallel to $u^\perp$. This contradiction verifies Theorem 7.7.5. ∎

Remark. Given the number of vertices and the volume of a polytope in $\mathbb{R}^d$, the use of Steiner symmetrization and some calculations yield the solution of the isoperimetric problem in the following cases. For $1 \leqslant i < n$ and $a, b > 0$, let $Q(i; a, b)$ be a polytope with $n + 2$ vertices that is the convex hull of an i-dimensional regular simplex T of edge length a and an $(n - i)$-dimensional regular simplex T' of edge length b where the centers of T and T' coincide, and their affine hulls are orthogonal. Among polytopes in $\mathbb{R}^n$ of at most k vertices and of given volume,

- $Q(\lfloor n/2 \rfloor; a, b)$ minimizes the surface area in the case of $k = n + 2$ vertices in $\mathbb{R}^n$ where $a/b = \sqrt{(\lfloor n/2 \rfloor + 1)/(\lceil n/2 \rceil + 1)}$ (cf. Böröczky, Böröczky [121]);
- $Q(2; a, a)$ minimizes the ith intrinsice volume $i = 1, 2, 3$ in the case of $k = 6$ vertices in $\mathbb{R}^4$ (cf. Böröczky, Böröczky [121]);
- $Q(2; a, b)$ minimizes the surface area in the case of $k = 5$ vertices in $\mathbb{R}^3$ where $a/b \approx 0.9451$ (cf. Böröczky, Böröczky [121]), and hence the vertices of the extremal polytope do not lie on a sphere;
- the regular octahedron minimizes the surface area in the case of $k = 6$ vertices in $\mathbb{R}^3$ (cf. Böröczky, Kovács [138]).

The three Platonic solids that are simple polytopes (three faces meet at each vertex), namely, the tetrahedron, cube and dodecahedron, minimize the isperimetric quotient among 3-polytopes of the same number of faces (cf. Theorem 7.7.5 and Theorem 7.7.8). In the case of the cube and the dodecahedron, L. Fejes Tóth's elegant argument verifying Theorem 7.7.8 needs not only Lindelöf's Theorem 7.7.1, but also the so-called Moment Theorem 7.7.6 proved also by L. Fejes Tóth [261, 264, 265]. We only quote this theorem, noting that $\mathcal{H}^2(S^2) = 4\pi$, and for an edge to edge tiling of S^2 by k spherically convex tiles, Euler's theorem yields that the mosaic has at most $3k - 6$ edges.

Theorem 7.7.6 (Moment Theorem). *Let $x_1, \dots, x_k \in S^2$, $k \geqslant 4$, be not contained in any open hemisphere, and let the spherical triangle $T = \mathrm{conv}_{S^2}\{a, p, q\}$ have angle $\frac{k \cdot 2\pi}{6k-12}$ at p, and satisfy that $d_{S^2}(a, p) = d_{S^2}(a, q)$ and $\mathcal{H}^2(T) = \frac{2\pi}{3k-6}$.*

(i) If $f : (0, \frac{\pi}{2}) \to [0, \infty)$ is monotone increasing, then

$$\int_{S^2} \min_{x_i : \langle u, x_i \rangle > 0} f\left(d_{S^2}(u, x_i)\right) \, du \geqslant (6k - 12) \int_T f\left(d_{S^2}(u, a)\right) \, du. \tag{7.54}$$

(ii) If $g : (0, \frac{\pi}{2}) \to [0, \infty)$ is monotone decreasing, then

$$\int_{S^2} \max_{x_i : \langle u, x_i \rangle > 0} g\left(d_{S^2}(u, x_i)\right) \, du \leqslant (6k - 12) \int_T g\left(d_{S^2}(u, a)\right) \, du. \tag{7.55}$$

Assuming that f and g are strictly monotone, equality holds if $x_1, \dots, x_k$ are vertices of an inscribed regular tetrahedron ($k = 4$), or octahedron ($k = 6$) or icosahedron ($k = 12$).

Remark 7.7.7. (i) A stability version of Theorem 7.7.6 is verified by Böröczky, G. Fejes Tóth [126].

(ii) The integrals in (7.54) and in (7.55) make sense even if $x_1, \dots, x_k \in S^2$ are contained in a closed hemisphere; namely, even if there exists a $u \in S^2$ with $d_{S^2}(u, x_i) \geqslant \frac{\pi}{2}$ for all x_i. The point is that $o \in \operatorname{conv}\{x_1, \dots, x_k\} = Q \subset \mathbb{R}^3$ as $x_1, \dots, x_k$ are not contained in any open hemisphere, and hence $o \in \partial Q$ and $u \in N_Q(o)$. However, $\dim N_Q(o) \leqslant 2$ as o is not a vertex (it is not among $x_1, \dots, x_k$); therefore, $N_K(o) \cap S^2 = \{v \in S^2 : d_{S^2}(v, x_i) \geqslant \frac{\pi}{2} \ \forall x_i\}$ satisfies that $\mathcal{H}^2(N_K(o) \cap S^2) = 0$.

We have already seen in Steiner's Theorem 7.7.5 that regular tetrahedra are optimal for the isoperimetric quotient. L. Fejes Tóth [261, 264, 265] proved that the other two Platonic solids that are simple polytopes are also extremal.

Theorem 7.7.8 (The Isoperimetric property of the 3-cube and the Dodecahedron). *For $k = 6, 12$, among 3-polytopes of given volume and having at most k faces, the ones with minimal surface area are the cube and dodecahedron, respectively.*

Proof. According to Lindelöf's Theorem 7.7.1, any polytope $Q_0 \subset \mathbb{R}^3$ with at most k faces minimizing the isoperimetric ratio $P(Q_0)/|Q_0|^{\frac{2}{3}}$ among polytopes with at most k facets is circumscribed around some ball B and has k faces. We may assume that $B = B^3$, and hence $P(Q_0)/|Q_0|^{\frac{2}{3}} = 3|Q_0|^{\frac{1}{3}}$ (cf. (7.53)).

Let the faces of Q_0 touch S^2 in exterior unit normals $u_1, \dots, u_k \in S^2$; therefore, the radial function of P is $\varrho_{P_0}(u) = \min\{\langle u, u_i\rangle^{-1} : \langle u, u_i\rangle > 0\}$ for $u \in S^2$, thus Lemma 1.11.6 yields

$$|Q_0| = \frac{1}{3}\int_{S^2} \varrho_{Q_0}(u)^3 \, du = \frac{1}{3}\int_{S^2} \min_{u_i : \langle u, u_i\rangle > 0} \langle u, u_i\rangle^{-3} \, du$$
$$= \frac{1}{3}\int_{S^2} \min_{u_i : \langle u, u_i\rangle > 0} \cos^{-3} d_{S^2}(u, u_i) \, du.$$

In turn, Theorem 7.7.8 follows from the Moment Theorem (7.54) with $f(t) = \cos^{-3} t$. ∎

Actually even a stability version of Theorem 7.7.8 is known, due to Böröczky, G. Fejes Tóth [126].

We recall that according to the Isoperimetric Inequality (cf. Theorem 2.4.1 or Theorem 4.1.6), if $K \subset \mathbb{R}^n$ is a convex body, then

$$\frac{P(K)}{|K|^{\frac{n-1}{n}}} \geqslant \frac{P(B^n)}{|B^n|^{\frac{n-1}{n}}} = n\omega_n^{\frac{1}{n}} \sim \sqrt{2e\pi} \cdot \sqrt{n}$$

where $f(n) \sim g(n)$ for positive $f(n), g(n)$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$. The paper Keith Ball, Böröczky, Naor [60] provides a bound on the isoperimetric quotient

of convex bodies with few facets. The argument uses the bounds in (6.54); namely, if $Q_{(k)} \supset B^n$ polytope of minimal volume with k facets, then

$$c_0^{-1}\left(\log\frac{k}{n}\right)^{\frac{-1}{2}} \leqslant |Q_{(k)}|^{\frac{1}{n}} \leqslant c_0\left(\log\frac{k}{n}\right)^{\frac{-1}{2}} \tag{7.56}$$

for an absolute constant $c_0 > 1$.

Theorem 7.7.9. *For an absolute constant $c > 0$, if $2n \leqslant k \leqslant 2^n$ and $Q_{(k)} \subset \mathbb{R}^n$ is a polytope with at most k facets, then*

$$\frac{P(Q_{(k)})}{|Q_{(k)}|^{\frac{n-1}{n}}} \geqslant \frac{cn}{\sqrt{\log\frac{k}{n}}}. \tag{7.57}$$

Remark. The estimate (7.57) is sharp, as there exist an absolute constant $\tilde{c} > 0$ and a polytope $\widetilde{Q}_{(k)} \subset \mathbb{R}^n$ with at most k facets such that

$$\frac{P(\widetilde{Q}_{(k)})}{|\widetilde{Q}_{(k)}|^{\frac{n-1}{n}}} \leqslant \frac{\tilde{c}n}{\sqrt{\log\frac{k}{n}}}. \tag{7.58}$$

Proof. It is equivalent to prove that the estimates of (7.57) and (7.58) hold for a polytope $Q_{(k)}$ of at most k facets minimizing the isoperimetric quotient $P(Q_{(k)})/|Q_{(k)}|^{\frac{n-1}{n}}$. According to Lindelöf's Theorem 7.7.1, we may assume that $Q_{(k)}$ is circumscribed around B^n, and hence $P(Q_{(k)})/|Q_{(k)}|^{\frac{n-1}{n}} = n|Q_{(k)}|^{\frac{1}{n}}$ (cf. (7.53)). In particular, (7.57) is verified by the lower bound in (7.56), while (7.58); namely, the optimality of (7.57), follows from using a $\widetilde{Q}_{(k)}$ witnessing the upper bound in (7.56). ∎

7.8 Comments to Section 7

The fact that the volume of non-negative linear combination of compact convex sets is a homogeneous polynomial in the coefficients, and the basic properties of mixed volumes were established by Minkowski [590, 591] around 1900, extending the ideas by Steiner [691] (cf. Theorem 7.1.2) about the volume of the parallel body around 1840. The intrinsic volume $V_i(K)$ of a compact convex set $K \subset \mathbb{R}^n$, the practical, dimension invariant renormalization of the mixed volumes of form $V(B^n, K; i)$ (traditionally called the quermassintegral $W_{n-i}(K)$), was introduced by McMullen [569] only in 1991. The integral formula in Theorem 7.1.2 representing the intrinsic volumes as mean projections is due to Kubota [489] around 1925.

Minkowski's classical inequality (7.26) and his "second inequality" (7.29) were proved around 1900 (cf. [590, 591]). Aleksandrov [5, 7, 11] himself provided two

proofs of the Aleksandrov-Fenchel Inequality (7.30) around 1937-38 (for additional arguments still based on Aleksandrov's ideas, see also Shenfeld, van Handel [678], Schneider [661], Section 7.3 and Cordero-Erausquin, Klartag, Merigot, Santambrogio [219]). Fenchel only stated the inequality, never actually provided a proof. Both arguments in this monograph, the one using strongly isomorphic polytopes in Section 7.A, and the one using the theory of elliptic operators in Section 8.5.2, are based on Aleksandrov's original ideas as developed further by Shenfeld, van Handel [678].

Bernstein [85] and Kouchnirenko [486] proved a very interesting connection between the mixed volumes and the intersection number of complex algebraic hypersurfaces. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we write $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ for $x = (x_1, \ldots, x_n) \in (\mathbb{C}\backslash\{o\})^n$. Now a Laurent polynomial is of the form $p(x) = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha x^\alpha$ where there are only finitely many $c_\alpha \neq 0$, and the Newton polytope associated to p is $Q(p) = \mathrm{conv}\{\alpha \in \mathbb{Z}^n : c_\alpha \neq 0\}$, and the associated complex hypsurface is $X(p) = \{x \in (\mathbb{C}\backslash\{o\})^n : p(x) = 0\}$. Given n-dimensional lattice polytopes $Q_1, \ldots, Q_n$ (convex hull of finitely many points of $\mathbb{Z}^n$), the Bernstein-Kouchnirenko theorem says that for generic Laurent polynomial $p_1, \ldots, p_n$ with $Q(p_i) = Q_i$, we have

$$n!V(Q_1, \ldots, Q_n) = \#(X(p_1) \cap \ldots \cap X(p_n)). \tag{7.59}$$

Here the condition "generic" ensures that $X(p_1), \ldots, X(p_n)$ are smooth hypersurfaces in $(\mathbb{C}\backslash\{o\})^n$ that intersect transversally in finitely many points.

The Bézout theorem stating that given integers $d_1, \ldots, d_n \geqslant 1$, the number of common roots of generic polynomials p_i of degree $d_i, i = 1, \ldots, n$, is $d_1 \cdot \ldots \cdot d_n$ is a special case of the Bernstein-Kouchnirenko theorem (7.59), as for $\Delta = \mathrm{conv}\{o, e_1, \ldots, e_n\}$ where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{Z}^n$, we take $Q_i = d_i\Delta$ in (7.59).

Another connection between the mixed volumes and the intersection number in algebraic geometry is via ample divisors on projective toric varieties (see Cox, Little, Schenck [222], Ewald [252], Fulton [308] and Oda [606]). In this setting, the Aleksandrov-Fenchel inequality is a consequence of the Hodge index theorem. In turn, the relations between the Aleksandrov-Fenchel inequality and the Hodge-Riemann theorem has been recently investigated in terms of valuations of convex bodies (see Section 8.D) by Kotrbatý [482] and Kotrbatý, Wannerer [484,485] and Bernig, Kotrbatý, Wannerer [84].

The Aleksandrov-Fenchel inequality (7.30) for mixed volumes also has important applications in combinatorics, for example concerning counting the number of linear extensions of a poset (see for example Chan, Pak [186], Kahn, Saks [452] and Stanley [690]). Here we sketch Stanley's result in [690] about linear extensions of a k element poset (partially ordered set) $P = \{p_0, \ldots, p_{k-1}\}$ for $k \geqslant 3$. A linear extension of P is an order preserving bijection $\pi : P \to \{1, \ldots, k\}$, and for $m = 1, \ldots, k$, let N_m be

the number of linear extensions of P where $\pi(p_0) = m$. For the polytopes

$$Q_0 = \{(x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1} : x_i \leqslant x_j \text{ if } p_i \prec p_j \text{ and } x_i = 0 \text{ if } p_0 \prec p_i\},$$
$$Q_1 = \{(x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1} : x_i \leqslant x_j \text{ if } p_i \prec p_j \text{ and } x_i = 1 \text{ if } p_0 \succ p_i\},$$

Stanley [690] verifies that N_m can be represenseted as a $(k-1)$-dimensional mixed volume

$$N_m = V(Q_1, i-1; Q_0, k-m),$$

and applies the Aleksandrov-Fenchel inequality (7.32) to prove that

$$N_m^2 \geqslant N_{m-1} N_{m+1} \tag{7.60}$$

for $m = 2, \ldots, k-1$; or in other words, the sequence $N_1, \ldots, N_k$ is log-concave. According to Shenfeld, van Handel [680] characterizing the equality case of the Aleksandrov-Fenchel inequality for polytopes, if $N_m > 0$ and $N_m^2 = N_{m-1} N_{m+1}$, then $N_m = N_{m-1} = N_{m+1}$.

For any convex body $K \subset \mathbb{R}^n$ and $i = 1, \ldots, n$, the form (7.31) of the Aleksandrov-Fenchel inequality yields that $V(K, -K; i) \geqslant |K|$ where equality holds for centrally symmetric bodies, which inequalities and expanding into mixed volumes imply the consequence $|K - K| \geqslant 2^n |K|$ of the Brunn-Minkowski inequality (cf. (1.36)). As a possible reverse form, the classical conjecture by Godbersen [327] in 1938 states that

$$V(K, -K; i) \leqslant \binom{n}{i} |K| \tag{7.61}$$

for any convex body $K \subset \mathbb{R}^n$ and $i = 1, \ldots, n-1$, where equality holds if K is a simplex. We observe that the Godbersen conjecture (7.61) for $i = 1, \ldots, n-1$ and expanding into mixed volumes imply the Rogers-Shephard inequality $|K - K| \leqslant \binom{2n}{n} |K|$ (cf. (1.36)).

Now, if K is centered, then $-K \subset nK$ (cf. Lemma 1.11.4); therefore, (7.61) holds if $i = 1$ or $i = n-1$ by the monotonicity of mixed volumes. On the other hand, the Godbersen conjecture (7.61) is still open when $n \geqslant 4$ and $i = 2, \ldots, n-2$. For partial results on the Godbersen conjecture and possible extensions, see, for example, Artstein-Avidan, Putterman [41], Artstein-Avidan, Sadovsky, Sanyal [42] and Kotrbatý [483].

Schneider [664] - clarifying earlier work by Esterov [250], who used representation of mixed volumes using toric varieties - expressed the mixed volumes of polytopes in terms of the family of their normal cones and the product of their support functions. For fixed n-dimensional polytopes $\widetilde{P}_1, \ldots, \widetilde{P}_n \subset \mathbb{R}^n$, let $\mathcal{P}$ denote the family of n-tuples $(P_1, \ldots, P_n)$ of n-polytopes such that the family of normal cones of P_i coincide with the family of normal cones of $\widetilde{P}_i$. In this case, Schneider [664] prove that the mixed volumes of the n-tuples in $\mathcal{P}$ depend only on the product of the n support functions.

A typical example when this theorem applies is when all polytopes are strongly isomorphic (cf. Section 1.8).

Concerning the isoperimetric problem in the spherical and hyperbolic spaces (see Section 4.A for their fundamental properties), the isoperimetric inequality for polygons of given number of vertices (sides) was extended to the spherical and the hyperbolic space by L. Fejes Tóth [264] and Bezdek [88] (see Theorem 4.C.1). In addition, L. Fejes Tóth [263] proved that among tetrahedra of given volume in H^3, the regular one has the minimal surface area. Isoperimetric-type problems for polytopes in two and three-dimensional spaces of constant curvature are discussed in L. Fejes Tóth [264], L. Fejes Tóth, G. Fejes Tóth, W. Kuperberg [265], Florian [291], and in higher dimensional spaces by Basit, Lángi [76].

Even if mixed volumes of arbitrary convex bodies have no analogues in the spherical space S^n and hyperbolic space H^n, mean projections and mean curvatures do have analogues, and some versions of the Aleksandrov-Fenchel inequalities are proved in these spaces. For S^n, see, for example, Hu, Li [415], and a close analogue by Guan, Ma, Trudinger, Zhu [358], and for H^n, see, for example, Andrews, Hu, Li [31], Hu, Li [415] and de Lima, Girão [519]. Minkowski type inequality has been proved by Natário [600] in H^3, and on the other hand, Chodosh, Eichmair, Koerber [204] show that some convexity assumptions are necessary in order to ensure that some analogues of the Minkowski inequality hold in S^3 even under strong symmetry assumptions.

7.A Supplement: Proof of the Aleksandrov-Fenchel inequality based on polytopes

This section proves of the Aleksandrov-Fenchel inequality for the mixed volumes based on the argument by Aleksandrov [6] using strongly isomorphic polytopes, as it was simplified by Shenfeld, van Handel [678] (see also Schneider [661], Section 7.3 for a version of this approach).

Theorem 7.A.1 (Aleksandrov-Fenchel Inequality). *If $n \geqslant 3$ and $C_1, \dots, C_{n-2}, K, L$ are compact convex sets in $\mathbb{R}^n$, then*

$$V(K, L, C_1, \dots, C_{n-2})^2 \geqslant V(K, K, C_1, \dots, C_{n-2})V(L, L, C_1, \dots, C_{n-2}). \quad (7.62)$$

7.A.1 Some properties of symmetric matrices

If $\mathcal{E}$ is a symmetric $d \times d$ matrix for $d \geqslant 2$, then there exist eigenvectors $x_1, \dots, x_d$ forming an orthonormal basis of $\mathbb{R}^d$ and eigenvalues $\lambda_1 \geqslant \dots \geqslant \lambda_d$ such that $\mathcal{E}x_i = \lambda_i x_i$ and if $j = 1, \dots, d-1$ and $x \in \mathbb{R}^d$ satisfy that $\langle x_i, x\rangle = 0$ for $i = 1, \dots, j$, then

$$\langle \mathcal{E}x, x\rangle \leqslant \lambda_{j+1}\langle x, x\rangle. \quad (7.63)$$

We also need the version Theorem 7.A.2 of the Perron-Frobenius theorem (for a proof, see Theorem 10.8.2 in the Appendix). Let $\mathcal{E} = [e_{ij}]$ be a $d \times d$ matrix. We say that $\mathcal{E}$ is non-negative, if $e_{ij} \geqslant 0$ for any $i, j = 1, \ldots, d$, and we say that the off-diagonal entries of $\mathcal{E}$ are non-negative, if $e_{ij} \geqslant 0$ when $i \neq j$. Assuming that the off-diagonal entries of $\mathcal{E}$ are non-negative, we say that $\mathcal{E}$ is irreducible, if for any $i \neq j$ there exist pairwise different $i_0, \ldots, i_k \in \{1, \ldots, d\}$ such that $i_0 = i$, $i_k = j$, and $e_{i_{m-1}, i_m} > 0$ for $m = 1, \ldots, k$.

Theorem 7.A.2 (Perron-Frobenius Theorem for irreducible symmetric matrices). *If $\mathcal{E}$ is symmetric non-negative irreducible $d \times d$ matrix with positive entries on the diagonal, and λ_1 is the largest eigenvalue, then*

- *$\lambda_1 > 0$ and λ_1 is a simple eigenvalue;*
- *there exists an eigenvector x_1 whose coordinates are all positive and $\mathcal{E}x_1 = \lambda_1 x_1$;*
- *any eigenvector x of $\mathcal{E}$ whose coordinates are all non-negative satisfy $x = r\, x_1$ for $r > 0$.*

If all we know about a symmetric $d \times d$ matrix $\mathcal{E}$ is that its off-diagonal entries of $\mathcal{E}$ are non-negative and $\mathcal{E}$ is irreducible, then there exists $\gamma > 0$ such that $\mathcal{E} + \gamma\, I_d$ is a non-negative irreducible $d \times d$ matrix with positive entries on the diagonal. Since λ is an eigenvalue of $\mathcal{E}$ with eigenvector x if and only if x is an eigenvector for $\mathcal{E} + \gamma\, I_d$ with eigenvalue $\lambda + \gamma$, we deduce the following consequence of the Perron-Frobenius Theorem 7.A.2:

Corollary 7.A.3. *If the off-diagonal entries of a symmetric $d \times d$ matrix $\mathcal{E}$ are non-negative and $\mathcal{E}$ is irreducible, and λ_1 is the largest eigenvalue, then*

- *λ_1 is a simple eigenvalue;*
- *there exists an eigenvector x_1 whose coordinates are all positive and $\mathcal{E}x_1 = \lambda_1 x_1$;*
- *any eigenvector x of $\mathcal{E}$ whose coordinates are all non-negative satisfy $x = r\, x_1$ for $r > 0$.*

If $\mathcal{E}$ is a symmetric matrix, its positive eigenspace is the subspace spanned by the eigeinvectors corresponding to positive eigenvalues.

Lemma 7.A.4 (Hyperbolic Quadratic Forms). *For a symmetric $d \times d$ matrix $\mathcal{E}$, $d \geqslant 2$, the following conditions are equivalent for $x, y \in \mathbb{R}^d$.*

(i) $\langle x, \mathcal{E}y\rangle^2 \geqslant \langle x, \mathcal{E}x\rangle\langle y, \mathcal{E}y\rangle$ if $\langle y, \mathcal{E}y\rangle \geqslant 0$.

(ii) The dimension of the positive eigenspace of $\mathcal{E}$ is at most one.

Proof. Let $\lambda_1 \geqslant \ldots \geqslant \lambda_d$ be the eigenvectors of $\mathcal{E}$, and let $x_1, \ldots, x_d$ be the eigenvectors forming an orthonormal basis of $\mathbb{R}^d$ such that $\mathcal{E}x_i = \lambda_i x_i$. We may assume

that $\lambda_1 > 0$ because otherwise $\langle x, \mathcal{E}x\rangle \leqslant 0$ for any $x \in \mathbb{R}^d$, and hence Lemma 7.A.4 readily holds.

Assuming (i), $0 \geqslant \lambda_1\lambda_2\langle x_1, x_1\rangle\langle x_2, x_2\rangle$ follows by applying (i) to $x = x_1$ and $y = x_2$; therefore, $\lambda_i \leqslant \lambda_2 \leqslant 0$ for $i \geqslant 2$.

Assuming (ii), we may also assume that $\langle y, \mathcal{E}y\rangle > 0$. Since $\lambda_i \leqslant \lambda_2 \leqslant 0$ by (ii), we deduce that $\langle z, \mathcal{E}z\rangle \leqslant 0$ holds for any $z \in \mathbb{R}^d$ satisfying $\langle z, \mathcal{E}x_1\rangle = 0$. In particular, $\langle y, \mathcal{E}x_1\rangle \neq 0$ follows from $\langle y, \mathcal{E}y\rangle > 0$. Since $\langle z, \mathcal{E}x_1\rangle = 0$ for $\alpha = \langle x, \mathcal{E}w\rangle/\langle y, \mathcal{E}w\rangle$ and $z = x - \alpha y$, it follows using also the condition that $\mathcal{E}$ is symmetric that

$$\begin{aligned} 0 \geqslant \langle z, \mathcal{E}z\rangle &= \langle x, \mathcal{E}x\rangle - 2\alpha\langle x, \mathcal{E}y\rangle + \alpha^2\langle y, \mathcal{E}y\rangle \\ &= \langle x, \mathcal{E}x\rangle - \frac{\langle x, \mathcal{E}y\rangle^2}{\langle y, \mathcal{E}y\rangle} + \langle y, \mathcal{E}y\rangle\left(\alpha - \frac{\langle x, \mathcal{E}y\rangle}{\langle y, \mathcal{E}y\rangle}\right)^2 \geqslant \langle x, \mathcal{E}x\rangle - \frac{\langle x, \mathcal{E}y\rangle^2}{\langle y, \mathcal{E}y\rangle}. \end{aligned}$$

In turn, we conclude $\langle x, \mathcal{E}y\rangle^2 \geqslant \langle x, \mathcal{E}x\rangle\langle y, \mathcal{E}y\rangle$. ∎

7.A.2 Mixed volumes of strongly isomorphic simple polytopes and multilinear forms

The sole goal of this section is to prove Theorem 7.A.7, which yields that mixed volumes of strongly isomorphic simple n-polytopes in $\mathbb{R}^n$ can be represented by symmetric multilinear forms of the vectors made up from the values of the support functions at the unit exterior normals of the facets. First, we recall some facts about simple and strongly isomorphic polytopes (cf. Section 1.8). For $d \geqslant 1$ and d-dimensional strongly isomorphic $M_1, \dots, M_d \subset \mathbb{R}^n$ such that their affine hulls are parallel to a linear d-space $L \subset \mathbb{R}^n$, we write $V(M_1, \dots, M_d)$ to denote their d-dimensional mixed volume (the dimension is determined by the number of variables). Writing $v_1, \dots, v_m \in L \cap S^{n-1}$ to denote the common normals to the $(d-1)$-dimensional faces, (7.23) says that

$$V(M_1, \dots, M_d) = \frac{1}{d}\sum_{i=1}^{m} h_{M_d}(v_i) \cdot V(F_{M_1}(v_i), \dots, F_{M_{d-1}}(v_i)) \tag{7.64}$$

where we set $V(F_{M_1}(v_i), \dots, F_{M_{d-1}}(v_i)) = 1$ if $d = 1$.

The polytopes $P_1, \dots, P_k \subset \mathbb{R}^n$ are strongly isomorphic if for any $1 \leqslant i < j \leqslant n$, P_i and P_j have exactly the same set of normal cones; or equivalently, $\dim F_{P_i}(u) = \dim F_{P_j}(u)$ for any $u \in S^{n-1}$. In this case, $\dim P_i = \dim P_j$ and the affine hulls of P_i and P_j are parallel if $\dim P_i < n$. A d-dimensional polytope $P \subset \mathbb{R}^n$ is simple, $1 \leqslant d \leqslant n$, if exactly d edges meet at any vertex of P. We note that any at least one-dimensional face of a simple polytope is simple. According to Lemma 1.8.6, a family of strongly isomorphic polytopes can be obtained by deforming a simple polytope.

Lemma 7.A.5. *For a simple n-polytope $P \subset \mathbb{R}^n$, there exists $\varepsilon_0 > 0$, such that if $u_1, \dots, u_n$ are the unit exterior normals of the facets of P, and $|t_i - h_P(u_i)| < \varepsilon_0$ for $i = 1, \dots, k$, then $P' = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant t_i\}$ is a simple n-polytope strongly isomorphic to P.*

For the rest of the section, we fix a simple n-polytope $K \subset \mathbb{R}^n$, $n \geqslant 2$, and write $\mathcal{P}_K$ to denote the family of simple polytopes in $\mathbb{R}^n$ strongly ismorphic to K, $\mathcal{F}_K^d$ to denote the family of d-dimensional faces K, $d = 0, \dots, n-1$, and $\mathcal{U}_K$ to denote the family of exterior unit normals to the facets $F \in \mathcal{F}_K^{n-1}$ of K. Any $P \in \mathcal{P}_K$ can be encoded by the vector $\bar{h}_P \in \mathbb{R}^{\mathcal{U}(K)}$ such that the coordinate of $\bar{h}_P$ corresponding to $u \in \mathcal{U}_K$ is $\bar{h}_P^{(u)} = h_P(u)$. Next, for any face $F \in \mathcal{F}_K^d$, $d = 1, \dots, n-1$, let $L_F = \operatorname{lin}(F - F)$ be the parallel linear d-subspace, let $\mathcal{P}_F$ be the family of simple d-polytopes in $\mathbb{R}^n$ strongly ismorphic to F, and let $\mathcal{U}_F \subset L_F \cap S^{n-1}$ be the family of exterior unit normals to the $(d-1)$-faces of F. Similarly as above, any polytope $P \in \mathcal{P}_F$ can be encoded by the vector $\bar{h}_P \in \mathbb{R}^{\mathcal{U}(F)}$ such that the coordinate of $\bar{h}_P$ corresponding to $u \in \mathcal{U}(F)$ is $\bar{h}_P^{(u)} = h_P(u)$. Let us show how these vectors of the form $\bar{h}_P$, $P \in \mathcal{P}_F$ generate $\mathbb{R}^{\mathcal{U}(F)}$.

Claim 7.A.6. *Let $G = K$, or $G \in \mathcal{F}_K^d$ for $d \in \{2, \dots, n-1\}$. For any $x \in \mathbb{R}^{\mathcal{U}(G)}$ and $P \in \mathcal{P}_G$, there exist $a > 0$ and $Q \in \mathcal{P}_G$ such that $x = \bar{h}_Q - a\bar{h}_P$.*

Proof. Choose $a > 0$ such $a^{-1}\|x\| < \varepsilon_0$ for the $\varepsilon_0 > 0$ in Lemma 7.A.5. Writing $x^{(u)}$ to denote the coordinate of x corresponding to a $u \in \mathcal{U}(G)$, Lemma 7.A.5 provides a $Q' \in \mathcal{P}$ such that if $u \in \mathcal{U}(G)$, then $h_{Q'}(u) = h_P(u) + a^{-1}x^{(u)}$, and hence $x = \bar{h}_Q - a\bar{h}_P$ for $Q = aQ'$. ∎

For any $F \in \mathcal{F}_K^d$, $d = 1, \dots, n-1$, and strongly isomorphic $P_1, \dots, P_d \in \mathcal{P}_F$, we write $V(P_1, \dots, P_d)$ to denote their mixed volume (the common dimension of the polytopes is determined by the number of variables). For the irreducibility of a symmetric matrix, see the discussion in front of Theorem 7.A.2

Theorem 7.A.7. *Using the notation as above for the fixed simple n-polytope $K \subset \mathbb{R}^n$, $n \geqslant 2$, there exists a symmetric (invariant under the permutations of the variables) real multilinear form $V(x_1, \dots, x_n)$ of $x_1, \dots, x_n \in \mathbb{R}^{\mathcal{U}(K)}$, and; in addition, for any $G \in \mathcal{F}_K^d$ for $d \in \{1, \dots, n-1\}$, there exists a symmetric real multilinear form $V(x_1, \dots, x_d)$ of $x_1, \dots, x_d \in \mathbb{R}^{\mathcal{U}(G)}$ with the following properties:*

(i) $V(\bar{h}_{P_1}, \dots, \bar{h}_{P_n}) = V(P_1, \dots, P_n)$ for $P_1, \dots, P_n \in \mathcal{P}_K$, and if $F \in \mathcal{F}_K^d$ for $d \in \{1, \dots, n-1\}$, then $V(\bar{h}_{P_1}, \dots, \bar{h}_{P_d}) = V(P_1, \dots, P_d)$ for $P_1, \dots, P_d \in \mathcal{P}_F$.

(ii) Let $G = K$, or $G \in \mathcal{F}_K^d$ for $d \in \{2, \dots, n-1\}$. For any $x \in \mathbb{R}^{\mathcal{U}(G)}$ and $u \in \mathcal{U}(G)$, there exists a "$(d-1)$-face" $F_x(u) \in \mathbb{R}^{\mathcal{U}(F_G(u))}$ of x such that

- *for fixed $u \in \mathcal{U}(G)$, $F_x(u)$ is a linear function of $x \in \mathbb{R}^{\mathcal{U}(G)}$;*
- *for fixed $u \in \mathcal{U}(G)$, $F_x(u) = \bar{h}_{F_P(u)}$ if $x = \bar{h}_P$ for $P \in \mathcal{P}_G$;*

- *if* $x_1, \dots, x_d \in \mathbb{R}^{\mathcal{U}(G)}$ *and* $x_d = (x_d^{(u)})_{u \in \mathcal{U}(G)}$, *then*

$$V(x_1, \dots, x_d) = \frac{1}{d} \sum_{u \in \mathcal{U}(G)} x_d^{(u)} \cdot V\left(F_{x_1}(u), \dots, F_{x_{d-1}}(u)\right). \tag{7.65}$$

(iii) Let $n = 2$, *or let* $n \geqslant 3$ *and* $P_1, \dots, P_{n-2} \in \mathcal{P}_K$. *Then there exist a symmetric irreducible matrix* $\mathcal{A} : \mathbb{R}^{\mathcal{U}(K)} \to \mathbb{R}^{\mathcal{U}(K)}$ *such that* $\mathcal{A}$ *has non-negative off-diagonal entries, and for* $x, y \in \mathbb{R}^{\mathcal{U}(K)}$, *the coordinate of* $\mathcal{A}x$ *corresponding to* $u \in \mathcal{U}(K)$ *is*

$$(\mathcal{A}x)^{(u)} = \begin{cases} V\left(F_x(u), F_{P_1}(u), \dots, F_{P_{n-2}}(u)\right) & \text{if } n \geqslant 3, \\ V(F_x(u)) & \text{if } n = 2, \end{cases} \tag{7.66}$$

and hence

$$\langle x, \mathcal{A}y \rangle = nV\left(x, y, P_1, \dots, P_{n-2}\right) \tag{7.67}$$

where $V\left(x, y, P_1, \dots, P_{n-2}\right) = V\left(x, y, \bar{h}_{P_1}, \dots, \bar{h}_{P_{n-2}}\right)$, *and* $\langle x, \mathcal{A}y \rangle = 2V(x, y)$ *if* $n = 2$.

Proof. We prove (i) and (ii) by induction on $d = 1, \dots, n$. If $d = 1$ and $F \in \mathcal{F}_K^1$ is an edge, then let $F = [p, q]$, and hence $u = \frac{p-q}{\|p-q\|}$ is the exterior unit normal at p and $\mathcal{U}(F) = \{u, -u\}$. We deduce that $V(F) = \langle u, p - q \rangle = h_F(u) + h_F(-u)$ (cf. (7.64)); therefore, $V(x) = x^{(u)} + x^{(-u)}$ for $x \in \mathbb{R}^{\mathcal{U}(F)}$.

Next let $G = K$, or $G \in \mathcal{F}_K^d$ for $d \in \{2, \dots, n-1\}$. Any $(d-2)$-dimensional face of G is contained in exactly two $(d-1)$-faces according to Proposition 1.4.3, thus let $\mathcal{Q}_G = \{(u, v) \in \mathcal{U}(G) \times \mathcal{U}(G) : \dim\left(F_G(u) \cap F_G(v)\right) = d - 2\}$ where the empty set has dimension -1. For $(u, v) \in \mathcal{Q}_G$ and $P \in \mathcal{P}_G$, let α_{uv} be the angle of u and v, let $F_P(u, v) = F_P(u) \cap F_P(v)$ the common $(d-2)$-face of $F_P(u)$ and $F_P(v)$, and let $w_{u,v} \in \mathcal{U}(F_P(u, v))$ be the exterior unit normal to the $(d-2)$-face $F_P(u, v)$ of $F_P(u)$ in $L_{F_P(u,v)}$ where $w_{u,v} \in \mathcal{U}(F_G(u, v))$ depends only on G, u, v and not on the choice of P. We observe that $\alpha_{uv} = \alpha_{vu}$ and $F_P(u, v) = F_P(v, u)$, but $w_{u,v} \neq w_{v,u}$ for $(u, v) \in \mathcal{Q}_G$. For $(u, v) \in \mathcal{Q}_G$, it follows that

$$w_{u,v} = \csc \alpha_{uv} \cdot v - \cot \alpha_{uv} \cdot u$$

for $\csc \alpha_{uv} = 1/\sin \alpha_{uv} > 0$, and hence choosing some $p_{uv} \in F_P(u, v)$, we have

$$h_{F_P(u)}(w_{u,v}) = \langle p_{uv}, w_{u,v} \rangle = \csc \alpha_{uv} \cdot h_P(v) - \cot \alpha_{uv} \cdot h_P(u). \tag{7.68}$$

Since $\mathcal{U}(F_G(u)) = \{w_{u,v} : (u, v) \in \mathcal{Q}_G\}$ for $u \in \mathcal{U}(G)$, we deduce from (7.64) that if $P_1, \dots, P_d \in \mathcal{P}_G$, then

$$V(F_{P_1}(u), \dots, F_{P_{d-1}}(u)) = \tag{7.69}$$
$$\frac{1}{d-1} \sum_{\{v:(u,v) \in \mathcal{Q}_G\}} \left(\csc \alpha_{uv} \cdot h_{P_{d-1}}(v) - \cot \alpha_{uv} \cdot h_{P_{d-1}}\right) V(F_{P_1}(u, v), \dots, F_{P_{d-2}}(u, v))$$

where we set $V(F_{P_1}(u,v),\dots,F_{P_{d-2}}(u,v)) = 1$ if $d = 2$. Applying again (7.64) yields

$$V(P_1,\dots,P_d) = \frac{1}{d}\sum_{u\in\mathcal{U}(G)} h_{P_d}(u)\cdot V(F_{P_1}(u),\dots,F_{P_{d-1}}(u)) = \tag{7.70}$$

$$\frac{1}{d(d-1)}\sum_{(u,v)\in\mathcal{Q}_G}\left(\csc\alpha_{uv}\cdot h_{P_d}(u)\cdot h_{P_{d-1}}(v) - \cot\alpha_{uv}\cdot h_{P_d}(u)\cdot h_{P_{d-1}}(u)\right)\cdot$$

$$\cdot V(F_{P_1}(u,v),\dots,F_{P_{d-2}}(u,v)).$$

Therefore, we define $F_x(u)$ for $x\in\mathbb{R}^{\mathcal{U}(G)}$ and $u\in\mathcal{U}(G)$ in a way such that the coordinate of $F_x(u)$ corresponding to $w_{u,v}$ provided $(u,v)\in\mathcal{Q}_G$ is

$$\csc\alpha_{uv}\cdot x^{(v)} - \cot\alpha_{uv}\cdot x^{(u)}.$$

We conclude the existence of a suitable multilinear function $V(x_1,\dots,x_d)$ of $x_1,\dots,x_d\in\mathbb{R}^{\mathcal{U}(G)}$ satisfying (i) and (ii) *via* combining (7.68), (7.69), (7.70) and the induction hypothesis. To show that $V(x_1,\dots,x_d)$ is symmetric in $x_1,\dots,x_d$, we note that $x_i = \bar{h}_{Q_i} - \bar{h}_{Q_i'}$ for some $Q_i, Q_i'\in\mathcal{P}_G$, $i = 1,\dots,d$, according to Claim 7.A.6, and we use the symmetry of the mixed volumes in its variables.

Finally, we consider (iii), where either $n = 2$, or $n\geqslant 3$ and we are given some $P_1,\dots,P_{n-2}\in\mathcal{P}_K$. We deduce from (7.69) and (7.70) that there exists a symmetric matrix $\mathcal{A}:\mathbb{R}^{\mathcal{U}(K)}\to\mathbb{R}^{\mathcal{U}(K)}$ such that $\mathcal{A}$ has non-negative off-diagonal entries, and $\mathcal{A}$ satisfies (7.66) and (7.67) for $x, y\in\mathbb{R}^{\mathcal{U}(K)}$.

Therefore, all we have to check is the irreducibility of $\mathcal{A}$. What we know is that the entry of $\mathcal{A}$ corresponding to a $(u,v)\in\mathcal{Q}_K$ is positive by (7.66). Now let $u,v\in\mathcal{U}(K)$, $u\neq v$ be arbitrary. Choose $p\in\operatorname{relint}F_K(u)$ and $q\in\operatorname{relint}F_K(v)$, and an affine 2-plane A containing p and q such that A does not intersect any $(n-3)$-dimensional face of K. It follows that $A\cap K$ is a polygon whose vertices are the intersections of A with certain $(n-2)$-faces of K, and the edges are intersections with facets of K. Therefore, taking a path on the relative boundary of $A\cap K$ from p to q, the facets of K containing the edges of the path generate an ordered list $u_0,\dots,u_m\in\mathcal{Q}_K$ with $u_0 = u$ and $u_m = v$ where the $(n-2)$-face of K corresponding to $(u_{i-1},u_i)\in\mathcal{Q}_K$ generates the ith vertex of the path. In turn, we conclude the irreducibility of $\mathcal{A}$. ∎

7.A.3 Proof of the Aleksandrov-Fenchel inequality using strongly isomorphic simple polytopes

The key step of the proof of the Aleksandrov-Fenchel inequality (7.62) is to verify the "reverse Cauchy-Schwarz" inequality in Proposition 7.A.8 for bilinear forms built from strongly isomorphic polytopes. We recall for a simple n-polytope $K\subset\mathbb{R}^n$, $\mathcal{P}_K$ is the family of polytopes strongly isomorphic to K, and $\mathcal{U}(K)$ is the family of exterior unit normals to the facets of K. A $P\in\mathcal{P}_K$ is encoded by the vector $\bar{h}_P\in\mathbb{R}^{\mathcal{U}(K)}$ where

the coordinate of $\bar{h}_P$ corresponding to a $u \in \mathcal{U}(K)$ is $h_P(u)$, and we use P and $\bar{h}_P$ interchangeably in the symmetric multilinear form $V(\cdot, \dots, \cdot)$ of Theorem 7.A.7. We also note that according to the Minkowski inequality (7.26), if $K, L \subset \mathbb{R}^2$ are polygons, then

$$V(K,L)^2 \geqslant V(K,K)V(L,L). \tag{7.71}$$

Proposition 7.A.8. *Let $K \subset \mathbb{R}^n$ be a simple n-polytope, $n \geqslant 2$, let $P \in \mathcal{P}_K$ and if $n \geqslant 3$, then let $C_1, \dots, C_{n-2} \in \mathcal{P}_K$ such that $o \in \operatorname{int} C_1$. If $x \in \mathbb{R}^{\mathcal{U}(K)}$, then*

$$V\left(x, P, C_1, \dots, C_{n-2}\right)^2 \geqslant V\left(x, x, C_1, \dots, C_{n-2}\right) V\left(P, P, C_1, \dots, C_{n-2}\right) \tag{7.72}$$

where the inequality is $V(x,P)^2 \geqslant V(x,x)V(P,P)$ in the case $n = 2$.

Proof. Let the family of exterior unit normals to the facets of K be $\mathcal{U}(K) = \{u_1, \dots, u_k\} \subset S^{n-1}$, and we identify $\mathbb{R}^{\mathcal{U}(K)}$ with $\mathbb{R}^k$. We write $e_1, \dots, e_k$ to denote the corresponding orthonormalt basis of $\mathbb{R}^k$ where e_i corresponds to u_i. We also note that in the case $n \geqslant 3$, $o \in \operatorname{int} C_1$ yields that

$$h_{C_1}(u_i) > 0 \ \text{ for } i = 1, \dots, k. \tag{7.73}$$

We prove Proposition 7.A.8 by induction on $n \geqslant 2$. If $n = 2$, then Claim 7.A.6 provides an $a > 0$ and a $Q \in \mathcal{P}_K$ such that $x + a\bar{h}_P = \bar{h}_Q$. We deduce from the Minkowski Inequlaity (7.71) that

$$V(x + a\bar{h}_P, \bar{h}_P)^2 = V(Q,P)^2 \geqslant V(Q,Q)\,V(P,P) = V(x + a\bar{h}_P, x + a\bar{h}_P)\,V(\bar{h}_P, \bar{h}_P),$$

which inequality is equivalent with $V(x, \bar{h}_P)^2 \geqslant V(x,x)V(\bar{h}_P, \bar{h}_P)$ by the linearity and symmetry of the bilinear form $V(x,y)$, $x, y \in \mathbb{R}^k$.

Next let $n \geqslant 3$. Now Theorem 7.A.7 provides a $k \times k$ irreducible symmetric matrix $\mathcal{A}$ with non-negative off-diagonal entries such that if $x, y \in \mathbb{R}^k$, then

$$\langle x, \mathcal{A}y \rangle = V\left(x, y, C_1, \dots, C_{n-2}\right).$$

We alter the definition of $\mathcal{A}$ in order to have better control of the eigenvalues, and also alter the scalar product accordingly: Let $\widetilde{\mathcal{A}} : \mathbb{R}^k \to \mathbb{R}^k$ be the linear transform satisfying that for $x \in \mathbb{R}^k$ and $i = 1, \dots, k$, the ith coordinate of $\widetilde{\mathcal{A}}x$ is (cf. (7.66))

$$\begin{aligned}(\widetilde{\mathcal{A}}x)_i &= \frac{h_{C_1}(u_i) \cdot V\left(F_x(u_i), F_{C_1}(u_i), \dots, F_{C_{n-2}}(u_i)\right)}{V\left(F_{C_1}(u_i), F_{C_1}(u_i), \dots, F_{C_{n-2}}(u_i)\right)} \\ &= (\mathcal{A}x)_i \cdot \frac{h_{C_1}(u_i)}{V\left(F_{C_1}(u_i), F_{C_1}(u_i), \dots, F_{C_{n-2}}(u_i)\right)},\end{aligned} \tag{7.74}$$

and hence

$$\widetilde{\mathcal{A}}\bar{h}_{C_1} = \bar{h}_{C_1}. \tag{7.75}$$

In addition, let $\langle \cdot, \rangle_0$ be the scalar product on $\mathbb{R}^k$ (cf. (7.73)) such that if $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $y = (y_1, \dots, y_k) \in \mathbb{R}^k$, then

$$\langle x, y \rangle_0 = \sum_{i=1}^{k} x_i y_i \cdot \frac{V\left(F_{C_1}(u_i), F_{C_1}(u_i), \dots, F_{C_{n-2}}(u_i)\right)}{h_{C_1}(u_i)},$$

and hence there exist $\theta_1, \dots, \theta_k > 0$ such that $\tilde{e}_i = \theta_i e_i$, $i = 1, \dots, k$ form an orthonormal basis of $\mathbb{R}^k$ with respect to $\langle \cdot, \rangle_0$, and

$$\langle x, \widetilde{\mathcal{A}} y \rangle_0 = \langle x, \mathcal{A} y \rangle = V(x, y, C_1, \dots, C_{n-2}). \tag{7.76}$$

We deduce that the $k \times k$ matrix of $\widetilde{\mathcal{A}}$ with respect to the basis $\tilde{e}_1, \dots, \tilde{e}_k$ is symmetric, irreducible, and has non-negative off-diagonal entries. Therefore (7.75), the positivity of $\bar{h}_{C_1}$ (cf. (7.73)) and the Perron-Frobenius theorem Corollary 7.A.3 yields that 1 is the maximal eigenvalue of $\widetilde{\mathcal{A}}$, and it is a simple eigenvaule.

Next we claim that if $x \in \mathbb{R}^k$, then

$$\langle \widetilde{\mathcal{A}} x, \widetilde{\mathcal{A}} x \rangle_0 \geqslant \langle x, \widetilde{\mathcal{A}}^2 x \rangle_0. \tag{7.77}$$

For $n \geqslant 4$, it follows from first applying (7.74) and the definition of $\langle \cdot, \cdot \rangle_0$, and then the known $(n-1)$-dimensional version of (7.72), and finally (7.65) that

$$\begin{aligned}
\langle \widetilde{\mathcal{A}} x, \widetilde{\mathcal{A}} x \rangle_0 &= \sum_{i=1}^{k} \frac{h_{C_1}(u_i) \cdot V\left(F_x(u_i), F_{C_1}(u_i), \dots, F_{C_{n-2}}(u_i)\right)^2}{V\left(F_{C_1}(u_i), F_{C_1}(u_i), \dots, F_{C_{n-2}}(u_i)\right)} \\
&\geqslant \sum_{i=1}^{k} h_{C_1}(u_i) \cdot V\left(F_x(u_i), F_x(u_i), F_{C_2}(u_i), \dots, F_{C_{n-2}}(u_i)\right) \\
&= nV(C_1, x, x, C_2, \dots, C_n).
\end{aligned}$$

Here $\langle x, \widetilde{\mathcal{A}} x \rangle_0 = nV(C_1, x, x, C_2, \dots, C_n)$ follows from the the symmetry of the multilinear form and (7.67), proving (7.77) if $n \geqslant 4$. The argument is the same if $n = 3$, only no $C_2, \dots, C_n$ are involved.

Finally, (7.77) yields that any eigenvalue λ of $\widetilde{\mathcal{A}}$ satisfies $\lambda^2 \geqslant \lambda$. Since we have already seen that 1 is the maximal eigenvaule and it is a simple eigenvalue, we deduce that the positive eigenspace of $\widetilde{\mathcal{A}}$ is one dimensional. Now (7.76) implies that $\langle \bar{h}_P, \widetilde{\mathcal{A}} \bar{h}_P \rangle_0 = V(P, P, C_1, \dots, C_{n-2}) > 0$, thus Lemma 7.A.4 implies (7.72). ∎

Before starting the argument to prove the Aleksandrov-Fenchel Inequality, let us point out that any finite family of compact convex sets can be simultaneously approximated by strongly isomorphic simple polytopes according to Proposition 1.8.5.

Proposition 7.A.9. *For compact convex sets $C_1, \dots, C_k \subset \mathbb{R}^n$ and $\varepsilon > 0$, there exist strongly isomorphic simple polytopes $P_1, \dots, P_k \subset \mathbb{R}^n$ such that $\delta_H(C_i, P_i) < \varepsilon$ for $i = 1, \dots, k$.*

Proof of the Aleksandrov-Fenchel Inequality (7.62). It follows from Proposition 7.A.9 that the Aleksandrov-Fenchel Inequality (7.62) is equivalent to the following statement: If $n \geqslant 3$ and $C_1, \ldots, C_{n-2}, K, L \subset \mathbb{R}^n$ are strongly isomorphic simple n-polytopes containing the origin in their interior, then

$$V(K, L, C_1, \ldots, C_{n-2})^2 \geqslant V(K, K, C_1, \ldots, C_{n-2}) V(L, L, C_1, \ldots, C_{n-2}). \quad (7.78)$$

In turn, (7.78) follows from Poroposition 7.A.8 by taking $L = P$ and $x = \bar{h}_K$ (cf. Theorem 7.A.7). ∎

7.B Supplement: The Simplex Mean Width conjecture and some related results

In this section, we mostly survey some extremal properties of the mean width of polytopes of bounded complexity. We recall that the first instrinsic volume $V_1(P)$ is proportional to the mean width of the polytope $P = \operatorname{conv}\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$, (cf. (7.6) and (7.7)), and

$$V_1(K) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \max_{i=1,\ldots,k} \langle v_i, u \rangle \, du. \quad (7.79)$$

Remark 7.B.1 (Some extremal properties of the Platonic solids). Let $k = 4, 6, 12$ and $i = 1, 2, 3$. For a 3-polytope $P \supset B^3$ and having at most k faces, the minimum of $V_i(P)$ is attained exactly if P is a circumscribed regular tetrahedron, cube and dodecahedron, respectively. For volume and surface area, this is due to L. Fejes Tóth [261, 264, 265] (see Theorem 7.7.8), for the mean width (or $V_1(P)$), this is due to Florian [290].

For a 3-polytope $P \subset B^3$ and having at most $k = 4, 6, 12$ vertices, the maximum of $V_i(P), i = 1, 2, 3$, is attained exactly if P is an inscribed regular tetrahedron, octahedron and icosahedron, respectively. For volume, this is due to L. Fejes Tóth [262, 264], for the surface area (or $V_2(P)$), this is due to Linhart [522], and for the mean width (or $V_1(P)$), this is due to Linhart [521] (see Theorem 7.B.2).

We present the argument for the following theorem due to Linhart [521] based on the Moment Theorem (cf. (7.55)).

Theorem 7.B.2 (Linhart). *For $k = 4, 6, 12$, among 3-polytopes of at most k vertices contained in B^3, the ones with maximal mean width are the inscribed regular tetrahedron, octahedron and icosahedron, respectively.*

Remark. A stability version is due to Böröczky, G. Fejes Tóth [126].

Proof. Let $P \subset B^3$ be a polytope having maximal mean width assuming that P has at most k vertices, and hence P has k vertices $v_1, \ldots, v_k \in S^{n-1}$, and $o \in P$. Therefore, Theorem 7.B.2 follows from (7.79) and applying (7.55) in the Moment Theorem with $g(t) = \cos t$. ∎

If B is ball in either in $\mathbb{R}^n$, or in the hyperbolic space H^n, or in the sphere S^n, then among simplices contained in B, the regular simplex has the maximal volume. This statement was probably known to Steiner in $\mathbb{R}^n$ (as his symmetrization method yields the statement), it is due to Böröczky [116] on S^n (who used spherical Steiner symmetrization), and it is due to Peyerimhoff [620] in H^n (who used Euclidean Steiner symmetrization in the Beltrami-Cayley-Klein model of H^n discussed in Section 4.A). Actually, Haagerup, Munkholm [370] (see Peyerimhoff [620] for a simpler proof) prove that among all simplices in H^n, $n \geqslant 3$, the regular ideal simplex has maximal volume (an ideal simplex in the Beltrami-Cayley-Klein model of H^n is a Euclidean simplex whose vertices are ideal points, and hence lie on S^{n-1}).

If we are to maximize the mean width and not the volume of a simplex contained in a ball in $\mathbb{R}^n$, then the problem becomes much more difficult.

Conjecture 7.B.3 (Inscribed Simplex Meanwidth Conjecture). *Among simplices contained B^n, $n \geqslant 4$, the inscribed regular simplices, and only them, maximize the mean width.*

Litvak [523] provides a survey about history, and states some equivalent formulations of Conjecture 7.B.3 in terms of random Gaussian processes and coding theory. We note that the conjecture was believed to hold by the information theory community (see for example Balakrishnan [51] in 1963), and even a false proof was published by Landau, Slepian [491] in 1966 applying similar ideas as in the 3-dimensional case Theorem 7.B.2.

If in oder to verify Conjecture 7.B.3, one tries to mimic the argument in the 3-dimensional case of the Simplex Mean Width Conjecture 7.B.3 using the Momen Theorem (7.55), then the missing result is the following one.

Conjecture 7.B.4. *For $p \in S^m$, $m \geqslant 3$ and $r \in (0, \frac{\pi}{2})$, among spherical simplices $Q \subset S^m$ of given m-volume, the regular simplices centered at p maximize $\mathcal{H}^m(Q \cap B_{S^m}(p, r))$.*

The case $m = 2$ of Conjecture 7.B.4 is due to L. Fejes Tóth [264, 265]. Conjecture 7.B.4 would yield the analogue of the Moment Theorem 7.7.6 on S^{n-1} for $n \geqslant 4$ and $k = n + 1$, and *via* this statement it would yield Conjecture 7.B.3 (see Litvak [523]).

Following Litvak [523], let us collect some observations concerning the Simplex Mean Width Conjecture 7.B.3 where Δ_n is the regular simplex inscribed into B^n.

- $V_1(\Delta_n) \sim 4\sqrt{\pi}\sqrt{\ln n}$ as n tends to infinity;
- if $Q \subset B^n$ is a simplex, then $V_1(Q) \leqslant (1 + \frac{c \ln \ln n}{\ln n}) V_1(\Delta_n)$ for an absolute constant $c > 0$;

- if $n = 2m - 1$, and C_m is a regular cross-polytope inscribed into $\mathbb{R}^m \cap B^n$, then $V_1(\Delta_n) \leqslant \sqrt{\frac{n+1}{n}}\, V_1(C_m)$. This shows how hard the Simplex Mean Width Conjecture is, as an inscribed polytope of dimension about $n/2$ has the same number of vertices and essentially the same mean width as the n-dimensional regular simplex.

Chapter 8

Convex bodies and Gaussian curvature

We recall from Section 2.1 that if $K \subset \mathbb{R}^n$ is a convex body, then ∂K is Lipschitz, and $\mathcal{H}^{n-1}$ a.e. point $y \in \partial K$ is regular; namely, there exists a unique exterior unit normal vector $\nu_K(y) \in S^{n-1}$ at y, and we write $\partial' K$ to denote the set of regular boundary points. However, ∂K is even twice differentiable at $\mathcal{H}^{n-1}$ a.e. point $y \in \partial K$ according to Aleksandrov's theorem. This chapter is concerned with notions and arguments using the Gaussian curvature at these points. These notions are naturally easiest to handle for convex bodies with C^2_+ boundaries; therefore, this is the case where we focus our attention. The topics in this chapter include approximation of any convex body by ones with C^∞_+ boundary that leads, for example, to verifying the weak convergence of surface area measure. While Chapter 7 has already introduced mixed volumes using Minkowski's original ideas based on polytopes, we build the theory independently based on Hilbert's and Aleksandrov's approach based on mixed discriminants of the Hessians of the support functions of convex bodies with C^2_+ boundary in this chapter. This other approach connects the Brunn-Minkowski theory to the realm of Minkowski type Monge-Ampère equations in PDE that are discussed in Chapter 9.

One advantage of considering convex bodies with C^2_+ boundaries is that the surface area measure can be nicely expressed in terms of the Hessian of the support function (cf. Section 8.2). This approach indicated already in Minkowsi's and Hilbert's work allows one to have Aleksandrov's elegant integral representation (8.31) of mixed volumes, and in turn to use elliptic operators to prove the Brunn-Minkowski inequality by Hilbert and the more general Aleksandrov-Fenchel inequality by Aleksandrov (cf. Section 8.5), and to verify the L_p-Brunn-Minkowski inequality for o-symmetric convex bodies when $p < 1$ is close to 1 (cf. Section 8.8). In addition, convex bodies with C^2_+ boundaries are the natural framework to verify a fundamental stability version of the Brunn-Minkowski inequality using optimal transport (cf. Section 8.6), and to discuss the equiaffine invariant notions of Affine Surface area (cf. Section 8.9).

8.1 Second order differentiability of the boundary

According to Aleksandrov's Theorem 10.6.2 (ii) on the second order differentiabily of convex functions, for $\mathcal{H}^{n-1}$ a.e. point on the boundary of a convex body in $\mathbb{R}^n$, the boundary is twice differentiable in the following sense:

Theorem 8.1.1 (Aleksandrov [8,11]). *If $K \subset \mathbb{R}^n$ is a convex body, then for $\mathcal{H}^{n-1}$ a.e. point $y \in \partial K$, we have the following property: $y \in \partial' K$ is a regular point, and writing $u = -\nu_K(y)$ and $x_0 = y|u^\perp$, the convex function $\varphi : (\mathrm{int}K)|u^\perp \to \mathbb{R}$ defined by the*

properties $y = x_0 + \varphi(x_0)u$ *and* $x + \varphi(x)u \in \partial K$ *for* $x \in (\mathrm{int}K)|u^\perp$ *satisfies that there exists a positive semi-definite quadratic form* Q_y *on* $u^\perp$ *such that*

$$\varphi(x) - \varphi(x_0) = \frac{1}{2} Q_y(x - x_0) + o(\|x - x_0\|^2)$$

as $x \in u^\perp$ *tends to* x_0. *In this case, we set*

$$\kappa(y) = \det Q_y \quad \textit{(Gaussian curvature).}$$

Moreover, there exists an orthonormal basis $v_1, \ldots, v_{n-1} \in u^\perp$ *(the "principal directions") such that*

$$Q_y\left(\sum_{i=1} t_i v_i\right) = \sum_{i=1} \kappa_i(y) t_i^2$$

for $\kappa_1(y), \ldots, \kappa_{n-1}(y) \geqslant 0$ *(the "principal curvatures" at* y*), and* $\kappa(y) = \prod_{i=1}^{n-1} \kappa_i(y)$.

In addition, for $\mathcal{H}^{n-1}$ *a.e.* $x \in (\mathrm{int}K)|u^\perp$, *we have*

$$\kappa(y) = \frac{\det D^2\varphi(x)}{(1 + \|D\varphi(x)\|^2)^{\frac{n+1}{2}}} \quad \textit{for } y = x + \varphi(x)u. \tag{8.1}$$

Remark. ∂K is C^2_+ if ∂K is C^2 and $\kappa(y) > 0$ for $y \in \partial K$. In this case, $\nu_K : \partial K \to S^{n-1}$ is the Gauss map, and as K is strictly convex (∂K contains no segment), and hence the support function h_K is C^1 on $\mathbb{R}^n \backslash \{o\}$ and $Dh_K(u) = x$ if $x \in \partial' K$ and $u = t\nu_K(x)$ for $t > 0$ (see Lemma 1.6.8 for these, and Theorem 8.1.5 for further properties).

Definition 8.1.2. If $1 \leqslant i \leqslant d$ and A is a $d \times d$ symmetric matrix, then $\sigma_i A$ is the i^{th} symmetric function of the eigenvalues of A. In particular, $\sigma_i A$ is the sum the determinants of the $i \times i$ principal minors of A.

Example 8.1.3 (Gaussian curvature for balls and polytopes).

Polytopes: If K is an n-polytope, then $\kappa(y) = 0$ for any $y \in \partial' K$.

Balls: Let $v_1, \ldots, v_n$ be any orthonormal basis of $\mathbb{R}^n$. For $z = (z_1, \ldots, z_n) = \sum_{i=1}^n z_i v_i$ and $R > 0$, we have

$$h_{RB^n}(z) = R\|z\| = R\sqrt{z_1^2 + \ldots + z_n^2}.$$

If $z \neq o$, then

$$\frac{d}{dz_i} h_{RB^n}(z) = \frac{Rz_i}{\sqrt{z_1^2 + \ldots + z_n^2}};$$

$$\frac{d^2}{dz_i^2} h_{RB^n}(z) = R \cdot \frac{(z_1^2 + \ldots + z_n^2) - z_i^2}{(z_1^2 + \ldots + z_n^2)^{\frac{3}{2}}};$$

$$\frac{d^2}{dz_i z_j} h_{RB^n}(z) = \frac{-Rz_i z_j}{(z_1^2 + \ldots + z_n^2)^{\frac{3}{2}}} \quad \text{for } i \neq j.$$

Since $v_n = (0, \ldots, 0, 1)$, it follows that

$$D^2 h_{RB^n}(v_n) = \operatorname{diag}[R, \ldots, R, 0]. \tag{8.2}$$

In particular, for any $x \in \partial RB^n$ and the exterior unit vector $u = x/R$, we have

$$\sigma_{n-1} D^2 h_{RB^n}(u) = R^{n-1};$$

$$\kappa(x) = R^{-(n-1)};$$

$$\sigma_i D^2 h_{RB^n}(u) = \binom{n-1}{i} R^i.$$

Definition 8.1.4 (Curvature function). If ∂K is C_+^2 for a convex body $K \subset \mathbb{R}^n$, then $f_K : S^{n-1} \to (0, \infty)$ is the curvature function where $f_K(\nu_K(y)) = \frac{1}{\kappa(y)}$ for $y \in \partial K$.

Theorem 8.1.5. *If ∂K is C_+^2 for a convex body $K \subset \mathbb{R}^n$, then h_K is C^2 and $D^2 h_K$ is positive definite on $\mathbb{R}^n \backslash \{o\}$. In addition, for any $y \in \partial K$, we have the following.*

(i) $D^2 h_K(\nu_K(y)) = \operatorname{diag}\left[\frac{1}{\kappa_1(y)}, \ldots, \frac{1}{\kappa_{n-1}(y)}, 0\right]$ *with respect to the orthonormal basis* $v_1, \ldots, v_{n-1}, \nu_K(y)$ *of* $\mathbb{R}^n$ *where* $v_1, \ldots, v_{n-1}$ *are the principal directions at* y*, and* $\kappa_1(y), \ldots, \kappa_{n-1}(y)$ *are the corresponding principal curvatures.*

(ii) $\dfrac{1}{\kappa(y)} = \sigma_{n-1} D^2 h_K(\nu_K(y)) = f_K(\nu_K(y))$.

(iii) If $g : S^{n-1} \to \mathbb{R}$ *is bounded and measurable, then*

$$\int_{S^{n-1}} g(u) d\mathcal{H}^{n-1}(u) = \int_{\partial K} g(\nu_K(x)) \kappa(x) \, d\mathcal{H}^{n-1}(x), \tag{8.3}$$

$$\int_{\partial K} g(\nu_K(x)) \, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} g(u) \sigma_{n-1} D^2 h_K(u) \, d\mathcal{H}^{n-1}(u), \tag{8.4}$$

$$= \int_{S^{n-1}} g(u) f_K(u) \, d\mathcal{H}^{n-1}(u). \tag{8.5}$$

(iv) $|K| = \dfrac{1}{n} \displaystyle\int_{S^{n-1}} h_K f_K \, d\mathcal{H}^{n-1} = \frac{1}{n} \int_{S^{n-1}} h_K \cdot \sigma_{n-1} D^2 h_K \, d\mathcal{H}^{n-1}$.

Proof. For $y \in \partial K$, we use the notation of Theorem 8.1.1.

For (i), first we observe that $\kappa(y) = \det Q_y > 0$ and $\kappa_1(y), \dots, \kappa_{n-1}(y) > 0$ in Theorem 8.1.1 (iii). Assume $y = x_0 = o$, and use the orthonormal basis $v_1, \dots, v_{n-1}, v_n$ of $\mathbb{R}^n$ where $v_n = \nu_K(y)$ and $v_1, \dots, v_{n-1}$ the principal directions at y, thus $Dh_K(tv_n) = y = Dh_K(v_n)$ for $t > 0$ yields

$$\partial_n \partial_i h_K(v_n) = 0 \text{ for } i = 1, \dots, n. \tag{8.6}$$

Using the notation of Theorem 8.1.1 and $\kappa_i = \kappa_i(y)$, we have

$$\varphi(x_1, \dots, x_{n-1}) = \frac{1}{2} \sum_{i=1}^{n-1} \kappa_i x_i^2 + o(\|x\|^2)$$

for $x = (x_1, \dots, x_{n-1}) \in v_n^\perp$ and C^2 function φ, thus

$$D\varphi(x_1, \dots, x_{n-1}) = (\kappa_1 x_1 + o(\|x\|), \dots, \kappa_{n-1} x_{n-1} + o(\|x\|)).$$

For $z = (x, \varphi(x)) \in \partial K$, we deduce that $\tilde{\nu}_z = (D\varphi(x), -1)$ is an exterior normal at z, and

$$\tilde{\nu}_z = (\kappa_1 x_1 + o(\|x\|), \dots, \kappa_{n-1} x_{n-1} + o(\|x\|), -1). \tag{8.7}$$

Combining (8.7) and $\kappa_1, \dots, \kappa_{n-1} > 0$ yields a local diffeomorphism $x \mapsto \tilde{\nu}_z$, $x \in v_n^\perp$, $\tilde{\nu}_z \in v_n + v_n^\perp$, $z = (x, \varphi(x))$ implying

$$Dh_K(\kappa_1 x_1, \dots, \kappa_{n-1} x_{n-1}, -1) = (x_1 + o(\|x\|), \dots, x_{n-1} + o(\|x\|), o(\|x\|^2)) \in \partial K.$$

It follows that for $w = (w_1, \dots, w_{n-1}) \in v_n^\perp$, we have

$$Dh_K(w_1, \dots, w_{n-1}, -1) = \left(\frac{w_1}{\kappa_1} + o(\|w\|), \dots, \frac{w_{n-1}}{\kappa_{n-1}} + o(\|w\|), o(\|w\|) \right),$$

and hence $D^2 h_K(\nu_K(y)) = \operatorname{diag}\left[\frac{1}{\kappa_1(y)}, \dots, \frac{1}{\kappa_{n-1}(y)}, 0 \right]$ (cf. (8.6)), verifying (i) .

Readily (i) implies (ii). For (iii), (8.7) yields that if $X \subset \partial K$ measurable, then

$$\mathcal{H}^{n-1}(\nu_K(X)) = \int_X \kappa(x)\, d\mathcal{H}^{n-1}(x).$$

We deduce (8.3), and in turn (8.3) yields (8.4) and (8.5).

Finally, (iv) follows from (2.3) taking $g = h_K$ in (8.4) and (8.5). ∎

We deduce from Theorem 8.1.5 that if ∂K is C^2_+ for a convex body, $t > 0$ and $u \in S^{n-1}$, then

$$D^2 h_K(tu) = \frac{1}{t} \cdot D^2 h_K(u). \tag{8.8}$$

Still, we usually consider the restriction of h_K to S^{n-1}. Let us discuss the various related notions of differentials of functions on the sphere:

Definition 8.1.6. For a C^2 function $h : S^{n-1} \to \mathbb{R}$, let $\tilde{h}(tu) = t \cdot h(u)$ and $\bar{h}(tu) = h(u)$ for $t > 0$ and $u \in S^{n-1}$, and hence $\tilde{h}, \bar{h} : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ are C^2 on $\mathbb{R}^n \backslash \{o\}$. In particular, $\mathbb{R}u$ is an eigenspace (with eigen value zero) of both $D^2\tilde{h}(u)$ and $D^2\bar{h}(u)$, and we define

$$\nabla h(u) = D\bar{h}(u)|_{u^\perp} = D\tilde{h}(u)|_{u^\perp} \tag{8.9}$$

$$\nabla^2 h(u) = D^2\bar{h}(u)|_{u^\perp} \tag{8.10}$$

$$\widetilde{D}^2 h(u) = \widetilde{D}^2\tilde{h}(u) = D^2\tilde{h}(u)|_{u^\perp}. \tag{8.11}$$

Remarks.

- ∇h is the spherical gradient and $\nabla^2 h$ is the spherical Hessian of h in Definition 8.1.6 with respect to a moving orthogonal frame in the sense of Riemannian geometry (see Schneider [661], Section 2.5) where if $u \in S^{n-1}$, then

$$D\tilde{h}(u) = \nabla h(u) + h(u)\, u, \tag{8.12}$$

  and combining (8.10), (8.11) and $\tilde{h}(x) = \|x\| \cdot \bar{h}(x)$ yield

$$\widetilde{D}^2 h(u) = \widetilde{D}^2\tilde{h}(u) = \nabla^2 h(u) + h(u) I_{n-1} \tag{8.13}$$

  on the tangent space $u^\perp$.

- For a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary and $u \in S^{n-1}$, $\mathrm{II} = (\widetilde{D} h_K(u))^{-1}$ is the *second fundamental form* on $u^\perp$ at the $x \in \partial K$ with $u = \nu_K(x)$, and hence the eigenvalues of II are the principal curvatures at x (cf. Theorem 8.1.5).

We recall that if $K \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int} K$, then $\varrho_K(u) \cdot u \in \partial K$ and $\varrho_K(u) = \|u\|_K^{-1}$ for $u \in S^{n-1}$. In particular (cf. Section 2.2),

$$\partial K \text{ is } C^2 \iff \varrho_K \text{ is } C^2 \text{ on } S^{n-1} \iff \|\cdot\|_K \text{ is } C^2 \text{ on } \mathbb{R}^n \backslash \{o\}.$$

In addition, $\|u\|_{K^*} = h_K$ and $\|u\|_K = h_{K^*}$ for the polar $K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle,\ y \in K\}$ of K (*cf.* Section 1.9).

Theorem 8.1.7. *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) ∂K is C^2_+ if and only if h_K is C^2 and $\widetilde{D}^2 h_K$ positive definite on $\in S^{n-1}$, which in turn holds if and only if $h = h_K|_{S^{n-1}}$ is C^2 and $\nabla^2 h$ is positive definite.

(ii) Assuming $o \in \operatorname{int} K$, ∂K is C^2_+ if and only if ∂K^* is C^2_+.

Remarks. Similar results hold if C^2_+ is replaced by C^∞_+. For the relation between the Gaussian curvatures at $x \in \partial K$ and $x^* \in \partial K^*$ with $\langle x, x^*\rangle = 1$, see Theorem 8.9.4.

Proof. We may assume $o \in \operatorname{int} K$. (i) and (ii) follow from verifying the following cycle of implications:

$$\partial K \text{ is } C^2_+ \Longrightarrow \widetilde{D}^2 h_K \text{ pos. def.} \Longrightarrow \partial K^* \text{ is } C^2_+ \Longrightarrow \widetilde{D}^2 h_{K^*} \text{ pos. def.} \Longrightarrow \partial K \text{ is } C^2_+.$$

Having Theorem 8.1.5, Theorem 8.1.7 (i) and (ii) follow by proving that if h_K is C^2 on $\mathbb{R}^n\backslash\{o\}$ and $D^2h_K(u)|_{u^\perp}$ positive definite $u \in S^{n-1}$, then ∂K^* is C^2_+. As $\|\cdot\|_{K^*} = h_K$, we already know that ∂K^* is C^2. Therefore, all we have to prove is that

$$\kappa_{\partial K^*}(z) > 0 \text{ if } z \in \partial K^*. \tag{8.14}$$

Since this property is invariant under linear transformations, we consider the (unique) $y \in \partial K$ such that $\langle y, z\rangle = 1$, and apply $\Phi \in \mathrm{GL}(n)$ such that $\tilde{y} = \Phi y \in S^{n-1}$, $\nu_{\widetilde{K}} = \tilde{y}$ for $\widetilde{K} = \Phi K$ and each principal curvature at $\tilde{y} \in \partial\widetilde{K}$ is one. It follows that there exists $a \in (0,1)$ such that for the ellipsoid E with one principal semi axis is $\mathrm{conv}\{0, y\}$, and the other principal semi axes are of length a, the "upper half of E" is contained in K; namely, $\{x \in E : \langle x, \tilde{y}\rangle \geqslant 0\} \subset K$.

On the other hand, $\widetilde{K}^* = \Phi^{-t}K^*$ satisfies that $\Phi^{-t}z = \tilde{y}$ and $\nu_{\widetilde{K}^*} = \tilde{y}$. Since any $w \in \widetilde{K}^*$ with $\langle w, \tilde{y}\rangle \geqslant 0$ satisfies $\langle w, x\rangle \leqslant 1$ for any $x \in E$ with $\langle x, \tilde{y}\rangle \geqslant 0$, we deduce that $\{w \in \widetilde{K}^* : \langle w, \tilde{y}\rangle \geqslant 0\} \subset E^*$, which in turn yields (8.14). ∎

Now we characterize C^2 functions on the sphere S^{n-1} that are restrictions of support functions.

Lemma 8.1.8. *For C^2 function $h : S^{n-1} \to \mathbb{R}$, let $\tilde{h}(tu) = t \cdot h(u)$ for $t \geqslant 0$ and $u \in S^{n-1}$, and hence $\tilde{h} : \mathbb{R}^n \to \mathbb{R}$ is C^2 on $\mathbb{R}^n\backslash\{o\}$.*

(i) $h = h_K|_{S^{n-1}}$ for a (strictly) convex body $K \subset \mathbb{R}^n$ if and only if $\widetilde{D}^2h(u)$ (and in turn $D^2\tilde{h}(u)$) is positive semi-definite for any $u \in S^{n-1}$;

(ii) ∂K in (i) is C^2_+ if and only if $\nabla^2 h = \widetilde{D}^2\tilde{h}(u) = D^2\tilde{h}(u)|_{u^\perp}$ is positive definite for any $u \in S^{n-1}$.

Remarks.

- The convex body K in (i) is strictly convex (∂K contains no segment), but may not have a C^1 boundary (cf. Lemma 1.9.6).
- If h is C^∞ and $\nabla^2 h$ is positive definite, then ∂K is C^∞_+ in (ii).
- It follows that for any $f \in C^2(S^{n-1})$ ($f \in C^\infty(S^{n-1})$), there exist $R > 0$ and a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary (with C^∞_+ boundary) such that

$$f = h_K - h_{R\,B^n}, \tag{8.15}$$

  we just choose R to be larger than the absolute value of any eigenvalue of $\widetilde{D}^2 f(u)$ for $u \in S^{n-1}$, and hence $\widetilde{D}^2 f + \widetilde{D}^2 h_{RB^n}$ is positive definite on S^{n-1} (cf. (8.2)).

Proof. We observe that $D^2\tilde{h}(x)$ is positive semi-definite for any $x \in \mathbb{R}^n\backslash\{o\}$ if and only if $\tilde{h}$ is convex on the half-space $\{x \in \mathbb{R}^n : \langle x, u\rangle > 0\}$ for any $u \in S^{n-1}$, and in turn $\tilde{h}$ is convex on $\mathbb{R}^n$ by the homogeneity of $\tilde{h}$. Therefore, again using the homogeneity of $\tilde{h}$, $\widetilde{D}^2h(u)$ is positive semi-definite for any $u \in S^{n-1}$ if and only if $D^2\tilde{h}(x)$ is positive

semi-definite for any $x \in \mathbb{R}^n \backslash \{o\}$, and in turn if and only if $\tilde{h} = h_K$ for a compact convex set (cf. Lemma 1.6.9). In this case, K is a strictly convex body by Lemma 1.6.8.

Finally, (ii) follows from Theorem 8.1.7. ∎

Next, (8.15) implies that the differences of support functions with C^∞_+ boundary are dense among continuous functions on S^{n-1}.

Proposition 8.1.9. *If $g : S^{n-1} \to \mathbb{R}$ is continuous and $\varepsilon > 0$, then there exist convex bodies $K, C \subset \mathbb{R}^n$ with C^∞_+ boundary such that*

$$\|g - (h_K - h_C)\|_\infty < \varepsilon; \text{ namely, } |g(u) - h_K(u) + h_C(u)| < \varepsilon \text{ for } u \in S^{n-1}.$$

Proof. According to the Stone-Weierstrass theorem Corollary 10.5.2, there exists a polynomial on $\mathbb{R}^n$ such that its restriction h to S^{n-1} satisfies $\|g - h\|_\infty < \varepsilon$. In particular, h is C^∞.

Now, (8.15) yields the existence of $R > 0$ and a convex body $K \subset \mathbb{R}^n$ with C^∞_+ boundary such that $h = h_K - h_{R\,B^n}$; therefore, we may choose $C = R\,B^n$. ∎

Firey [288] solved the problem of approximating by convex bodies with C^∞_+ boundary in an elegant manner.

Theorem 8.1.10 (Approximation by smooth convex bodies). *If $C \subset \mathbb{R}^n$ is compact convex and $\varepsilon > 0$, then there exists a convex body $K \supset C$ in $\mathbb{R}^n$ with C^∞_+ boundary such that $\delta_H(K, C) < \varepsilon$, where K is o-symmetric if C is o-symmetric.*

Remark. It is equivalent to saying that h_K is C^∞ on $\mathbb{R}^n \backslash \{o\}$ such that $D^2 h_K(u)|_{u^\perp}$ is positive definite and $0 \leqslant h_K(u) - h_C(u) < \varepsilon$ for $u \in S^{n-1}$.

Proof. Let P be an n-polytope such that $C \subset \operatorname{int} P$ and $\delta_H(C, P) < \frac{\varepsilon}{3}$ (cf. (1.18) where P is o-symmetric if C is o-symmetric). It is sufficient to construct a convex body $M \supset C$ such that h_M is C^∞ and $\delta_H(P, M) \leqslant \frac{\varepsilon}{3}$ because $K = M + \frac{\varepsilon}{3} B^n$ works for Theorem 8.1.10.

Assume that $o \in \operatorname{int} P$ and $v_1, \ldots, v_k$ are the vertices of P. For $i = 1, \ldots, k$, E_i is an ellipsoid centered at $v_i/2$ such that

$$\operatorname{conv}\{o, v_i\} \subset \operatorname{int} E_i \subset \operatorname{conv}\{o, v_i\} + \frac{\varepsilon}{3} B^n$$

(e.g. E_i has semi axes $\frac{\|v_i\|}{2} + \frac{\varepsilon}{3}, \frac{\varepsilon}{3}, \ldots, \frac{\varepsilon}{3}$). As E_i is an ellipsoid, $h_{E_i}(x) = \sqrt{\langle A_i x, x\rangle} + \langle \frac{v_i}{2}, x\rangle$ for a positive definite $n \times n$ matrix A_i, thus $h_{E_i} > 0$ is C^∞ on $\mathbb{R}^n \backslash \{o\}$.

Let M_p be the convex body for $p \in [1, \infty]$ with

$$h_{M_\infty} = \max_{i=1,\ldots,k} h_{E_i} \quad \text{and} \quad h_{M_p} = \left(\frac{1}{k} \sum_{i=1}^k h^p_{E_i} \right)^{\frac{1}{p}} \text{ for } p \in [1, \infty)$$

where h_{M_p} is convex by the Minkowski inequality (10.10) and is C^∞ on $\mathbb{R}^n\backslash\{o\}$ if $p < \infty$. For large $p \in (1,\infty)$, we have $P \subset M_p \subset M_\infty \subset P + \frac{\varepsilon}{3} B^n$, and hence $\delta_H(P, M_p) \leqslant \frac{\varepsilon}{3}$.

Finally, let $K = M_p + \frac{\varepsilon}{3} B^n$ for large p. Since $h_{M_p}|_{S^{n-1}}$ is C^∞, it follows that $h_K|_{S^{n-1}}$ is C^∞_+, and $\delta_H(C, K) \leqslant \varepsilon$. ∎

8.2 Surface area measure and the curvature function

Surface area measure, introduced by Aleksandrov [4,11], has been already discussed in Section 2.5. Here we focus on properties that are related to the second differentiability of the boundary. First, we recall the definition of the surface area measure.

Definition 8.2.1 (Surface area measure). For a compact convex set $K \subset \mathbb{R}^n$, the surface area measure S_K is the following Borel measure on S^{n-1}.

- If K is convex body, then

$$S_K(\omega) = \mathcal{H}^{n-1}\left(\left\{x \in \partial' K : \nu_K(x) \in \omega\right\}\right);$$

namely, if $g : S^{n-1} \to \mathbb{R}$ is bounded measurable, then

$$\int_{S^{n-1}} g\, dS_K = \int_{\partial K} g \circ \nu_K \, d\mathcal{H}^{n-1}. \tag{8.16}$$

- If $\dim K = n-1$ and $K \subset x + u^\perp$ for $u \in S^{n-1}$ and $x \in \mathbb{R}^n$, then $\operatorname{supp} S_K = \{u, -u\}$ and $S_K(\{u\}) = S_K(\{-u\}) = \mathcal{H}^{n-1}(K)$.
- If $\dim K \leqslant n-2$, then $S_K \equiv 0$.

Example 8.2.2.

- If K is an n-dimensional polytope with facets $F_1, \ldots, F_m$ and exterior unit normals $u_1, \ldots, u_m$, then

$$\operatorname{supp} S_K = \{u_1, \ldots, u_m\} \text{ and } S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i),\ i = 1,.$$

- If $K = rB^n$ for $r > 0$, then $S_K = r^{n-1} \cdot \mathcal{H}^{n-1}$.
- If ∂K is C^2_+, then

$$dS_K = f_K d\mathcal{H}^{n-1} \tag{8.17}$$

where $f_K(\nu_K(x)) = 1/\kappa(x)$, for $x \in \partial K$, is the so called curvature function on S^{n-1} according to Theorem 8.1.5. In other words (cf. (8.13)),

$$\begin{aligned} dS_K &= \sigma_{n-1} D^2 h_K \, d\mathcal{H}^{n-1} = \det \widetilde{D}^2 h_K \, d\mathcal{H}^{n-1} \\ &= \det\left(\nabla^2 h + h I_{n-1}\right) d\mathcal{H}^{n-1} \end{aligned} \tag{8.18}$$

for the C^2 function $h = h_K|_{S^{n-1}}$.

Using Aleksandrov's Theorem 10.6.2 and Theorem 8.1.1 on the second order differentiabily of convex functions and hypersurfaces, the formula (8.18) for convex bodies with C^2_+ boundary can be partially generalized to any convex body, as it is proved in Lemma 3.4 and Theorem 3.5 by Hug [425].

Remark 8.2.3 (Hug). For any convex body $K \subset \mathbb{R}^n$, $S_K = S^a_K + S^s_K$ on S^{n-1} where

(i) $dS^a_K = f_K\, d\mathcal{H}^{n-1}$ is the absolutely continuous part where $f_K(u) = \sigma_{n-1} D^2 h_K(u)$ is the "generalized" curvature function for $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$;

(ii) S^s_K is a singular Borel measure (i.e. there exists $X_K \subset S^{n-1}$ such that $\mathcal{H}^{n-1}(X_K) = 0$ and $S^s_K(S^{n-1}\backslash X_K) = 0$) and S^s_K is regular (cf. Theorem 10.1.3);

(iii) for the measurable $Y_K = \{y \in \partial K : \kappa(y) \text{ exists and } \kappa(y) > 0\}$ (possibly $Y_K = \varnothing$), we have (see also Lemma 8.9.5)

$$\begin{aligned} \nu_K(Y_K) \cap X_K &= \varnothing, \\ f_K(u) &= 0 \quad \text{for a.e. } u \in S^{n-1}\backslash \nu_K(Y_K), \\ f_K(\nu_K(y)) \cdot \kappa(y) &= 1 \quad \text{for } y \in Y_K. \end{aligned} \tag{8.19}$$

Example 8.2.4. Let $K \subset \mathbb{R}^n$ be a convex body.

- If K is a polytope, then $S_K = S^s_K$.
- If ∂K is C^2_+, then $dS_K = dS^a_K = f_K\, d\mathcal{H}^{n-1}$.

We deduce from (8.16) and Remark 8.2.3 the following formula for integrals on the boundary depending on the Gaussian curvature.

Proposition 8.2.5. *If $K \subset \mathbb{R}^n$ is a convex body, and $\psi : S^{n-1} \to [0, \infty)$ is measurable and satisfies that $\psi(\nu_K(x)) = 0$ whenever $\kappa(x)$ exists and $\kappa(x) = 0$, then*

$$\int_{\partial K} \psi(\nu_K(x))\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} \psi \cdot f_K\, d\mathcal{H}^{n-1}.$$

Let us list some fundamental properties of the surface area measure that are discussed in this book.

Basic properties of S_K if $K, C \subset \mathbb{R}^n$ are convex bodies.

- $S_K(S^{n-1}) = P(K)$;
- $S_{\lambda K} = \lambda^{n-1} \cdot S_K$;
- S_K Borel measure, "first variation of the volume" (see Aleksandrov's lemma Theorem 7.5.2); in particular (cf. Lemma 2.5.7 and Proposition 2.5.9),

$$|K| = \frac{1}{n} \int_{S^{n-1}} h_K\, dS_K \quad \text{and} \quad \lim_{\varrho \to 0^+} \frac{|K + \varrho\, C|}{\varrho} = \int_{S^{n-1}} h_C\, dS_K; \tag{8.20}$$

- $S_K = S_C$ if and only if K and C are translates (cf. Theorem 2.5.11);

- S_K is weakly continuous in K (see Proposition 2.6.13 and Proposition 8.4.1);
- $\int_{S^{n-1}} u\, dS_K(u) = o$ (cf. Lemma 2.5.7) and supp S_K is not contained in a closed hemisphere (cf. Lemma 2.5.6), which properties characterize the surface area measure of a convex body (see Theorem 9.2.3).

8.3 Mixed volumes and smooth convex bodies

Mixed volumes have already been considered in Chapter 7, and their study there was based on polytopes. In this section, our discussion of the mixed volumes is independent of Chapter 7, and is based on Theorem 8.1.5 for a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary. Let us summarize the corresponding properties of convex bodies with smooth boundary following Theorem 8.1.5:

Remark 8.3.1. For a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary and $u \in S^{n-1}$, $\widetilde{D}^2 h_K(u) = D^2 h_K(u)|_{u^\perp}$ is a $(n-1) \times (n-1)$ positive definite matrix, and u is an eigenvector of $D^2 h_K(u)$ with eigenvalue zero. In particular, if $i = 1, \dots, n-1$, then

$$\sigma_i(\widetilde{D}^2 h_K(u)) = \sigma_i(D^2 h_K(u)). \tag{8.21}$$

If $u = \nu_K(x)$ for $x \in \partial K$, then

$$\det \widetilde{D}^2 h_K(u) = \sigma_{n-1} D^2 h_K(u) = f_K(u) = \kappa(x)^{-1}; \tag{8.22}$$

$$|K| = \frac{1}{n} \int_{S^{n-1}} h_K \det \widetilde{D}^2 h_K \, d\mathcal{H}^{n-1}. \tag{8.23}$$

If $C \subset \mathbb{R}^n$ is a convex body with C^2_+ boundary and $\alpha, \beta > 0$, then

$$\widetilde{D}^2 h_{\alpha K + \beta C} = \alpha\, \widetilde{D}^2 h_K + \beta\, \widetilde{D}^2 h_C, \tag{8.24}$$

and it follows from (8.18) and (8.20) that

$$\lim_{\varrho \to 0^+} \frac{|K + \varrho C|}{\varrho} = \int_{S^{n-1}} h_C \cdot \det \widetilde{D}^2 h_K \, d\mathcal{H}^{n-1}. \tag{8.25}$$

If $h = h_K|_{u^\perp}$, then the spherical Hessian $\nabla^2 h$ satisfies (cf. (8.13))

$$\widetilde{D}^2 h_K(u) = \nabla^2 h(u) + h(u) I_{n-1} \ \text{ on the tangent space } u^\perp. \tag{8.26}$$

Based on (8.23) and (8.24), the idea is that first we understand the determinant of linear combination of matrices in order to verify that the volume of a positive linear combination of smooth convex bodies is polynomial in the coefficient.

Definition 8.3.2 (Mixed Discriminant). If $A_1, \ldots, A_d$ are $d \times d$ real matrices, $A_i = [a_1^{(i)}, \ldots, a_d^{(i)}]$ for $a_j^{(i)} \in \mathbb{R}^d$, $d \geqslant 1$, then

$$\mathcal{D}(A_1, \ldots, A_d) = \frac{1}{d!} \sum_{\pi:\{1,\ldots,d\}\to\{1,\ldots,d\} \text{ bijection}} \det[a_1^{(\pi(1))}, \ldots, a_d^{(\pi(d))}].$$

Remark. $\mathcal{D}(A_1, \ldots, A_d)$ is symmetric and linear in its variables $A_1, \ldots, A_d$; namely, if $\beta_1, \ldots, \beta_k \in \mathbb{R}$ and $B_1, \ldots, B_k$ are $d \times d$ real matrices, then

$$\mathcal{D}\left(A_1, \ldots, A_{d-1}, \sum_{j=1}^{k} \beta_j B_j\right) = \sum_{j=1}^{k} \beta_j \mathcal{D}(A_1, \ldots, A_{d-1}, B_j), \tag{8.27}$$

and if $A_1, \ldots, A_m$ are $d \times d$ real matrices and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, then

$$\det\left(\sum_{j=1}^{m} \lambda_j A_j\right) = \sum_{i_1,\ldots,i_d \in \{1,\ldots,m\}} \mathcal{D}(A_{i_1}, \ldots, A_{i_d}) \lambda_{i_1}, \ldots, \lambda_{i_d}. \tag{8.28}$$

Lemma 8.3.3. *If $A_1, \ldots, A_d$ are $d \times d$ positive semi-definite symmetric matrices, $d \geqslant 2$, then $\mathcal{D}(A_1, \ldots, A_d) \geqslant 0$, and even $\mathcal{D}(A_1, \ldots, A_d) > 0$ if $A_1, \ldots, A_d$ are positive definite.*

Remark. More precisely, given positive semi-definite $A_1, \ldots, A_d$, $\mathcal{D}(A_1, \ldots, A_d) > 0$ if and only if there exist independent $v_1, \ldots, v_d$ such that v_i is an eigenvector of A_i corresponding to a positive eigenvalue. In particular, if $A_1 \neq 0$ is positive semi-definite, and $A_2, \ldots, A_d$ are positive definite, then

$$\mathcal{D}(A_1, \ldots, A_d) > 0. \tag{8.29}$$

Proof. As any semi-definite matrix is a non-negative linear combination of matrixes of the form $v\, v^t$ for $v \in S^{d-1}$, we may assume that $A_i = v_i\, v_i^t$ for $v_i \in S^{d-1}$.

Let $e_1, \ldots, e_d$ form an orthonormal basis, and let $v_i = \Phi e_i$ for $d \times d$ matrix Φ and hence

$$\det\left(\sum_{i=1}^{d} \lambda_i A_i\right) = \det\left[\Phi\left(\sum_{i=1}^{d} \lambda_i e_i e_i^t\right)\Phi^t\right] = (\det \Phi)^2 \lambda_i \cdot \ldots \cdot \lambda_d.$$

Therefore, $\mathcal{D}(A_1, \ldots, A_d) = \det[v_1, \ldots, v_d]^2/d! \geqslant 0$, and $\mathcal{D}(A_1, \ldots, A_d) > 0$ if $v_1, \ldots, v_n$ are independent. ∎

The elegant integral representation (8.31) of mixed volumes within Theorem 8.3.4 is due to Aleksandrov [4]. For the notion of i^{th} *intrinsic volume* $V_i(K)$ of a compact convex $K \subset \mathbb{R}^n$ occurring in (x), see the Steiner formula Theorem 1.10.

Theorem 8.3.4 (Mixed volumes for smooth convex bodies). *If $K_1, \dots, K_m$ and $C_1, \dots, C_l$ are convex bodies in $\mathbb{R}^n$ with C^2_+ boundaries, then*

$$V\left(\sum_{i=1}^{m} \lambda_i K_i\right) = \sum_{i_1,\dots,i_n \in \{1,\dots,m\}} V(K_{i_1}, \dots, K_{i_n})\lambda_{i_1}, \dots, \lambda_{i_n}.$$

(i) $\displaystyle\sum_{j=1}^{l} \varrho_j V(K_1, \dots, K_{n-1}, C_j) = V\left(K_1, \dots, K_{n-1}, \sum_{j=1}^{l} \varrho_j C_j\right)$ *for any* $\varrho_j \geqslant 0$.

(ii) $\displaystyle n!V(K_1, \dots, K_n) = \sum_{i=1}^{n} (-1)^{n-i} \sum_{1 \leqslant j_1 < \dots < j_i \leqslant n} |K_{j_1} + \dots + K_{j_i}|$.

(iii) $V(K_1, \dots, K_n)$ *is continuous and symmetric in* $K_1, \dots, K_n$.

(iv) $V(K_1 + z_1, \dots, K_n + z_n) = V(K_1, \dots, K_n)$ *for* $z_1, \dots, z_n \in \mathbb{R}^n$.

(v) $V(\Phi K_1, \dots, \Phi K_n) = V(K_1, \dots, K_n)$ *for* $\Phi \in \mathrm{SL}(n)$.

(vi) Setting $V(\overbrace{K_1, \dots, K_1}^{n-i}, \overbrace{K_2, \dots, K_2}^{i}) = V(K_1, K_2; i)$, *we have*

$$V(\lambda_1 K_1 + \lambda_2 K_2) = \sum_{i=0}^{n} \binom{n}{i} V(K_1, K_2; i)\lambda_1^{n-i}\lambda_2^i. \tag{8.30}$$

(vii) For an absolutely continuous so called mixed area measure $S_{K_1,\dots,K_{n-1}}$ *on* S^{n-1},

$$\begin{aligned} V(K_1, \dots, K_n) &= \frac{1}{n} \int_{S^{n-1}} h_{K_n} \mathcal{D}\left(\widetilde{D}^2 h_{K_1}, \dots, \widetilde{D}^2 h_{K_{n-1}}\right) d\mathcal{H}^{n-1} \\ &= \frac{1}{n} \int_{S^{n-1}} h_{K_n}\, dS_{K_1,\dots,K_{n-1}}. \end{aligned} \tag{8.31}$$

(viii) $V(K, \dots, K) = |K|$ *and* $S_{K,\dots,K} = S_K$.

(ix) $V(K_1, \dots, K_n) > 0$ *and* $V(K_1, \dots, K_n) \geqslant V(C_1, \dots, C_n)$ *if* $C_i \subset K_i$.

(x) Mean curvatures: if $i = 1, \dots, n-1$ *and* ∂K *is* C^2_+, *then*

$$\begin{aligned} n\binom{n-1}{i} \cdot V(B^n, K; i) &= (n-i)\omega_{n-i} \cdot V_i(K) = \int_{S^{n-1}} \sigma_i D^2 h_K(u)\, du \\ &= \int_{\partial K} \sigma_{n-1-i}(\kappa_1(x), \dots, \kappa_{n-1}(x))\, dx. \end{aligned} \tag{8.32}$$

Proof. We may assume that $o \in \operatorname{int} K_i$, thus $h_{K_i}(u) > 0$ for $u \in S^{n-1}$ and $i = 1, \dots, m$. According to (8.23) and (8.24), if $K = \sum_{i=1}^m \lambda_i K_i$, then

$$\begin{aligned} V(K) &= \frac{1}{n} \int_{S^{n-1}} h_K(u) \widetilde{D}^2 h_K(u)\, du; \\ h_K &= \sum_{i=1}^m \lambda_i h_{K_i}; \\ \widetilde{D}^2 h_K &= \sum_{i=1}^m \lambda_i \widetilde{D}^2 h_{K_i}. \end{aligned}$$

Therefore, (8.27), (8.28) and (8.31) yield (i)-(vi), and we deduce from Lemma 8.3.3 that $V(K_1, \dots, K_n) > 0$ in (ix). It follows from (8.25) that if $M, C \subset \mathbb{R}^n$ are convex bodies with C^2_+ boundaries, then

$$V(M, \dots, M, C) = \frac{1}{n} \int_{S^{n-1}} h_C \cdot (\det \widetilde{D}^2 h_M)\, d\mathcal{H}^{n-1}.$$

Now considering $M = \sum_{i=1}^{n-1} \lambda_i K_i$ for $\lambda_1, \dots, \lambda_{n-1} \geqslant 0$, the linearity of the mixed volume and the mixed discriminant (cf. (i) and (8.27)) implies (vii). In turn, (vii) yields (viii) and (ix).

For (x), $\binom{d}{i} \mathcal{D}(\overbrace{I_d, \dots, I_d}^{d-i}, \overbrace{A, \dots, A}^{i}) = \sigma_i(A)$ for any $d \times d$ positive definite matrix A, and hence (8.21) and (vii) imply

$$\begin{aligned} V(B^n, K; i) &= \frac{1}{n} \int_{S^{n-1}} h_{B^n} \cdot \mathcal{D}(\overbrace{\widetilde{D}^2 h_{B^n}, \dots, \widetilde{D}^2 h_{B^n}}^{n-1-i}, \overbrace{\widetilde{D}^2 h_K, \dots, \widetilde{D}^2 h_K}^{i})\, d\mathcal{H}^{n-1} \\ &= \frac{1}{n} \binom{n-1}{i}^{-1} \int_{S^{n-1}} \sigma_i(\widetilde{D}^2 h_K)\, d\mathcal{H}^{n-1}. \end{aligned}$$

■

Theorem 8.3.5. *The mixed volumes can be defined for any convex compact sets in $\mathbb{R}^n$ in a way such that Theorem 8.3.4 (i)-(vi) hold; moreover, $V(K, \dots, K) = |K|$, $V(K_1, \dots, K_n) \geqslant 0$ and $V(K_1, \dots, K_n) \geqslant V(C_1, \dots, C_n)$ if $C_i \subset K_i$.*

(a) If $K \subset \mathbb{R}^n$ is convex compact and $C \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} C$, then $P(K) = nV(K, B^n; 1)$ and $P_C(K) = nV(K, C; 1)$ are continuous in K and C.

(b) For any convex compact sets $K_1, \dots, K_{n-1} \subset \mathbb{R}^n$, there exists a (unique) finite Borel measure $S_{K_1, \dots, K_{n-1}}$ on S^{n-1} called mixed surface area measure such that for any compact convex $C \subset \mathbb{R}^n$, we have

$$V(K_1, \dots, K_{n-1}, C) = \frac{1}{n} \int_{S^{n-1}} h_C\, dS_{K_1, \dots, K_{n-1}} \tag{8.33}$$

where $S_{K,\dots,K} = S_K$.

(c) $S_{K_1,\dots,K_{n-1}}$ *is symmetric and positive linear in each of its variables, and for any* $\Phi \in \mathrm{O}(n)$ *and* $x_1, \dots, x_{n-1} \in \mathbb{R}^n$, *we have*

$$S_{\Phi K_1 + x_1, \dots, \Phi K_{n-1} + x_{n-1}} = \Phi_* S_{K_1,\dots,K_{n-1}}.$$

Proof. We may assume that all convex compact sets $K_1, \dots, K_m, C_1, \dots, C_m$ are contained in $\operatorname{int} RB^n$ for some $R > 0$. We can approximate them by convex bodies $\widetilde{K}_1, \dots, \widetilde{K}_m, \widetilde{C}_1, \dots, \widetilde{C}_m$ with C^2_+ boundary (cf. Theorem 8.1.10), and each mixed volume is at most $V(RB^n, \dots, RB^n) = R^n \omega_n$ by the monotonicity property (ix) and the continuity of the volume (cf. Lemma 1.7.5). Therefore each formula holds by approximation.

For (a), use Theorem 8.3.4 (vi) and $P_C(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho C| - |K|}{\varrho}$.

To construct the mixed surface area measure $S_{K_1,\dots,K_{n-1}}$ for convex compact sets $K_1, \dots, K_{n-1} \subset \mathbb{R}^n$ in (b), the idea is to consider a positive linear operator $L : C(S^{n-1}) \to \mathbb{R}$ such that for any compact convex $M \subset \mathbb{R}^n$,

$$L\left(h_M|_{S^{n-1}}\right) = V(K_1, \dots, K_{n-1}, M). \tag{8.34}$$

Let $\mathcal{V} \subset C(S^{n-1})$ be the real vector vectorspace generated by the restrictions of the support functions of compact convex sets to S^{n-1}. As the mixed volume is positive linear (see Theorem 8.3.4 (i)), the definition of L as in (8.34) extends to a linear operator on $\mathcal{V}$. It follows from the monotonicity of the mixed volumes (see Theorem 8.3.4 (ix)) that L is a positive operator; namely, $L(f) \geqslant 0$ for non-negative $f \in \mathcal{V}$, and hence also $L(|f|) \leqslant \|f\|_\infty \cdot L(\mathbf{1})$ where $L(\mathbf{1}) = V(K_1, \dots, K_{n-1}, B^n)$ for the constant 1 function $\mathbf{1}$ on S^{n-1}. Since $\mathcal{V}$ is dense in $C(S^{n-1})$ according to Proposition 8.1.9, L can be extended into a positive linear operator on $C(S^{n-1})$. Then $S_{K_1,\dots,K_{n-1}}$ is the unique Borel measure on S^{n-1} representing L provided by the Riesz Representation Theorem 10.1.5.

(c) follows from corresponding properties of the mixed volumes and the uniqueness of the mixed surface area measure. ∎

We repeat the statement and proof of Lemma 7.3.6.

Lemma 8.3.6. *For convex compact* $K_1, \dots, K_n \subset \mathbb{R}^n$, $V(K_1, \dots, K_n) > 0$ *if and only if there exists* $x_i, y_i \in K_i$ *such that* $x_1 - y_1, \dots, x_n - y_n$ *are independent.*

Proof. If $x_1 - y_, \dots, x_n - y_n$ are independent, then for $s_i = [x_i, y_i]$, we have

$$V(K_1, \dots, K_n) \geqslant V(s_1, \dots, s_n) = |\det[x_1 - y_1, \dots, x_n - y_n]|/n! > 0.$$

If there exist no suitable $x_i, y_i \in K_i$, then after possibly translating and reindexing, there exist $1 \leqslant m \leqslant n$ and linear $(m-1)$-plane L such that $K_1, \dots, K_m \subset L$. Thus

there exist compact convex sets $C \subset L$ and $M \subset L^{\perp}$ such that $K_i \subset C$ if $i \leqslant m$ and $K_j \subset C + M$ if $j > m$. It follows that

$$V(K_1,\dots,K_n) \leqslant V(C,m;C+M,n-m) = \sum_{j=0}^{n-m} \binom{n-m}{j} V(C,m+j;M,n-m-j) = 0.$$

■

Next we show that the centroid of the mixed surface area measure $S_{K_1,\dots,K_{n-1}}$ on S^{n-1} is the origin in $\mathbb{R}^n$.

Lemma 8.3.7. *If $K_1,\dots,K_{n-1} \subset \mathbb{R}^n$ are convex and compact, then*

$$\int_{S^{n-1}} u \, dS_{K_1,\dots,K_{n-1}}(u) = o. \tag{8.35}$$

Proof. For any $z \in \mathbb{R}^n$, $h_{\{z\}}(u) = \langle z,u\rangle$, and hence Proposition 7.3.6 and (8.33) yield

$$0 = V(K_1,\dots,K_{n-1},\{z\}) = \frac{1}{n}\int_{S^{n-1}} \langle z,u\rangle \, dS_{K_1,\dots,K_{n-1}}(u),$$

implying (8.35). ■

Finally, we recall the Kubota formula for the mean projections.

Remark 8.3.8 (Mean projections, Intrinsic volumes). For a compact convex set $K \subset \mathbb{R}^n$ and $i = 1,\dots,n-1$, the intrinsic volumes $V(B^n,K;i)$ can also represented as mean projections: If $\mathcal{L}^{i,n}$ is the space of linear i-planes (Grassmanian), and $\varrho_{i,n}$ is the Haar probability measure on $\mathcal{L}^{i,n}$ (invariant under $O(n)$), then the Kubota formula (7.5) says

$$V(B^n,K;i) = \frac{\omega_n}{\omega_i} \cdot \int_{\mathcal{L}^{i,n}} \mathcal{H}^i(K|L)\, d\varrho_{i,n}(L).$$

We note that a practical renormalization of $V(B^n,K;i)$ is the *intrinsic volume*

$$V_i(K) = \frac{\binom{n}{i}}{\omega_{n-i}} V(B^n,K;i),$$

because, for example, $V_i(K) = \mathcal{H}^i(K)$ if $\dim K = i$.

8.4 Minkowski inequality, the Surface area measure and the other Area measures

In this section, we use the theory of Mixed Volumes to establish two fundamental properties of the surface area measure; namely, weak continuity and characterization of uniqueness. In addition, we introduce and discuss the so-called area measures.

As a nice application of the continuity of mixed volumes (cf. Theorem 8.3.4) and the fact that differences of support functions with C^2_+ boundary are dense among continuous functions on S^{n-1} (cf. Proposition 8.1.9), first we show that the surface area measure is weakly continuous on the space of compact convex sets.

Proposition 8.4.1 (Aleksandrov [4,11]). *If compact convex sets $K_m \subset \mathbb{R}^n$ tend to the compac convex set K, then S_{K_m} tends weakly to S_K; namely, $\lim_{m\to\infty} \int_{S^{n-1}} g\, dS_{K_m} = \int_{S^{n-1}} g\, dS_K$ for any continuous $g : S^{n-1} \to \mathbb{R}$.*

Proof. Since K_m tends to K, we have

$$\int_{S^{n-1}} h_C\, dS_{K_m} = nV(K_m, C; 1) \to nV(K, C; 1) = \int_{S^{n-1}} h_C\, dS_K$$

holds for any convex body $C \subset \mathbb{R}^n$ by the continuity of mixed volumes (cf. Theorem 8.3.5), and as any $g \in C(S^{n-1})$ can be approximated uniformly by differences of support functions by Proposition 8.1.9, we also have $\int_{S^{n-1}} g\, dS_{K_m} \to \int_{S^{n-1}} g\, dS_K$. ∎

As we will shortly see, the Minkowski inequality (8.37) follows from the Brunn-Minkowski inequality (cf. Theorem 1.12.3) stating that if $K, C \subset \mathbb{R}^n$ are convex bodies and $\alpha, \beta > 0$, then

$$|\alpha K + \beta C|^{\frac{1}{n}} \geqslant \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}} \tag{8.36}$$

with equality if and only if $K = \gamma C + z$ for $\gamma > 0$ and $z \in \mathbb{R}^n$. Here we provide a proof of the Minkowski inequality that is actually shorter than the one already provided in Theorem 7.4.2. We use the formulation of the mixed volumes in terms of surface area measures (cf. (8.33)), and note that the Minkowski inequality (8.37) is actually equivalent with the Brunn-Minkowski inequality according to Remark 7.4.4.

Theorem 8.4.2 (Minkowski inequality). *If $K, C \subset \mathbb{R}^n$ are convex bodies, then*

$$\frac{1}{n} \int_{S^{n-1}} h_C\, dS_K = V(K, C; 1) \geqslant |K|^{\frac{n-1}{n}} |C|^{\frac{1}{n}} \tag{8.37}$$

with equality if and only if K and C are homothetic.

Proof. We may assume that $|K| = |C|$, and hence (8.37) is equivalent to $V(K, C; 1) \geqslant |K|$ with equality if and only if K and C are translates.

The function $f(t) = |K + t\, C|^{\frac{1}{n}}$ is concave for $t \in [0, 1]$ by the Brunn-Minkowski inequality (8.36) because $M_t = K + tC$ satisfies $M_{\frac{1}{2}t + \frac{1}{2}s} = \frac{1}{2} M_t + \frac{1}{2} M_s$; therefore, the representation of $|K + t\, C|$ in terms of mixed volumes (cf. Theorem 8.3.4) and the Brunn-Minkowski inequality (8.36) lead to

$$V(K, C; 1) \cdot |K|^{\frac{1-n}{n}} = f'(0) \geqslant f(1) - f(0) = |K + C|^{\frac{1}{n}} - |K|^{\frac{1}{n}} \geqslant |K|^{\frac{1}{n}},$$

yielding the Minkowski inequality (8.37).

If equality holds in the Minkowski inequality (8.37), then $|K+C|^{\frac{1}{n}} = 2|K|^{\frac{1}{n}}$, and hence K and C are translates. ∎

Now, we characterize the cases when two surface area measures coincide.

Proposition 8.4.3. *For convex bodies $K, C \subset \mathbb{R}^n$, $S_K = S_C$ if and only if K and C are translates.*

Proof. It follows from $S_K = S_C$ and the Minkowski inequality (8.37) that

$$|K| = \frac{1}{n}\int_{S^{n-1}} h_K \, dS_K = \frac{1}{n}\int_{S^{n-1}} h_K \, dS_C = V(C,K;1) \geqslant |C|^{\frac{n-1}{n}}|K|^{\frac{1}{n}}; \quad (8.38)$$

therefore, $|K| \geqslant |C|$. Reversing the role of K and C in (8.38) yields $|C| \geqslant |K|$, and hence $|C| = |K|$. In turn, equality in (8.38) implies equality in the Minkowski inequality (8.37), and hence we deduce from $|C| = |K|$ that K and C are translates. ∎

According to Theorem 8.3.5, the surface area measure of a compact convex set $K \subset \mathbb{R}^n$ can be expressed as a mixed area measure $S_{K,\dots,K} = S_K$. For $i = 1, \dots, n-1$, in analogue of intrinsic volumes, we consider the *ith area measure* $S_i(K,\cdot)$ - that is a finite Borel measure on S^{n-1} introduced by Aleksandrov [4] and Fenchel, Jessen [268] - such that

$$S_i(K,\cdot) = S_{\underbrace{B^n,\dots,B^n}_{n-1-i},\underbrace{K,\dots,K}_{i}}, \quad (8.39)$$

and hence $S_{n-1}(K,\cdot) = S_K$. We recall that if $K \subset \mathbb{R}^n$ is a convex body with C^2_+ boundary, then for the C^2 function $h = h_K|_{S^{n-1}}$ and $u \in S^{n-1}$ (cf. Definition 8.1.6 and (8.13)) and the differential operators

$$\nabla^2 h(u) + h(u) I_{n-1} = \widetilde{D}^2 h_K(u) = D^2 h_K(u)|_{u^\perp}.$$

Proposition 8.4.4 (Area Measures). *Let $i = 1, \dots, n-1$.*

(i) $S_i(K,\cdot)$ is a weakly continuous function of a compact convex set $K \subset \mathbb{R}^n$.

(ii) If $K \subset \mathbb{R}^n$ is a compact convex set, then

$$\int_{S^{n-1}} u \, dS_i(K,u) = o.$$

(iii) If $K, C \subset \mathbb{R}^n$ are compact convex sets, then

$$n\binom{n-1}{i} V(\overbrace{K,\dots,K}^{i}, \overbrace{B^n,\dots,B^n}^{n-1-i}, C) = \int_{S^{n-1}} h_C(u) \, dS_i(K,u).$$

(iv) $S_i(K,\cdot)$ is translation invariant and $O(n)$ equivariant.

(v) If $K \subset \mathbb{R}^n$ is a convex body with C^2_+ boundary, then

$$dS_i(K,\cdot) = \sigma_i(\widetilde{D}^2 h_K)\, d\mathcal{H}^{n-1} = \sigma_i(D^2 h_K)\, d\mathcal{H}^{n-1} = \sigma_i\left(\nabla^2 h + h I_{n-1}\right) d\mathcal{H}^{n-1}$$

where $\sigma_i(A)$ is the ith symmetric function of the eigenvalues of a $d \times d$ symmetric matrix A and $h = h_K|_{S^{n-1}}$.

Remark. In particular, if K is a convex body with C^2_+ boundary, then the density function of $S_i(K,\cdot)$ is the ith symmetric function of the radii of curvatures corresponding to the outer unit normals.

Proof. We deduce (i) from Proposition 8.4.1 as Theorem 8.3.5 (c) yields

$$S_{K+rB^n} = r^{n-1} S_{B^n} + \sum_{i=1}^{n-1} \binom{n-1}{i} r^{n-1-i} S_i(K,\cdot)$$

for any $r \geqslant 0$.

(ii) follows from Lemma 8.3.7, and (iii) and (iv) from Theorem 8.3.5. Finally, (8.31) and (8.32) in Theorem 8.3.4 yield (v). ∎

To determine a convex body from a prescribed ith area measure is the Christoffel-Minkowski problem (cf. Section 9.5.1), where the case $i = 1$ - when the sum of the radii of curvatures is prescribed - is the Christoffel problem dating back to 1865.

8.5 The Hilbert-Aleksandrov operator and the Aleksandrov-Fenchel inequality

Our main goal is to sketch the proof of the Aleksandrov-Fenchel inequality for the mixed volumes based on the paper Shenfeld,van Handel [678], which puts the classical approach by Hilbert and Aleksandrov on Brunn-Minkowski type inequalities into a new perspective.

Theorem 8.5.1 (Aleksandrov-Fenchel Inequality). *If $n \geqslant 3$ and $C_1, \ldots, C_{n-2}, K, L$ are compact convex sets in $\mathbb{R}^n$, then*

$$V(K,L,C_1,\ldots,C_{n-2})^2 \geqslant V(K,K,C_1,\ldots,C_{n-2})V(L,L,C_1,\ldots,C_{n-2}). \quad (8.40)$$

It was David Hilbert who provided a proof of the Brunn-Minkowski inequality based on the theory of elliptic differential operators at the beginning of 20th century (his argument is sketched in the 1934 classic monograph Bonnesen, Fenchel [107]), and Aleksandrov developed further Hilbert's approach in the 1930's. Let us summarize the main idea: Given convex bodies $C_1, \ldots, C_{n-2} \subset \mathbb{R}^n$ with C^∞_+ boundary and $o \in$

int C_i, one considers the elliptic differential operator (cf. Section 8.1 for the notions in (8.41), and Section 8.5.1 below for the notion of elliptic differential operator)

$$\mathcal{A}f = \mathcal{D}(\widetilde{D}^2 f, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}}), \tag{8.41}$$

for $f \in C^\infty(S^{n-1})$, which, according to (8.31), satifies that

$$\begin{aligned} V(K, L, C_1, \dots, C_{n-2}) &= \frac{1}{n} \int_{S^{n-1}} h_K \cdot \mathcal{D}\left(\widetilde{D}^2 h_L, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1} \\ &= \frac{1}{n} (h_K, \mathcal{A} h_L) \end{aligned} \tag{8.42}$$

for convex bodies $K, L \subset \mathbb{R}^n$ with C^2_+ boundary and $o \in \operatorname{int} K, o \in \operatorname{int} L$. Here for $\varphi, \psi \in L^2(S^{n-1}, \mathcal{H}^{n-1}) \supset C^2(S^{n-1})$, we consider the inner product

$$(\varphi, \psi) = \int_{S^{n-1}} \varphi\psi \, d\mathcal{H}^{n-1}.$$

The representation (8.42) of the mixed volume suggests that in order to understand the Hilber-Aleksandrov operator (8.41), we need some properties of mixed discriminants. We note that if $A_1 \neq 0$ is a symmetric positive semi-definite matrix, and $A_2, \dots, A_d$ are symmetric positive definite matrices, then

$$\mathcal{D}(A_1, A_2, \dots, A_d) > 0 \tag{8.43}$$

according to (8.29). The other key property of mixed discriminants is the following Aleksandrov-Fenchel type inequality, proved in Section 8.A as Theorem 8.A.1 using Gromov's elegant argument in [343].

Theorem 8.5.2 (Aleksandrov's Mixed Discriminant inequality). *If $d \geqslant 2$, A is any symmetric $d \times d$ matrix, and $B, M_1, \dots, M_{d-2}$ are positive semi-definite symmetric $d \times d$ matrices, then*

$$\mathcal{D}(A, B, M_1, \dots, M_{d-2})^2 \geqslant \mathcal{D}(A, A, M_1, \dots, M_{d-2}) \mathcal{D}(B, B, M_1, \dots, M_{d-2}) \tag{8.44}$$

where no $M_1, \dots, M_{d-2}$ occur in the case of $d = 2$.

In the upcoming Section 8.5.1, we collect some fundamental properties of elliptic operators defined on $C^\infty(S^{n-1})$, and we prove the Aleksandrov-Fenchel Inequality (8.40) in Section 8.5.2.

8.5.1 Self-adjoint Elliptic linear operators and Hyperbolic Quadratic Forms

For properties of self-adjoint elliptic linear operators, see Section 10.7 (based on Evans [251], Chapter 6). If $\mathcal{E}$ is a self-adjoint elliptic operator, then its positive eigenspace is the subspace spanned by the eigeinfunctions corresponding to positive eigenvalues.

Lemma 8.5.3 (Hyperbolic Quadratic Forms). *Let $\mathcal{E}$ be a self-adjoint elliptic operator densily defined on $C^2(S^{n-1}) \subset L^2(S^{n-1}, \mu)$ for an absolutely continuous measure μ on S^{n-1} having a positive C^∞ density function, and let $(\cdot,\cdot)_\mu$ be the standard inner product on $L^2(S^{n-1}, \mu)$. Then the following are equivalent:*

(i) $(\varphi, \mathcal{E}\psi)_\mu^2 \geqslant (\varphi, \mathcal{E}\varphi)_\mu (\psi, \mathcal{E}\psi)_\mu$ *if* $\varphi, \psi \in C^\infty(S^{n-1})$ *and* $(\psi, \mathcal{E}\psi)_\mu \geqslant 0$.

(ii) The dimension of the positive eigenspace of $\mathcal{E}$ is at most one.

Proof. Let $\lambda_1 > \lambda_2 \geqslant \dots$ be the eigenvectors of $\mathcal{E}$, and let $\varphi_1, \varphi_2, \dots \in C^\infty(S^{n-1})$ form an orthogonal basis of $L^2(S^{n-1}, \mu)$ where $\mathcal{E}x_i = \lambda_i x_i$. We may assume that $\lambda_1 > 0$ because otherwise $(\varphi, \mathcal{E}\varphi)_\mu \leqslant 0$ for any $\varphi \in C^\infty(S^{n-1})$, and hence Lemma 8.5.3 readily holds.

If (i) holds, then $0 \geqslant \lambda_1\lambda_2(\varphi_1, \varphi_1)_\mu(\varphi_2, \varphi_2)_\mu$ follows by applying (i) to $\varphi = \varphi_1$ and $\psi = \varphi_2$; therefore, $\lambda_j \leqslant \lambda_2 \leqslant 0$ for $j \geqslant 2$, verifying (ii).

Assuming (ii), we observe that $(\xi, \mathcal{E}\xi)_\mu \leqslant 0$ for any $\xi \in C^\infty(S^{n-1})$ with $(\xi, \mathcal{E}\varphi_1)_\mu = 0$ by (10.24). We may assume that $(\psi, \mathcal{E}\psi)_\mu > 0$, and hence $(\psi, \mathcal{E}\varphi_1)_\mu \neq 0$. It follows that $(\xi, \mathcal{E}\varphi_1)_\mu = 0$ for $\alpha = (\varphi, \mathcal{E}\varphi_1)_\mu / (\psi, \mathcal{E}\varphi_1)_\mu$ and $\xi = \varphi - \alpha\psi$; therefore, the condition that $\mathcal{E}$ is symmetric yields

$$\begin{aligned} 0 \geqslant (\xi, \mathcal{E}\xi)_\mu &= (\varphi, \mathcal{E}\varphi)_\mu - 2\alpha(\varphi, \mathcal{E}\psi)_\mu + \alpha^2(\psi, \mathcal{E}\psi)_\mu \\ &= (\varphi, \mathcal{E}\varphi)_\mu - \frac{(\varphi, \mathcal{E}\psi)_\mu^2}{(\psi, \mathcal{E}\psi)_\mu} + (\psi, \mathcal{E}\psi)_\mu \left(\alpha - \frac{(\varphi, \mathcal{E}\psi)_\mu}{(\psi, \mathcal{E}\psi)_\mu}\right)^2 \geqslant (\varphi, \mathcal{E}\varphi)_\mu - \frac{(\varphi, \mathcal{E}\psi)_\mu^2}{(\psi, \mathcal{E}\psi)_\mu}. \end{aligned}$$

In turn, we conclude $(\varphi, \mathcal{E}\psi)_\mu^2 \geqslant (\varphi, \mathcal{E}\varphi)_\mu (\psi, \mathcal{E}\psi)_\mu$. ∎

8.5.2 The Aleksandrov-Fenchel Inequality via Elliptic operators

In this section, we prove the Aleksandrov-Fenchel Inequality (8.40) using an argument due to Shenfeld, van Handel [678] based on Hilbert's and Aleksandrov's ideas.

Since any compact convex set can be approximated by convex bodies with C_+^∞ boundary (cf. Theorem 8.1.10), we may assume that $C_1, \dots, C_{n-2}, K, L \subset \mathbb{R}^n$ are convex bodies with C_+^∞ boundary and containing the origin in their interior, and hence $h_{C_i}(u) > 0$ and $\widetilde{D}^2 h_{C_i}(u)$ is positive definite for $i = 1, \dots, n-2$ and $u \in S^{n-1}$. We consider the differential operator

$$\begin{aligned} \mathcal{A}f &= \mathcal{D}\left(\widetilde{D}^2 f, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}}\right) \\ &= \mathcal{D}\left(\nabla^2 f, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}}\right) + \mathcal{D}\left(f \cdot I_{n-1}, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}}\right) \end{aligned} \tag{8.45}$$

for $f \in C^\infty(S^{n-1})$.

Next we claim that if $f, g, h \in C^\infty(S^{n-1})$, then

$$\begin{aligned}&\int_{S^{n-1}} f \cdot \mathcal{D}\left(\widetilde{D}^2 g, \widetilde{D}^2 h, \widetilde{D}^2 h_{C_2}, \dots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}} g \cdot \mathcal{D}\left(\widetilde{D}^2 f, \widetilde{D}^2 h, \widetilde{D}^2 h_{C_2}, \dots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1}\end{aligned} \tag{8.46}$$

which reads as $\int_{S^2} f \cdot \mathcal{D}\left(\widetilde{D}^2 g, \widetilde{D}^2 h\right) d\mathcal{H}^2 = \int_{S^2} g \cdot \mathcal{D}\left(\widetilde{D}^2 f, \widetilde{D}^2 h\right) d\mathcal{H}^2$ if $n = 3$. According to Lemma 8.1.8, there exist convex bodies $K, K', L, L', M, M' \subset \mathbb{R}^n$ with C_+^∞ boundary such that $f = h_K - h_{K'}$, $g = h_L - h_{L'}$ and $h = h_M - h_{M'}$. Since for any convex bodies $P, Q, R \subset \mathbb{R}^n$ with C_+^∞ boundary, (8.31) and the symmetry of mixed volumes (cf. Theorem 8.3.4 (iii)) yield

$$\begin{aligned}&\int_{S^{n-1}} h_P \cdot \mathcal{D}\left(\widetilde{D}^2 h_Q, \widetilde{D}^2 h_R, \widetilde{D}^2 h_{C_2}, \dots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1} \\ &= nV(P, Q, R, C_2, \dots, C_{n-2}) = nV(Q, P, R, C_2, \dots, C_{n-2}) \\ &= \int_{S^{n-1}} h_Q \cdot \mathcal{D}\left(\widetilde{D}^2 h_P, \widetilde{D}^2 h_R, \widetilde{D}^2 h_{C_2}, \dots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1},\end{aligned}$$

and the mixed discriminant is linear in each of its variables, we conclude (8.46).

$\mathcal{A}$ is a uniformly elliptic differential operator because if $\mathcal{M}$ is a non-trivial $(n-1) \times (n-1)$ positive semi-definite matrix, then $\mathcal{D}(\mathcal{M}, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}}) > 0$ holds by (8.43) (see (10.22) in Section 10.7). In addition, (8.46) yields that $\mathcal{A}$ is also symmetric with respect to $L^2(S^{n-1}, \mathcal{H}^{n-1})$; namely,

$$(f, \mathcal{A}g) = (f, \mathcal{A}g) \quad \text{for any } f, g \in C^\infty(S^{n-1}). \tag{8.47}$$

According to (8.42), the Aleksandrov-Fenchel Inequality (8.40) is equivalent to the inequality $(h_K, \mathcal{A}h_L)^2 \geqslant (h_K, \mathcal{A}h_K)(h_L, \mathcal{A}h_L)$; in particular, it is sufficient to verify that the bilinear form $(\cdot, \mathcal{A}\cdot)$ extended to $L^2(S^{n-1}, \mathcal{H}^{n-1})$ is hyperbolic (cf. Lemma 8.5.3 (i)). In order to achieve this goal, we provide another interpretation of this bilinear form on $C^\infty(S^{n-1})$. We consider the absolutely continuous measure

$$d\mu = \frac{\mathcal{D}(\widetilde{D}^2 h_{C_1}, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}})}{h_{C_1}} d\mathcal{H}^{n-1}$$

with positive C^∞ density function, and the differential operator

$$\widetilde{\mathcal{A}} f = \frac{h_{C_1} \cdot \mathcal{D}(\widetilde{D}^2 f, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}})}{\mathcal{D}(\widetilde{D}^2 h_{C_1}, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}})}$$

for $f \in C^2(S^{n-1})$. Similarly, as for $\mathcal{A}$, we deduce that $\widetilde{\mathcal{A}}$ is uniformly elliptic. For the inner product $(\varphi, \psi)_\mu = \int_{S^{n-1}} \varphi\psi \, d\mu$ of $\varphi, \psi \in L^2(S^{n-1}, \mu)$, we have

$$(f, \mathcal{A}g) = (f, \widetilde{\mathcal{A}}g)_\mu \quad \text{for any } f, g \in C^\infty(S^{n-1}) \tag{8.48}$$

by definition; therefore, (8.47) yields that $\widetilde{\mathcal{A}}$ is symmetric (and hence self-adjoint) with respect to the inner product $(\cdot,\cdot)_\mu$.

The reason to work with $\widetilde{\mathcal{A}}$ instead of $\mathcal{A}$ is that h_{C_1} is obviously an eigenfunction of $\widetilde{\mathcal{A}}$ with eigenvalue 1. As $h_{C_1} > 0$, we are automatically ensured that 1 is the maximal principal eigenvalue of $\widetilde{\mathcal{A}}$; namely, if λ is an eigenvalue of $\widetilde{\mathcal{A}}$ with eigenvector g, then $\lambda \leqslant 1$, and $\lambda = 1$ yields that $g \equiv t h_{C_1}$ for some $t \neq 0$ (cf. (10.23)).

Lemma 8.5.4. *If $f \in C^\infty(S^{n-1})$, then using the notation as above, we have*

$$(\widetilde{\mathcal{A}} f, \widetilde{\mathcal{A}} f)_\mu \geqslant (f, \widetilde{\mathcal{A}} f)_\mu.$$

Proof. It follows from applying first Aleksandrov's Mixed Discriminant inequality (8.44), then (8.46), and finally (8.48) that

$$\begin{aligned}(\widetilde{\mathcal{A}} f, \widetilde{\mathcal{A}} f)_\mu &\geqslant \int_{S^{n-1}} h_{C_1} \cdot \mathcal{D}(\widetilde{D}^2 f, \widetilde{D}^2 f, \widetilde{D}^2 h_{C_2}, \dots, \widetilde{D}^2 h_{C_{n-2}})\, d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}} f \cdot \mathcal{D}(\widetilde{D}^2 h_{C_1}, \widetilde{D}^2 f, \widetilde{D}^2 h_{C_2}, \dots, \widetilde{D}^2 h_{C_{n-2}})\, d\mathcal{H}^{n-1} \\ &= (f, \widetilde{\mathcal{A}} f)_\mu.\end{aligned}$$

■

Proof of the Aleksandrov-Fenchel Inequality (8.40). It follows by approximation (cf. Theorem 8.1.10) that we may assume that $C_1, \dots, C_{n-2}, K, L \subset \mathbb{R}^n$ are convex bodies with C^∞_+ boundary and containing the origin in their interior, and hence $h_{C_i} > 0$ and $\widetilde{D}^2 h_{C_i}$ is positive definite for $i = 1, \dots, n-2$. We use the elliptic differential operator $\widetilde{\mathcal{A}}$ as above that is self-adjoint with respect to $(\cdot,\cdot)_\mu$, and satifies that

$$(h_P, \widetilde{\mathcal{A}} h_Q)_\mu = nV(P, Q, C_1, \dots, C_{n-2}) \tag{8.49}$$

for convex bodies $P, Q \subset \mathbb{R}^n$ with C^∞_+ boundary by (8.42) and (8.48).

Now, $h_{C_1} > 0$ is an eigenfunction of $\widetilde{\mathcal{A}}$ with eigenvalue 1, thus 1 is the simple maximal principal eigenvalue of $\widetilde{\mathcal{A}}$ (cf. Proposition 10.7.1). If λ is any other eigenvalue of $\widetilde{\mathcal{A}}$ with eigenfunction $\varphi \in C^\infty(S^{n-1})$, then we deduce from Lemma 8.5.4 that $\lambda^2 \geqslant \lambda$, which property combined with $\lambda < 1$ yields that $\lambda \leqslant 0$. It follows that the positive eigenspace of $\widetilde{\mathcal{A}}$ is one-dimensional, thus Lemma 8.5.3 implies that $(h_K, \widetilde{\mathcal{A}} h_L)^2_\mu \geqslant (h_K, \widetilde{\mathcal{A}} h_K)_\mu (h_L, \widetilde{\mathcal{A}} h_L)_\mu$, yielding the Aleksandrov-Fenchel Inequality (8.40) by (8.49).

■

8.6 Stability of the (Anistropic) Isoperimetric and the Brunn-Minkowski inequalities for convex bodies

The Anisotropic Perimeter and the Anisotropic Isoperimetric Inequality has been discussed for convex bodies in Section 2.4, for sets with Lipschitz boundary in Section 4.3, and most generally, for sets of finite perimeter in Section 5.2.1. In this section, we provide stability versions of the Brunn-Minkowski Inequality and the Anisotropic Isoperimetric Inequality for convex bodies based on estimates in the case of convex bodies with C^2_+ boundary. In turn, we conclude the stability of the Isoperimetric Inequality for convex bodies. We do not discuss the case of measurable sets in detail in this book, but Figalli, van Hintum, Tiba [273] provide an essentially optimal stability version of the Brunn-Minkowski Inequality for them (cf. Theorem 8.6.7).

While we derived the Anisotropic Isoperimetric inequality from the Brunn-Minkowski inequality in Section 4.3, in this section, we use the reverse path: First we prove the stability of the Anisotropic Isoperimetric Inequality using optimal transport and the divergence theorem for convex bodies with C^2_+ boundary, and then deduce the corresponding stability results for the Brunn-Minkowski inequality for general convex bodies. This section essentially follows the arguments of Figalli, Maggi, Pratelli [277, 278], and for the improvements on the factor θ_n in Theorem 8.6.1, we borrow ideas from Segal [675] and Kolesnikov, E. Milman [480].

Let us recall that if $K, E \subset \mathbb{R}^n$ are convex bodies with $o \in \operatorname{int} K$, then the Anisotropic Perimeter of E in terms of K is

$$P_K(E) = \int_{\partial E} h_K(\nu_E)\, d\mathcal{H}^{n-1} = \int_{\partial E} \|\nu_E\|_{K^*}\, d\mathcal{H}^{n-1} = \lim_{\varrho \to 0^+} \frac{|E + \varrho K| - |E|}{\varrho}. \tag{8.50}$$

The last formula yields that $P_K(E) = nV(E, K; 1)$ is continuous in E and K, and

$$P_{\Phi K}(\Phi E) = |\det \Phi| \cdot P_K(E) \quad \text{for } \Phi \in \mathrm{GL}(n). \tag{8.51}$$

In this setting, the natural distance of convex bodies is in terms of the volume of the symmetric difference. To define the "homothetic distance" $A(K, E)$ of convex bodies $K, E \subset \mathbb{R}^n$, let $\alpha = |K|^{\frac{-1}{n}}$ and $\beta = |E|^{\frac{-1}{n}}$, and let

$$A(K, E) = \min\left\{|\alpha K \Delta (x + \beta E)| : x \in \mathbb{R}^n\right\}.$$

$A(\cdot, \cdot)$ is actually a metric on the homothety classes of convex bodies (see Claim 8.6.14). Figalli, Maggi, Pratelli [277,278] proved the following estimate with the optimal exponent in terms of $A(K, E)$, and the best estimate for the factor θ_n has been obtained by Kolesnikov, E. Milman [480].

Theorem 8.6.1. *For $\theta_n = cn^{-5}(\log n)^{-1}$ where $c \in (0, 1)$ is an absolute constant, if $K, E \subset \mathbb{R}^n$ are convex bodies with $o \in \operatorname{int} K$, then*

$$P_K(E) \geqslant n|K|^{\frac{1}{n}}|E|^{\frac{n-1}{n}} \left[1 + \theta_n \cdot A(K, E)^2\right]. \tag{8.52}$$

Remark. Here the exponent 2 of $A(K,E)^2$ is optimal, and θ_n can't be larger than $36n^{-2}$ (see Remark 8.6.6).

Theorem 8.6.1 yields directly the corresponding stability version of the Isoperimetric Inequality, due to Fusco, Maggi, Pratelli [311]. We verify a version where the factor θ_n is slightly better than in Theorem 8.6.1 if E is close to be a ball:

Theorem 8.6.2. *If $E \subset \mathbb{R}^n$ is a convex body, then*

$$P(E) \geqslant n\omega_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}} \left[1 + \min\{\theta_n A(B^n, E)^2, \delta_n\}\right] \tag{8.53}$$

where $\theta_n = 2^{-12} n^{-4}$ and $\delta_n > 0$ depends on n and can be explicitly calculated.

To have a stability version of the Isoperimetric Inequality in terms of the Hausdorff metric, we note that if $\delta_H(E, B^n) \geqslant \delta$ for $\delta \in (0, \frac{1}{2})$, then either there exists $u \in S^{n-1}$ such that $\langle x, u\rangle \leqslant 1 - \delta$ for $x \in E$, and hence $B^n \backslash K$ contains a circular cone of height δ and of base of radius $\sqrt{1-(1-\delta)^2} > \sqrt{\delta}$; or $(1-\delta)B^n \subset K$ and there exists $z \in K$ with $\|z\| \geqslant 1 + \delta$, and hence $K \backslash B^n$ contains a circular cone of height δ and of base of radius $\delta \cdot \frac{1-\delta}{\sqrt{(1+\delta)^2-(1-\delta)^2}} > \frac{1}{4}\sqrt{\delta}$; therefore,

$$|E \Delta B^n| \geqslant \frac{\omega_{n-1}}{n 4^{n-1}} \cdot \delta^{\frac{n+1}{2}}. \tag{8.54}$$

In turn, we deduce the following estimate from Theorem 8.6.2:

Corollary 8.6.3. *For $\delta \in [0, \frac{1}{2})$, $r > 0$ and centered convex body $E \subset \mathbb{R}^n$ with $|E| = |rB^n|$, if $\delta_H(E, rB^n) \geqslant r\delta$, then*

$$P(E) \geqslant n\omega_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}} \left[1 + \eta_n \cdot \delta^{n+1}\right] \tag{8.55}$$

where $\eta_n > 0$ depends only on n and can be explicitly calculated.

Remark. The optimal error term in (8.55) is of order δ^2 if $n = 2$ by Bonnesen [110] (cf. (8.56)), $\delta^{(n+1)/2}$ if $n \geqslant 4$ by Fuglede [307], and $\delta^2/|\log \delta|$ if $n = 3$ by Fuglede [307] (see Groemer [338] for a comprehensive survey). In the planar case, we note the following beautiful inequality due to Bonnesen [110]: For any planar convex body $K \subset \mathbb{R}^2$, there exist $z \in K$ and $R \geqslant r > 0$ such that $z + r\,B^2 \subset K \subset z + R\,B^2$, and

$$P(K)^2 - 4\pi|K| \geqslant \pi^2 (R - r)^2. \tag{8.56}$$

In order to consider the stability of the Brunn-Minkowski inequality , let

$$\sigma(K, E) = \max\left\{\frac{|E|}{|K|}, \frac{|K|}{|E|}\right\} \geqslant 1$$

for convex bodies $K, E \subset \mathbb{R}^n$.

We note that Figalli, Maggi, Pratelli [278] proved Theorem 8.6.1 with the explicit factor $\theta_n^* = (\frac{(2-2^{\frac{n-1}{n}})^{\frac{3}{2}}}{122n^7})^2$ that has somewhat worst order as n tends to infinity. Theorem 8.6.1 yields the following (essentially optimal) estimate:

Theorem 8.6.4. *For $\theta_n = cn^{-5}(\log n)^{-1}$ where $c \in (0,1)$ is an absolute constant, if $K, E \subset \mathbb{R}^n$ are convex bodies, then*

$$|K+E|^{\frac{1}{n}} \geqslant (|K|^{\frac{1}{n}} + |E|^{\frac{1}{n}}) \left[1 + \frac{\theta_n}{\sigma(K,E)^{\frac{1}{n}}} \cdot A(K,E)^2\right]. \tag{8.57}$$

In turn, Theorem 8.6.4 can be written in the following form:

Corollary 8.6.5. *For $\theta_n = cn^{-4}(\log n)^{-1}$ where $c \in (0,1)$ is an absolute constant, if $K, E \subset \mathbb{R}^n$ are convex bodies with $|E| = |K|$ and $\tau \leqslant \lambda \leqslant 1-\tau$ for $\tau \in (0, \frac{1}{2}]$, then*

$$|(1-\lambda)K + \lambda E| \geqslant |K| \left[1 + \theta_n \tau \cdot A(K,E)^2\right]. \tag{8.58}$$

Remark. According to Theorem 8.6.7 due to Figalli, van Hintum, Tiba [273], $A(K,E)$ in (8.58) can be replaced by $|E|^{-1} \min_{z \in \mathbb{R}^n} |M_z \backslash E|$ for $M_z = \operatorname{conv}\{E \cup (K-z)\}$, while in this case, the factor $\theta_n > 0$ depending on n might be much smaller.

Remark 8.6.6 (Constants in the stability results for convex bodies).

- If the absolute constant $c = \tilde{c}$ works in Theorem 8.6.1, then (8.85) in the proof of Theorem 8.6.4 shows that $c = \frac{1}{4}\tilde{c}$ works in Theorem 8.6.4 and Corollary 8.6.5.
- The exponent 2 of $A(K,E)^2$ and the exponent 1 of τ are optimal, and θ_n can't be larger than $36n^{-2}$ in (8.52) and (8.57) as the following example by Harutyunyan [380] shows: It is sufficient to verify that θ_n can't be larger than $9n^{-2}$ in (8.57). For small $\varepsilon > 0$, let $m = [n/2]$, $K = [-1,1]^n$ and for $t_\varepsilon = 1+\varepsilon$, let

$$E_\varepsilon = \begin{cases} [-t_\varepsilon, t_\varepsilon]^m \times [-t_\varepsilon^{-1}, t_\varepsilon^{-1}]^m & \text{if } n = 2m \\ [-t_\varepsilon, t_\varepsilon]^m \times [-t_\varepsilon^{-1}, t_\varepsilon^{-1}]^m \times [-1,1] & \text{if } n = 2m+1. \end{cases}$$

It follows that $|E_\varepsilon| = |K| = 2^n$, and as E, K are o-symmetric we have

$$A(E_\varepsilon, K) = 2^{-n}|E_\varepsilon \Delta K| = t_\varepsilon^{-m}(t_\varepsilon^m - 1) + (1 - t_\varepsilon^{-m}) \geqslant m\,\varepsilon \geqslant (n/3)\,\varepsilon$$

if $\varepsilon > 0$ is small enough. On the other hand, again for $\varepsilon > 0$ small enough,

$$\frac{|K+E_\varepsilon|^{\frac{1}{n}}}{|K|^{\frac{1}{n}} + |E_\varepsilon|^{\frac{1}{n}}} = \frac{1}{2} \cdot 2^{\frac{n-2m}{n}} \left(1 + \varepsilon + (1+\varepsilon)^{-1}\right)^{\frac{2m}{n}} = \left(\frac{1 + \varepsilon + (1+\varepsilon)^{-1}}{2}\right)^{\frac{2m}{n}}$$
$$\leqslant 1 + \varepsilon^2 \leqslant 1 + 9n^{-2} A(E,K)^2.$$

- As it was observed by Segal [675], Dar's conjecture (1.41) would imply that one can choose $\theta_n \leqslant c/n^2$ for an absolute constant $c > 0$ in (8.52) and (8.57).

Figalli, van Hintum, Tiba [273] have recently proved an optimal stability version of the Brunn-Minkowski Inequality for linear combinations of measurable sets:

Theorem 8.6.7 (Stability of the Brunn-Minkowski Inequality for measurable sets, [273]). *For $n \geqslant 2$ and $t \in (0, 1/2]$, there exist $c_n, d_{n,t} > 0$ depending on n and n, t such that if the measurable sets $X, Y \subset \mathbb{R}^n$ satisfy that*

$$|(1-t)X + tY| \leqslant (1+\delta)|X| \text{ and } |X| = |Y| > 0 \tag{8.59}$$

for $\delta \in (0, d_{n,t})$, then there exists a convex body K such that $X, Y - y \subset K$ for some $y \in \mathbb{R}^n$, and

$$|K \backslash X| = |K \backslash (Y - y)| \leqslant c_n t^{-1/2} \delta^{1/2} |X|. \tag{8.60}$$

Remarks.

- The exponents of t and δ are optimal in (3.3) according to Remark 8.6.6.
- The condition that $\delta < d_{n,t}$ in Theorem 3.1.5 for some $d_{n,t} > 0$ depending on n and t is necessary. For example, take $X = [0,1]^n \cap \{p\}$ and $Y = [0,1]^n$ where $\|p\| > 2n$. Then $|X| = |Y| = 1$ and $|(1-t)X + tY| = (1+t^n)|X|$ but $|\operatorname{conv} X|$ can be arbitrarily large, and hence $d_{n,t} \leqslant t^n$. Actually, $d_{n,t} = t^n$ according to van Hintum, Keevash [392].
- Figalli, van Hintum, Tiba [273] verified an even stronger estimate if we compare the X and Y satisfying (3.2) to their respective convex hulls:

  $$|\operatorname{conv} X \backslash X| \leqslant c_{n,t}\delta\,|X| \quad \text{and} \quad |\operatorname{conv} Y \backslash Y| \leqslant c_{n,t}\delta\,|Y|$$

  where $c_{n,t} > 0$ depends on n, t.

8.6.1 The Anisotropic Isoperimetric Inequality in the case of C^2_+ boundary

In this section, we provide provide a simple proof of the Anisotropic Isoperimetric Inequality for convex bodies with C^2_+ boundaries using the same idea as in Section 5.2.1 in the case of sets of finite perimeter. The fundamental tool is optimal transport, and we heavily use Theorem 8.6.8 by Caffarelli [171] (see also Villani [711], Theorem 4.14). For a convex body $E \subset \mathbb{R}^n$ and $\alpha \in (0,1)$, we say that a function $F : E \to \mathbb{R}^m$ is $C^{k,\alpha}$ for $k \in \mathbb{N}$ if F is C^k in int_E, and all of its partial derivatiaves up to order k extend to a $C^{0,\alpha}$ function on E.

Theorem 8.6.8 (Caffarelli). *Let $E, K \subset \mathbb{R}^n$ be convex bodies with C^2_+ boundaries, and let $f : E \to (0,\infty)$ and $g : K \to (0,\infty)$ be $C^{0,\alpha}$ probability densities. Then there exists a*

$C^{2,\alpha}$ convex function φ on E such that $T = D\varphi : E \to K \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ diffeomorphism satisfying that $DT(x)$ is a positive definite symmetric matrix for $x \in \operatorname{int} E$ and

$$f(x) = g(T(x)) \cdot \det DT(x). \tag{8.61}$$

Theorem 8.6.9 is a special case of Theorem 5.2.4 for sets of finite perimeter:

Theorem 8.6.9. *If $K, E \subset \mathbb{R}^n$ are convex bodies with C^2_+ boundary and $o \in \operatorname{int} K$, then*

$$P_K(E) \geqslant n|K|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}, \tag{8.62}$$

with equality if and only if K and E are homothetic.

Proof. We may assume that $|E| = |K| = 1$, and then we show that the equality case is characterized by E and K being translates. Applying Theorem 8.6.8 to the functions $f = \mathbf{1}_E$ and $g = \mathbf{1}_K$, there exists a C^1 diffeomorphism $T : E \to K$ such that $DT(x)$ is a positive definite symmetric matrix and $\det DT(x) = 1$ for $x \in E$. For $\operatorname{div} T = \operatorname{tr} DT$, the AM-GM inequality for the eigenvalues of DT yields that $\operatorname{div} T(x) \geqslant n(\det DT(x))^{\frac{1}{n}} = n$ for $x \in E$, with equality if and only if each eigenvalue of $DT(x)$ is 1; or equivalently, if $DT(x) = I_n$. We deduce from the Divergence Theorem 2.1.4, and as $T(x) \in K$ yields $\langle T(x), \nu_E(x)\rangle \leqslant h_K(\nu_E(x))$ for $x \in \partial E$ that

$$P_K(E) = \int_{\partial E} h_K(\nu_E(x))\, d\mathcal{H}^{n-1}(x) \geqslant \int_{\partial E} \langle T(x), \nu_E(x)\rangle\, d\mathcal{H}^{n-1}(x) \tag{8.63}$$

$$= \int_E \operatorname{div} T(x) dx \geqslant \int_E n\, dx = n|E| = n|K|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}, \tag{8.64}$$

which proves the anisotropic isoperimetric inequality (8.62). Equality in (8.64) yields that $\operatorname{div} T(x) = n(\det DT(x))^{\frac{1}{n}} = n$ for $x \in \operatorname{int} E$; therefore, $DT \equiv I_n$ by the equality case of AM-GM inequality. We conclude that $T(x) = x + z$ for a $z \in \mathbb{R}^n$, and hence $K = E + z$. ∎

8.6.2 Stability of the Anisotropic Isoperimetric inequality for convex bodies

In order to estimate the error in (8.64) within the proof of the Anisotropic Isoperimetric inequality Theorem 8.6.9, we consider the error in the AM-GM inequality. The argument leading to the stability version Lemma 8.6.10 of the AM-GM inequality is due to Harutyunyan [380]. We note that the optimal factor is $2n$ instead of $4n$ in (8.65) (see Alzer [22]).

Lemma 8.6.10. *If $\lambda_1, \ldots, \lambda_n \geqslant 0$, $\lambda_A = \frac{1}{n}(\lambda_1 + \ldots + \lambda_n)$ is the arithmetic mean, and $\lambda_G = (\lambda_n \ldots \lambda_1)^{1/n}$ is the geometric mean, then*

$$\sum_{i=1}^{n} (\lambda_i - \lambda_G)^2 \leqslant 4n \left(\max_{i=1,\ldots,n} \lambda_i\right) \cdot (\lambda_A - \lambda_G). \tag{8.65}$$

Proof. We may assume that $\lambda_n \geqslant \ldots \geqslant \lambda_1 > 0$. As $\frac{1}{n}\sum_{i=1}^n \sqrt{\lambda_i} \geqslant \sqrt{\lambda_G}$ and $\left(\sqrt{\lambda_i} + \sqrt{\lambda_G}\right)^2 \leqslant 2(\lambda_i + \lambda_G)$ by the AM-GM inequality, we deduce that

$$\begin{aligned}\lambda_A \geqslant & \lambda_G + \frac{1}{n}\sum_{i=1}^n \left(\lambda_i - 2\sqrt{\lambda_i}\sqrt{\lambda_G} + \lambda_G\right) = \lambda_G + \frac{1}{n}\sum_{i=1}^n \left(\sqrt{\lambda_i} - \sqrt{\lambda_G}\right)^2 \\ \geqslant & \lambda_G + \frac{1}{n}\sum_{i=1}^n \frac{(\lambda_i - \lambda_G)^2}{2(\lambda_i + \lambda_G)},\end{aligned}$$

which in turn yields (8.65) as $\lambda_i + \lambda_G \leqslant 2\lambda_n$. ∎

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. For a symmetric $n \times n$ matrix A, its Hilbert-Schmidt norm is $\|A\| = \sqrt{\sum_{i=1}^n \|Ae_i\|^2}$. As the Hilbert-Schmidt norm (see Section 10.8) is independent of the orthonomal basis of $\mathbb{R}^n$, we have

$$\|A\| = \sqrt{\sum_{i=1}^n \mu_i^2} \tag{8.66}$$

where $\mu_1, \ldots, \mu_n$ are the eigenvalues of A. We deduce from (8.65) (applied to the eigenvalues of Φ) and (8.66) (applied to $\Phi - I_n$) that if $\Phi \in \mathrm{SL}(n)$ is a symmetric positive definite matrix with eigenvalues $\lambda_1, \ldots, \lambda_n > 0$ (and hence $\max_{i=1}^n \lambda_i < \operatorname{tr}\Phi$), then

$$\|\Phi - I_n\| = \sqrt{\sum_{i=1}^n (\lambda_i - 1)^2} \leqslant 2\sqrt{\operatorname{tr}\Phi}\sqrt{\operatorname{tr}\Phi - n}. \tag{8.67}$$

We also need the Poincaré type inequality (6.24) in Proposition 6.4.18 due to Kolesnikov, E. Milman [480] that we quote as (8.68). According to Section 6.4, for a centered convex body $E \subset \mathbb{R}^n$ with $|E| = 1$ ($\sigma_E = o$ holds for the centroid), one finds a $\Phi \in \mathrm{SL}(n)$ such that $\widetilde{E} = \Phi E$ is in isotropic position; namely, there exists $\lambda > 0$ such that $\int_{\widetilde{E}} \langle x, u\rangle^2\, dx = \lambda$ for any $u \in S^{n-1}$. For a continuous $f : E \to \mathbb{R}$, we recall that the median m_f with respect to E satisfies $|\{f > m_f\}| \leqslant \frac{1}{2}$ and $|\{f < m_f\}| \leqslant \frac{1}{2}$.

Proposition 8.6.11 (Kolesnikov-Milman). *If $E \subset \mathbb{R}^n$ is a centered convex body with $|E| = 1$ in isotropic position, and $f : E \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial E} |f - m_f|\, d\mathcal{H}^{n-1} \leqslant cn\sqrt{\log n} \cdot \int_E \|Df\|\, d\mathcal{H}^n \tag{8.68}$$

for an absolute constant $c > 0$.

Finally, we need an estimate (8.69) for the volume of the difference of two convex bodies due to Figalli, Maggi, Pratelli [278]. We recall that for a convex body $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $\Pi_K(x)$ is the closest point of K to x, and $\|\Pi_K(x) - \Pi_K(y)\| \leqslant \|x - y\|$ for $x, y \in \mathbb{R}^n$ according to Lemma 1.2.13.

Lemma 8.6.12. *If $E, K \subset \mathbb{R}^n$ are convex bodies and $E \not\subset K$, then*

$$|E\backslash K| \leqslant \int_{(\partial E)\backslash K} |\Pi_K(x) - x| \, d\mathcal{H}^{n-1}(x). \tag{8.69}$$

Proof. We consider $\mathbb{R}^n$ embedded into $\mathbb{R}^{n+1}$ as $f_0^\perp$ for $f_0 \in S^n \subset \mathbb{R}^{n+1}$, and hence $X = ((\partial E)\backslash K) + (0,1)f_0 \subset \mathbb{R}^{n+1}$ is an embedded Lipschitz n-manifold (see Remark 10.4.8), whose points we write as $(x,t) \in X$ for $x \in (\partial E)\backslash K$ and $t \in (0,1)$.

We claim that the function $F : X \to \mathbb{R}^n$ defined by

$$F(x,t) = (1-t)x + x\Pi_K(x) = x + t(\Pi_K(x) - x) \tag{8.70}$$

is Lipschitz and satisfies $(\operatorname{int} E)\backslash K \subset F(X)$. To prove this, we note that Π_K is a contraction, thus if $x, y \in (\partial E)\backslash K$ and $t \in (0,1)$, then

$$\|F(x,t) - F(y,t)\| = \|(1-t)(x-y) + t\,(\Pi_K(x) - \Pi_K(y))\| \leqslant \|x-y\|, \tag{8.71}$$

and hence F is Lipschitz with factor $1 + \max_{x\in E} \|\Pi_K(x) - x\|$. If $z \in (\operatorname{int} E)\backslash K$, then $x = s(z - \Pi_K(z)) \in \partial E$ for some $s > 1$ where $z - \Pi_K(z)$ is an exterior normal at $\Pi_K(z) \in \partial K$, and hence $z = F(x,t)$ for $t = \frac{s-1}{s}$, completing the proof of the claim about the $F(x,t)$ in (8.70).

Next, we estimate $|\det DF(x,t)|$ for $x \in (\partial' E)\backslash K$ and $t \in (0,1)$, thus X is differentiable at (x,t). We consider an orthonomal basis $f_0, f_1, \ldots, f_{n-1}$ of the tangent space $\nu_E(x)^\perp \subset \mathbb{R}^{n+1}$ at (x,t), and write $\partial_i F$ to denote the partial derivative in the direction of $i = 0, \ldots, n-1$. It follows from (8.70) that if $x \in (\partial' E)\backslash K$ and $t \in (0,1)$, then $\partial_0 F(x,t) = \Pi_K(x) - x$, and from (8.71) that $\|\partial_i F(x,t)\| \leqslant 1$ for $i = 1, \ldots, n-1$; therefore, Hadamard's inequality (10.26) yields

$$|\det DF(x,t)| \leqslant \prod_{i=0}^{n-1} \|\partial_i F(x,t)\| \leqslant \|\Pi_K(x) - x\|.$$

As $(\operatorname{int} E)\backslash K \subset F(X)$ by the claim above, we conclude that

$$|E\backslash K| \leqslant \int_{(\partial E)\backslash K} \int_0^1 |\det DF(x,t)| \, dt \, d\mathcal{H}^{n-1}(x) \leqslant \int_{(\partial E)\backslash K} \|\Pi_K(x) - x\| \, d\mathcal{H}^{n-1}(x).$$

■

Proof of Theorem 8.6.1. Since $P_K(E) = nV(E,K;1)$ is continuous in E and K, we may assume that E and K have C^2_+ boundary. According to (8.51), we may also assume that $|E| = |K| = 1$, and E is a centered convex body in isotropic position. In particular, the Anisotropic Isotropic inequality (8.62) is equivalent to $P_K(E) \geqslant n$. We set

$$\delta(E,K) = \frac{P_K(E)}{n} - 1,$$

and hence Theorem 8.6.1 is equivalent with the existence of a $z \in \mathbb{R}^n$ such that

$$|E\Delta(K-z)| \leqslant cn^{\frac{5}{2}} \log n \sqrt{\delta(E,K)} \tag{8.72}$$

for an absolute constant $c > 0$. Since $|E\Delta(K-z)| \leqslant 2$ for any $z \in \mathbb{R}^n$, we may assume that

$$\delta(E,K) \leqslant 1. \tag{8.73}$$

Applying Caffarelli's Theorem 8.6.8 to the functions $f = \mathbf{1}_E$ and $g = \mathbf{1}_K$, there exists a C^1 diffeomorphism $T : E \to K$ such that $DT(x)$ is a positive definite symmetric matrix and $\det DT(x) = 1$ for $x \in E$ where $\operatorname{div} T = \operatorname{tr} DT$. We deduce from (8.64) in the proof of the Anisotropic Isoperimetric inequality that

$$\delta(E,K) \geqslant \frac{1}{n} \int_E (\operatorname{div} T(x) - n)\, dx. \tag{8.74}$$

In addition, the Divergence Theorem, $T(x) \in K$ for $x \in E$ and (8.73) yield that

$$\begin{aligned} \int_E \operatorname{div} T(x)\, dx &= \int_{\partial E} \langle T(x), \nu_E(x) \rangle\, d\mathcal{H}^{n-1}(x) \leqslant \int_{\partial E} h_K(\nu_E(x))\, d\mathcal{H}^{n-1}(x) \\ &= P_K(E) = n(\delta(E,K)+1) \leqslant 2n. \end{aligned} \tag{8.75}$$

As $\det DT(x) = 1$ and $DT(x)$ is positive definite for $x \in E$, we deduce from (8.67), the Hölder inequality, $\operatorname{tr} DT(x) = \operatorname{div} T(x)$, (8.74) and (8.75) that

$$\begin{aligned} \int_E \|DT(x) - I_n\|\, dx &\leqslant 2 \int_E \sqrt{\operatorname{tr} DT(x)} \sqrt{\operatorname{tr} DT(x) - n}\, dx \\ &\leqslant 2 \cdot \sqrt{\int_E \operatorname{div} T(x)\, dx} \cdot \sqrt{\int_E (\operatorname{div} T(x) - n)\, dx} \\ &\leqslant 4n \sqrt{\delta(E,K)}. \end{aligned} \tag{8.76}$$

On the other hand, writing $T = (T_1, \ldots, T_n)$ for the C^1 functions $T_i : E \to \mathbb{R}$ for $i = 1, \ldots, n$, we have $\|DT(x) - I_n\| = \sqrt{\sum_{i=1}^n \|T_i(x) - e_i\|^2}$ for $x \in \operatorname{int} E$. To verify (8.72), we translate K in a way such that $m_{f_i} = 0$ holds for the median of $f_i(x) = T_i(x) - \langle e_i, x \rangle$, $i = 1, \ldots, n$. We deduce by applying the inequality between the quadratic mean and the arithmetic mean, then Proposition 8.6.11 to $f_i(x) = T_i(x) - \langle e_i, x \rangle$, and finally

the triangle inequality that

$$\begin{aligned}\int_E \|DT(x) - I_n\|\,dx &\geqslant \frac{1}{\sqrt{n}}\int_E \sum_{i=1}^n \|DT_i(x) - e_i\|\,dx \\ &\geqslant \frac{1}{c_1 n^{\frac{3}{2}}\sqrt{\log n}}\int_{\partial E}\sum_{i=1}^n \|T_i(x) - \langle e_i, x\rangle\|\,dx \\ &\geqslant \frac{1}{c_1 n^{\frac{3}{2}}\sqrt{\log n}}\int_{\partial E}\|T(x) - x\|\,dx \qquad (8.77)\end{aligned}$$

for some absolute constant $c_1 > 0$. Here $|E\Delta K| = 2|E\backslash K|$ follows from $|E| = |K|$, and hence Lemma 8.6.12, $T(x) \in K$, and combining (8.76) and (8.77) yield

$$\begin{aligned}|E\Delta K| = 2|E\backslash K| &\leqslant 2\int_{(\partial E)\backslash K}\|\Pi_K(x) - x\|\,d\mathcal{H}^{n-1}(x) \\ &\leqslant 2\int_{\partial E}\|T(x) - x\|\,dx \leqslant 8c_1 n^{\frac{5}{2}}\sqrt{\log n}\sqrt{\delta(E,K)}.\end{aligned}$$

We conclude (8.72), and in turn Theorem 8.6.1. ∎

8.6.3 Improvement in the case of the Isoperimetric inequality for convex bodies

If $K = B^n$ in the argument above and E is close to B^n, then we use the following improvement of Proposition 8.6.11 (cf. Proposition 6.4.20):

Proposition 8.6.13. *If $E \subset \mathbb{R}^n$ is a convex body with $|E| = |B^n|$ and $|E\Delta B^n| \leqslant (4n)^{-2n}|B^n|$, and $f : E \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial E}|f - m_f|\,d\mathcal{H}^{n-1} \leqslant 8\sqrt{n}\cdot\int_E \|Df\|\,d\mathcal{H}^n.$$

Proof of Theorem 8.6.2. The argument is based on the proof of Theorem 8.6.1 in the previous section. We set K to be the centered Euclidean ball of volume 1, and assume that $|E| = 1$. It follows from Theorem 8.6.1 that there exists an explicit $\delta_n > 0$ such that if $\delta(E,K) \leqslant \delta_n$, then $|(E-w)\Delta K| \leqslant (4n)^{-2n}$ for a suitable $w \in \mathbb{R}^n$. Therefore, Proposition 8.6.13 yields that

$$\int_{\partial E}|f - m_f|\,d\mathcal{H}^{n-1} \leqslant 8\sqrt{n}\cdot\int_E \|Df\|\,d\mathcal{H}^n. \qquad (8.78)$$

for any Lipschitz function $f : E \to \mathbb{R}$.

Using (8.78) instead of Proposition 8.6.11 in the calculations leading to (8.77) implies

$$\begin{aligned}\int_E \|DT(x) - I_n\|\,dx &\geqslant \frac{1}{\sqrt{n}} \int_E \sum_{i=1}^n \|DT_i(x) - e_i\|\,dx \\ &\geqslant \frac{1}{8n} \int_{\partial E} \sum_{i=1}^n \|T_i(x) - \langle e_i, x\rangle\|\,dx \\ &\geqslant \frac{1}{8n} \int_{\partial E} \|T(x) - x\|\,dx,\end{aligned}$$

and hence Lemma 8.6.12, $T(x) \in K$, and (8.76) yield

$$\begin{aligned}|E\Delta K| = 2|E\backslash K| &\leqslant 2 \int_{(\partial E)\backslash K} \|\Pi_K(x) - x\|\; d\mathcal{H}^{n-1}(x) \\ &\leqslant 2 \int_{\partial E} \|T(x) - x\|\,dx \leqslant 64n^2 \sqrt{\delta(E,K)},\end{aligned}$$

proving Theorem 8.6.2. ∎

8.6.4 Stability of the Brunn-Minkowski Inequality for convex bodies

We prepare the proof of Theorem 8.6.4 by some simple observations. First (2.3) yields that

$$P_M(M) = \int_{\partial M} h_M(\nu_M(x))\, d\mathcal{H}^{n-1}(x) = n\,|M|. \tag{8.79}$$

holds for any convex body $M \subset \mathbb{R}^n$. In addition, since, $h_{K+E} = h_K + h_E$ for convex bodies $K, E \subset \mathbb{R}^n$, we have

$$P_{K+E}(M) = P_K(M) + P_E(M). \tag{8.80}$$

We also note that $A(\cdot,\cdot)$ is a metric on the homothety classes of convex bodies because it is obviously symmetric, and satisfies the triangle inequality.

Claim 8.6.14. *If $K, E, M \subset \mathbb{R}^n$ are convex bodies, then*

$$A(K,E) \leqslant A(K,M) + A(E,M).$$

Proof. We may assume that $|K| = |E| = |M| = 1$, when the claim follows from $K\Delta E \subset (K\Delta M) \cup (E\Delta M)$ as if $x \in K\backslash E$, then either $x \in M$, and hence $x \in E\Delta M$, or $x \notin M$, and hence $x \in K\Delta M$. ∎

Proof of Theorem 8.6.4. According to Theorem 8.6.1, there exists and absolute constant $\tilde{c} > 0$ such that for $\tilde{\theta}_n = \tilde{c}n^{-5}(\log n)^{-2}$, we have

$$\begin{aligned} P_K(K+E) &\geqslant n|K|^{\frac{1}{n}}|E+K|^{\frac{n-1}{n}}\left(1+\tilde{\theta}_n \cdot A(K,K+E)^2\right); \\ P_E(K+E) &\geqslant n|E|^{\frac{1}{n}}|E+K|^{\frac{n-1}{n}}\left(1+\tilde{\theta}_n \cdot A(E,K+E)^2\right). \end{aligned} \tag{8.81}$$

When adding the two estimates in (8.81), we use first that $P_K(K+E)+P_E(K+E) = n|K+E|$ by (8.79) and (8.80), after that the estimate

$$2\sigma(K,E)^{\frac{1}{n}} \geqslant \max\left\{\frac{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}}, \frac{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}}{|E|^{\frac{1}{n}}}\right\}$$

in (8.82) and (8.83), and finally the inequality between the quadratic and arithmetic means and then Claim 8.6.14 in (8.84) to obtain

$$\frac{|K+E|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}} \geqslant \frac{|K|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}}\left(1+\tilde{\theta}_n \cdot A(K,K+E)^2\right) \tag{8.82}$$

$$+\frac{|E|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}}\left(1+\tilde{\theta}_n \cdot A(E,K+E)^2\right) \tag{8.83}$$

$$\geqslant 1+\frac{\tilde{\theta}_n \cdot \left(A(K,K+E)^2+A(E,K+E)^2\right)}{2\sigma(K,E)^{\frac{1}{n}}} \tag{8.84}$$

$$\geqslant 1+\frac{\tilde{\theta}_n \cdot A(K,E)^2}{4\sigma(K,E)^{\frac{1}{n}}}. \tag{8.85}$$

■

8.7 The Logarithmic Brunn-Minkowski and the Logarithmic Minkowski conjectures

This section discusses the Logarithmic (L_0) Brunn-Minkowski Conjecture (8.89) and the Logarithmic (L_0) Minkowski conjecture (8.90) that strengthen the Brunn-Minkowski inequality and the Minkowski inequality in the case of o-symmetric convex bodies. We recall that for $\lambda \in (0,1)$ and convex bodies $K, C \subset \mathbb{R}^n$, the Minkowski linear combination is $(1-\lambda)K + \lambda C = \{x \in \mathbb{R}^n : \langle x,u\rangle \leqslant (1-\lambda)h_K(u) + \lambda h_C(u)\ \forall u \in S^{n-1}\}$, and according to the Brunn-Minkowski inequality(cf. Lemma 1.12.2), we have

$$|(1-\lambda)K + \lambda C| \geqslant |K|^{1-\lambda}|C|^{\lambda}, \tag{8.86}$$

with equality if and only if K and C are translates. In addition, the Minkowski's inequality (8.37) can written in the form that if $|K| = |C|$, then

$$\int_{S^{n-1}} h_C \, dS_K \geqslant \int_{S^{n-1}} h_K \, dS_K, \tag{8.87}$$

with equality if and only if K and C are translates.

For $\lambda \in (0,1)$ and o-symmetric convex bodies $K, C \subset \mathbb{R}^n$, Böröczky, Lutwak, Yang Zhang [140] define their logarithmic or L_0 combination as the Wulff shape

$$(1-\lambda)K +_0 \lambda C = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_K(u)^{1-\lambda} h_C(u)^{\lambda} \text{ for } u \in S^{n-1}\}, \quad (8.88)$$

and hence $(1-\lambda)K +_0 \lambda C \subset (1-\lambda)K + \lambda C$. In some sense, the L_0 combination generalizes the coordinatewise product of unconditional convex bodies (see Section 8.7.6). The Log Brunn-Minkowski Conjecture 8.7.1 is the strengthening of (8.86) for origin symmetric convex bodies, stating that

$$|(1-\lambda)K +_0 \lambda C| \geqslant |K|^{1-\lambda}|C|^{\lambda} \quad (8.89)$$

where assuming that K and C have C^1 boundary, equality holds only if K and C are dilates. The equivalent Logarithmic Minkowski Conjecture 8.7.3 claims that if $|K| = |C|$, then

$$\int_{S^{n-1}} \log h_C \, dV_K \geqslant \int_{S^{n-1}} \log h_K \, dV_K \quad (8.90)$$

where assuming that K and C have C^1 boundary, equality holds only if $K = C$.

The importance of these conjectures is exhibited by some equivalent formulations and possible consequences. As it is dicussed in Section 9.4, the Log Minkowski conjecture (8.90), including the characterization of equality case for o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with C^{∞}_+ boundary and $|K| = |C|$, is equivalent with the uniqueness, conjectured by Lutwak [550], of the solution of the Monge-Ampère equation

$$h \det(\nabla^2 h + h\, I_{n-1}) = f \quad (8.91)$$

for a given positive even C^{∞} function f on S^{n-1}.

According to Saroglou [647], the Log Brunn-Minkowski Conjecture (8.89) yields the "B-conjecture" for any even log-concave measure μ on $\mathbb{R}^n$ and any o-symmetric convex body $K \subset \mathbb{R}^n$ stating that $t \mapsto \mu(e^t K)$ is a log-concave function of $t \in \mathbb{R}$. Concerning the Gaussian probability measure γ_n on $\mathbb{R}^n$ with density function $e^{-\pi\|x\|^2}$, it was an earlier celebrated "B-inequality" by Cordero-Erausquin, Fradelizi, Maurey [216] (see Herscovici, Livshyts, Rotem, Volberg [390] for a stability version), and the case when μ is a rotationally symmetric log-concave even measure has been verified by Cordero-Erausquin, Rotem [221].

Colesanti, Livshyts, Marsiglietti [212] conjectured - which is another possible consequence of the Log Brunn-Minkowski Conjecture according to Livshyts, Marsiglietti, Nayar, Zvavitch [532]) - the following generalization of the Gardner-Zvavitch conjecture. If μ is an even log-concave measure on $\mathbb{R}^n$, then

$$\mu((1-\lambda)K + \lambda C)^{\frac{1}{n}} \geqslant (1-\lambda)\mu(K)^{\frac{1}{n}} + \lambda \mu(C)^{\frac{1}{n}} \quad (8.92)$$

holds for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$. In the case when μ is a Gaussian density (the case of the original Gardner-Zvavitch conjecture stated by Gardner, Zvavitch [318]), the conjecture was finally verified by Eskenazis, Moschidis [249]. In addition, Cordero-Erausquin, Rotem [221] proved (8.92) if μ is a rotationally symmetric log-concave measure. On the other hand, Livshyts [530] verified that (8.92) holds for any even log-concave measure on $\mathbb{R}^n$ and o-symmetric convex bodies K and C if the exponents $\frac{1}{n}$ in (8.92) are changed into $n^{-4-o(1)}$.

For o-symmetric convex bodies, the L_p-Minkowski Conjecture 8.8.1 for $p \in (0,1)$ - that is a consequence of the Log-Brunn-Minkowski Conjecture 8.7.1 by Remark 8.8.5 - has been proved when $p < 1$ is close to 1 by Chen, Huang, Li, Liu [192] based on the local result by Kolesnikov, E. Milman [480] (see Theorem 8.8.6).

8.7.1 Basic properties of the L_0 sum, and some equivalent formulations of the Logarithmic Brunn-Minkowski and Logarithmic Minkowski conjectures

Let us list some basic properities of the L_0 linear combination of convex bodies $K, C \subset \mathbb{R}^n$ containing o in their interior (see Section 8.7.5):

- $(1-\lambda) \cdot K +_0 \lambda \cdot C$ is a convex body containing the origin its interior.
- Since $h_{\Phi M}(u) = h_M(\Phi^t u)$ holds for any compact convex set $M \subset \mathbb{R}^n$, $\Phi \in \mathrm{GL}(n)$ and $u \in \mathbb{R}^n$, we deduce that

$$(1-\lambda) \cdot \Phi(K) +_0 \lambda \cdot \Phi(C) = \Phi\left((1-\lambda) \cdot K +_0 \lambda \cdot C\right). \tag{8.93}$$

- $(1-\lambda) \cdot (\alpha K) +_0 \lambda \cdot (\beta C) = \alpha^{1-\lambda}\beta^{\lambda}\Big((1-\lambda) \cdot K +_0 \lambda \cdot C\Big)$ for $\alpha, \beta > 0$.
- The L_0 linear combination of polytopes is a polytope (cf. Lemma 8.7.8), but the boundary of the L_0 linear combination of convex bodies with C^2_+ boundaries may not be even C^1 (cf. Example 8.7.9).
- If $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for compact convex sets $K_1, \ldots, K_m$ and $C_1, \ldots, C_m$ of dimension at least one that contain the origin and satisfy $C_i = \theta_i K_i$ for $\theta_i > 0$, $i = 1, \ldots, m$ and $\sum_{i=1}^m \dim K_i = n$, then $(1-\lambda)K +_0 \lambda C = \sum_{i=1}^m \theta_i^{\lambda} K_i$ (cf. Lemma 8.7.7).

Böröczky, Lutwak, Yang, Zhang [140] conjectured the following strengthening of the Brunn-Minkowski inequality for origin symetric convex bodies, and Martin Henk proposed the version with centered convex bodies (see also [137]). We recall from Section 1.11 that a convex compact set is centered if its centroid (with respect to its affine hulll) is the origin.

Conjecture 8.7.1 (Log Brunn-Minkowski conjecture). *If $\lambda \in (0,1)$ and $K, C \subset \mathbb{R}^n$ are centered convex bodies, then*

$$|(1-\lambda) \cdot K +_0 \lambda \cdot C| \geqslant |K|^{1-\lambda}|C|^{\lambda} \tag{8.94}$$

with equality if and only if $K = K_1 + \ldots + K_m$ *and* $C = C_1 + \ldots + C_m$ *for centered compact convex sets* $K_1, \ldots, K_m, C_1, \ldots, C_m$ *of dimension at least one where* $\sum_{i=1}^m \dim K_i = n$ *and* K_i *and* C_i *are dilates,* $i = 1, \ldots, m$.

Remark. The conjecture is $\mathrm{GL}(n)$ invariant according to (8.93).

As it was observed by Nayar, Tkocz [601], (8.94) may not hold if $K, C \subset \mathbb{R}^n$ are arbitrary convex bodies with $o \in \operatorname{int} K$ and $o \in \operatorname{int} C$. For example, if $K = [-\frac{1}{2}, \frac{1}{2}]^n$, then $|(1-\lambda)K +_0 \lambda(K - z)| < 1$ if $z \in \operatorname{int} K$ and $z \neq o$.

The following example shows that $(1-\lambda)K +_0 \lambda C$ might be much smaller than $(1-\lambda)K + \lambda C$:

Example 8.7.2. If $a > 1$ is large, $n = 2$, $K = [\frac{-1}{a}, \frac{1}{a}] \times [-a, a]$ and $C = [-a, a] \times [\frac{-1}{a}, \frac{1}{a}]$, then (cf. Lemma 8.7.7)

$$\begin{array}{rcl} \frac{1}{2} \cdot K +_0 \frac{1}{2} \cdot C & = & [-1,1]^2 \\ \frac{1}{2} \cdot K + \frac{1}{2} \cdot C & = & \left[-\frac{1}{2}(a + \frac{1}{a}), \frac{1}{2}(a + \frac{1}{a})\right]^2 . \end{array} \tag{8.95}$$

The importance of the L_0 linear combination is exhibited by the observation (cf. (8.109)) that if $\lambda \in (0,1)$ and K and C are centered convex bodies in $\mathbb{R}^n$, then

$$4^{-n}|K|^{1-\lambda}|C|^{\lambda} \leqslant |(1-\lambda) \cdot K +_0 \lambda \cdot C| \leqslant n^{3n/2}|K|^{1-\lambda}|C|^{\lambda}. \tag{8.96}$$

Naturally, the Log Brunn-Minkowski Conjecture 8.7.1 states that 4^{-n} can be replaced by 1. The upper bound in (8.96) shows that the Log Brunn-Minkowski Conjecture 8.7.1 is significantly stronger than the Brunn-Minkowski inequality as $|(1-\lambda)K + \lambda C|$ might be arbitrary large if $|K| = |C| = 1$ according to Example 8.7.2.

Let us state the Logarithmic Minkowski conjecture due to Böröczky, Lutwak, Yang, Zhang [140] for origin symmetric bodies, and to Böröczky, Kalantzopoulos [137] for centered convex bodies.

Conjecture 8.7.3 (Log-Minkowski conjecture). *If* $K, C \subset \mathbb{R}^n$ *are centered convex bodies, then*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \, dV_K \geqslant \frac{|K|}{n} \log \frac{|C|}{|K|} \tag{8.97}$$

with equality if and only if $K = K_1 + \ldots + K_m$ *and* $C = C_1 + \ldots + C_m$ *for centered compact convex sets* $K_1, \ldots, K_m, C_1, \ldots, C_m$ *of dimension at least one where* $\sum_{i=1}^m \dim K_i = n$ *and* K_i *and* C_i *are dilates,* $i = 1, \ldots, m$.

Remarks.

- The conjecture is $\mathrm{GL}(n)$ invariant by the invariance properties of the cone volume measure (cf. Proposition 2.6.15 and Lemma 2.6.14).

- An equivalent form of Conjecture 8.7.3 is that if $|C| = |K|$ for centered C and K, then
$$\int_{S^{n-1}} \log h_C \, dV_K \geqslant \int_{S^{n-1}} \log h_K \, dV_K \tag{8.98}$$
where the case of equality is like in Conjecture 8.7.3.
- We note that the choice of the right translates of K and C are important in Conjecture 8.7.3 as it is shown by the example - due to Nayar, Tkocz [601] - when $K = [-1, 1]^n$ and $C \neq K$ is a translate of K.

In Lemma 8.7.4, we say that the centered convex bodies $K, C \subset \mathbb{R}^n$ have dilated summands if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for centered compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ that are of dimension at least one, $\sum_{i=1}^m \dim K_i = n$ and K_i and C_i are dilates, $i = 1, \ldots, m$.

Lemma 8.7.4. *For a family $\mathcal{F}$ of centered convex bodies in $\mathbb{R}^n$ that is closed under L_0 combination, the following are equivalent if they hold for all $K, C \in \mathcal{F}$.*

(i) $|(1-\lambda) \cdot K +_0 \lambda \cdot C| \geqslant |K|^{1-\lambda}|C|^{\lambda}$ *with equality if and only if K and C have dilated summands;*

(ii) $f_{K,C}(\lambda) = |(1-\lambda) \cdot K +_0 \lambda \cdot C|$ *is log-concave where* $\log f_{K,C}$ *is linear if and only if K and C have dilated summands;*

(iii) $\int_{S^{n-1}} \log \frac{h_C}{h_K} \, dV_K \geqslant \frac{|K|}{n} \log \frac{|C|}{|K|}$ *with equality if and only if K and C have dilated summands.*

Remark. The typical $\mathcal{F}$ is the family of convex bodies invariant under a subgroup $G \subset \mathrm{GL}(n)$ (cf. (8.93)), like origin symmetric convex bodies (when $G = \{I_n, -I_n\}$).

Proof. (i) $\Longrightarrow$ (ii): Let $M_\lambda = (1-\lambda) \cdot K +_0 \lambda \cdot C$. According to (i), (ii) follows if for $t, s, \alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$, we have $\alpha \cdot M_t +_0 \beta \cdot M_s \subset M_{\alpha t + \beta s}$. By Lemma 2.5.6, it is sufficient to observe $h_{M_{\alpha t+\beta s}}(\nu_{M_{\alpha t+\beta s}}(z)) \geqslant h_N(\nu_{M_{\alpha t+\beta s}}(z))$ for $z \in \partial' M_{\alpha t+\beta s}$ and $N = \alpha \cdot M_t +_0 \beta \cdot M_s$ as Lemma 7.5.1 about regular boundary points of a Wulff shape implies the first equality for $u = \nu_{M_{\alpha t+\beta s}}(z)$ in the series

$$\begin{aligned} h_{M_{\alpha t+\beta s}}(u) &= h_K(u)^{1-\alpha t-\beta s} h_C(u)^{\alpha t+\beta s} = \left(h_K(u)^{1-t} h_C(u)^t\right)^{\alpha} \left(h_K(u)^{1-s} h_C(u)^s\right)^{\beta} \\ &\geqslant h_{M_t}(u)^{\alpha} h_{M_s}(u)^{\beta} \geqslant h_N(u). \end{aligned}$$

We conclude $\alpha \cdot M_t +_0 \beta \cdot M_s \subset M_{\alpha t+\beta s}$, and in turn, the concavity of $\log f_{K,C}$. In particular, $\log f_{K,C}$ is linear if and only if $f_{K,C}(\frac{1}{2}) = f_{K,C}(0)^{\frac{1}{2}} f_{K,C}(1)^{\frac{1}{2}}$, and hence (ii) has the same equality conditions as (i).

(ii) $\Longrightarrow$ (iii): We may assume that $|K| = |C| = 1$ and $R^{-1}B^n \subset K, C \subset RB^n$ for some $R > 3$, and hence $f_{K,C}(\lambda) \geqslant 1$ for $\lambda \in (0,1)$ and $R^{-2} \leqslant h_C/h_K \leqslant R^2$. As

$e^t = 1 + t + O(t^2)$ provided $|t| \leqslant \frac{1}{2}$ where the implied constant in $O(\cdot)$ is an absolute constant, we deduce that if $0 < \lambda < (4 \log R)^{-1}$ and $u \in S^{n-1}$, then

$$\begin{aligned} h_K(u)^{1-\lambda} h_C(u)^{\lambda} &= h_K(u) \exp\left(\lambda \cdot \log \frac{h_C}{h_K}\right) \\ &= h_K(u) + \lambda \cdot h_K(u) \log \frac{h_C}{h_K} + O(\lambda^2 R (\log R)^2). \end{aligned}$$

It follows from (ii) and Aleksandrov's Lemma 7.5.2 for the Wulff shape that

$$0 \leqslant f'_{K,C}(0) = \int_{S^{n-1}} h_K \cdot \log \frac{h_C}{h_K} \, dS_K = n \int_{S^{n-1}} \log \frac{h_C}{h_K} \, dV_K, \tag{8.99}$$

proving (iii). Equality in (8.99) yields $f'_{K,C}(0) = 0$, and hence $\log f_{K,C}(0)$ is linear.

(iii) $\Longrightarrow$ (i): Using again the notation $M_\lambda = (1-\lambda)K +_0 \lambda C$, we deduce from Lemma 2.5.6 and Lemma 7.5.1 that $h_{M_\lambda}(u) = h_K(u)^{1-\lambda} h_C(u)^{\lambda}$ for S_{M_λ} a.e. $u \in S^{n-1}$; therefore, (iii) implies

$$\begin{aligned} 0 &= \int_{S^{n-1}} \log \frac{h_K^{1-\lambda} h_C^{\lambda}}{h_{M_\lambda}} \, dV_{M_\lambda} = (1-\lambda) \int_{S^{n-1}} \log \frac{h_K}{h_{M_\lambda}} \, dV_{M_\lambda} + \lambda \int_{S^{n-1}} \log \frac{h_C}{h_{M_\lambda}} \, dV_{M_\lambda} \\ &\geqslant \frac{(1-\lambda)|M_\lambda|}{n} \log \frac{|K|}{|M_\lambda|} + \frac{\lambda |M_\lambda|}{n} \log \frac{|C|}{|M_\lambda|} = \frac{|M_\lambda|}{n} \log \frac{|K|^{1-\lambda}|C|^{\lambda}}{|M_\lambda|}. \end{aligned}$$

We conclude that $|M_\lambda| \geqslant |K|^{1-\lambda}|C|^{\lambda}$, and equality in (i) yields equality in (iii). ∎

Remark 8.7.5 (Equivalent formulations of the Log Minkowski conjecture for o-symmetric convex bodies without the case of equality).

Log Minkowski strengthening Minkowski's second inequality: Kolesnikov, E. Milman [480] and Putterman [629] prove that the Log Minkowski conjecture (8.98) for o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ is equivalent to the conjectured inequality

$$\frac{V(K, C; 1)^2}{|K|} \geqslant \frac{n-1}{n} V(K, C; 2) + \frac{1}{n} \int_{S^{n-1}} \frac{h_C^2}{h_K^2} \, dV_K, \tag{8.100}$$

again for all o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ where the implication that the Log Minkowski conjecture (8.98) yields (8.100) was proved earlier by Colesanti, Livshyts, Marsiglietti [212]. Here (8.100) is a strengthened form of Minkowski's second inequality (7.29) because

$$\begin{aligned} n|K| \int_{S^{n-1}} \frac{h_C^2}{h_K} \, dS_K &= \int_{S^{n-1}} h_K \, dS_K \cdot \int_{S^{n-1}} \frac{h_C^2}{h_K} \, dS_K \\ &\geqslant \left(\int_{S^{n-1}} h_C \, dS_K \right)^2 = n^2 V(K, C; 1)^2 \end{aligned}$$

by the Hölder's inequality (10.4).

Actually, Kolesnikov, E. Milman [480] calculated the second derivative of $\lambda \mapsto |(1-\lambda)K +_0 \lambda C|$ at $\lambda = 0$ for o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with C^2_+ boundary to show the equivalence of (8.100) with a local version of the Log Minkowski conjecture, and Putterman [629] extended the result to the global version.

Spectral gap formulation of Log Minkowski: A related equivalent formulation using the Hilbert-Brunn-Minkowski operator is due to Kolesnikov, E. Milman [480], and is discussed in Section 8.8.1 in detail. The elliptic and symmetric Hilbert-Brunn-Minkowski operator $\mathcal{L}_K$ for a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary is defined for a $\varphi \in C^2(S^{n-1})$ as

$$\mathcal{L}_K \varphi = \frac{\mathcal{D}(\widetilde{D}^2(\varphi\, h_K), \widetilde{D}^2 h_K, \dots, \widetilde{D}^2 h_K)}{\mathcal{D}(\widetilde{D}^2 h_K, \dots . \widetilde{D}^2 h_K)} - \varphi. \tag{8.101}$$

The normalization is chosen by Kolesnikov, E. Milman [480] in a way such that eigenvalues of $-\mathcal{L}_K$ are non-negative, and the positive eigenvalues are at least 1, which property - in line with Hilbert's original approach - is actually equivalent with the Brunn-Minkowski inequality for convex bodies. If K is o-symmetric, then let $\lambda_{1,e}(-\mathcal{L}_K)$ be the minimal positive eigenvalue when we restrict $-\mathcal{L}_K$ to even functions. We note that the operator $\mathcal{L}_K$ and the operator $\mathcal{A}\varphi = \mathcal{D}\left(\widetilde{D}^2\varphi, \widetilde{D}^2 h_K, \dots, \widetilde{D}^2 h_K\right)$ used in Section 8.5.2 in the proof of the Aleksandrov-Fenchel Inequality are essentially equivalent.

According to Kolesnikov, E. Milman [480], the Log Brunn-Minkowski Conjecture 8.7.1 for o-symmetric convex bodies $C, K \subset \mathbb{R}^n$ is equivalent to

$$\lambda_{1,e}(-\mathcal{L}_K) \geqslant \frac{n}{n-1} \tag{8.102}$$

for any o-symmetric convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary; more precisely, (8.102) is directly equivalent to (8.100). What known is that an estimate by Kolesnikov, E. Milman [480] combined with the bound of Theorem 4.7.14 due to Klartag [468] on the Cheeger constant in the KLS constant yields that

$$\lambda_{1,e}(-\mathcal{L}_K) \geqslant 1 + \frac{c}{n^2 \log n}$$

for an absolute constant $c > 0$.

Log Minkowski and Monge-Ampère equations: As it is dicussed in Section 9.4, the Log Minkowski Conjecture 8.7.3 for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with C^∞_+ boundary is equivalent to saying that for any positive even C^∞ function f on S^{n-1}, the Monge-Ampère equation

$$h \det(\nabla^2 h + h\, I_{n-1}) = f \tag{8.103}$$

on the sphere S^{n-1} has a unique even solution (cf. Remark 9.4.7). In this case, we deduce the Log Minkowski inequality (8.97) for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ by approximation.

The conjectured "B-property" due to Saroglou [646]*:* For any $N \geqslant 2$ and o-symmetric convex body $K \subset \mathbb{R}^N$ and $N \times N$ positive definite diagonal matrix Φ, the function $s \mapsto |[-1,1]^n \cap \Phi^s K|$ of $s \in \mathbb{R}$ is log-concave.

The conjectured "strong B-property" due to Nayar, Tkocz [602]*:* For any $N > n$ and n-dimensional linear subspace L of $\mathbb{R}^N$, the n-volume of $L \cap \prod_{i=1}^N [-e^{t_i}, e^{t_i}]$ is a log-concave function of $(t_1, \dots, t_N) \in \mathbb{R}^N$.

Log Brunn Minkowski for log-concave measures due to Saroglou [647]*:* The even Log Brunn-Minkowski Conjecture 8.7.1 yields that for any even log-concave measure μ on $\mathbb{R}^n$ and $\lambda \in (0,1)$, we have

$$\mu((1-\lambda) \cdot K +_0 \lambda \cdot C) \geqslant \mu(K)^{1-\lambda} \mu(C)^{\lambda} \tag{8.104}$$

for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$. In turn, if (8.104) holds for one fixed even log-concave measure $d\mu = f \, d\mathcal{H}^{n-1}$ with $f(o) > 0$ and f differentiable at o (for example, for the Gaussian γ_n), then the Log Brunn-Minkowski Conjecture 8.7.1 for origin symmetric convex bodies follows.

The Reverse Log-Brunn-Minkowski Conjecture. For $\lambda \in (0,1)$ and o-symmetric convex bodies $K_0, K_1 \subset \mathbb{R}^n$, Xi [722] proposes the Reverse log-Brunn-Minkowski Conjecture

$$\left| \widetilde{K}_\lambda \right| \leqslant |K_0|^{1-\lambda} |K_1|^{\lambda} \tag{8.105}$$

where $dS_{K_i}/d(S_{K_0} + S_{K_1})$ denotes the Radon-Nikodym derivative for $i = 1, 2$ and

$$dS_{\widetilde{K}_\lambda} = \left(\frac{dS_{K_0}}{d(S_{K_0} + S_{K_1})} \right)^{1-\lambda} \left(\frac{dS_{K_1}}{d(S_{K_0} + S_{K_1})} \right)^{\lambda} d(S_{K_0} + S_{K_1}).$$

Xi [722] prove that the Reverse log-Brunn-Minkowski Conjecture (8.105) is equivalent to the Log Brunn-Minkowski Conjecture 8.7.1 for o-symmetric convex bodies.

Log Brunn Minkowski in terms of optimal transport due to Kolesnikov [475]*:* The paper [475] provides an equivalent formulation of the Log Brunn-Minkowski Conjecture for o-symmetric convex bodies in terms of the displacement convexity of certain functional of probability measures on the sphere.

8.7.2 Some known cases of the Log Brunn-Minkowski and the Log Minkowski conjectures

The Log Brunn-Minkowski Conjecture 8.7.1 and the Log Minkowski Conjecture 8.7.3 are still open but have been verified in various cases. In $\mathbb{R}^2$, Conjectures 8.7.1 and 8.7.3 are verified by Böröczky, Lutwak, Yang, Zhang [140] for origin symmetric convex bodies, but it is still open for general centered planar convex bodies.

For unconditional convex bodies, Conjectures 8.7.1 and 8.7.3 have been verified by Saroglou [646] (see Section 8.7.6). This result was extended to convex bodies invariant under reflections through n independent linear hyperplanes by Böröczky, Kalantzopoulos [137], and even stability versions are provided by Böröczky, De [124, 125] in this case. The two conjectures for convex bodies are also verified for complex bodies by Rotem [635].

The Log Brunn-Minkowski Conjecture 8.7.1 and the Log Minkowski Conjecture 8.7.3 also holds for origin symmetric convex bodies in a neighbourhood of a fixed centered ellipsoid E; more precisely, for origin symmetric K and C provided $E \subset K, C \subset (1 + c_n)E$ where $c_n > 0$ depends only on n. In this form, the statement is due to Chen, Huang, Li, Liu [192] extending the local estimate by Kolesnikov, E. Milman [480] (an analogues result holds for linear images of l_q balls for $q > 2$ if the dimension n is high enough according to [480] and the method of [192]). If $\mathbb{R}^3$, some additional partial results are obtained by Chen, Feng, Liu [190], and some earlier partial results are due to Colesanti, Livshyts, Marsiglietti [212].

Xi, Leng [723] considered a version of the Log Brunn-Minkowski Conjecture 8.7.1 where the convex bodies K and C in $\mathbb{R}^n$ are translated by vectors depending in both K and C. We set $r(K, C) = \max\{t > 0 : \exists x,\ x + tC \subset K\}$ and $R(K, C) = \min\{t > 0 : \exists x,\ K \subset x + tC\}$, and say that K and C are in dilated position if $o \in K \cap C$ and $r(K, C)\, C \subset K \subset R(K, C)\, C$. We observe that $r(C, K)\, K \subset C \subset R(C, K)\, K$ in this case. Now for any convex bodies K and C there exist $z \in K$ and $w \in C$ such that $K - z$ and $C - w$ are in dilated position. If $n = 2$ and K and C are in dilated position, then Xi, Leng [723] proved Conjecture 8.7.1 including the characterization of equality.

8.7.3 Some cases when only the Logarithmic Minkowski conjecture is known

There are some cases when the Logarithmic Minkowski Conjecture 8.7.3 is known; namely, given an o-symmetric convex body $K \subset \mathbb{R}^n$,

$$\int_{S^{n-1}} \log h_C \, dV_K \geqslant \int_{S^{n-1}} \log h_K \, dV_K \tag{8.106}$$

holds for any o-symmetric convex body $C \subset \mathbb{R}^n$ with $|K| = |C|$; however, no corresponding version of the Log Brunn-Minkowski inequality exists because L_0 combinations are not in the family. These results are still very important as (8.106) is the inequality intimately connected to the uniqueness of the solution of the Monge-Ampère equation (8.103) called even Logarithmic Minkowski problem (see Section 9.4).

The most elementary case where Conjecture 8.7.3; or equivalently, (8.106) is known is when K is an o-symmetric ellipsoid, as it is verified by Guan, Ni [361] as follows. We may assume that $K = B^n$ and $|C| = |B^n|$, and hence the Jensen inequality

(10.6) and the Blaschke-Santaló inequality (6.32) yield

$$\exp\left(\int_{S^{n-1}} \log h_C \cdot \frac{1}{|B^n|} dV_K\right) = \exp\left(\int_{S^{n-1}} \log h_C \cdot \frac{1}{n|B^n|}\, d\mathcal{H}^{n-1}\right)$$
$$\geqslant \left(\int_{S^{n-1}} h_C^{-n} \cdot \frac{1}{n|B^n|}\, d\mathcal{H}^{n-1}\right)^{\frac{-1}{n}} \geqslant 1. \quad (8.107)$$

Using much more involved methods, (8.106) is verified

- under rather generous explicit curvature pinching bounds for ∂K by E. Milman [579] and Ivaki, E. Milman [442];
- if there exists a centered ellipsoid E such that $E \subset K \subset (1 + c_n)E$ where $c_n > 0$ depends on n according to Chen, Huang, Li, Liu [192] building on earlier local estimates by Kolesnikov, E. Milman [480];
- if K is a zonoid by van Handel [710] (with equality case only clarified when K has C^2_+ boundary).

Let $q > 2$. Then the unit ball $B_q^n \subset \mathbb{R}^n$ of the ℓ_q-norm $\|\cdot\|_q$ is a zonoid, and hence (8.106) holds when $K = B_q^n$. Combining the methods of Kolesnikov, E. Milman [480] and Chen, Huang, Li, Liu [192], one finds a threshold $N_q \geqslant 2$ and a constant $c_{n,q} > 0$ for $n \geqslant N_q$ such that (8.106) holds if $\Phi B_q^n \subset K \subset \Phi(1 + c_{n,q})B_q^n$ where $n \geqslant N_q$ and $\Phi \in \mathrm{GL}(n)$.

While the cases when Logarithmic Minkowski Conjecture 8.7.3 is known seem to be rather sparse in the space of all o-symmetric convex bodies, E. Milman [579] proved that for any o-symmetric convex body M, there exists an o-symmetric convex body K with C^∞_+ boundary such that $M \subset K \subset 8M$ and (8.106) holds for any o-symmetric convex body C.

8.7.4 Reverse Logarithmic Minkowski type inequality for the polars

Saroglou [647] proved the following variant of the Logarithmic Brunn-Minkowski Conjecture 8.7.1. If $\lambda \in (0, 1)$ and K and C are convex bodies in $\mathbb{R}^n$, then

$$\left|((1 - \lambda) \cdot K +_0 \lambda \cdot C)^*\right| \leqslant |K^*|^{1-\lambda}|C^*|^\lambda. \quad (8.108)$$

In turn, V. Milman, Rotem [586] observed that (8.108) leads to the following estimates.

Lemma 8.7.6. *If $\lambda \in (0, 1)$ and K and C are centered convex bodies in $\mathbb{R}^n$, then*

$$4^{-n}|K|^{1-\lambda}|C|^\lambda \leqslant |(1 - \lambda) \cdot K +_0 \lambda \cdot C| \leqslant n^{3n/2}|K|^{1-\lambda}|C|^\lambda. \quad (8.109)$$

Proof. For the lower bound, G. Kuperberg's Reverse Blaschke-Santaló inequality (6.38), (8.108) and the Blaschke-Santaló inequality (6.31) yield that

$$\begin{aligned} 4^{-n}|B^n|^2 &\leqslant |(1-\lambda)\cdot K +_0 \lambda\cdot C|\cdot\left|((1-\lambda)\cdot K +_0 \lambda\cdot C)^*\right| \\ &\leqslant |(1-\lambda)\cdot K +_0 \lambda\cdot C|\cdot |K^*|^{1-\lambda}|C^*|^{\lambda} \\ &\leqslant |(1-\lambda)\cdot K +_0 \lambda\cdot C|\cdot \frac{|B^n|^2}{|K|^{1-\lambda}|C|^{\lambda}}. \end{aligned}$$

For the upper bound, Lenma 1.11.5 due to Kannan, Lovász, Simonovits [454] yields the existence of centered ellipsoids $E'\subset K$ and $E\subset C$ such that $K\subset nE'$ and $C\subset nE$. After a linear transform, we may assume that $E'=B_2^n$ and E is unconditional. Let $a_1,\dots,a_n$ be the half axes of E, and hence $C\subset \widetilde{C}=\prod_{i=1}^n[-na_i,na_i]$ and $K\subset \widetilde{K}=[-n,n]^n$ with $|\widetilde{C}|\leqslant n^{3n/2}|C|$ and $|\widetilde{K}|\leqslant n^{3n/2}|K|$. Since $|(1-\lambda)\cdot\widetilde{K}+_0\lambda\cdot\widetilde{C}|=|\widetilde{K}|^{1-\lambda}|\widetilde{C}|^{\lambda}$, we conclude the upper bound in (8.109). ∎

Additional local versions of the Logarithmic Brunn-Minkowski Conjecture 8.7.3 are due to Colesanti, Livshyts, Marsiglietti [212], Kolesnikov, Livshyts [478], Hosle, Kolesnikov, Livshyts [399] and Kolesnikov, Livshyts, Rotem [479].

8.7.5 L_0 sum of polytopes and lower dimensional compact convex sets

Let $K,C\subset\mathbb{R}^n$ be convex bodies with $o\in\operatorname{int}K$, $\operatorname{int}C$, and let $\lambda\in(0,1)$. We recall that according to Lemma 7.5.1, if $z\in\partial'((1-\lambda)K+_0\lambda C)$ and u is the exterior unit normal at z, then

$$h_{(1-\lambda)\cdot K+_0\lambda\cdot C}(u)=h_K(u)^{1-\lambda}h_C(u)^{\lambda}. \tag{8.110}$$

We also use that $M=\{x\in\mathbb{R}^n:\langle x,\nu_M(y)\rangle\leqslant h_M(\nu_M(y))\ \forall y\in\partial' M\}$ holds for any convex body $M\subset\mathbb{R}^n$ by Lemma 2.5.6.

Lemma 8.7.7. *If $K=K_1+\ldots+K_m\subset\mathbb{R}^n$ and $C=C_1+\ldots+C_m\subset\mathbb{R}^n$ are convex bodies for compact convex sets $K_1,\dots,K_m,C_1,\dots,C_m$ of dimension at least one and containing the origin where $\sum_{i=1}^m\dim K_i=n$ and $C_i=\theta_iK_i$ for $\theta_i>0$, $i=1,\dots,m$, $m\geqslant 1$, then $(1-\lambda)K+_0\lambda C=\sum_{i=1}^m\theta_i^{\lambda}K_i$.*

Proof. If $m=1$, then the lemma readily holds; therefore, we assume that $m\geqslant 2$. Let $L_i=\operatorname{lin}K_i$, $i=1,\dots,m$. According to linear invariance (cf. (8.93)), we may assume that $L_1,\dots,L_m$ are pairwise orthogonal.

Lemma 8.7.7 follows if for any $z\in\partial'((1-\lambda)K+_0\lambda C)$ with $u=\nu_{(1-\lambda)K+_0\lambda C}(z)$, we have $u\in L_i$ for an $i\in\{1,\dots,m\}$. Let $x\in\partial K$ such that u is an exterior normal at x. Then $x=x_+\ldots+x_m$ for $x_i\in K_i$. We may assume that there exists index $p\geqslant 1$ such that $x_i\in\operatorname{relbd}K_i$ for $i\leqslant p$ and $x_i\in\operatorname{relint}K_i$ for $i>p$. In particular, there exists exterior unit normal $u_i\in L_i$ to K_i at x_i for $i\leqslant p$ such that $u=\sum_{i=1}^p\alpha_iu_i$ for $\alpha_1,\dots,\alpha_p\geqslant 0$. We may assume that for some index q with $1\leqslant q\leqslant p$, $\alpha_i>0$ if $i\leqslant q$, and $\alpha_i=0$ if

$i > q$. We observe that $u = \sum_{i=1}^{q} \alpha_i u_i$ is an exterior normal at $y = \sum_{i=1}^{m} \lambda_i x_i \in \partial C$, $h_K(u_i) = \langle x, u_i \rangle$ and $h_C(u_i) = \langle y, u_i \rangle$ for $i = 1, \dots, q$. As $z \in \partial'((1-\lambda) \cdot K +_0 \lambda \cdot C)$, (8.110) and the Hölder inequality (10.5) yield that

$$\begin{aligned}\langle z, u \rangle = h_{(1-\lambda)\cdot K +_0 \lambda \cdot C}(u) = h_K(u)^{1-\lambda} h_C(u)^{\lambda} = \left(\sum_{i=1}^{q} \alpha_i \langle x, u_i \rangle \right)^{1-\lambda} \left(\sum_{i=1}^{q} \alpha_i \langle y, u_i \rangle \right)^{\lambda} \\ \geqslant \sum_{i=1}^{q} \alpha_i \langle x, u_i \rangle^{1-\lambda} \langle y, u_i \rangle^{\lambda} = \sum_{i=1}^{q} \alpha_i h_K(u_i)^{1-\lambda} h_C(u_i)^{\lambda} \\ \geqslant \sum_{i=1}^{q} \alpha_i h_{(1-\lambda)K +_0 \lambda C}(u_i) \geqslant \sum_{i=1}^{q} \alpha_i \langle z, u_i \rangle = \langle z, u \rangle. \qquad (8.111)\end{aligned}$$

We deduce that $h_{(1-\lambda)K +_0 \lambda C}(u_i) = \langle z, u_i \rangle$ for $i = 1, \dots, q$, and hence $u_1, \dots, u_q$ are exterior unit normals at $z \in \partial'((1-\lambda)K +_0 \lambda C)$. We conlude that $q = 1$; therefore, $u \in L_1$. ∎

Similar argument yields that the L_0 combination of polytopes is a polytope. This property is heavily used in Putterman's argument in [629] that (8.100) yields the Log Minkowski conjecture (8.98) for any pairs of o-symmetric convex bodies $K, C \subset \mathbb{R}^n$.

Lemma 8.7.8. *If $K, C \subset \mathbb{R}^n$ are n-polytopes, then $(1-\lambda)K +_0 \lambda C$ is an n-polytope whose facet exterior unit normals are facet exterior unit normals to the Minkowski sum $K + C$, as well.*

Proof. Let $u_1, \dots, u_m$ be the exterior unit normals to the facets of the polytope $K + C$ (cf. (1.7)). Lemma 8.7.8 follows if whenever $u = \nu_{(1-\lambda)K +_0 \lambda C}(z)$ for a $z \in \partial'((1-\lambda)K +_0 \lambda C)$, then $u \in \{u_1, \dots, u_m\}$. Let w be a vertex of $K + C$ such that u is an exterior normal at w to $K + C$. According to Lemma 1.4.10, the normal cone at w is the positive hull of a subset of $\{u_1, \dots, u_m\}$; therefore, we may assume that $u = \sum_{i=1}^{q} \alpha_i u_i$ for $\alpha_1, \dots, \alpha_q \geqslant 0$, $q \geqslant 1$, where $u_1, \dots, u_q$ are exterior normals at w to $K + C$. Now $w = x + y$ for a vertex x of K and a vertex y of C, and hence $u, u_1, \dots, u_q$ are exterior normals at x to K and at y to C, and $h_K(u_i) = \langle x, u_i \rangle$ and $h_C(u_i) = \langle y, u_i \rangle$ for $i = 1, \dots, q$. Exactly the same calculations as in (8.111) yield that $h_{(1-\lambda)K +_0 \lambda C}(u_i) = \langle z, u_i \rangle$ for $i = 1, \dots, q$, and hence $u_1, \dots, u_q$ are exterior unit normals at $z \in \partial'((1-\lambda)K +_0 \lambda C)$. We conclude that $q = 1$; therefore, $u = u_1$. ∎

Example 8.7.9 (The boundary of the L_0 combination of convex bodies with C^2_+ boundaries may not be C^1). Let e_1, e_2 be the orthonormal basis of $\mathbb{R}^2$. For $a > 2$, and consider the ellipses $E_a = \operatorname{diag}[a, a^{-1}]B^2$ and $\widetilde{E}_a = \operatorname{diag}[a^{-1}, a]B^2$. Since $E_a \subset P_a = [a, -a] \times [\frac{1}{a}, \frac{-1}{a}]$ and $\widetilde{E}_a \subset \widetilde{P}_a = [\frac{1}{a}, \frac{-1}{a}] \times [a, -a]$, we deduce using Lemma 8.7.7 that

$$\tfrac{1}{2} \cdot E_a +_0 \tfrac{1}{2} \cdot \widetilde{E}_a \subset \tfrac{1}{2} \cdot P_a +_0 \tfrac{1}{2} \cdot \widetilde{P}_a = [-1, 1] \times [-1, 1] = W.$$

For $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, we have

$$h_{E_a}(u)^{\frac{1}{2}} h_{\widetilde{E}_a}(u)^{\frac{1}{2}} > \langle ae_1, u\rangle^{\frac{1}{2}} \langle ae_2, u\rangle^{\frac{1}{2}} = a/\sqrt{2} > h_W(u) \geqslant h_{\frac{1}{2}\cdot E_a +_0 \frac{1}{2}\cdot \widetilde{E}_a}(u);$$

therefore, u is not an exterior normal at a regular boundary point of $\frac{1}{2}\cdot E_a +_0 \frac{1}{2}\cdot \widetilde{E}_a$ by (8.110).

8.7.6 The Logarithmic Brunn-Minkowski and Logarithmic Minkowski conjectures for unconditional convex bodies

This section discusses the L_0 combination of unconditional convex bodies. Here a convex body $K \subset \mathbb{R}^n$ is unconditional if $(\pm x_1, \dots, \pm x_n) \in K$ for $(x_1, \dots, x_n) \in K$. From Section 3.6, we recall the notion of coordinatewise product of unconditional convex bodies: If $K, C \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0,1)$, then

$$K^{1-\lambda} \cdot C^{\lambda} = \left\{ \left(\pm |x_1|^{1-\lambda} |y_1|^{\lambda}, \dots, \pm |x_n|^{1-\lambda} |y_n|^{\lambda}\right) : (x_1, \dots, x_n) \in K \ \& \ (y_1, \dots, y_n) \in C \right\}.$$

The coordinatewise product is a classical notion of Functional Analysis, while the L_0 combination was only defined in the 2012 paper Böröczky, Lutwak, Yang, Zhang [140]; still the two notions are closely related as

$$K^{1-\lambda} \cdot C^{\lambda} \subset (1-\lambda)\cdot K +_0 \lambda \cdot C. \tag{8.112}$$

To prove (8.112), it is sufficient to verify that if $x = (x_1, \dots, x_n) \in K$, $y = (y_1, \dots, y_n) \in C$ and $u = (u_1, \dots, u_n) \in S^{n-1}$ with $x_i, y_i, u_i \geqslant 0$, then $\langle z, u\rangle \leqslant \langle x, u\rangle^{1-\lambda} \langle y, u\rangle^{\lambda}$ for $z = (x_1 y_1, \dots, x_n y_n)$. In turn, this property follows from the Hölder inequality using the measure μ on $\{1, \dots, n\}$ with $\mu(\{i\}) = u_i$.

According to Theorem 3.6.4, the coordinatewise product satisfies the following Brunn-Minkowski type inequality:

Theorem 8.7.10. *If $K, C \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0,1)$, then*

$$\left|K^{1-\lambda} \cdot C^{\lambda}\right| \geqslant |K|^{1-\lambda} |C|^{\lambda}. \tag{8.113}$$

Equality holds if and ony if $K = \Phi C$ for a positive definit diagonal matrix Φ.

In turn, (8.112) directly yields the Log Brunn-Minkowski Conjecture 8.7.1 for unconditional convex bodies, as it was observed by Saroglou [646]. The non-trivial part is to characterize equality, which was also done by Saroglou [646] (with a slight correction in [137]).

Theorem 8.7.11. *If $K, C \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0,1)$, then*

$$|(1-\lambda)\cdot K +_0 \lambda \cdot C| \geqslant |K|^{1-\lambda} |C|^{\lambda}, \tag{8.114}$$

with equality if and only if $K = K_1 \oplus \ldots \oplus K_m$ *and* $L = L_1 \oplus \ldots \oplus L_m$ *for unconditional compact convex sets* $K_1, \ldots, K_m, L_1, \ldots, L_m$ *of dimension at least one,* $m \geqslant 1$ *where* $\sum_{i=1}^m \dim K_i = n$ *and* K_i *and* L_i *are dilates,* $i = 1, \ldots, m$.

Proof. Combining Theorem 8.7.10 and (8.112) yields (8.114). If K and C are as described after (8.114), then we have equality in (8.114) according to Lemma 8.7.7.

On the other hand, if equality holds in (8.114), then we deduce from Theorem 8.7.10 and (8.112) that $K^{1-\lambda} \cdot C^{\lambda} = (1-\lambda) \cdot K +_0 \lambda \cdot C$ and $|K^{1-\lambda} \cdot C^{\lambda}| = |K|^{1-\lambda}|C|^{\lambda}$. According to Theorem 8.7.10, there exists a positive definite diagonal matrix Φ such that $C = \Phi K$. Let $e_1, \ldots, e_n$ be the corresponding orthonormal basis of $\mathbb{R}^n$, and for $t \in \mathbb{R}$, consider the diagonal matrix Φ^t defined by $\langle \Phi^t e_i, e_i \rangle = \langle \Phi e_i, e_i \rangle^t$ for $i = 1, \ldots, n$. Since the definition of the coordinatewise product yields that $K^{1-\lambda} \cdot (\Phi K)^{\lambda} = \Phi^{\lambda} \left(K^{1-\lambda} \cdot K^{\lambda} \right)$, we have

$$\Phi^{\lambda} K = (1-\lambda) \cdot K +_0 \lambda \cdot (\Phi K). \tag{8.115}$$

If all eigenvalues of Φ are the same $\theta > 0$, then $C = \theta K$, and hence the characterization of equality follows with $m = 1$. Therefore, we may assume that $L_1, \ldots, L_m$, $m \geqslant 2$, are the eigenspaces of Φ with eigenvalues $\theta_1, \ldots . \theta_m$ with $\theta_i \neq \theta_j$ for $i \neq j$ Since $K = \{x \in \mathbb{R}^n : \langle x, \nu_K(y) \rangle \leqslant h_K(\nu_K(y)) \ \forall y \in \partial' K\}$ according to Lemma 2.5.6, Theorem 8.7.11 follows from the claim

$$\nu_K(y) \in \bigcup_{i=1}^m L_i \text{ for } y \in \partial' K. \tag{8.116}$$

We prove (8.116) by contradiction; therefore, we suppose that $\nu_K(y) \notin \cup_{i=1}^m L_i$ for a $y = (y_1, \ldots, y_n) \in \partial' K$, and set $u = (u_1, \ldots, u_n) = \nu_K(y)$. Since K is unconditional, we may assume that each $u_i \geqslant 0$ and each $y_i \geqslant 0$. Possibly after reindexing, we may also assume that $e_1 \in L_1$ and $u_1 > 0$, and $e_n \in L_m$ and $u_n > 0$ by the indirect hypethesis. Using the unconditionality of K, $u_1 > 0$, $u_n > 0$ and $y \in \partial' K$, we also deduce that $y_1 > 0$ and $y_n > 0$ As $\langle \Phi^t u, e_1 \rangle = \theta_1^t \langle u, e_1 \rangle > 0$ and $\langle \Phi^t u, e_n \rangle = \theta_m^t \langle u, e_n \rangle > 0$ where $\theta_1 \neq \theta_m$, we deduce that

$$\text{neither } u \text{ nor } \Phi^{-1} u \text{ is parallel to } v = \Phi^{-\lambda} u. \tag{8.117}$$

We observe that $v = \Phi^{-\lambda} u$ is an exterior normal at the regular boundary point $\Phi^{\lambda} y$ of $\Phi^{\lambda} K$. Combining this property with (8.115) and Lemma 7.5.1 yields that

$$\langle v, \Phi^{\lambda} y \rangle = h_K(v)^{1-\lambda} h_{\Phi K}(v)^{\lambda}. \tag{8.118}$$

On the other hand, since $\nu_K(y) = u$ for $y \in \partial' K$ and $\nu_{\Phi K}(\Phi x) = \Phi^{-1} u$ for $\Phi y \in \partial'(\Phi K)$, we deduce from (8.117) that

$$\langle v, y \rangle < h_K(v) \text{ and } \langle v, \Phi y \rangle < h_{\Phi K}(v).$$

In particular, the Hölder inequality (10.5) yields that

$$\langle v, \Phi^{\lambda} y\rangle = \sum_{i=1}^{n} u_i y_i \leqslant \left(\sum_{i=1}^{n} \theta_i^{-\lambda} u_i y_i\right)^{1-\lambda} \left(\sum_{i=1}^{n} \theta_i^{1-\lambda} u_i y_i\right)^{\lambda} = \langle v, y\rangle^{1-\lambda} \langle v, \Phi y\rangle^{\lambda}$$
$$< h_K(v)^{1-\lambda} h_{\Phi K}(v)^{\lambda}.$$

This contradicts (8.118), and completes the proof of Proposition 8.7.11. ∎

8.8 The L_p-Brunn-Minkowski and the L_p-Minkowski conjectures and inequalities when $p \in (0,1)$

This section discusses possible L_p versions of the Brunn-Minkowski inequality and the Minkowski inequality for $p \in (0,1)$ and centered convex bodies,; namely, the L_p-Brunn-Minkowski and the L_p-Minkowski conjectures/inequalities. They connect on the one hand, the L_0 (logarithmic) Brunn-Minkowski Conjecture 8.7.1 and the L_0 (logarithmic) Minkowski Conjecture 8.7.3, and on the other hand, the Brunn-Minkowski inequality and the Minkowski inequality (the L_1 case) and their L_p generalizations for $p > 1$ by Firey [284] from 1962 (see Section 7.6).

For $\alpha, \beta \geqslant 0$, Böröczky, Lutwak, Yang, Zhang [140] defined the L_p linear combination of the convex bodies $K, C \subset \mathbb{R}^n$ containing the origin in their interior as

$$\alpha \cdot K +_p \beta \cdot C = \left\{x \in \mathbb{R}^n : \langle x, u\rangle^p \leqslant \alpha\, h_K(u)^p + \beta\, h_C(u)^p,\ u \in S^{n-1}\right\}, \quad (8.119)$$

which is a convex body containing the origin in its interior. This extends the definition of Firey's L_p addition for $p \geqslant 1$ (see (7.43)), and the definition of the L_0 sum (see (8.88)). Given $p \in (0,1)$, let us list some basic properties of the L_p combination of convex bodies $K, C \subset \mathbb{R}^n$ containing the origin in their interior where (8.120) follows from the relation $h_{\Phi M}(u) = h_M(\Phi^t u)$ for any convex body M, (8.121) follows from the Jensen inequality (10.6), and the last property is proved in Section 8.8.2:

- The L_p combination is linear invariant; namely, if $\Phi \in \mathrm{GL}(n)$ and $\alpha, \beta > 0$, then

$$\alpha \cdot \Phi(K) +_p \beta \cdot \Phi(C) = \Phi\left(\alpha \cdot K +_p \beta \cdot C\right). \quad (8.120)$$

- If $q > r \geqslant 0$ and $\lambda \in (0,1)$, then

$$(1-\lambda) \cdot K +_r \lambda \cdot C \subset (1-\lambda) \cdot K +_q \lambda \cdot C. \quad (8.121)$$

- The L_p combination of polytopes is a polytope (cf. Lemma 8.8.13), but the boundary of the L_p combination of convex bodies with C^2_+ boundaries may not be even C^1 (cf. Example 8.8.14).

Böröczky, Lutwak, Yang, Zhang [140] conjectured the following strengthening of the Brunn-Minkowski inequality for origin symetric convex bodies, and Martin Henk proposed the version with centered convex bodies (see also [137]).

Conjecture 8.8.1 (L_p-Brunn-Minkowski conjecture for $p \in (0,1)$). *If $K, C \subset \mathbb{R}^n$ are centered convex bodies, then*

$$|\alpha \cdot K +_p \beta \cdot C|^{\frac{p}{n}} \geqslant \alpha |K|^{\frac{p}{n}} + \beta |C|^{\frac{p}{n}} \tag{8.122}$$

for $\alpha, \beta > 0$ with equality if and only if K and C are dilates, and

$$|(1-\lambda) \cdot K +_p \lambda \cdot C| \geqslant |K|^{1-\lambda} |C|^{\lambda}. \tag{8.123}$$

for $\lambda \in (0,1)$ with equality if and only if $K = C$.

Remark 8.8.2. [The forms (8.122) and (8.123) of the L_p-Brunn-Minkowski conjecture are equivalent] On the one hand, (8.122) yields (8.123) by the AM-GM inequality. On the other hand, writing $K = \alpha_0 K_0$ and $C = \beta_0 C_0$ for $\alpha_0 = |K|^{1/n}$ and $\beta_0 = |C|^{1/n}$, applying (8.123) to K_0, C_0 and $\lambda = \frac{\beta\beta_0}{\alpha\alpha_0 + \beta\beta_0}$ implies (8.122).

As it was observed by Nayar, Tkocz [601], Conjectiure 8.8.1 may not hold if $K, C \subset \mathbb{R}^n$ are arbitrary convex bodies with $o \in \operatorname{int} K$ and $o \in \operatorname{int} C$. For example, if $K = [-\frac{1}{2}, \frac{1}{2}]^n$, then $|(1-\lambda) \cdot K +_p \lambda \cdot (K - z)| < 1$ if $z \in \operatorname{int} K$ and $z \neq o$.

Next we state the L_p-Minkowski conjecture due to Böröczky, Lutwak, Yang, Zhang [140] for origin symmetric bodies, and to Böröczky, Kalantzopoulos [137] for centered convex bodies.

Conjecture 8.8.3 (L_p-Minkowski conjecture for $p \in (0,1)$). *If $K, C \subset \mathbb{R}^n$ are centered convex bodies, then*

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K \geqslant |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}}, \tag{8.124}$$

with equality if and only if K and C are dilates.

Remarks.

- The conjecture is $\mathrm{GL}(n)$ invariant by the invariance properties of the cone volume measure (cf. Proposition 2.6.15 and Lemma 2.6.14).
- It is equivalent saying that if $|C| = |K|$ for centered convex bodies $C, K \subset \mathbb{R}^n$, then
$$\int_{S^{n-1}} \left(\frac{h_C}{h_K} \right)^p |K|^{-1} dV_K \geqslant 1 \tag{8.125}$$
for the probability measure $|K|^{-1} dV_K$ with equality if and only if $K = C$.
- We note that the choice of the right translates of K and C are important in Conjecture 8.7.3 as it is shown by the example - due to Nayar, Tkocz [601] - when $K = [-1,1]^n$ and $C \neq K$ is a translate of K.

In the case of o-symmetric convex bodies and $p \in (0,1)$, Lemma 8.8.4 shows that the L_p-Brunn-Minkowski Conjecture 8.8.1 and the L_p-Minkowski Conjecture 8.8.3 are equivalent.

Lemma 8.8.4. *For $p \in (0,1)$, and family $\mathcal{F}$ of centered convex bodies in $\mathbb{R}^n$ that is closed under L_p combination, the following are equivalent if they hold for all $K, C \in \mathcal{F}$.*

(i) $|\alpha \cdot K +_p \beta \cdot C| \geqslant \alpha |K|^{\frac{p}{n}} + \beta |C|^{\frac{p}{n}}$ for $\alpha, \beta > 0$ with equality if and only if K and C are dilates;

(ii) The function $f_{K,C}(\lambda) = |(1-\lambda) \cdot K +_p \lambda \cdot C|^{\frac{p}{n}}$ is concave on $[0,1]$, and is linear if and only if K and C are dilates;

(iii) $\int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K \geqslant |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}}$ with equality if and only if K and C are dilates.

Remark. The typical $\mathcal{F}$ is the family of convex bodies invariant under a subgroup $G \subset \mathrm{GL}(n)$ (cf. (8.120)), like origin symmetric convex bodies (when $G = \{I_n, -I_n\}$).

Proof. The proof of (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are essentially the same as in the L_0 case in Lemma 8.7.4; therefore, we do not present the argument.

(i) $\Longrightarrow$ (ii): Let $M_\lambda = (1-\lambda) \cdot K +_p \lambda \cdot C$ for $\lambda \in [0,1]$. According to (i), (ii) follows if for any $t, s, \alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$, we have $\alpha \cdot M_t +_p \beta \cdot M_s \subset M_{\alpha t + \beta s}$. By Lemma 2.5.6, it is sufficient to observe $h_{M_{\alpha t+\beta s}}(\nu_{M_{\alpha t+\beta s}}(z)) \geqslant h_N(\nu_{M_{\alpha t+\beta s}}(z))$ for $z \in \partial' M_{\alpha t+\beta s}$ and $N = \alpha \cdot M_t +_p \beta \cdot M_s$ as Lemma 7.5.1 about regular boundary points of a Wulff shape implies the first equality for $u = \nu_{M_{\alpha t+\beta s}}(z)$ in the series

$$\begin{aligned} h_{M_{\alpha t+\beta s}}(u)^p &= (1-\alpha t-\beta s) h_K(u)^p + (\alpha t + \beta s) h_C(u)^p \\ &= \alpha \left((1-t) h_K(u)^p + t h_C(u)^p \right) + \beta \left((1-s) h_K(u)^p + s h_C(u)^p \right) \\ &\geqslant \alpha h_{M_t}(u)^p + \beta h_{M_s}(u)^p \geqslant h_N(u)^p. \end{aligned}$$

We conclude $\alpha \cdot M_t +_p \beta \cdot M_s \subset M_{\alpha t+\beta s}$, and in turn, the concavity of $f_{K,C}$. In particular, $f_{K,C}$ is linear if and only if $f_{K,C}(\frac{1}{2}) = f_{K,C}(0)^{\frac{1}{2}} f_{K,C}(1)^{\frac{1}{2}}$, and hence (ii) has the same equality conditions as (i).

(ii) $\Longrightarrow$ (iii): We may assume that $|K| = |C| = 1$ and $R^{-1}B^n \subset K, C \subset RB^n$ for some $R > 1$, and hence $f_{K,C}(\lambda) \geqslant 1$ for $\lambda \in (0,1)$ and $R^{-2} \leqslant h_C/h_K \leqslant R^2$. As $(1+t)^{\frac{1}{p}} = 1 + \frac{1}{p} \cdot t + O(t^2)$ provided $|t| \leqslant \frac{1}{2}$ where the implied constant in $O(\cdot)$ depends on p, we deduce that if $0 < \lambda < (4 \log R)^{-1}$ and $u \in S^{n-1}$, then

$$\begin{aligned} \left((1-\lambda) h_K(u)^p + \lambda h_C(u)^p\right)^{\frac{1}{p}} &= h_K(u) \left(1 + \lambda \cdot \left(\left(\frac{h_C}{h_K}\right)^p - 1\right)\right)^{\frac{1}{p}} \\ &= h_K(u) + \frac{\lambda}{p} \cdot h_K(u) \left(\left(\frac{h_C}{h_K}\right)^p - 1\right) + O(\lambda^2 R^{1+4p}) \end{aligned}$$

where the implied constant in $O(\cdot)$ depends on p, but not on u. It follows from (ii) and Aleksandrov's Lemma 7.5.2 for the Wulff shape that

$$0 \leqslant f'_{K,C}(0) = \frac{1}{p}\int_{S^{n-1}} h_K \left(\left(\frac{h_C}{h_K}\right)^p - 1\right) dS_K = \frac{n}{p}\int_{S^{n-1}} \left(\left(\frac{h_C}{h_K}\right)^p - 1\right) dV_K, \tag{8.126}$$

proving (iii). Equality in (8.126) yields $f'_{K,C}(0) = 0$, and hence $f_{K,C}(0)$ is linear.

(iii) $\Longrightarrow$ (i): Using the notation $M_\lambda = (1-\lambda)\cdot K +_p \lambda \cdot C$ for $\lambda \in (0,1)$, we deduce from Lemma 2.5.6 and Lemma 7.5.1 that $h_{M_\lambda}(u) = ((1-\lambda)h_K(u)^p + \lambda\, h_C(u)^p)^{\frac{1}{p}}$ for S_{M_λ} (and hence V_{M_λ}) a.e. $u \in S^{n-1}$; therefore, (iii) implies

$$\begin{aligned} |M_\lambda| &= \int_{S^{n-1}} \frac{(1-\lambda)h_K^p + \lambda\, h_C^p}{h_{M_\lambda}^p}\, dV_{M_\lambda} \\ &= (1-\lambda)\int_{S^{n-1}} \frac{h_K^p}{h_{M_\lambda}^p}\, dV_{M_\lambda} + \lambda \int_{S^{n-1}} \frac{h_C^p}{h_{M_\lambda}^p}\, dV_{M_\lambda} \\ &\geqslant (1-\lambda)|M_\lambda|^{1-\frac{p}{n}}|K|^{\frac{p}{n}} + \lambda |M_\lambda|^{1-\frac{p}{n}}|C|^{\frac{p}{n}}. \end{aligned}$$

We conclude that $|M_\lambda|^{\frac{p}{n}} \geqslant (1-\lambda)|K|^{\frac{p}{n}} + \lambda\,|C|^{\frac{p}{n}}$, which in turn yields (i) by Remark 8.8.2. In addition, having equality in (i) with $\alpha = 1-\lambda$ and $\beta = \lambda$ and equality in (iii) are equivalent. ∎

Remark 8.8.5 (The L_p-Minkowski conjecture is "monotone in p"). The Jensen inequality (10.6) yields the following where $1 > q > p \geqslant 0$ and $K, C \subset \mathbb{R}^n$ are centered convex bodies with $|K| = |C| = 1$.

(i) If the inequality in the L_p-Brunn-Minkowki inequality holds for K and C; namely, $|(1-\lambda)\cdot K +_p \lambda \cdot C| \geqslant 1$ (cf. (8.94) and (8.123)), then $|(1-\lambda)\cdot K +_q \lambda \cdot C| \geqslant 1$.

(ii) If the inequality in the L_p Minkowki inequality holds for K and C; namely, $\int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K \geqslant 0$ if $p = 0$ (cf. (8.97)) or $\int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K \geqslant 1$ if $p > 0$ (cf. (8.124)), then

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K \geqslant 1, \tag{8.127}$$

where equality in (8.127) yields $C = K$ provided ∂K is C^1.

For the characterization of equality, what the Jensen inequality (10.6) directly yields is that there exists $t > 0$ such that $h_C(u) = t \cdot h_K(u)$ for S_K a.e. $u \in S^{n-1}$. Since $\operatorname{supp} S_K = S^{n-1}$ if ∂K is C^1, we deduce that $C = t \cdot K$, and hence $|K| = |C|$ implies that $C = K$.

Given "the monotonicity of the L_p-Minkowski conjecture" as in Remark 8.8.5, it is not surprising that one of the most major result concerning the L_p-Minkowski

conjecture is about the case when $p < 1$ is close to 1. This result is due to Chen, Huang, Li, Liu [192], based on the local result by Kolesnikov, E. Milman [480]. Another argument for the local-to-global step is provided by Putterman [629].

Theorem 8.8.6. *If $1 - \frac{c}{n \log n} < p < 1$ for an absolute constant $c > 0$, then the L_p-Brunn-Minkowki Conjecture 8.8.1 and the L_p-Minkowski Conjecture 8.8.3 hold for any origin symmetric convex bodies K, C and $\lambda \in (0,1)$.*

According to Remark 8.8.5, the L_p-Brunn-Minkowski Conjecture 8.8.1 and the L_p-Minkowski Conjecture 8.8.3 follow for $p \in (0,1)$ in all cases when the (L_0) Log Brunn-Minkowski and the Log Minkowski conjectures are known

Remark 8.8.7 (Some known cases of the L_p-Brunn-Minkowski (LpBM) and the L_p-Minkowski (LpM) conjectures for o-symmetric convex bodies for $p \in [0,1)$ without characterization of equality).

- $n = 2$ (both (LpBM) and (LpM), Böröczky, Lutwak, Yang, Zhang [140]).
- K, C are unconditional (Saroglou [646]), and more generally, K, C are invariant under reflections through n independent linear hyperplanes (both (LpBM) and (LpM), Böröczky, Kalantzopoulos [137]).
- Complex bodies (both (LpBM) and (LpM), Rotem [635]).
- $E \subset K, C \subset (1 + c_n)E$ where $c_n > 0$ depends only on n and E is a centered ellipsoid (both (LpBM) and (LpM), Chen, Huang, Li, Liu [192] and Kolesnikov, E. Milman [480]).
- K is a zonoid and C is any o-symmetric convex body (only (LpM), van Handel [710]).
- Under rather generous explicit curvature pinching bounds for ∂K by E. Milman [579] and Ivaki, E. Milman [442].

8.8.1 Equivalent formulations of the L_p-Minkowski Conjecture 8.8.3 for $p \in (0,1)$ and o-symmetric convex bodies

If $p \in (0,1)$, then the following statements are equivalent if they hold for all o-symmetric convex bodies $K, C \subset \mathbb{R}^n$. Here the equivalence of (i), (ii) and (iii) are proved in Lemma 8.8.4, the equivalence of (iii) and (iv) is discussed in Remark 9.4.7. The equivalence of (ii) (without characterization of linearity of the concave function) and (v) follows from taking the second derivative of $\lambda \mapsto |(1-\lambda)K +_p \lambda C|$ at $\lambda = 0^+$ and some additional arguments according to Kolesnikov, E. Milman [480] (see also Putterman [629]), and we discuss the equivalence of (v) and (vi) - due to Kolesnikov, E. Milman [480] - below. The implication that (i) yields (v) was proved earlier by Colesanti, Livshyts, Marsiglietti [212].

(i) $|(1-\lambda)\cdot K +_p \lambda\cdot C| \geqslant |K|^{1-\lambda}|C|^{\lambda}$ with equality if and only if K and C are dilates;

(ii) The function $\lambda \mapsto |(1-\lambda)\cdot K +_p \lambda\cdot C|^{\frac{p}{n}}$ is concave on $[0,1]$, and is linear if and only if K and C are dilates;

(iii) $\displaystyle\int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K \geqslant |K|^{\frac{n-p}{n}}|C|^{\frac{p}{n}}$, with equality if and only if K and C are dilates.

(iv) The inequality in (iii) for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with C_+^∞ boundary is equivalent to saying that for any positive even C^∞ function f on S^{n-1}, the Monge-Ampère equation

$$h^{1-p}\det(\nabla^2 h + h\, I_{n-1}) = f$$

on the sphere S^{n-1} has a unique even solution. The inequality in (iii) for any o-symmetric K, C follows from the smooth case by approximation.

(v) $\displaystyle\frac{V(K,C;1)^2}{|K|} \geqslant \frac{n-1}{n-p}V(K,C;2) + \frac{1-p}{n-p}\int_{S^{n-1}} \frac{h_C^2}{h_K^2}\, dV_K$, which is a strengthened form of Minkowski's second inequality (7.29) because the Hölder's inequality (10.4) implies $|K|\cdot \frac{1}{n}\int_{S^{n-1}} \frac{h_C^2}{h_K}\, dS_K \geqslant (\frac{1}{n}\int_{S^{n-1}} h_C\, dS_K)^2 = V(K,C;1)^2$.

(vi) The inequalities above without characterization of equality are equivalent to

$$\lambda_{1,e}(-\mathcal{L}_K) \geqslant \frac{n-p}{n-1} \tag{8.128}$$

for any o-symmetric convex body $K \subset \mathbb{R}^n$ with C_+^∞ boundary where $\mathcal{L}_K$ is the elliptic self-adjoint Hilbert-Brunn-Minkowski operator of Kolesnikov, E. Milman [480] (see below for definition and properties), and $\lambda_{1,e}(-\mathcal{L}_K)$ is the minimal positive eigenvalue when we restrict $-\mathcal{L}_K$ to even C^∞ functions on S^{n-1}.

In the rest of the section, we discuss the Hilbert-Brunn-Minkowski operator $\mathcal{L}_K$ in (vi). For a convex body $K \subset \mathbb{R}^n$ with C_+^∞ boundary, Kolesnikov, E. Milman [480] - extending Hilbert's work around 1910 - defines the Hilbert-Brunn-Minkowski operator $\mathcal{L}_K$ as

$$\mathcal{L}_K\varphi = \frac{\mathcal{D}(\widetilde{D}^2(\varphi\, h_K), \widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)}{\mathcal{D}(\widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)} - \varphi \tag{8.129}$$

for a $\varphi \in C^\infty(S^{n-1})$ where $\mathcal{D}(\cdot,\ldots,\cdot)$ is the mixed discriminant of $n-1$ matrices of size of $(n-1)\times(n-1)$ (cf. Definition 8.3.2), and we frequently write simply h_K to denote $h_K|_{S^{n-1}}$, as well.

First we discuss the elementary properties of Hilbert-Brunn-Minkowski operator. Let us recall (cf. Definition 8.1.6) that for any $\psi \in C^\infty(S^{n-1})$, writing $\tilde{\psi} : \mathbb{R}^n\backslash\{o\}$ and $\bar{\psi} : \mathbb{R}^n\backslash\{o\}$ to denote the functions $\tilde{\psi}(tu) = t\cdot\psi(u)$ and $\bar{\psi}(tu) = \psi(u)$ for $t > 0$ and

$u \in S^{n-1}$, we have

$$\nabla^2\psi(u) = D^2\bar{\psi}(u)|_{u^\perp} \text{ and } \widetilde{D}^2\psi(u) = D^2\tilde{\psi}(u)|_{u^\perp} = \nabla^2\psi(u) + \psi(u)\cdot I_{n-1}. \quad (8.130)$$

For basic properties of self-adjoint elliptic operators, see Section 10.7.

Lemma 8.8.8 (Kolesnikov, Milman). *Let $K \subset \mathbb{R}^n$ be a convex body with C^∞_+ boundary and $o \in \text{int}K$.*

(i) $\mathcal{L}_K$ is elliptic.

(ii) $\int_{S^{n-1}} \varphi(\mathcal{L}_K\psi)\, dV_K = \int_{S^{n-1}} (\mathcal{L}_K\varphi)\psi\, dV_K$ for $\varphi, \psi \in C^\infty(S^{n-1})$, and hence $\mathcal{L}_K$ has a self-adjoint extension to $L^2(S^{n-1}, V_K)$.

(iii) $\mathcal{L}_K\varphi \equiv 0$ for any constant function φ.

(iv) $-\mathcal{L}_K\ell_{K,w} = \ell_{K,w}$ for any $w \in \mathbb{R}^n$ where $\ell_{K,w}(u) = \frac{\langle w,u\rangle}{h_K(u)}$.

Proof. For (iv), we have $\ell_{K,w}(u)h_K(u) = \langle w,u\rangle$ is a linear function, and hence $\widetilde{D}^2(\ell_{K,w}h_K)$ is the zero matrix.

To understand $\mathcal{L}_K$, using the notation of (8.130) yields

$$\widetilde{D}^2(\varphi\, h_K)(u) = D^2\left(\bar{\varphi}h_K)(u)\right)|_{u^\perp} = \varphi(u)\cdot\widetilde{D}^2h_K(u) + h_K(u)\cdot\nabla^2\varphi(u) + \mathcal{G}_K\varphi(u)$$

where $\mathcal{G}_K\varphi(u) = \left[D\bar{\varphi}(u)Dh_K(u)^t\right]|_{u^\perp} + \left[Dh_K(u)D\bar{\varphi}(u)^t\right]|_{u^\perp}$, or in other words, $\mathcal{G}_K\varphi = (\nabla\varphi)(\nabla h_K)^t + (\nabla h_K)(\nabla\varphi)^t$. It follows that

$$\mathcal{L}_K\varphi = h_K\cdot\frac{\mathcal{D}(\nabla^2\varphi, \widetilde{D}^2h_K, \ldots, \widetilde{D}^2h_K)}{\mathcal{D}(\widetilde{D}^2h_K, \ldots, \widetilde{D}^2h_K)} + \frac{\mathcal{D}(\mathcal{G}_K\varphi, \widetilde{D}^2h_K, \ldots, \widetilde{D}^2h_K)}{\mathcal{D}(\widetilde{D}^2h_K, \ldots, \widetilde{D}^2h_K)}. \quad (8.131)$$

In particular, $\mathcal{L}_K$ is elliptic (cf. (8.29) and (10.22) in Section 10.7).

If φ is a constant, then $\bar{\varphi}$ is also a constant, and hence $\nabla\varphi(u) = D\bar{\varphi}(u)|_{u^\perp}$ and $\nabla^2\varphi(u) = D^2\bar{\varphi}(u)|_{u^\perp}$ are zero. We deduce from (8.131) that $\mathcal{L}_K\varphi \equiv 0$.

By now, all we are left to do is to prove (ii). We recall (cf. (8.45)) the differential operator

$$\mathcal{A}\varphi = \mathcal{D}(\widetilde{D}^2\varphi, \widetilde{D}^2h_K, \ldots, \widetilde{D}^2h_K) \quad (8.132)$$

for $\varphi \in C^\infty(S^{n-1})$, which is symmetric with respect to $L^2(S^{n-1}, \mathcal{H}^{n-1})$ according to (8.47); namely,

$$\int_{S^{n-1}} \varphi\cdot\mathcal{A}\psi\, d\mathcal{H}^{n-1} = \int_{S^{n-1}} \psi\cdot\mathcal{A}\varphi\, d\mathcal{H}^{n-1} \text{ for any } \varphi, \psi \in C^\infty(S^{n-1}). \quad (8.133)$$

For any $f \in C(S^{n-1})$, we have (cf. (8.18))

$$\int_{S^{n-1}} f\, dV_K = \frac{1}{n}\int_{S^{n-1}} f\cdot h_K\, dS_K = \frac{1}{n}\int_{S^{n-1}} f\cdot h_K\cdot\mathcal{D}(\widetilde{D}^2h_K, \ldots, \widetilde{D}^2h_K)\, d\mathcal{H}^{n-1}.$$

We deduce that

$$\int_{S^{n-1}} \varphi(\mathcal{L}_K\psi)\,dV_K = \frac{1}{n}\int_{S^{n-1}} (\varphi h_K)\cdot \mathcal{A}(\psi h_K)\,d\mathcal{H}^{n-1}$$
$$= \frac{1}{n}\int_{S^{n-1}} (\psi h_K)\cdot \mathcal{A}(\varphi h_K)\,d\mathcal{H}^{n-1} = \int_{S^{n-1}} \psi(\mathcal{L}_K\varphi)\,dV_K,$$

completing the proof of Lemma 8.8.8. ∎

The following is Hilbert's result (see [391], Chapter XIX) about the Brunn-Minkowski inequality as reinterpreted by Kolesnikov, E. Milman [480]:

Theorem 8.8.9 (Hilbert). *For a convex body $K \subset \mathbb{R}^n$ with C^∞_+ boundary and $o \in \operatorname{int} K$, all eigenvalues of $-\mathcal{L}_K$ are non-negative, the eigenvalue 0 is simple, the eigenvalue 1 has multiplicity n where the corresponding eigenspace consists of all $\ell_{K,w}$, $w \in \mathbb{R}^n$, and the rest of the eigenvalues are larger than 1.*

Remark. Hilbert's key observation was (in the setting of Kolesnikov, E. Milman [480]) that the positive eigenvalues of $-\mathcal{L}_K$ are at least 1, and this fact is equivalent with the Brunn-Minkowski inequality.

Next, let K be an o-symmetric convex body with C^∞_+ boundary, and let $\lambda_{1,e}(-\mathcal{L}_K)$ be the minimal positive eigenvalue when $-\mathcal{L}_K$ is restricted to even C^2 functions on S^{n-1}. Since the eigenfunctions $\ell_{K,w}$, $w \in \mathbb{R}^n$, of $-\mathcal{L}_K$ are odd functions, we have $\lambda_{1,e}(-\mathcal{L}_K) > 1$. Kolesnikov, E. Milman [480] proved that any uniform improvement on this spectral gap estimate is equivalent to the L_p-Minkowski Conjecture 8.8.3 for a suitable $p \in [0,1)$.

Theorem 8.8.10 (Kolesnikov, E. Milman). *Let $p \in [0,1)$, and let $K \subset \mathbb{R}^n$ be an o-symmetric convex body with C^∞_+ boundary. The estimate $\lambda_{1,e}(-\mathcal{L}_K) \geqslant \frac{n-p}{n-1}$ is equivalent with (v) above for any o-symmetric convex body $C \subset \mathbb{R}^n$.*

Remark. In particular, for given $p \in [0,1)$, the estimate $\lambda_{1,e}(-\mathcal{L}_K) \geqslant \frac{n-p}{n-1}$ for any o-symmetric convex body $K \subset \mathbb{R}^n$ with C^∞_+ boundary is equivalent to saying that the the L_p-Minkowski Conjecture 8.8.3 holds for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ (without the characterization of the case of equality).

As a spectacular achievement, Kolesnikov, E. Milman [480] managed to provide a uniform positive lower bound for $\lambda_{1,e}(-\mathcal{L}_K) - 1$, which, combined with the bound of Theorem 4.7.14 due to Klartag [468] on the Cheeger constant in the KLS constant, reads as follows.

Theorem 8.8.11 (Kolesnikov, Milman). *If $K \subset \mathbb{R}^n$ is an o-symmetric convex body with C^∞_+ boundary, then*

$$\lambda_{1,e}(-\mathcal{L}_K) \geqslant 1 + \frac{c}{n^2 \log n}$$

for an absolute constant $c > 0$.

Combining Theorems 8.8.10 and 8.8.11 yields Theorem 8.8.6; namely, if $1 - \frac{c}{n \log n} < p < 1$ for an absolute constant $c > 0$, then the L_p-Brunn-Minkowki Conjecture 8.8.1 and the L_p-Minkowski Conjecture 8.8.3 hold for any origin symmetric convex bodies K, C and $\lambda \in (0,1)$.

Remark 8.8.12.

- According to E. Milman [579], $\Delta_K = (n-1)\mathcal{L}_K$ is the centro-affine Laplacian for a convex body $K \subset \mathbb{R}^n$ with C^∞_+ boundary and $o \in \operatorname{int} K$.
- While in this section, we have been discussing the case when ∂K is C^∞_+ in order to match the set up of most books on elliptic operators, the theory works for convex bodies with C^2_+ boundary, as well.

8.8.2 L_p combination of polytopes when $p \in (0,1)$

For $p \in (0,1)$, we show that the L_p combination of n-polytopes is a polytope, but the boundary of the L_p combination of convex bodies with C^2_+ boundaries may not be even C^1.

Lemma 8.8.13. *If $p \in (0,1)$ and $K, C \subset \mathbb{R}^n$ are n-polytopes, then $(1-\lambda)K +_p \lambda C$ is an n-polytope whose facet exterior unit normals are facet exterior unit normals to the Minkowski sum $K + C$, as well.*

Proof. Let $u_1, \dots, u_m$ be the exterior unit normals to the facets of the polytope $K + C$ (cf. (1.7)). As a convex body is determined by the exterior normals at the regular boundary points (cf. Lemma 2.5.6), Lemma 8.8.13 follows if whenever $u = \nu_{(1-\lambda)K +_p \lambda C}(z)$ for a $z \in \partial'((1-\lambda)K +_p \lambda C)$, then $u \in \{u_1, \dots, u_m\}$. Let w be a vertex of $K + C$ such that u is an exterior normal at w to $K + C$. According to Lemma 1.4.10, the normal cone at w is the positive hull of a subset of $\{u_1, \dots, u_m\}$; therefore, we may assume that $u = \sum_{i=1}^q \alpha_i u_i$ for $\alpha_1, \dots, \alpha_q \geqslant 0$, $q \geqslant 1$, where $u_1, \dots, u_q$ are exterior normals at w to $K + C$. Now $w = x + y$ for a vertex x of K and a vertex y of C, and hence $u, u_1, \dots, u_q$ are exterior normals at x to K and at y to C, and $h_K(u_i) = \langle x, u_i \rangle$ and $h_C(u_i) = \langle y, u_i \rangle$ for $i = 1, \dots, q$. It follows from the fact that the function defining the Wulff shape agrees with the support function at the exterior normals at the regular

boundary points (cf. Lemma 7.5.1) that

$$\begin{aligned}\langle z,u\rangle = h_{(1-\lambda)\cdot K+_p\lambda\cdot C}(u) &= ((1-\lambda)h_K(u)^p + \lambda\, h_C(u)^p)^{\frac{1}{p}}\\ &= \left((1-\lambda)\left(\sum_{i=1}^q \alpha_i\langle x,u_i\rangle\right)^p + \lambda\left(\sum_{i=1}^q \alpha_i\langle y,u_i\rangle\right)^p\right)^{\frac{1}{p}}\\ &= \left((1-\lambda)\left(\sum_{i=1}^q \alpha_i h_K(u_i)\right)^p + \lambda\left(\sum_{i=1}^q \alpha_i h_C(u_i)\right)^p\right)^{\frac{1}{p}}.\end{aligned}$$

We apply the Minkowski inequality (10.10) with $f(u_i) = (1-\lambda)h_K(u_i)^p$, $g(u_i) = \lambda\, h_C(u_i)^p$ and $\mu(\{u_i\}) = a_i$ and with parameter $1/p > 1$ instead of p to conclude that

$$\begin{aligned}\langle z,u\rangle &\geqslant \sum_{i=1}^q \alpha_i\,((1-\lambda)h_K(u_i)^p + \lambda\, h_C(u_i)^p)^{\frac{1}{p}}\\ &\geqslant \sum_{i=1}^q \alpha_i h_{(1-\lambda)K+_p\lambda C}(u_i) \geqslant \sum_{i=1}^q \alpha_i\langle z,u_i\rangle = \langle z,u\rangle.\end{aligned}$$

We deduce that $h_{(1-\lambda)K+_0\lambda C}(u_i) = \langle z,u_i\rangle$ for $i = 1,\dots,q$, and hence $u_1,\dots,u_q$ are exterior unit normals at $z \in \partial'((1-\lambda)K +_0 \lambda C)$. We conclude that $q = 1$; therefore, $u = u_1$. ∎

Example 8.8.14 (For $p \in (0,1)$, the boundary of the L_p combination of convex bodies with C^2_+ boundaries may not be C^1). Let $p \in (0,1)$, let e_1, e_2 be the orthonormal basis of $\mathbb{R}^2$, and let $a > 2$ be large enough such that $b = \frac{a}{2^{1/p}}(1+a^{-2p})^{1/p} < \frac{a}{2}$. We consider the ellipses $E_a = \operatorname{diag}[a,a^{-1}]B^2$ and $\widetilde{E}_a = \operatorname{diag}[a^{-1},a]B^2$. Since $E_a \subset P_a = [a,-a]\times[\frac{1}{a},\frac{-1}{a}]$ and $\widetilde{E}_a \subset \widetilde{P}_a = [\frac{1}{a},\frac{-1}{a}]\times[a,-a]$, we deduce using Lemma 8.8.13 that

$$\tfrac{1}{2}\cdot E_a +_p \tfrac{1}{2}\cdot\widetilde{E}_a \subset \tfrac{1}{2}\cdot P_a +_p \tfrac{1}{2}\cdot\widetilde{P}_a = [-b,b]\times[-b,b] = W.$$

On the other hand, for $u = (\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, we have

$$\begin{aligned}\left(\frac{1}{2}\,h_{E_a}(u)^p + \frac{1}{2}\,h_{\widetilde{E}_a}(u)^p\right)^{\frac{1}{p}} &> \left(\frac{1}{2}\,\langle ae_1,u\rangle^p + \frac{1}{2}\,\langle ae_2,u\rangle^p\right)^{\frac{1}{p}}\\ &= \frac{a}{\sqrt{2}} > h_W(u) \geqslant h_{\frac{1}{2}\cdot E_a+_p\frac{1}{2}\cdot\widetilde{E}_a}(u);\end{aligned}$$

therefore, u is not an exterior normal at a regular boundary point of $\frac{1}{2}\cdot E_a +_p \frac{1}{2}\cdot\widetilde{E}_a$ by Lemma 7.5.1.

8.9 L_p-Affine Surface Areas for $p > 0$

For a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary, the affine surface area

$$\Omega(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}}\, d\mathcal{H},$$

and if, in addition, $o \in \operatorname{int} K$, then the centro-affine surface area

$$\widetilde{\Omega}(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{2}}}{\langle x, \nu_K(x)\rangle^{\frac{n-1}{2}}}\, d\mathcal{H}^{n-1}(x)$$

- that are the main subjects of this section - were investigated as part of Affine Differential Geometry in the first half of the 20th century. Their importance comes from the fact that $\Omega(K)$ is equiaffine invariant (invariant under volume preserving affine transformations), while $\widetilde{\Omega}(K)$ is $\mathrm{GL}(n)$ invariant, and they satisfy the so-called affine isoperimetric and the centro-affine isoperimetric inequalities. The 1980's and 1990's saw various approaches - that are reviewed in the Comments Section 8.10 to Chapter 8 - to define affine surface area of convex bodies with non-smooth boundary, which notions eventually turned out to be equivalent, and lead to the proof of upper semi-continuity. In order to discuss these notions - and their extension, the so-called L_p-affine surface area for $p > 0$ - for convex bodies for non-smooth boundaries, as well, we need to deal with various technical statements concerning the boundary of a convex body.

8.9.1 Affine Surface Area and Centro-Affine Surface Area

Let $K \subset \mathbb{R}^n$, $n \geqslant 2$, be a convex body. We deduce from Theorem 8.1.1 on the second order differentiabily of ∂K and Aleksandrov's Theorem 10.6.2 on the second order differentiabily of convex functions that ∂K is $\mathcal{H}^{n-1}$ a.e. twice differentiable, and (restricted) support function $h_K|_{S^{n-1}}$ is also $\mathcal{H}^{n-1}$ a.e. twice differentiable on S^{n-1} in the Aleksandrov sense. In particular, we can speak about the generalized Gaussian curvature $\kappa(x) \in [0, \infty)$ at $\mathcal{H}^{n-1}$ a.e. $x \in \partial K$, and the *generalized curvature function*

$$f_K(u) = \sigma_{n-1} D^2 h_K(u) = \det\left(D^2 h_K(u)|_{u^\perp}\right)$$

at $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$ where the equality between the two formulations follows from the homogeneity of h_K as a function on $\mathbb{R}^n$. According to Remark 8.2.3 due to Hug [425], the surface area measure S_K on S^{n-1} can be written as $S_K = S^a_K + S^s_K$ where

- $dS^a_K = f_K d\mathcal{H}^{n-1}$ is the absolutely continuous part and $f_K(u) = \sigma_{n-1} D^2 h_K(u)$ is the generalized curvature function for $\mathcal{H}^{n-1}$ at a.e. $u \in S^{n-1}$;
- S^s_K is a singular Borel measure ion S^{n-1} (i.e. there exists $X_K \subset S^{n-1}$ such that $\mathcal{H}^{n-1}(X_K) = 0$ and $S^s_K(S^{n-1} \backslash X_K) = 0$) and S^s_K is regular (cf. Theorem 10.1.3);

- the measurable subset $Y_K = \{y \in \partial' K : \exists \kappa(y) > 0\}$ of $\partial' K$ (possibly $Y_K = \varnothing$) satisfies that $\nu_K(Y_K) \cap X_K = \varnothing$, $f_K(u) = 0$ for a.e. $u \in S^{n-1} \backslash \nu_K(Y_K)$ and

$$f_K(\nu_K(y)) \cdot \kappa(y) = 1 \quad \text{for } y \in Y_K. \tag{8.134}$$

According to Proposition 8.2.5, if $\psi : S^{n-1} \to [0, \infty)$ is measurable and satisfies that $\psi(\nu_K(x)) = 0$ whenever $\kappa(x)$ exists and $\kappa(x) = 0$, then

$$\int_{\partial K} \psi(\nu_K(x))\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} \psi \cdot f_K \, d\mathcal{H}^{n-1}. \tag{8.135}$$

The following definitions of the two core versions of affine surface area were initiated around 1920 in the case of convex bodies with C^2_+ boundaries in $\mathbb{R}^2$ and $\mathbb{R}^3$. The equivalence of the two types of definitions in Definition 8.9.1 follows from Theorem 8.1.5 if the convex body K has C^2_+ boundary, and from (8.134) and (8.135) in general (the statement in the general case is proved by Schütt [670] and Dolzmann, Hug [236]).

Definition 8.9.1. Let $K \subset \mathbb{R}^n$ be a compact convex set.

Affine Surface Area: If $\dim K \leqslant n-1$, then $\Omega(K) = 0$, and if $\operatorname{int} K \neq \varnothing$, then

$$\Omega(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}}\, d\mathcal{H}^{n-1}(u).$$

Centro-Affine Surface Area: Assuming that $o \in \operatorname{int} K$, let

$$\widetilde{\Omega}(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{2}}}{\langle x, \nu_K(x)\rangle^{\frac{n-1}{2}}}\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{1}{2}}}{h_K(u)^{\frac{n-1}{2}}}\, d\mathcal{H}^{n-1}(u).$$

Remarks.

- $\Omega(K)$ is translation invariant; namely, $\Omega(K) = \Omega(K + w)$ for $w \in \mathbb{R}^n$;
- $\Omega(\lambda K) = \lambda^{\frac{n(n-1)}{n+1}} \Omega(K)$ and $\widetilde{\Omega}(\lambda K) = \widetilde{\Omega}(K)$ for $\lambda > 0$;
- If $K \subset \mathbb{R}^n$ is a convex body with C^2_+ boundary and $o \in \operatorname{int} K$, then $\Omega(K) > 0$ and $\widetilde{\Omega}(K) > 0$;
- If $K \subset \mathbb{R}^n$ is a polytope with $o \in \operatorname{int} K$, then $\Omega(K) = \widetilde{\Omega}(K) = 0$.
- The finiteness of the integrals in Definition 8.9.1 follow, for example, from the Affine and Centro-Affine Isoperimetric Inequalities in Theorem 8.9.10 and Theorem 8.9.14.

One of the core properties of affine surface area and the centro-affine surface area is their invariance under certain linear maps. We remark that in spite of its name, the affine surface area is not invariant under all affine transformations, only under the

volume preserving ones (the so-called equiaffine transformations, cf. Theorem 8.9.7). In addition, centro-affine surface area is invariant under $\mathrm{GL}(n)$ (cf. Theorem 8.9.7).

In order to discuss invariance under certain linear maps, we need the notion of the $\mathrm{SL}(n)$ invariant centro-affine curvature.

8.9.2 The Centro-Affine Curvature

The invariance of the centro-affine curvature under volume preserving linear transformations (at least in $\mathbb{R}^2$) was already observed by Tzitzéica [705] in 1908.

Definition 8.9.2 (Centro-Affine Curvature, Tzitzéica). If $K \subset \mathbb{R}^n$ is a convex body with $o \in \mathrm{int}K$, and ∂K is twice differentiable at $x \in \partial K$, then the centro-affine curvature at x is

$$\kappa_0(K,x) = \frac{\kappa_K(x)}{\langle x, \nu_K(x)\rangle^{n+1}} = \frac{\kappa_K(x)}{h_K(\nu_K(x))^{n+1}}.$$

Proposition 8.9.3 (Tzitzéica). *If $K \subset \mathbb{R}^n$ is a convex body with $o \in \mathrm{int}K$, ∂K is twice differentiable at $x \in \partial K$, and $\Phi \in \mathrm{GL}(n)$, then*

$$\kappa_0(\Phi K, \Phi x) = (\det \Phi)^2 \kappa_0(K,x).$$

Remark. It follows that $\kappa_0(K,x)^{-\frac{1}{2}}$ is proportional with the volume of the osculating o-symmetric ellipsoid at x.

Proof. We may assume that $\det \Phi = 1$. It is equivalent to prove that for any $\lambda \in (1,2)$, we have

$$\lambda^{-1}\kappa_0(K,x) \leqslant \kappa_0(\Phi K, \Phi x) \leqslant \lambda \kappa_0(K,x). \tag{8.136}$$

Fix $\lambda \in (1,2)$. First we verify some formulas for a convex body $M \subset \mathbb{R}^n$ with $o \in \mathrm{int}M$ assuming that ∂M is twice differentiable at a $y \in \partial M$. For $u = -\nu_K(y)$ and $z_0 = y|u^\perp$, let $\varphi : (\mathrm{int}M)|u^\perp \to \mathbb{R}$ be the convex function such that $z + \varphi(z)u \in \partial K$ and $y = z_0 + \varphi(z_0)u$. As ∂M is twice differentiable at y, there exists a positive semi-definite quadratic form Q_y with $\kappa(y) = \det Q_y$ on $u^\perp$ such that

$$\varphi(z) - \varphi(z_0) = \frac{1}{2} Q_y(z - z_0) + o(\|z - z_0\|^2).$$

If $0 < t < \langle y, \nu_M(y)\rangle$, then we consider the cap

$$C(M,y,t) = \{w \in M : \langle w, \nu_M(y)\rangle \geqslant \langle y, \nu_M(y)\rangle - t\}.$$

First let $\kappa(y) > 0$. There exists $\theta(M,y) \in (0,1)$ such that if $0 < t < \theta(M,y)\langle y, \nu_M(y)\rangle$, then as subsets of $u^\perp$,

$$\lambda^{\frac{-1}{2(n-1)}} \cdot \left\{\tfrac{1}{2} Q_y \leqslant t\right\} \leqslant \{\varphi \leqslant t\} \leqslant \lambda^{\frac{1}{2(n-1)}} \cdot \left\{\tfrac{1}{2} Q_y \leqslant t\right\},$$

which in turn yields that

$$\lambda^{\frac{-1}{2}} \cdot \frac{2^{\frac{n-1}{2}} t^{\frac{n-1}{2}}}{\kappa(y)^{\frac{1}{2}}} \leqslant \mathcal{H}^{n-1}\Big(\{\varphi \leqslant t\}\Big) \leqslant \lambda^{\frac{1}{2}} \cdot \frac{2^{\frac{n-1}{2}} t^{\frac{n-1}{2}}}{\kappa(y)^{\frac{1}{2}}}.$$

Therefore if $\kappa(y) > 0$ and $0 < s < \theta(M, y)$, then

$$\lambda^{\frac{-1}{2}} \cdot \frac{2^{\frac{n+1}{2}} s^{\frac{n+1}{2}}}{n+1} \cdot \frac{\langle y, \nu_M(y)\rangle^{\frac{n+1}{2}}}{\kappa(y)^{\frac{1}{2}}} \leqslant |C(M, y, s\langle y, \nu_M(y)\rangle)| \leqslant \lambda^{\frac{1}{2}} \cdot \frac{2^{\frac{n+1}{2}} s^{\frac{n+1}{2}}}{n+1} \cdot \frac{\langle y, \nu_M(y)\rangle^{\frac{n+1}{2}}}{\kappa(y)^{\frac{1}{2}}} \tag{8.137}$$

where $\frac{\langle y, \nu_M(y)\rangle^{\frac{n+1}{2}}}{\kappa(y)^{\frac{1}{2}}} = \kappa_0(M, y)^{\frac{-1}{2}}$.

If $\kappa(y) = 0$, then similar argument as above yields

$$\lim_{s \to 0^+} s^{-\frac{n+1}{2}} |C(M, y, s\langle y, \nu_M(y)\rangle)| = \infty. \tag{8.138}$$

Since $\Phi(\nu_K(x)^\perp) = \nu_{\Phi K}(\Phi x)^\perp$, we deduce that

$$\Phi\, C(K, x, s\langle x, \nu_K(x)\rangle) = C(\Phi K, \Phi x, s\langle \Phi x, \nu_{\Phi K}(\Phi x)\rangle)$$

for small $s > 0$, and hence $\Big|C(K, x, s\langle x, \nu_K(x)\rangle)\Big| = \Big|C(\Phi K, \Phi x, s\langle \Phi x, \nu_{\Phi K}(\Phi x)\rangle)\Big|$. Therefore, combining (8.137) and (8.138) yields (8.136). ∎

As we have seen in Section 1.9, the polar $K^* = \{y \in \mathbb{R}^n : \langle y, x\rangle \leqslant 1\}$ of a convex body $K \subset \mathbb{R}^n$ with $o \in \mathrm{int}K$ satisfies that $(\Phi K)^* = \Phi^{-t} K^*$ for $\Phi \in \mathrm{GL}(n)$, and if $y \in \mathbb{R}^n$ is an exterior normal $x \in \partial K$ with $\langle y, x\rangle = 1$, then x is an exterior normal at $y \in \partial K^*$. The following statement was already known at the beginning of the 20th century in the case of C^2_+ boundary, and is due to Hug [424] for general convex bodies.

Lemma 8.9.4 (Hug [424]). *Let $K \subset \mathbb{R}^n$ be convex body with $o \in \mathrm{int}K$. If ∂K is twice differentiable at $x \in \partial K$ with $\kappa_{\partial K}(x) > 0$, then ∂K^* is twice differentiable at $x^* \in \partial K^*$ with $\kappa_{\partial K^*}(x^*) > 0$ and*

$$\kappa_0(K, x) \cdot \kappa_0(K^*, x^*) = 1 \tag{8.139}$$

where x^* is the the exterior normal at x with $\langle x^*, x\rangle = 1$.

Proof. Since this property is invariant under linear transformations by Proposition 8.9.3, we may assume that $x = \nu_K(x) = x^* = \nu_{K^*}(x^*)$. In addition, we may assume that each principal curvature at $x \in \partial K$ is one.

For any $\varepsilon > 0$, we consider the ellipsoid E_ε whose one semi axis is $\mathrm{conv}\{0, x\}$, and the other semi axes are of length $1 + \varepsilon$. It follows that $\mathrm{conv}\{0, x\}$ is a semi axis

of E_ε^*, as well, and the other semi axes of E_ε^* are of length $(1+\varepsilon)^{-1}$. In addition, for any $\varepsilon > 0$, there exists $s \in (0,1)$ depending on ε and K such that

$$\{z \in E_\varepsilon^* : \langle z, x\rangle \geqslant 1-s\} \subset \{z \in K : \langle z, x\rangle \geqslant 1-s\} \subset E_\varepsilon.$$

We deduce the existence of $t \in (0,s)$ (depending on ε, K and s) such that

$$\{z \in E_\varepsilon^* : \langle z, x\rangle \geqslant 1-t\} \subset \{z \in K^* : \langle z, x\rangle \geqslant 1-t\} \subset E_\varepsilon,$$

which in turn yields (8.139) by the arbitraryness of $\varepsilon > 0$. ∎

Similar argument like the one for Lemma 8.9.4 yields the following statement:

Lemma 8.9.5 (Hug [424]). *Let $K \subset \mathbb{R}^n$ be convex body, and let $x \in \partial K$. Then ∂K is twice differentiable at x with $\kappa_{\partial K}(x) > 0$ if and only if h_K is twice differentiable at $\nu_K(x)$ with $\sigma_{n-1} D^2 h_K(\nu_K(x)) > 0$, and in this case, we have*

(i) $f_K(\nu_K(x)) = \sigma_{n-1} D^2 h_K(\nu_K(x)) = \kappa_{\partial K}(x)^{-1}$;

(ii) $\kappa_0(K,x)^{-1} = f_K(\nu_K(x)) \cdot h_K(\nu_K(x))^{n+1}$.

Remark. Based on (ii), $f_K(u) \cdot h_K(u)^{n+1}$, $u \in S^{n-1}$, is frequently called the centro-affine curvature function.

Proof. We may assume that $o \in \operatorname{int} K$. Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ where $e_n = \nu_K(x)$, either $e_1, \dots, e_{n-1}$ are the principal directions at $x \in \partial K$ (if ∂K is twice differentiable at x) or $e_1, \dots, e_n$ are the principal directions for $D^2 h_K(\nu_K(x))$ (if h_K is twice differentiable at $\nu_K(x)$). In particular, if h_K is twice differentiable at $\nu_K(x)$, then any second partial derivative involving ∂_n is zero. In the argument, we use Proposition 8.9.3 and the fact that $h_{\Phi K} = h_K \circ \Phi^t$ for any convex body K and $\Phi \in \mathrm{GL}(n)$.

First applying a linear transform with $e_n \mapsto e_n + \sum_{i=1}^{n-1} \alpha_i e_i$ and $e_i \mapsto e_i$ for $i \leqslant n-1$, we may assume that $x = h_K(\nu_K(x)) \cdot \nu_K(x)$. Then applying a linear transform with $e_i \mapsto e_i$ for $i \leqslant n-1$ and $e_n \mapsto \beta e_n$, we may assume that $h_K(e_n) = 1$. Finally, applying a a linear transform with $e_i \mapsto \tau_i e_i$ for $i \leqslant n-1$ and $e_n \mapsto e_n$, we may assume that either each principal curvature at $x = e_n$, or each eigenvalue of $D^2 h_K(e_n)$ corresponding to e_i, $i \leqslant n-1$, is 1.

Since $h_K(u) = \|u\|_{K^*}$, we complete the argument for (i) using the ellipsoids E_ε and E_ε^* as in the proof of Lemma 8.9.4. In addition, (ii) follows from (i). ∎

8.9.3 L_p-Affine Surface Areas for $p > 0$ and their linear invariance

For $p \geqslant 1$, Lutwak [552] provided a rather natural common generalization of the notions of affine surface area and centro-affine surface area in the form of the $\mathrm{SL}(n)$ invariant L_p-affine surface area. The new notion of L_p-affine surface area was extended to $p \geqslant 0$ by Hug [423], who also verified the equivalence of the two types ((8.142)

and (8.143)) of definitions for any convex body (see also Schütt, Werner [673] and Werner [715] for equivalent definitions). Ludwig, Reitzner [546] provided the definition (8.141) of the L_p-affine surface area exhibiting the $\mathrm{SL}(n)$ invariance, which definition uses the notion of auxiliary cone volume measure $\widetilde{V}_K$ for a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ introduced in Section 2.6. In particular, if $\Xi \subset \partial K$ is Borel, then (cf. (2.35))

$$\widetilde{V}_K(\Xi) = \frac{1}{n} \int_{\Xi} \langle \nu_K(x), x \rangle \, d\mathcal{H}^{n-1}(x),$$

and if $\Phi \in \mathrm{GL}(n)$, then (cf. Proposition 2.6.15)

$$\widetilde{V}_{\Phi K}(\Phi \Xi) = |\det \Phi| \cdot \widetilde{V}_K(\Xi). \tag{8.140}$$

In turn, we deduce the equivalence of (8.141) and (8.142) in Definition 8.9.6 of the L_p affine surface area, and the equivalence of (8.142) and (8.143) follows from (8.134) and (8.135). The finitenesss of the integral follows, for example, from the L_p-Affine Isoperimetric inequality in Theorem 8.9.15.

Definition 8.9.6 (L_p-affine surface area). If $p \geqslant 0$ and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then

$$\Omega_p(K) = n \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}} \, d\widetilde{V}_K(x) \tag{8.141}$$

$$= \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, \nu_K(x) \rangle^{\frac{n(p-1)}{n+p}}} \, dx \tag{8.142}$$

$$= \int_{S^{n-1}} f_K^{\frac{n}{n+p}} \cdot h_K^{\frac{n(1-p)}{n+p}} \, d\mathcal{H}^{n-1}. \tag{8.143}$$

In addition, if $C \subset \mathbb{R}^n$ is a compact convex set with $o \in C$, then $\Omega_p(C) = 0$.

Remark. $\Omega_0(K) = n|K|$, $\Omega_1(K) = \Omega(K)$ and $\Omega_n(K) = \widetilde{\Omega}(K)$.

The following invariance properties of the affine surface areas directly follow from (8.140) and Proposition 8.9.3.

Theorem 8.9.7. *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) $\Omega(\Phi K + w) = \Omega(K)$ *if* $\Phi \in \mathrm{GL}(n)$, $\det \Phi = \pm 1$ *and* $w \in \mathbb{R}^n$.

(ii) $\widetilde{\Omega}(\Phi K) = \widetilde{\Omega}(K)$ *if* $o \in \operatorname{int} K$ *and* $\Phi \in \mathrm{GL}(n)$.

(iii) $\Omega_p(\Phi K) = \Omega(K)$ *for* $p \geqslant 0$ *if* $o \in \operatorname{int} K$ *and* $\Phi \in \mathrm{GL}(n)$ *with* $\det \Phi = \pm 1$.

8.9.4 Upper semi-continuity, Affine Isoperimetric Inequality

We recall that $|K^*| = \frac{1}{n} \int_{S^{n-1}} \varrho_{K^*}(u)^n \, du = \frac{1}{n} \int_{S^{n-1}} h_K(u)^{-n} \, du$ if K is a convex body with $o \in \operatorname{int} K$ (see Section 1.9). We now prove a useful upper bound on the volume of the L_p-affine surface area in terms of the convex body and its polar.

Proposition 8.9.8. *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int} K$, and let $0 < p < n$.*

(i) $\widetilde{\Omega}(K) \leqslant n\sqrt{|K| \cdot |K^*|}$.

(ii) $\Omega_p(K) \leqslant n^{\frac{n-p}{n+p}} \widetilde{\Omega}(K)^{\frac{2p}{n+p}} |K|^{\frac{n-p}{n+p}} \leqslant n|K^*|^{\frac{p}{n+p}} |K|^{\frac{n}{n+p}}$.

Remark. In particular, $\Omega(K) \leqslant n^{\frac{n-1}{n+1}} \widetilde{\Omega}(K)^{\frac{2}{n+1}} |K|^{\frac{n-1}{n+1}} \leqslant n|K^*|^{\frac{1}{n+1}} |K|^{\frac{n}{n+1}}$.

Proof. Applying Hölder inequality to (8.143) and to (8.141), we deduce

$$\begin{aligned}
\widetilde{\Omega}(K) &= \int_{S^{n-1}} (h_K f_K)^{\frac{1}{2}} h_K^{\frac{-n}{2}} \, d\mathcal{H}^{n-1} \leqslant \left(\int_{S^{n-1}} h_K f_K \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \left(\int_{S^{n-1}} h_K^{-n} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} \\
&\leqslant \left(\int_{S^{n-1}} h_K \, dS_K \right)^{\frac{1}{2}} \left(\int_{S^{n-1}} h_K^{-n} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} = (n|K|)^{\frac{1}{2}} (n|K^*|)^{\frac{1}{2}} \\
\Omega_p(K) &= n \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}} \, d\widetilde{V}_K(x) \\
&\leqslant n \left(\int_{\partial K} \kappa_0(K,x)^{\frac{1}{2}} \, d\widetilde{V}_K(x) \right)^{\frac{2p}{n+p}} \left(\int_{\partial K} 1 \, d\widetilde{V}_K(x) \right)^{\frac{n-p}{n+p}} \\
&= n^{\frac{n-p}{n+p}} \widetilde{\Omega}(K)^{\frac{2p}{n+p}} |K|^{\frac{n-p}{n+p}}.
\end{aligned}$$

■

The affine surface area, that was defined at the beginning of 20th century for convex bodies with C^2_+ boundary, was long conjectured to be upper semi-continuous (see Theorem 8.9.10). Finally, Lutwak [549] provided the following representation of the affine surface area for convex bodies with C^2_+ boundary in 1991 that yields automatically the upper semi-continuity, and the representation was extended to any convex body by Dolzmann, Hug [236]. Here we prove Theorem 8.9.9 for convex bodies with C^2_+ boundary and for lower dimensional compact convex sets, and the general case is handled as Theorem 8.B.2 in Section 8.B.

Theorem 8.9.9 (Lutwak, Dolzmann-Hug). *If $K \subset \mathbb{R}^n$ is a compact convex set, then*

$$\Omega(K) = \inf_{\substack{g: S^{n-1} \to (0,\infty) \\ g \text{ continuous}}} \left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1} \right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1} \, dS_K \right)^{\frac{n}{n+1}}. \tag{8.144}$$

Proof of Theorem 8.9.9 if ∂K is C^2_+ or $\dim K \leqslant n-1$. If $\dim K \leqslant n-1$, then we may assume that $K \subset v^\perp$ for a $v \in S^{n-1}$, and hence $\operatorname{supp} S_K \subset \{v, -v\}$. Therefore, the infimum on the right hand side of (8.144) is 0.

If K is a convex body and ∂K is C^2_+, then $dS_K = f_K d\mathcal{H}^{n-1}$ where f_K is continuous and positive (cf. (8.17)). On the one hand, if $g : S^{n-1} \to (0,\infty)$ is continuous, then

the Hölder inequality yields

$$\left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1} f_K \, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+1}} \geqslant \int_{S^{n-1}} f_K^{\frac{n}{n+1}} \, d\mathcal{H}^{n-1} = \Omega(K).$$

On the other hand, taking $g = f_K^{\frac{1}{n+1}}$ finishes the proof of (8.144). ∎

The Affine Isoperimetric Inequality below is due to Blaschke [97] in dimensions $n = 2, 3$ around 1920 and to Santaló [642] in any dimensions in 1949, and the equality case was clarified by Petty [619].

Theorem 8.9.10 (Main properties of the Affine Surface Area). *Let $K, K_m \subset \mathbb{R}^n$, $m \in \mathbb{N}$, be convex compact sets.*

- $\Omega(\lambda K) = \lambda^{\frac{n(n-1)}{n+1}} \Omega(K)$ *for* $\lambda > 0$.
- $\Omega(z + \Phi K) = \Omega(K)$ *if* $z \in \mathbb{R}^n$ *and* $\Phi \in \mathrm{GL}(n)$, $\det \Phi = \pm 1$.
- **Affine Isoperimetric inequality.** $\Omega(K) \leqslant n\omega_n^{\frac{2}{n+1}} |K|^{\frac{n-1}{n+1}}$, *with equality if and only if K is an ellipsoid or* $\dim K \leqslant n-1$.
- **Upper semi-continuity.** $\Omega(K) \geqslant \limsup_{m\to\infty} \Omega(K_m)$ *if* $\lim_{m\to\infty} K_m = K$.

Remarks.

- Upper semi-continuity shows that if $\dim K \leqslant n-1$, then $\lim_{m\to\infty} \Omega(K_m) = 0$.
- Upper semi-continuity is proved directly by a nice argument in Ludwig [537].

Proof. Homogeneity of degree $\frac{n(n-1)}{n+1}$ follows from the definition, and equiaffine invariance from Theorem 8.9.7.

For the Affine Isoperimetric inequality, we may assume that K is a convex body and $\sigma_K = o$ by the translation invariance of $\Omega(K)$, and then we use the Blaschke-Santaló inequality (6.31) and Proposition 8.9.8 (ii) with $p = 1$.

For the upper semi-continuity, given a continuous $g : S^{n-1} \to (0, \infty)$, the function $K \mapsto \left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1} \, dS_K\right)^{\frac{n}{n+1}}$ of convex bodies is continuous by the weak continuity of the surface area measure (cf. Proposition 8.4.1). Since $\Omega(K)$ is the infimum of continuous functions, it is upper semi-continuous. ∎

Remark 8.9.11 (Some additional properties of the Affine Surface Area).

- The affine surface area is not decreased by Steiner symmetrization, see Section 8.C.
- The affine surface area is related to the floating body and plays a very important role in approximation by polytopes of high complexity (see the Comments Section 8.10 to Chapter 8).
- Ludwig [535] ($n = 2$) and Ludwig, Reitzner [545] ($n \geqslant 2$) characterized the affine surface area as an equiaffine invariant and upper semi-continuous valuation (finitely

additive measure on the space of compact convex sets, see Section 8.D) that vanishes on polytopes. Ludwig [535] actually used this characterization to obtain asymptotic expressions for polytopal approximation of planar convex bodies.

Turning to the case of L_p-affine surface area for $p > 0$, we start with the analogue of Theorem 8.9.9 verified by Lutwak [552] when the boundary is C^2_+, and by Hug [423] in general (see Theorem 8.B.2 in Section 8.B for the argument).

Theorem 8.9.12 (Lutwak, Hug). *If $p > 0$ and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$\Omega_p(K) = \inf_{\substack{g: S^{n-1} \to (0,\infty) \\ g \text{ continuous}}} \left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1} \right)^{\frac{p}{n+p}} \left(\int_{S^{n-1}} g^{-p} h_K^{1-p} \, dS_K \right)^{\frac{n}{n+p}}.$$

Remark. The measure $h_K^{1-p} \, dS_K$ on S^{n-1} in Theorem 8.9.12 is the so-called L_p-surface area measure (see Section 9.3).

Proof of Theorem 8.9.12 if ∂K is C^2_+. We have $dS_K = f_K d\mathcal{H}^{n-1}$ where f_K is continuous and positive. If $g : S^{n-1} \to (0, \infty)$ is continuous, then the Hölder inequality yields

$$\begin{aligned} \Omega_p(K) &= \int_{S^{n-1}} f_K^{\frac{n}{n+p}} h_K^{\frac{n(1-p)}{n+p}} f_K \, d\mathcal{H}^{n-1} \\ &\leqslant \left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1} \right)^{\frac{p}{n+p}} \left(\int_{S^{n-1}} g^{-p} h_K^{1-p} f_K \, d\mathcal{H}^{n-1} \right)^{\frac{n}{n+p}}. \end{aligned}$$

Finally, $\Omega_p(K)$ is attained when $g = f_K^{\frac{1}{n+p}} h_K^{\frac{1-p}{n+p}}$. ∎

It follows from (2.35) that if $M \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int} M$ and $f : \partial M \to [0, \infty)$ is measurable, then

$$\int_{\partial M} f \, d\widetilde{V}_M = \frac{1}{n} \int_{S^{n-1}} f(\varrho_M(u) \cdot u) \cdot \varrho_M(u)^n \, d\mathcal{H}^{n-1}(u). \tag{8.145}$$

The following statement due to Hug [424] relates the L_p-affine surface areas of a convex body and its polar. In the case of smooth convex bodies in $\mathbb{R}^n$ and $p = n$, the fact that the centro-affine surface are of a convex body and its polar coincides was already discussed by Salkowski [644], §34, in 1934.

Lemma 8.9.13 (Hug). *If $p > 0$ and $K \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int} K$, then*

$$\Omega_p(K) = \Omega_{n^2/p}(K^*). \tag{8.146}$$

Proof. Let $\Xi \subset S^{n-1}$ be the measurable set of all $u \in S^{n-1}$ such that h_K is twice differentiable at u with $f_K(u) = \sigma_{n-1} D^2 h_K(u) > 0$ (see Theorem 10.6.2). In order to simplify some formulas, we set $r_{K^*}(u) = \varrho_{K^*}(u) \cdot u$ for $u \in S^{n-1}$. According to Lemma 8.9.4 and Lemma 8.9.5, $u \in \Xi$ if and only if ∂K^* is twice differentiable at $r_{K^*}(u) \in \partial K^*$ and $\kappa_{\partial K^*}(r_{K^*}(u)) > 0$. It follows by (8.141) (applied to K^*) and (8.143) (applied to K) that $\Omega_p(K) > 0$ if and only if $\Omega_{n^2/p}(K^*) > 0$, which are in turn, equivalent to $\mathcal{H}^{n-1}(\Xi) > 0$. In this case, Lemma 8.9.4 and Lemma 8.9.5 yield that if $u \in \Xi$, then

$$\frac{\kappa_{\partial K^*}(r_{K^*}(u))}{h_{K^*}\left(\nu_{K^*}(r_{K^*}(u))\right)^{n+1}} = f_K(u) h_K(u)^{n+1}.$$

We conclude from (8.141), (8.145), (8.143) and $\frac{n^2/p}{n+(n^2/p)} = \frac{n}{n+p}$ that

$$\begin{aligned}
\Omega_{n^2/p}(K^*) &= n \int_{\partial K^*} \kappa_0(K^*, x)^{\frac{n}{n+p}} \, d\widetilde{V}_{K^*}(x) \\
&= \int_{\Xi} \frac{\kappa_{\partial K^*}(r_{K^*}(u))^{\frac{n}{n+p}}}{h_{K^*}\left(\nu_{K^*}(r_{K^*}(u))\right)^{\frac{n(n+1)}{n+p}}} \varrho_{K^*}(u)^n \, d\mathcal{H}^{n-1}(u) \\
&= \int_{\Xi} f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(n+1)}{n+p}} h_K(u)^{-n} \, d\mathcal{H}^{n-1}(u) = \Omega_p(K).
\end{aligned}$$

■

The upper semi-continuity of the centro-affine surface area and the Centro-Affine Isoperimetric Inequality are due to Lutwak [552].

Theorem 8.9.14 (Main properties of the Centro-Affine Surface Area). *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int} K$.*

Linear invariance: $\widetilde{\Omega}(\Phi K) = \widetilde{\Omega}(K)$ *if* $\Phi \in \mathrm{GL}(n)$*;*

Upper semi-continuity: $\widetilde{\Omega}(K) \geqslant \limsup_{m\to\infty} \widetilde{\Omega}(K_m)$ *if* $\lim_{m\to\infty} K_m = K$*;*

Invariance under polarity: $\widetilde{\Omega}(K) = \widetilde{\Omega}(K^*)$*;*

Centro-Affine Isoperimetric inequality *If* $\sigma_K = o$*, then*

$$\widetilde{\Omega}(K) \leqslant n\omega_n$$

with equality if and only if K is an o-symmetric ellipsoid.

Remark. The centro-affine surface area can be characterized as a $\mathrm{GL}(n)$ invariant and upper semi-continuous valuation (see Section 8.D), and it is also related to polytopal approximation (see the Comments Section 8.10 to Chapter 8).

Proof. $\mathrm{GL}(n)$ invariance follows from (8.140) and Proposition 8.9.3, and upper semi-continuity follows from Theorem 8.9.12 as in the case of the affine surface area.

Next (8.146) yields $\widetilde{\Omega}(K) = \widetilde{\Omega}(K^*)$. For the Centro-Affine Isoperimetric inequality, we use the Blaschke-Santaló inequality (6.31) and Proposition 8.9.8 (i). ∎

Finally, we consider the L_p-affine surface area for $p > 0$. If $p > 1$, then the upper semi-continuity of the L_p-surface area and the L_p-Affine Isoperimetric inequality are due to Lutwak [552], and if $p \in (0, 1)$, then the upper semi-continuity of the L_p surface area is due to Hug [423], and the L_p-Affine Isoperimetric inequality is due to Werner, Ye [718].

Theorem 8.9.15 (Main properties of the L_p-Affine Surface Area). *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int} K$, and let $p > 0$.*

- $\Omega_p(\Phi K) = \Omega_p(K)$ *if* $\Phi \in \mathrm{GL}(n)$ *with* $|\det \Phi| = 1$.
- **Upper semi-continuity**
  $\Omega_p(K) \geqslant \limsup_{m\to\infty} \Omega_p(K_m)$ *if* $\lim_{m\to\infty} K_m = K$;
- $\Omega_p(K) = \Omega_{n^2/p}(K^*)$;
- L_p**-Affine Isoperimetric inequality** *If* $\sigma_K = o$, *then*

$$\Omega_p(K) \leqslant n|B^n|^{\frac{2p}{n+p}}|K|^{\frac{n-p}{n+p}}$$

with equality if and only if K is an o-symmetric ellipsoid.

Proof. Theorem 8.9.7 yields the invariance under volume preserving linear maps, and upper semi-continuity follows from Theorem 8.9.12 as in the case of the affine surface area.

Next, $\Omega_p(K) = \Omega_{n^2/p}(K^*)$ is just (8.146). For the L_p-Affine Isoperimetric inequality, if $0 < p \leqslant n$, then we use the Blaschke-Santaló inequality (6.31) and Proposition 8.9.8. If $p > n$, then (8.146) and the case $0 < p < n$ yield

$$\Omega_p(K) = \Omega_{n^2/p}(K^*) \leqslant n|K|^{\frac{n}{n+p}}|K^*|^{\frac{p}{n+p}} \leqslant n|B^n|^{\frac{2p}{n+p}}|K|^{\frac{n-p}{n+p}},$$

with equality if and only if K is a centered ellipsoid. ∎

We note that the notion of L_p-affine surface area discussed above for $p > 0$ have been extended to the Orlicz case by Ludwig [540] in following way: for any concave function $\varphi : (0, \infty) \to (0, \infty)$ such that $\lim_{t\to 0^+} \varphi(t) = \lim_{t\to\infty} \varphi(t)/t = 0$, let

$$\Omega_{(\varphi)}(K) = n \int_{\partial K} \varphi\left(\kappa_0(K, x)\right)\, d\widetilde{V}_K(x). \tag{8.147}$$

In particular, $\Omega_{(\varphi_p)} = \Omega_p$ where $\varphi_p(t) = t^{\frac{p}{n+p}}$ for $p > 0$. Ludwig [540] proved all the results for the extended notions that are analogous to the ones about L_p-affine surface area in this section. These notions are valuations (cf. Section 8.D), and come up in the characterization of $\mathrm{SL}(n)$ invariant and upper semi-continuous valuations on $\mathcal{K}^n_{(o)}$

by Reitzner, Ludwig [546] (when vanishing on polytopes) and by Haberl, Parapatits [372] (in general, see Theorem 8.D.7).

Werner, Ye [718] extended the notion of L_p-affine surface area to the case $p < 0$, and verified various analogous properties. In particular, if $-n < p < 0$, then the L_p-affine surface area is lower semi-continuous, and this case was also extended to the Orlicz setting by Ludwig [540].

8.10 Comments to Chapter 8

The main properties of related to the second differentiability of convex bodies were worked out by Aleksandrov [4–6, 11]. Originally, Minkowski himself proposed a way to approximate any compact convex set by convex bodies with C^∞_+ boundary (see Bonnesen, Fenchel [107]), but his argument contained a gap. In the proof of Theorem 8.1.10, we present the probably simplest construction due to Firey [288].

To approximate compact convex sets by convex bodies with C^∞_+ boundary, Schneider [661] introduces a useful operator. Let $K \subset R\,B^n$, $R > 0$, be a compact convex set. Given $\varepsilon \in (0,1)$, Schneider (see [661], Theorem 3.4.1) constructed a convex body $T_\varepsilon K$ such that $h_{T_\varepsilon K}$ is C^∞ on $\mathbb{R}^n\backslash\{o\}$ and $\delta_H(K, T_\varepsilon K) \leqslant R\varepsilon$. In addition, the map $K \mapsto T_\varepsilon K$ satisfies that $\Phi T_\varepsilon K = T_\varepsilon \Phi K$ if Φ is an isometry of $\mathbb{R}^n$, and hence $T_\varepsilon K$ is a ball if K is a ball, and in general, $T_\varepsilon(\alpha K + \beta C) = \alpha T_\varepsilon K + \beta T_\varepsilon C$ and $\delta_H(T_\varepsilon K, T_\varepsilon C) \leqslant (1+\varepsilon)\delta_H(K,C)$ for $\alpha, \beta > 0$ and compact convex sets K, C. In particular, $T_\varepsilon K + \varepsilon\, B^n$ is a good approximation of K with C^∞_+ boundary, and if K is of constant width $D > 0$ (*i.e.* $K - K = DB^n$), then $T_\varepsilon K + \varepsilon\, B^n$ is of constant width, as well (see Martini, Montejano, Oliveros [562] for properties of convex bodies of constant width).

The fact that non-negative linear combination of compact convex sets is a homogeneous polynomial in the coefficients, and the basic properties of mixed volumes, and the relation of mean curvatures to mixed volumes were established by Minkowski [590, 591]. The representation of the intrinsic volumes as mean projections is due to Kubota [489].

See Florentin, V. Milman, Schneider [289] for a characterization and related properties of the mixed discriminant of positive definite matrices, and V. Milman, Schneider [588] for some additional characterizations of mixed volumes.

For various properties of mixed area measures, including a Kubota-type formula, see Hug, Mussnig, Ulivelli [428].

Minkowski's first and second inequalities are due to Minkowski [590,591] around 1900, and he also proved the Aleksandrov-Fenchel inequality in $\mathbb{R}^3$. Aleksandrov [5,7, 11] provided two proofs of the Aleksandrov-Fenchel Inequality in higher dimensions around 1937-38 (for additional arguments still based on Aleksandrov's ideas, see also Shenfeld, van Handel [678], Schneider [661], Section 7.3 and D. Cordero-Erausquin, B. Klartag, Q. Merigot, F. Santambrogio [219]). Fenchel only stated the inequality

and sketched some ideas, never actually provided a detailed proof. Both arguments in this monograph, the one using strongly isomorphic polytopes in Section 7.A, and the one using the theory of elliptic operators in Section 8.5.2, are based on Aleksandrov's original ideas as developed further by Shenfeld, van Handel [678]. For various problems in algebraic geometry or combinatorics, etc, related to the Aleksandrov-Fenchel inequality, see Section 7.8. Concerning the equality case of the Aleksandrov-Fenchel inequality, Hug, Reichert [429] provided a common generalization of the results by Shenfeld, van Handel [680] in the case of polytopes, and by Schneider [658] in the case of convex bodies with C^1 boundary.

For convex bodies K and C, the first stability forms of the Brunn-Minkowski inequality were due to Minkowski himself (see Groemer [338]). If the distance of the convex bodies K and C is measured in terms of the Hausdorff distance, then Diskant [235] and Groemer [337] provided close to optimal stability versions (see Groemer [338]). However, the natural distance is in terms of the volume of the symmetric difference, and the optimal result is due to the work of Figalli, Maggi, Pratelli [277,278] and Kolesnikov, E. Milman [480] (see Section 8.6).

The L_p version of the Brunn-Minkowski inequality for $p > 1$ was proved by Firey [284] in 1962 (see Section 7.6) as a consequence of the classical Brunn-Minkowski inequality, the $p = 1$ case. The right extension of the L_p combination for $p \in [0, 1)$ was eventually developed by Böröczky, Lutwak, Yang, Zhang [140] within Erwin Lutwak's L_p-Brunn-Minkowski theory, which started with the seminal paper Lutwak [550] in 1993. The L_p-Brunn-Minkowski conjecture for origin symmetric convex bodies in $\mathbb{R}^n$ for $p \in [0, 1)$ was stated by Böröczky, Lutwak, Yang, Zhang [140], who verified the the conjecture when $n = 2$. The conjecture has been verified if $p < 1$ is close to 1 in any dimension by Chen, Huang, Li, Liu [192] based on the local results by Kolesnikov, E. Milman [480] (see Puttermann [629] for another local-to-global approach based on results by Kolesnikov, E. Milman [480]). For $p \in [0, 1)$, the L_p-Brunn-Minkowski conjecture is deeply related to many open problems in Convex Geometric Analysis, for example, it is intimately connected the uniqueness of the solution of the even L_p-Minkowski problem, a Minkowski type Monge-Ampère equation on the sphere. The L_p-Brunn-Minkowski conjecture is discussed from the point of view of the Brunn-Minkowski theory and Elliptic operators in Section 8.7 (the fundamental case $p = 0$) and Section 8.8 (the case $p \in (0, 1)$), and from the point of view of Monge-Ampère equations on the sphere in Section 9.4. Functional analogues of the L_0-addition are presented by Crasta, Fragalà [226] and Malliaris, Melbourne, Roberto, Roysdon [564] (actually, [564] even handles L_p addition for $p > 0$).

The centro-affine curvature in the planar case was already considered by Tzitzéica [705] in 1908. The notion of affine surface area for convex bodies with C^2_+ boundary was developed by Blaschke [97] in $\mathbb{R}^2$ and $\mathbb{R}^3$ around 1920, who established the

equiaffine invariance, the connection to the Blaschke-Santaló inequality and the affine isoperimetric inequality. These results were extended to any dimension by Santaló [642].

It was a highly non-trivial task to extend the notion of affine surface area to convex bodies of C^2_+ boundary to any convex bodies. Leichtweiss' attempt in [505] was based on Dupin's floating body in 1986, and Schütt, Werner [672] provided a more satisfactory definition based on the convex floating body defined by Bárány, Larman [64] and Schütt, Werner [672] (see below for the definition of these two notions of floating body). The definition of affine surface area using the extremal problem (8.144) is due Lutwak [549], who showed that his notion coincides with the classical definition in the case of convex bodies with C^2_+ boundary. Lutwak's approach verified the long conjectured upper semi-continuity of the affine surface area, which property in turn had an important role in the solution of the affine Plateau problem by Trudinger and Wang [703]. Finally, Dolzmann, Hug [236] proved that all these notions of affine surface area (Lutwak's approach, and the ones using centroaffine curvature or convex floating body) are equivalent for any convex body. For a simplified unified approach, see Ludwig [540], and for a survey on affine surface area, see Schütt, Werner [674].

Let us discuss some properties of the *Affine Surface Area* that are not listed in Theorem 8.9.10 and Theorem 8.D.5:

Best approximation with respect to the volume, C^2 boundary: Fundamental earlier work by Schneider, Gruber and Ludwig in the case of C^2_+ boundary lead to the result, that if $K \subset \mathbb{R}^n$ is a convex body such that ∂K is C^2, $P_m \subset K$ polytope with at most m vertices and maximal volume, and $P_{(m)} \supset K$ polytope with at most m facets and minimal volume, then (see Gruber [345,346] when the boundary is C^2_+, and Böröczky [117] when the boundary is C^2)

$$\lim_{m\to\infty} m^{\frac{2}{n-1}} |K \backslash P_m| = \frac{\mathrm{del}_{n-1}}{2} \cdot \Omega(K)^{\frac{n+1}{n-1}} \tag{8.148}$$

$$\lim_{m\to\infty} m^{\frac{2}{n-1}} |P_{(m)} \backslash K| = \frac{\mathrm{div}_{n-1}}{2} \cdot \Omega(K)^{\frac{n+1}{n-1}} \tag{8.149}$$

where $\mathrm{del}_1 = \frac{1}{6}$, $\mathrm{div}_1 = \frac{1}{12}$, $\mathrm{del}_2 = \frac{1}{2\sqrt{3}}$, and $\mathrm{div}_2 = \frac{5}{18\sqrt{3}}$; moreover, $\lim_{n\to\infty} \frac{\mathrm{del}_{n-1}}{n} = \lim_{n\to\infty} \frac{\mathrm{div}_{n-1}}{n} = (2\pi e)^{-1}$ (see Hoehner, Kur [396] for more exact estimates, which paper also discusses the history of this problem). In the planar case $n = 2$, Ludwig [535] proved (8.148) and (8.149) for any convex body $K \subset \mathbb{R}^2$.
Ludwig [536] provided similar formulas in the case of approximation of a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary by "general polytopes" (neither inscribed or circumscribed) with respect to the volume of the symmetric difference, and Böröczky [117] handled the case when ∂K is assumed to be only C^2.

Random approximation: Schütt [671] proved that if $K \subset \mathbb{R}^n$ is any convex body, and Q_m is a the convex hull of m random points of K acccording to the uniform

propability measure, then

$$\lim_{m\to\infty} m^{\frac{2}{n-1}} \cdot \mathbb{E}|K\backslash Q_m| = \gamma_n |K|^{\frac{2}{n+1}} \cdot \Omega(K)$$

where $\gamma_n > 0$ depends on n (see Böröczky, Fodor, Hug [129] for the value of γ_n and for a clarification concerning the argument).

Convex floating body: Let $K \subset \mathbb{R}^n$ be a convex body, let $0 < \delta < \frac{1}{e|K|}$, and for any $u \in S^{n-1}$, we consider $t_{u,\delta} \in \mathbb{R}$ satisfying that $|C_{u,\delta}| = \delta$ for the cap $C_{u,\delta} = \{x \in K : \langle x, u\rangle \geqslant t_{u,\delta}\}$. Now Bárány, Larman [64] and Schütt, Werner [672] defined the convex floating body K_δ by

$$K_\delta = \{x \in K : \langle x, u\rangle \leqslant t_{u,\delta} \ \forall\, u \in S^{n-1}\} = \mathrm{cl}\left(K\backslash \cup_{u\in S^{n-1}} C_{u,\delta}\right).$$

We note that the centroid $\sigma_K \in K_\delta$ by $\delta < \frac{1}{e|K|}$ and Lemma 1.11.4. Schütt, Werner [672] proved (see Prochno, Schütt, Werner [628] or Werner [717] for a clear formulation of the statement) that

$$\Omega(K) = 2\left(\frac{\omega_{n-1}}{n+1}\right)^{\frac{2}{n+1}} \lim_{\delta\to 0^+} \frac{|K| - |K_\delta|}{\delta^{\frac{2}{n+1}}}.$$

In 1822, Dupin [240] considered the floating body that is the envelope of the centroids of the $(n-1)$-dimensional sections $\{x \in K : \langle x, u\rangle = t_{u,\delta}\}$ for $u \in S^{n-1}$. Dupin's floating may not be convex, but it is convex when ∂K is C^2 and δ is small, and in this case, it agrees with the convex floating body K_δ. This floating body was investigated in detail, for example, by Blaschke [97] and Leichweiss [505].

For the Affine Isoperimetric inequality, Blaschke's and Santaló's approach was to prove it for convex bodies with C^2_+ boundaries via Steiner symmetrization with characterization of the equality, and then deduce the Blaschke-Santaló inequality. Meyer, Pajor [575] proved the Blaschke-Santaló inequality via Steiner symmetrization with a characterization of the equality case for all convex bodies, and Lutwak [551] explained how the Affine Isoperimetric inequality and the Blaschke-Santaló inequality are equivalent, even including the equality case.

We note that Hug [423] managed to characterize equality in the Affine Isoperimetric inequality for any convex body via Steiner symmetrization. The curvature relation (8.139) is significantly generalized by Hug [426], Theorem 5.1.

Extending the work of Tzitzéica [705], the *centro-affine surface* area was introduced by Blaschke [97] for convex bodies with C^2_+ boundaries in $\mathbb{R}^2$ and $\mathbb{R}^3$. Due to its $\mathrm{GL}(n)$ invariance, the centro-affine surface area shows up in the asymptotic formula in the case best approximation by polytopes with respect to the Banach-Mazur distance. For o-symmetric convex bodies $K, C \subset \mathbb{R}^n$, their Banach-Mazur distance $\delta_{\mathrm{BM}}(K, C)$ is the minimum of $\log\lambda$ where there exists $\Phi \in \mathrm{GL}(n)$ such that $K \subset \Phi C \subset \lambda K$, which is a metric on the equivalence classes of o-symmetric convex bodies with respect to

linear transformations. Now, if $K \subset \mathbb{R}^n$ convex body, ∂K is C^2, and P_{2m} is an o-symmetric polytope with at most $2m$ vertices minimizing $\delta_{\mathrm{BM}}(K, P_{2m})$, and $P_{(2m)}$ is an o-symmetric polytope with at most $2m$ facets minimizing $\delta_{\mathrm{BM}}(K, P_{(2m)})$, then (see Gruber [348] when the boundary is C^2_+, and Böröczky [117] when the boundary is C^2)

$$\lim_{m\to\infty} (2m)^{\frac{2}{n-1}} \cdot \delta_{\mathrm{BM}}(K, P_{2m}) = \frac{1}{2}\left(\frac{\vartheta_{n-1}}{\omega_{n-1}}\right)^{\frac{2}{n-1}} \cdot \widetilde{\Omega}(K)^{\frac{2}{n-1}}$$

$$\lim_{m\to\infty} (2m)^{\frac{2}{n-1}} \cdot \delta_{\mathrm{BM}}(K, P_{(2m)}) = \frac{1}{2}\left(\frac{\vartheta_{n-1}}{\omega_{n-1}}\right)^{\frac{2}{n-1}} \cdot \widetilde{\Omega}(K)^{\frac{2}{n-1}}$$

where ϑ_{n-1} is the covering density; namely, the minimal density of a covering of $\mathbb{R}^{n-1}$ by equal balls. We note that $\lim_{n\to\infty} \vartheta_{n-1}^{\frac{2}{n-1}} = 1$ as $\vartheta_d \leqslant 4d \log d$ according to Erdős, Rogers [248].

Lutwak [549] introduced L_p-affine surface area for $p \geqslant 1$, which notion was extended to $p > 0$ by Hug [423], and even further to $p < 0$ by Werner, Ye [718] and Ludwig [540]. If $-n < p < 0$ and $K \subset \mathbb{R}^n$ is a centered convex body ($\sigma_K = o$), then Werner, Ye [718] proved that

$$\Omega_p(K) \geqslant n|B^n|^{\frac{2p}{n+p}}|K|^{\frac{n-p}{n+p}}, \tag{8.150}$$

with equality if and only if K is a centered ellipsoid. In addition, Ludwig [540] verified that if $-n < p < 0$, then $\Omega_p(K)$ is lower semi-continuous on the space of convex bodies containing the origin in their interior. Ludwig [540] actually extended the L_p-affine surface area to a broad Orlicz setting, which extended affine surface area shows up in the characterization of upper or lower semi-continuous $\mathrm{SL}(n)$ invariant valuations on the space of convex bodies containing the origin in their interior by Ludwig, Reitzner [546]. Concerning the L_p-affine surface area, Steiner-type formulas are proved and investigated by Tatarko, Werner [700, 701]. Some additional extensions of the notion of L_p-affine surface area discussed by Werner [716] and Caglar, *et al* [174].

8.A Supplement: Aleksandrov's Mixed Discriminant Inequality

The sole goal of this section is to prove the following inequality for mixed discriminants (see below for their definition) based on Gromov [343] (as communicated by van Handel) where $\mathcal{M}^d$ is the vectorspace of $d \times d$ real symmetric matrices for $d \geqslant 2$ (see Shenfeld, van Handel [678] or Schneider [661], Section 5.5 for additional arguments).

Theorem 8.A.1 (Aleksandrov's Mixed Discriminant Inequality). *For $d \geqslant 2$ and matrices $A, B, M_1, \ldots, M_{d-2} \in \mathcal{M}^d$, if $B, M_1, \ldots, M_{d-2}$ are positive semi-definite, then*

$$\mathcal{D}(A, B, M_1, \ldots, M_{d-2})^2 \geqslant \mathcal{D}(A, A, M_1, \ldots, M_{d-2})\mathcal{D}(B, B, M_1, \ldots, M_{d-2}) \tag{8.151}$$

where no $M_1, \dots, M_{d-2}$ occur in the case of $d = 2$.

Let us collect some some basic properties of the mixed discriminant of $A_1, \dots, A_d \in \mathcal{M}^d$ for $d \geqslant 2$. According to Definition 8.3.2, if $A_i = [a_1^{(i)}, \dots, a_d^{(i)}]$ for $a_j^{(i)} \in \mathbb{R}^d$ in an orthonormal basis of $\mathbb{R}^d$, then

$$\mathcal{D}(A_1, \dots, A_d) = \frac{1}{d!} \sum_{\pi:\{1,\dots,d\}\to\{1,\dots,d\} \text{ bijection}} \det[a_1^{(\pi(1))}, \dots, a_d^{(\pi(d))}]. \quad (8.152)$$

In particular, for any permutation $\pi : \{1, \dots, d\} \to \{1, \dots, d\}$, we have

$$\mathcal{D}(A_1, \dots, A_d) = \mathcal{D}(A_{\pi(1)}, \dots, A_{\pi(d)}), \quad (8.153)$$

and if $M, N \in \mathcal{M}^d$ and $t, s \in \mathbb{R}$, then

$$\mathcal{D}(A_1, \dots, A_{d-1}, tM + sN) = t\mathcal{D}(A_1, \dots, A_{d-1}, M) + s\mathcal{D}(A_1, \dots, A_{d-1}, N); \quad (8.154)$$

moreover, if $\lambda_1, \dots, \lambda_d \in \mathbb{R}$, then

$$\det\left(\sum_{i=1}^{d} \lambda_i A_i\right) = \sum_{i_1,\dots,i_d \in \{1,\dots,d\}} \mathcal{D}(A_{i_1}, \dots, A_{i_d})\lambda_{i_1}, \dots, \lambda_{i_d}. \quad (8.155)$$

We deduce from (8.155) that if $U, V \in \mathrm{GL}(d)$, then

$$\mathcal{D}(UA_1V, \dots, UA_dV) = \det(UV) \cdot \mathcal{D}(A_1, \dots, A_d). \quad (8.156)$$

Next for $x \in \mathbb{R}^d \backslash \{o\}$, writing $A_i^{\perp} = A_i|_{x^{\perp}}$ for $i = 1, \dots, d-1$, we have

$$\mathcal{D}(A_1, \dots, A_{d-1}, xx^t) = \frac{\|x\|^2}{d} \cdot \mathcal{D}\left(A_1^{\perp}, \dots, A_{d-1}^{\perp}\right), \quad (8.157)$$

where (8.157) follows from (8.152) writing the the matrices in terms of an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^d$ with $x = \|x\|e_n$, which we may do according to (8.156).

We note that if $A_1, \dots, A_d$ are positive definite, then

$$\mathcal{D}(A_1, A_2, \dots, A_d) > 0 \quad (8.158)$$

according to Lemma 8.3.3.

The following characterization of hyperbolic quadratic forms has been proved as Lemma 7.A.4, but the argument is essentially the same as in the case of Lemma 8.5.3:

Lemma 8.A.2 (Hyperbolic Quadratic Forms). *For a symmetric bilinear form $Q(x, y)$ of $x, y \in \mathbb{R}^m$, $m \geqslant 2$, the following conditions are equivalent for $x, y \in \mathbb{R}^m$.*

(i) $Q(x, y)^2 \geqslant Q(x, x)Q(y, y)$ if $Q(y, y) \geqslant 0$.

(ii) There exists a $w \in \mathbb{R}^m \backslash \{o\}$ such that $Q(x, x) \leqslant 0$ if $Q(x, w) = 0$.

(iii) Either Q is negative semi-definite, or has exactly one positive eigenvalue, and this eigenvalue has multiplicity one.

Now the main idea of the proof of Theorem 8.A.1 is that if $M_1, \dots, M_{d-2} \in \mathcal{M}^d$ are positive definite, then the quadratic form $Q(A, B) = \mathcal{D}(A, B, M_1, \dots, M_{d-2})$ of $A, B \in \mathcal{M}^d$ is hyperbolic. We will use on the one hand, condition (ii) in Lemma 8.A.2 where w is any positive definite matrix, and on the other hand, induction of the size of the matrix based on the following technical observation forming the core of the argument.

Lemma 8.A.3. *For $d \geqslant 3$, if $M, A_1, \dots, A_{d-1} \in \mathcal{M}^d$ satisfy that M is positive definite and $\mathcal{D}(A_1, \dots, A_{d-1}, M) = 0$, then*

$$M = \textstyle\sum_{i=1}^d x_i x_i^t \text{ for } x_i \in \mathbb{R}^d \backslash \{o\} \text{ where } \mathcal{D}(A_1, \dots, A_{d-1}, x_i x_i^t) = 0 \text{ for } i = 1, \dots, d.$$

Proof. Lemma 8.A.3 readily follows from the following stronger statement by induction on the rank r of N: If $N \in \mathcal{M}^d$ is positive semi-definite with rank $r \geqslant 1$, and $\mathcal{D}(A_1, \dots, A_{d-1}, N) = 0$, then there exists $z \in \mathbb{R}^d \backslash \{o\}$ such that

$$N - zz^t \text{ is positive semi-definite with rank } r - 1, \text{ and } \mathcal{D}(A_1, \dots, A_{d-1}, zz^t) = 0. \tag{8.159}$$

We may assume that $r \geqslant 2$. Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of N such that $Ne_i = \lambda_i e_i$ for $\lambda_i > 0$ if $i \leqslant r$. It follows that $N = \sum_{i=1}^r \lambda_i e_i e_i^t$, and hence $\mathcal{D}(A_1, \dots, A_{d-1}, N) = 0$ and the linearity (8.154) of the mixed discriminant yields that we may assume that $\mathcal{D}(A_1, \dots, A_{d-1}, e_1 e_1^t) \geqslant 0$ and $\mathcal{D}(A_1, \dots, A_{d-1}, e_2 e_2^t) \leqslant 0$. Therefore, there exists an $\alpha \in [0, \frac{\pi}{2}]$ such that $\tilde{e}_1 = e_1 \cos\alpha + e_2 \sin\alpha \in S^{n-1}$ satisfies that

$$\mathcal{D}(A_1, \dots, A_{d-1}, \tilde{e}_1 \tilde{e}_1^t) = 0. \tag{8.160}$$

For $L = \operatorname{lin}\{e_1, e_2\}$, let $\tilde{e}_2 \in S^{n-1} \cap L$ be orthogonal to $\tilde{e}_1$, and let the positive definite symmetric matrix of $N|_L$ be $[n_{ij}]$ for $i, j = 1, 2$ with respect to the basis $\tilde{e}_1, \tilde{e}_2$.

If $n_{12} = n_{21} = 0$, then we may choose $z = \sqrt{n_{11}} \cdot \tilde{e}_1$ in (8.159) by (8.160).

If $n_{12} = n_{21} \neq 0$, then $\det[n_{ij}] = n_{11} n_{22} - n_{12}^2 > 0$ yields the existence $t \in (0, 1)$ such that $\det\left([n_{ij}] - t n_{11} \tilde{e}_1 \tilde{e}_1^t\right) = 0$. Since the symmetric 2×2 matrix $[n_{ij}] - s n_{11} \tilde{e}_1 \tilde{e}_1^t$ is positive definite for $s \in (0, t)$, we deduce that the $d \times d$ matrix $N - t n_{11} \tilde{e}_1 \tilde{e}_1^t$ is positive semi-definite with rank $r - 1$, and hence we may choose $z = \sqrt{t n_{11}} \cdot \tilde{e}_1$ in (8.159) by (8.160). ∎

Proof of Theorem 8.A.1. We may assume that the $B, M_1, \dots, M_{d-2}$ in Theorem 8.A.1 are positive definite by adding εI_n to them for small $\varepsilon > 0$.

We deduce from (8.153) and (8.154) that $Q(A, B)$ of $A, B \in \mathcal{M}^d$ is a symmetric bilinear form where either $d = 2$ and $Q(A, B) = \mathcal{D}(A, B)$, or $d \geqslant 3$ and $Q(A, B) = \mathcal{D}(A, B, M_1, \dots, M_{d-2})$ for fixed $M_1, \dots, M_{d-2} \in \mathcal{M}^d$. In addition, (8.158) yields

that if B is positive definite in the case $d = 2$, and $B, M_1, \dots, M_{d-2}$ are positive definite in the case $d \geqslant 3$, then $Q(B, B) > 0$. Therefore, according to Lemma 8.A.2, Theorem 8.A.1 follows from the following statement: If either $d = 2$ and $A, B \in \mathcal{M}^2$ such that B is positive definite and $\mathcal{D}(A, B) = 0$, then

$$\mathcal{D}(A, A) \leqslant 0, \tag{8.161}$$

or $d \geqslant 3$ and $A, B, M_1, \dots, M_{d-2} \in \mathcal{M}^d$ such that $B, M_1, \dots, M_{d-2}$ are positive definite and $\mathcal{D}(A, B, M_1, \dots, M_{d-2}) = 0$, then

$$\mathcal{D}(A, A, M_1, \dots, M_{d-2}) \leqslant 0. \tag{8.162}$$

We prove (8.161) and (8.162) by induction on $d \geqslant 2$.

If $d = 2$, then we may assume by (8.156) that B is diagonal. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ for $i, j = 1, 2$, and hence (8.152) yields that $0 = \mathcal{D}(A, B) = \frac{1}{2}(a_{11}b_{22} + a_{22}b_{11})$ where $b_{ii} > 0$ for $i = 1, 2$. We deduce that $\mathcal{D}(A, A) = \det A \leqslant a_{11}a_{22} \leqslant 0$, proving (8.161).

Now, let $d \geqslant 3$. It follows from $\mathcal{D}(A, B, M_1, \dots, M_{d-2}) = 0$ and Lemma 8.A.3 that $M_{d-2} = \sum_{j=1}^{d} x_j x_j^t$ for $x_j \in \mathbb{R}^d \backslash \{o\}$ where

$$\mathcal{D}(A, B, M_1, \dots, M_{d-3}, x_j x_j^t) = 0 \text{ for } j = 1, \dots, d \tag{8.163}$$

and no M_i occurs in the formula if $d = 3$ (which will be the convention in the upcoming discussion). For each $j = 1, \dots, d$, let $A^{(j)}, B^{(j)}, M_1^{(j)}, \dots, M_{d-3}^{(j)}$ be the restrictions of $A, B, M_1, \dots, M_{d-3}$ to $x_j^{\perp}$ where $B^{(j)}, M_1^{(j)}, \dots, M_{d-3}^{(j)}$ are positive definite, and (8.157) and (8.163) yield that

$$\mathcal{D}(A^{(j)}, B^{(j)}, M_1^{(j)}, \dots, M_{d-3}^{(j)}) = 0 \text{ for } j = 1, \dots, d.$$

We deduce by induction that

$$\mathcal{D}(A^{(j)}, A^{(j)}, M_1^{(j)}, \dots, M_{d-3}^{(j)}) \leqslant 0 \text{ for } j = 1, \dots, d; \tag{8.164}$$

therefore, (8.157) and $M_{d-2} = \sum_{j=1}^{d} x_j x_j^t$ imply $\mathcal{D}(A, A, M_1, \dots, M_{d-2}) \leqslant 0$ as required by (8.162). ∎

8.B Supplement: The upper semi-continuity of the L_p-Affine Surface Area for general convex bodies

The upper semi-continuity of the L_p affine surface area - proved by Lutwak [552] in the C_+^2 case - is one of its most fundamental properties (see the proofs of Theorem 8.9.9 and Theorem 8.9.12 for the elegant argument provided by Lutwak [552]). However,

the case of general convex bodies requires a refined approach, as developed by Hug [423], and discussed in this section. Another nice argument verifying the upper semi-continuity of the L_p-affine surface area for $p > 0$ is provided by Ludwig [537].

The following observation is needed in the argument for Theorem 8.B.2.

Lemma 8.B.1. *If $\eta > 0$ and $f \geqslant \eta$ for $f \in L^1(S^{n-1})$, then there exists a sequence $h_m \in C^\infty(S^{n-1})$ such that $h_m \geqslant \frac{\eta}{2^{n-1}}$ for each m and $\lim_{m\to\infty} \int_{S^{n-1}} |h_m - f|\, d\mathcal{H}^{n-1} = 0$.*

Remark. Similar statement can be proved using spherical convolution (see Feng Dai, Yuan Xu [229], Chapter 2).

Proof. For an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$, let $e_{i+n} = -e_i$ for $i = 1, \dots, n$, therefore, $e_1, \dots, e_{2n}$ are the exterior unit normals of the facets $F_1, \dots, F_{2n}$, respectively, of the cube $W = \sum_{i=1}^n [-1, 1]e_i$. For $i = 1, \dots, 2n$, let $F_i' = F_i|e_i^\perp = W \cap e_i^\perp$, let Θ_i be the radial projection of F_i onto S^{n-1}, and let $\pi_i : e_i^\perp \to S^{n-1}$ be defined by $\pi_i(x) = \frac{x+e_i}{\|x+e_i\|}$, and hence $\pi_i(F_i') = \Theta_i$ and the spherically convex sets $\Theta_1, \dots, \Theta_{2n}$ cover S^{n-1} but their spherical interiors are pairwise disjoint.

For $i = 1, \dots, 2n$, let $f_{(i)} = f \cdot \mathbf{1}_{\Theta_i}$ on S^{n-1}, let $\tilde{f}_{(i)} : e_i^\perp \to [0, \infty)$ satisfy $\tilde{f}_{(i)}(x) = f(\pi_i(x))$ if $x \in F_i'$ and $\tilde{f}_{(i)}(x) = 0$ if $x \notin F_i'$. In addition, let $k_i(x) = \gamma\varphi(1 - \|x\|^2)$ for $x \in e_i^\perp$ be a $\mathbb{C}^\infty$ function where $\varphi(t) = 0$ if $t \leqslant 0$, and $\varphi(t) = e^{\frac{-1}{t^2}}$ if $t > 0$, and the constant $\gamma > 0$ is chosen in a way such that $\int_{e_i^\perp} k_i d\mathcal{H}^{n-1} = 1$. For $m \geqslant 1$, we consider the approximate identity $k_{(i),m}(z) = m^{n-1}k(mz)$ and the approximating function $\tilde{h}_{(i),m} = \tilde{f}_{(i)} * k_{(i),m}$ on $e_i^\perp$ (cf. Lemma 10.3.3 in the Appendix), and hence $\tilde{h}_{(i),m} \geqslant 0$, $\lim_{m\to\infty} \int_{e_i^\perp} |\tilde{h}_{(i),m} - \tilde{f}_{(i)}|\, d\mathcal{H}^{n-1} = 0$ and if $x \in F_i'$, then

$$\begin{aligned}\tilde{h}_{(i),m}(x) &= \int_{e_i^\perp} \tilde{f}_{(i)}(y) k_{(i),m}(x - y)\, d\mathcal{H}^{n-1}(y)\\ &\geqslant \eta \int_{F_i'} k_{(i),m}(x - y)\, d\mathcal{H}^{n-1}(y) \geqslant \frac{\eta}{2^{n-1}}\end{aligned}$$

as F_i' is an $(n-1)$-cube. Now we define $h_m = \sum_{i=1}^{2n} h_{(i),m}$ where $h_{(i),m}(\pi_i(x)) = \tilde{h}_{(i),m}(x)$ for $x \in e_i^\perp$ and $h_{(i),m}(u) = 0$ if $u \in S^{n-1}$ with $\langle u, e_i\rangle \leqslant 0$. Since each π_i is a contraction, and $f(u) = \sum_{i=1}^{2n} f_{(i)}(u)$ for $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$, we conclude Lemma 8.B.1. ∎

We also need the following consequence of the Mean Value Theorem: If $\eta > 0$ and $a, b \geqslant \eta$ and $p > 0$, then

$$|a^{\frac{n}{n+p}} - b^{\frac{n}{n+p}}| \leqslant \eta^{\frac{-p}{n+p}}|a - b|. \tag{8.165}$$

In the case $p = 1$, the following statement was indicated by Leichtweiss [506], and was proved by Lutwak [549] for convex bodies with C_+^2 boundary.

Theorem 8.B.2 (Hug [423]). *If $K \subset \mathbb{R}^n$ is a convex body and $p > 0$, then*

$$\Omega_p(K) = \inf_{\substack{g: S^{n-1} \to (0,\infty) \\ g \text{ continuous}}} \left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1} \right)^{\frac{p}{n+p}} \left(\int_{S^{n-1}} g^{-p} h_K^{1-p} \, dS_K \right)^{\frac{n}{n+p}}. \tag{8.166}$$

Proof. We recall that $dS_K = f_K d\mathcal{H}^{n-1} + dS_K^s$ on S^{n-1} where $f_K = \sigma_{n-1} D^2 h_K$ is measurable and the singular part S_K^s is a regular Borel measure (cf. Theorem 10.1.3) such that there exists a $\mathcal{H}^{n-1}$-measurable set $X \subset S^{n-1}$ satisfying that $\mathcal{H}^{n-1}(X) = 0$ and $S_K^s(S^{n-1} \backslash X) = 0$. We only provide the argument in detail when $p = 1$ (the case of classical affine surface area), and indicate the necessary changes in argument for any $p > 0$ at the end.

The right hand side RHS of (8.166) is estimated by the fact that $f_K d\mathcal{H}^{n-1}$ is the absolutely continuous part of dS_K and by the Hölder inequality:

$$\begin{aligned} \text{RHS} &\geqslant \inf_{\substack{g: S^{n-1} \to (0,\infty) \\ g \text{ continuous}}} \left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1} \right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1} \cdot f_K \, d\mathcal{H}^{n-1} \right)^{\frac{n}{n+1}} \\ &\geqslant \int_{S^{n-1}} f_K^{\frac{n}{n+1}} \, d\mathcal{H}^{n-1} = \Omega(K). \end{aligned}$$

Therefore, all we have to prove is that for any $\varepsilon > 0$, there exists a continuous function $g : S^{n-1} \to (0, \infty)$ such that

$$\left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1} \right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1} \, dS_K \right)^{\frac{n}{n+1}} < \int_{S^{n-1}} f_K^{\frac{n}{n+1}} \, d\mathcal{H}^{n-1} + \varepsilon. \tag{8.167}$$

For $\eta \in (0,1)$, let $f_\eta = \max\{\eta, f_K\} \in L^1(S^{n-1})$. Our first step towards verifying (8.167) is that for any $\eta \in (0,1)$, there exists a continuous $g_\eta : S^{n-1} \to (0, \infty)$ such that

$$\left(\int_{S^{n-1}} g_\eta^n \, d\mathcal{H}^{n-1} \right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g_\eta^{-1} f_\eta \, d\mathcal{H}^{n-1} \right)^{\frac{n}{n+1}} < \int_{S^{n-1}} f_\eta^{\frac{n}{n+1}} \, d\mathcal{H}^{n-1} + \eta. \tag{8.168}$$

To prove (8.168), we consider a sequence of continuous $\theta_m : S^{n-1} \to [\frac{\eta}{2^{n-1}}, \infty)$ provided by Lemma 8.B.1 such that $\lim_{m\to\infty} \int_{S^{n-1}} |\theta_m - f_\eta| \, d\mathcal{H}^{n-1} = 0$. It follows from (8.165) that

$$\lim_{m\to\infty} \int_{S^{n-1}} \left| \theta_m^{\frac{n}{n+1}} - f_\eta^{\frac{n}{n+1}} \right| d\mathcal{H}^{n-1} = 0. \tag{8.169}$$

For $\psi_m = \theta_m^{\frac{1}{n+1}}$, $\psi_m^{-1} \leqslant \frac{2}{\eta^{\frac{1}{n+1}}}$ yields

$$\lim_{m\to\infty} \int_{S^{n-1}} \left| \psi_m^{-1} f_\eta - \theta_m^{\frac{n}{n+1}} \right| d\mathcal{H}^{n-1} = \lim_{m\to\infty} \int_{S^{n-1}} \psi_m^{-1} |f_\eta - h_m| \, d\mathcal{H}^{n-1} = 0,$$

which estimate combined with (8.169) implies that we can take $g_\eta = \psi_m$ for large m in (8.168).

According to Lebesgue's Dominated Convergence Theorem 10.1.4, we can fix a small $\eta \in (0,1)$ in (8.168) such that

$$\left(\int_{S^{n-1}} g_\eta^n \, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g_\eta^{-1} f_K \, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+1}} < \int_{S^{n-1}} f_K^{\frac{n}{n+1}} \, d\mathcal{H}^{n-1} + \frac{\varepsilon}{2}. \tag{8.170}$$

Fix a $G > 1$ such that $G^{-1} < g_\eta < G$.

To deduce (8.167) from (8.170), we note that there exist compact $C_m \subset X$ and open $U_m \supset X$ such that

$$S_K^s(S^{n-1}\backslash C_m) < \frac{1}{m} \quad \text{and} \quad \mathcal{H}^{n-1}(U_m) < \frac{1}{m^{n+1}}$$

as S_K^s and $\mathcal{H}^{n-1}$, being finite Borel mesures on S^{n-1}, are regular (cf. Theorem 10.1.3). For $m > G$, let $\theta_m : S^{n-1} \to [0,m]$ be continuous such that $\theta_m(x) = m$ if $x \in C_m$ and $\theta_m(x) = 0$ if $x \notin U_m$, and let $\varphi_m = \max\{g_\eta, \theta_m\}$. In particular,

$$\begin{aligned}\int_{S^{n-1}} \varphi_m^n \, d\mathcal{H}^{n-1} &\leqslant \int_{S^{n-1}\backslash U_m} g_\eta^n \, d\mathcal{H}^{n-1} + \int_{U_m} m^n \, d\mathcal{H}^{n-1} \\ &\leqslant \int_{S^{n-1}} g_\eta^n \, d\mathcal{H}^{n-1} + \frac{1}{m},\end{aligned}$$

and as $\varphi_m^{-1} \leqslant g_\eta^{-1} \leqslant G$ on S^{n-1} and $\varphi_m^{-1} = \frac{1}{m}$ on C_m, we also have

$$\begin{aligned}\int_{S^{n-1}} \varphi_m^{-1} \, dS_K &\leqslant \int_{S^{n-1}} g_\eta^{-1} f_K \, d\mathcal{H}^{n-1} + \int_{U_m\backslash C_m} G \, dS_K^s + \int_{C_m} \frac{1}{m} \, dS_K^s \\ &\leqslant \int_{S^{n-1}} g_\eta^{-1} f_K \, d\mathcal{H}^{n-1} + \frac{G}{m} + \frac{S_K^s(S^{n-1})}{m}.\end{aligned}$$

Therefore, (8.170) yields that we can choose $g = \varphi_m$ for large m in (8.167), completing the argument for Theorem 8.B.2 when $p = 1$.

For arbitrary $p > 0$, the role of f_K is played by $f_K h_K^{1-p}$ where we use that there exist $C > 1$ such that $C^{-1} < h_K^{1-p} < C$. For example, to prove the analogue of (8.168) for $f_\eta = \max\{\eta, f_K\}$, we choose the continuous function θ_m provided by Lemma 8.B.1 in a way such that $\lim_{m\to\infty} \int_{S^{n-1}} |\theta_m - f_\eta h_K^{1-p}| \, d\mathcal{H}^{n-1} = 0$, and hence

$$\lim_{m\to\infty} \int_{S^{n-1}} \left| \theta_m^{\frac{n}{n+p}} - f_\eta^{\frac{n}{n+p}} h_K^{\frac{n(1-p)}{n+p}} \right| d\mathcal{H}^{n-1} = 0,$$

and we consider $\psi_m = \theta_m^{\frac{1}{n+p}}$ in the argument. ∎

8.C Supplement: Affine Isoperimetric Inequality via Steiner symmetrization

It was Santaló [642] in 1949 who verified first the affine isoperimetric inequality for convex bodies with C^2_+ boundaries in any dimensions, including the characterization of the equality case. The work of Petty, Leichtweiss, Lutwak, Schütt, Weiner and Hug in the late 1980s and early 1990s lead to extending the notion of affine surface area to any convex body, and proving the affine isoperimetric inequality for any convex bodies together with the case of equality (see the Comments Section 8.10). Hug [424] even managed to prove the monotonicity of the affine surface area of any convex body under Steiner symmetrization, including the characterization of the equality case. In this section, we provide the original argument by Santaló [642].

Lemma 8.C.1 (Minkowski's determinantal inequality). *If A, B are positive semi-definite symmetric $n \times n$ matrices and $p \geqslant n$, then*

$$\frac{1}{2}(\det A)^{\frac{1}{p}} + \frac{1}{2}(\det B)^{\frac{1}{p}} \leqslant \det\left(\frac{1}{2}(A+B)\right)^{\frac{1}{p}}. \tag{8.171}$$

Assuming that A, B are positive definite, equality holds if and only if

- *$B = \lambda A$ for some $\lambda > 0$ provided $p = n$;*
- *$A = B$ provided $p > n$.*

Proof. According to the Jensen inequality (10.6), if $0 < \alpha < \frac{1}{n}$ and $t, s > 0$, then

$$\left(\frac{1}{2}t^{\alpha} + \frac{1}{2}s^{\alpha}\right)^{\frac{1}{\alpha}} \leqslant \left(\frac{1}{2}t^{\frac{1}{n}} + \frac{1}{2}s^{\frac{1}{n}}\right)^{n} \tag{8.172}$$

with equality if and only if $t = s$. To prove (8.171), we may assume that $p = n$ by (8.172), and also that A, B positive definite, and hence also that both A, B are diagonal with eigenvalues $a_1, \dots, a_n > 0$ and $b_1, \dots, b_n > 0$, respectively. In this case, the inequality (8.171) is

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} b_i\right)^{\frac{1}{n}} \leqslant \left(\prod_{i=1}^{n}(a_i + b_i)\right)^{\frac{1}{n}},$$

which follows from the AM-GM inequality as

$$\left(\prod_{i=1}^{n}\frac{a_i}{a_i+b_i}\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n}\frac{b_i}{a_i+b_i}\right)^{\frac{1}{n}} \leqslant \frac{1}{n}\left(\sum_{i=1}^{n}\frac{a_i}{a_i+b_i}\right) + \frac{1}{n}\left(\sum_{i=1}^{n}\frac{b_i}{a_i+b_i}\right) = 1.$$

Let us assume that A and B are positive definte and equality holds in (8.171). If $p = n$, then the argument above and the equality case in the AG-GM inequality yield

the existence of $\lambda > 0$ such that $\frac{b_1}{a_1} = \ldots = \frac{b_n}{a_n} = \lambda$, and hence $B = \lambda A$. If $p > n$, then equality holds in (8.171) in the case $p = n$, as well, and hence $B = \lambda A$ for a $\lambda > 0$, and in addition, equality holds in the Jensen inequality (8.172) for $t = \det A$ and $s = \det B$; therefore, $\lambda = 1$. ∎

Let us recall the well-known formula for the Gaussian curvature for a graph of a convex function. For (relatively) open and convex $\Omega \subset u^\perp$, $u \in S^{n-1}$, and for convex $\varphi : \Omega \to \mathbb{R}$, let $Z = \{x + \varphi(x)u : x \in \Omega\} \subset \mathbb{R}^n$ be the graph of φ. If φ is twice differentiable at $x \in \Omega$, then the Gauss curvature $\kappa(z)$ at $z = x + \varphi(x)u$ is

$$\kappa(z) = \frac{\det D^2 f(x)}{\left(1 + \|Df(x)\|^2\right)^{\frac{n+1}{2}}}. \tag{8.173}$$

The following statement is due to Santaló [642].

Theorem 8.C.2 (Santaló). *If $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then*

$$\Omega(\Theta_{u^\perp} K) \geqslant \Omega(K). \tag{8.174}$$

Assuming in addition that ∂K is C^2_+, equality holds in (8.174) if and only if the midpoints of the secants of K parallel to u are contained in a hyperplane.

Proof. There exist continuous concave functions f and g on $K|u^\perp$ such that

$$K = \left\{x + tu : -g(x) \leqslant t \leqslant f(x) \text{ for } x \in K|u^\perp\right\}.$$

Let $X \subset (\mathrm{int}K)|u^\perp$ be the set of points $x \in (\mathrm{int}K)|u^\perp$ where both f and g twice differentiable, and hence $\mathcal{H}^{n-1}\left((K|u^\perp)\backslash X\right) = 0$. For $Z^+ = \{x + f(x)u : x \in X\} \subset \partial K$ and $Z^- = \{x + g(x)u : x \in X\} \subset \partial K$, we have

$$\begin{aligned}\Omega(K) &= \int_{Z^+} \kappa(z)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(z) + \int_{Z^-} \kappa(z)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(z)\\ &= \int_X (\det D^2 f(x))^{\frac{1}{n+1}} + (\det D^2 g(x))^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(x).\end{aligned}$$

Since

$$\Theta_{u^\perp} K = \left\{x + tu : -\frac{f(x) + g(x)}{2} \leqslant t \leqslant \frac{f(x) + g(x)}{2} \text{ for } x \in K|u^\perp\right\},$$

we deduce that

$$\Omega(\Theta_{u^\perp} K) = 2\int_X \left(\det \frac{D^2 f(x) + \det D^2 g(x)}{2}\right)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(x).$$

As $D^2 f(x)$ and $D^2 g(x)$ are positive semi-definite for $x \in X$, Lemma 8.C.1 yields (8.174).

Now let us assume that ∂K is C^2_+ and equality holds in (8.174). As $D^2 f(x)$ and $D^2 g(x)$ are positive definite and continuous functions of x, Lemma 8.C.1 yields that $D^2 f(x) = D^2 g(x)$ for all $x \in (\mathrm{int} K)|u^\perp$. Therefore, there exist $v \in u^\perp$ and $\gamma \in \mathbb{R}$ such that if $x \in K|u^\perp$, then $f(x) = g(x) + \langle v, x\rangle + \gamma$, and hence

$$\frac{1}{2}\left((x + f(x)u) + (x - g(x)u)\right) = \left\langle \frac{v}{2}, x \right\rangle + \frac{\gamma}{2}.$$

■

In turn, we deduce the Affine Isoperimetric Inequality following Santaló [642].

Theorem 8.C.3 (Affine Isoperimetric inequality, Santaló). *If $K \subset \mathbb{R}^n$ convex body, then*

$$\Omega(K) \leqslant n\omega_n^{\frac{2}{n+1}} |K|^{\frac{n-1}{n+1}}. \tag{8.175}$$

Assuming that ∂K is C^2_+, equality holds in (8.175) if and only if K is an ellipsoid.

Proof. Since starting from K, a sequence of Steiner symmetrizations lead to a ball of volume $|K|$ (cf. Theorem 1.10.7 or Theorem 1.C.3), and the affine surface area is upper semi-continuous (cf. Theorem 8.9.10), we deduce (8.175) from Theorem 8.C.2.

Now let $K \subset \mathbb{R}^n$ be a convex body that is not an ellipsoid such that ∂K is C^2_+. According to Theorem 6.2.1, there exists a $u \in S^{n-1}$ such that the midpoints of the secants of K parallel to u are not contained in some hyperplane. In this case, strict inequality holds in (8.175), namely $\Omega(K_0) > \Omega(K)$ and $|K_0| = |K|$ for $K_0 = \Theta_{u^\perp} K$. As we have already seen that $\Omega(K_0) \leqslant n\omega_n^{\frac{2}{n+1}} |K_0|^{\frac{n-1}{n+1}}$, we conclude that $\Omega(K) < n\omega_n^{\frac{2}{n+1}} |K|^{\frac{n-1}{n+1}}$. ■

8.D Supplement: Valuations on convex bodies

We recall that $\mathcal{K}^n$ is the space of compact convex sets in $\mathbb{R}^n$ equipped with the Hausdorff metric, and observe that the Minkowski addition makes $\mathcal{K}^n$ a cancellative abelian semigroup. In addition, let $\mathcal{K}^n_{(o)}$ be the subspace of $\mathcal{K}^n$ consisting of convex bodies K with $o \in \mathrm{int}\, K$.

Definition 8.D.1 (Valuations on convex compact sets). For a cancellative abelian semigroup $\mathcal{A}$, a function $Z : \mathcal{K}^n \to \mathcal{A}$ (or $Z : \mathcal{K}^n_{(o)} \to \mathcal{A}$) is called a valuation if for any $K, C \in \mathcal{K}^n$ (or $K, C \in \mathcal{K}^n_{(o)}$) satisfying that $K \cup C \in \mathcal{K}^n$, we have

$$Z(K \cup C) + Z(K \cap C) = Z(K) + Z(C). \tag{8.176}$$

We observe that if $K, C \in \mathcal{K}^n$ satisfy that $K \cup C$ is convex, and hence lies in $\mathcal{K}^n$, then

$$h_{K\cup C} = \max\{h_K, h_C\} \text{ and } h_{K\cap C} = \min\{h_K, h_C\}. \tag{8.177}$$

In this section, we only consider some fundamental properties of continuous or semi-continuous valuations on compact convex sets. As we will see, a crucial property of a valuation is how it intertwines with some subgroup of affine transformations of $\mathbb{R}^n$. We call a valuation *simple* if it vanishes on lower dimensional compact convex sets. For surveys on various related aspects of the theory of valuations, see, for example, the monograph Alesker [13], the survey papers McMullen, Schneider [571], Schneider [662], Ludwig [543], Ludwig, Mussnig [544], and Section 6 of Schneider [661].

Examples of valuations on $\mathcal{K}^n$

(a) $Z(K) = \gamma$ for a constant $\gamma \in \mathbb{R}$.

(b) $Z(K) = |K|$ is a continuous simple valuation which satisfies $Z(\Phi K + x) = Z(K)$ for $\Phi \in \mathrm{SL}(n)$ and $x \in \mathbb{R}^n$.

(c) Given $M \in \mathcal{K}^n$, $Z(K) = |K + M|$ is a continuous valuation as $(K + M) \cup (C + M) = (K \cup C) + M$ and $(K + M) \cap (C + M) = (K \cap C) + M$ if $K \cup C \in \mathcal{K}^n$ (cf. (8.177)). Here $Z(K + x) = Z(K)$ for $x \in \mathbb{R}^n$, and if $M = B^n$, then even $Z(\Phi K) = Z(K)$ for $\Phi \in \mathrm{O}(n)$.

(d) Given $i = 1, \dots, n-1$, $Z(K) = V_i(K)$ is a continuous valuation (this follows from (b) and the Kubota formula (7.5) representing $V_i(K)$ as a mean projection). Here $Z(\Phi K + x) = Z(K)$ for $\Phi \in \mathrm{O}(n)$ and $x \in \mathbb{R}^n$.

(e) Generalizing (d), given $C_1, \dots, C_m \in \mathcal{K}^n$ where $1 \leqslant m < n$,

$$Z(K) = V(C_1, \dots, C_m, K, \dots, K) \tag{8.178}$$

is a valuation (this follows from (c) and Theorem 7.3.1 (iii)). Here $Z(K + x) = Z(K)$ for $x \in \mathbb{R}^n$.

(f) *Difference body:* $Z(K) = K - K$ (here $\mathcal{A} = \mathcal{K}^n$) is a continuous valuation which satisfies $Z(\Phi K + x) = \Phi Z(K)$ for $\Phi \in \mathrm{GL}(n)$ and $x \in \mathbb{R}^n$.

(g) *Affine surface area:* $Z(K) = \Omega(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(x)$ if K is a convex body, and $Z(K) = 0$ if $K \in \mathcal{K}^n$ with $\dim K \leqslant n-1$ is a simple upper-semicontinuous valuation which satisfies $Z(\Phi K + x) = Z(K)$ for $\Phi \in \mathrm{SL}(n)$ and $x \in \mathbb{R}^n$ (cf. Theorem 8.9.10).

(h) *Moment vector:* $Z(K) = \int_K x\, d\mathcal{H}^n(x)$ (here $\mathcal{A} = \mathbb{R}^n$) is a simple continuous valuation which satisfies $Z(\Phi K) = \Phi Z(K)$ for $\Phi \in \mathrm{SL}(n)$ and $x \in \mathbb{R}^n$.

Examples of valuations on $\mathcal{K}^n_{(o)}$

(i) *Polar body:* $Z(K) = K^*$ (here $\mathcal{A} = \mathcal{K}^n_{(o)}$) is a continuous valuation which satisfies $Z(\Phi K) = \Phi^{-t} Z(K)$ for $\Phi \in \mathrm{GL}(n)$, and hence $Z(\Phi K) = \Phi Z(K)$ for $\Phi \in \mathrm{O}(n)$.

(ii) *L_p Affine Surface area:* Given $p > 0$, $Z(K) = \Omega_p(K) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, \nu_K(x)\rangle^{\frac{n(p-1)}{n+p}}}\, d\mathcal{H}^{n-1}(x)$ (in particular, $Z(K) = \widetilde{\Omega}(K)$ if $p = n$) is an upper semi-continuous valuation which satisfies $Z(\Phi K) = Z(K)$ for $\Phi \in \mathrm{SL}(n)$ (cf. Theorem 8.9.15).

For references concerning properties of continuous valuations discussed in this section, see Alesker [13] and Section 6 of Schneider [661]. Any continuous valuation $Z : \mathcal{K}^n \to \mathbb{R}^d$, $d \geqslant 1$, (or $Z : \mathcal{K}^n \to \mathcal{K}^n$) satisfies the inclusion-exclusion principle; namely, if $K_1 \cup \ldots \cup K_m \in \mathcal{K}^n$ for $K_1, \ldots, K_m \in \mathcal{K}^n$, then

$$\begin{aligned} Z(K_1 \cup \ldots \cup K_m) &= \sum_{i=1}^{m} Z(K_i) - \sum_{1 \leqslant i < j \leqslant m} Z(K_i \cap K_j) + \ldots \\ &= \sum_{i=1}^{m} (-1)^{i-1} \sum_{1 \leqslant j_1 < \ldots < j_i \leqslant m} Z(K_{j_1} \cap \ldots \cap K_{j_i}). \end{aligned}$$

Remark. According to Groemer [336], if a valuation Z satisfies the inclusion-exclusion principle, then Z can be extended to be a finitely additive measure on the finite unions of convex compact sets. For example, the constant 1 valuation on $\mathcal{K}^n$ extends to the Euler characteristic of finite unions of convex compact sets (a result due to Hadwiger [377]).

Let us review some results about valuations that characterize quantities related to this book. If G is a subgroup of the group of affine transformations of $\mathbb{R}^n$ (any element of G is of the form $x \mapsto \Phi x + w$ for $\Phi \in \mathrm{GL}(n)$ and $w \in \mathbb{R}^n$) and $\mathcal{A}$ is cancellative abelian semigroup, then we say that a valuation $Z : \mathcal{K}^n \to \mathcal{A}$ is G invariant if $Z(gK) = Z(K)$ for $g \in G$ and $K \in \mathcal{K}^n$, and a valuation $Z : \mathcal{K}^n \to \mathbb{R}^n$ or a valuation $Z : \mathcal{K}^n \to \mathcal{K}^n$ is G equivariant if $Z(gK) = gZ(K)$ for $g \in G$ and $K \in \mathcal{K}^n$. The modern theory of valuations started with Hadwiger's characterization theorem in 1957 (see Klain [462] for a simpler proof).

Theorem 8.D.2 (Hadwiger [377]). *A continuous valuation $Z : \mathcal{K}^n \to \mathbb{R}$ is invariant under isometries of $\mathbb{R}^n$ if and only if there exist $\gamma_0, \ldots, \gamma_n \in \mathbb{R}$ such that*

$$Z(K) = \sum_{i=0}^{n} \gamma_i V_i(K).$$

Remark. It is sufficient that Z is $\mathrm{SO}(n)$ and translation invariant instead of isometry invariance. A variant of the statement is that the continuity of Z is replaced by monotoncity ($Z(K) \leqslant Z(C)$ if $K \subset C$), and in that case, $\gamma_i \geqslant 0$ for $i \geqslant 1$ must be assumed.

We say that a valuation $Z : \mathcal{K}^n \to \mathbb{R}$ is homogeneous of degree $q \in \mathbb{R}$ if $Z(\lambda K) = \lambda^q Z(K)$ for $\lambda > 0$. Concerning volume, Hadwiger [377] verified that any translation

invariant and n-homogeneous valuation $Z : \mathcal{K}^n \to \mathbb{R}$ is of the form

$$Z(K) = \gamma |K| \quad \text{for a } \gamma \in \mathbb{R}, \tag{8.179}$$

and Klain [462] proved the same conlusion for any continuous, translation invariant and even ($Z(-K) = Z(K)$) simple valuation $Z : \mathcal{K}^n \to \mathbb{R}$. In addition, Schneider [660] characterizes any continuous, translation invariant simple valuation $Z : \mathcal{K}^n \to \mathbb{R}$; namely, there exist $\gamma \in \mathbb{R}$ and an odd continuous function $g : S^{n-1} \to \mathbb{R}$ ($g(-u) = -g(u)$ for $u \in S^{n-1}$) such that

$$Z(K) = \gamma |K| + \int_{S^{n-1}} g \, dS_K.$$

Next, Haberl and Parapatits [374, 375] characterized the moment vector among vector valued valuations, and Ludwig [539] characterized the difference body among Minkowski (compact convex set valued) valuations. Let us remark that the centroid of a convex body is not a valuation.

Theorem 8.D.3 (Haberl, Parapatits [374, 375]). *$Z : \mathcal{K}^n_{(o)} \to \mathbb{R}^n$ is a continuous and $\mathrm{SL}(n)$ equivariant valuation for $n \geqslant 3$ if and only if there exists $\gamma \in \mathbb{R}$ such that*

$$Z(K) = \gamma \int_K x \, dx.$$

Theorem 8.D.4 (Ludwig [539]). *$Z : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, $\mathrm{SL}(n)$ equivariant and translation invariant valuation if and only if there exists $\gamma \geqslant 0$ such that*

$$Z(K) = \gamma(K - K).$$

Projection bodies (cf. Section 2.B) have been characterized by Ludwig [538, 541] as Minkowski valuations (compact convex set valued) with certain properties.

Extending the result by Ludwig [535] in $\mathbb{R}^2$, Ludwig, Reitzner [545] characterized the Affine Surface area (see Section 8.9 for the notions of Affine Surface area, Centro-Affine Surface area, and in general, L_p-Affine Surface area) among valuations as follows.

Theorem 8.D.5 (Ludwig, Reitzner [545]). *$Z : \mathcal{K}^n \to \mathbb{R}$ is an upper semi-continuous, translation and $\mathrm{SL}(n)$ invariant valuation if and only if there exist $\gamma_0, \gamma_1 \in \mathbb{R}$ and $\gamma_2 \geqslant 0$ such that*

$$Z(K) = \gamma_0 + \gamma_1 |K| + \gamma_2 \Omega(K).$$

Remark. $Z : \mathcal{K}^n \to \mathbb{R}$ is an upper semi-continuous, translation and $\mathrm{SL}(n)$ invariant valuation that vanishes on polytopes if and only if $Z(K) = \gamma \Omega(K)$ for $\gamma \geqslant 0$.

In the planar case $n = 2$, Ludwig [535] used Theorem 8.D.5 to prove the asymptotic formulas (8.148) and (8.149) for best approximation of any convex body $K \subset \mathbb{R}^2$ by polygons.

Ludwig, Reitzner [546] provided the following characterization of the centro-affine surface area.

Theorem 8.D.6 (Ludwig, Reitzner [546]). *$Z : \mathcal{K}^n_{(o)} \to \mathbb{R}$ is an upper semi-continuous, and $\mathrm{GL}(n)$ invariant valuation if and only if there exist $\gamma_0 \in \mathbb{R}$ and $\gamma_1 \geqslant 0$ such that*

$$Z(K) = \gamma_0 + \gamma_1 \widetilde{\Omega}(K).$$

Remark. In particular, $Z : \mathcal{K}^n_{(o)} \to \mathbb{R}$ is an upper semi-continuous and $\mathrm{GL}(n)$ invariant valuation that vanishes on polytopes if and only if $Z(K) = \gamma \widetilde{\Omega}(K)$ for $\gamma \geqslant 0$.

For $0 < p < n$, Ludwig, Reitzner [546] characterized the L_p-surface area $\Omega_p(K)$ as upper semi-continuous and $\mathrm{SL}(n)$ invariant valuation $Z : \mathcal{K}^n_{(o)} \to \mathbb{R}$ that is homogeneous of degree $q = n(n-p)/(n+p)$ (recall that $\Omega(K) = \Omega_1(K)$). In general, the "centro-affine Hadwiger theorem" due to Haberl and Parapatits characterizes all upper semi-continuous and $\mathrm{SL}(n)$ invariant valuations on $\mathcal{K}^n_{(o)}$.

Theorem 8.D.7 (Haberl, Parapatits [372]). *Upper semi-continuous and $\mathrm{SL}(n)$ invariant valuations $Z : \mathcal{K}^n_{(o)} \to \mathbb{R}$ are the ones of the form*

$$Z(K) = c_0 + c_1|K| + c_2|K^*| + \Omega_{(\varphi)}(K)$$

where $c_0, c_1, c_2 \in \mathbb{R}$, and $\Omega_{(\varphi)}$ is defined as in (8.147) for a concave function $\varphi : (0,\infty) \to (0,\infty)$ such that $\lim_{t\to 0^+} \varphi(t) = \lim_{t\to\infty} \varphi(t)/t = 0$.

Let us provide some hints about the flourishing theory of the structure of the Banach space $\mathrm{Val}(\mathbb{R}^n)$ of continuous and translation invariant valuations, which space is the main topic of the monograph Alesker [13]. According to McMullen's decomposition theorem in 1977, the existence of mixed volumes depends on the properties that the volume of a compact convex set is a continuous translation invariant valuation that is homogeneous of degree n.

Theorem 8.D.8 (McMullen's decomposition [567]). *Let $Z : \mathcal{K}^n \to \mathbb{R}$ be a continuous and translation invariant valuation.*

(i) Z can be written as $Z = Z_0 + Z_1 + \ldots + Z_n$ where Z_i is a continuous and translation invariant valuation that is homogeneous of degree i.

(ii) Given $K_1, \ldots, K_m \in \mathcal{K}^n$, $Z(\lambda_1 K_1 + \ldots + \lambda_m K_m)$ is a polynomial in $\lambda_1, \ldots, \lambda_m \geqslant 0$ of degree at most n.

Observe that McMullen's decomposition is in line with Hadwiger's Theorem 8.D.2.

The next crucial step towards understanding continuous and translation invariant valuations is Alesker's theory introducing representation theory as the main tool around 2000. We write $\mathrm{Val}(\mathbb{R}^n)$ to denote the real (or complex) topological vector space of continuous and translation invariant valuations on $\mathcal{K}^n$ that is actually a Banach space. There is a natural $\mathrm{GL}(n)$ action on $\mathrm{Val}(\mathbb{R}^n)$ defined by $\Phi Z(K) = Z(\Phi^{-1}K)$ for $\Phi \in \mathrm{GL}(n)$ and $K \in \mathcal{K}^n$. In addition, any valuation $Z : \mathcal{K}^n \to \mathbb{R}$ can be written (uniquely) as $Z = Z^+ + Z^-$ where Z^+ is even ($Z^+(-K) = Z^+(K)$) and Z^- is odd ($Z^-(-K) = -Z^-(K)$); namely; $Z^+(K) = \frac{1}{2}(Z(K) + Z(-K))$ and $Z^-(K) = \frac{1}{2}(Z(K) - Z(-K))$. According to McMullen's decomposition Theorem 8.D.8,

$$\mathrm{Val}(\mathbb{R}^n) = \bigoplus_{i=0}^{n} \left(\mathrm{Val}_i^+(\mathbb{R}^n) \oplus \mathrm{Val}_i^-(\mathbb{R}^n)\right)$$

where $\mathrm{Val}_i^+(\mathbb{R}^n)$ ($\mathrm{Val}_i^-(\mathbb{R}^n)$) is the subspace of even (odd) continuous and translation invariant valuations in the space $\mathrm{Val}_i(\mathbb{R}^n)$ of i-homogeneous valuations. We note that $\mathrm{Val}_0(\mathbb{R}^n) = \mathbb{R}\, V_0$ and $\mathrm{Val}_n(\mathbb{R}^n) = \mathbb{R}\, V_n$ by Hadwiger's theorem (8.179). According to Alesker's irreducibility theorem (see [13]), the irreducible closed subspaces with respect to the natural action of $\mathrm{GL}(n)$ on $\mathrm{Val}(\mathbb{R}^n)$ are exactly $\mathrm{Val}_i^+(\mathbb{R}^n)$ and $\mathrm{Val}_i^-(\mathbb{R}^n)$, $i = 0, \ldots, n$. As a consequence, Alesker proved McMullen's conjecture that the mixed volumes (see Example (e)) are dense among continuous and translation invariant valuations.

The dense subspace $\mathrm{Val}^\infty(\mathbb{R}^n)$ of $\mathrm{Val}(\mathbb{R}^n)$ consists of the so-called smooth valuations where $Z \in \mathrm{Val}^\infty(\mathbb{R}^n)$ if the map $\mathrm{GL}(n) \to \mathrm{Val}^\infty(\mathbb{R}^n)$, $\Phi \mapsto \Phi Z$ is C^∞. Here $\mathrm{Val}^\infty(\mathbb{R}^n) = \bigoplus_{i=0}^n \mathrm{Val}_i^\infty(\mathbb{R}^n)$ where $\mathrm{Val}_i^\infty(\mathbb{R}^n)$ is the subspace of i-homogeneous smooth valuations. For example, the valuation $K \mapsto |K + M|$ in Example (c) is smooth if M is a convex body with C_+^∞ boundary, and the mixed volume in (8.178) in Example (e) is a smooth $(n-m)$-homogeneous valuation if $C_1, \ldots, C_m$ are convex bodies with C_+^∞ boundary. Knoerr [487] proves that any i-homogeneous smooth valuation, $i = 1, \ldots, n-1$, can be written as finite linear combination of smooth valuations of the type as in (8.178) in Example (e) with $m = n - i$.

The graded Banach space $\mathrm{Val}^\infty(\mathbb{R}^n)$ is equipped with two natural products $\mathrm{Val}^\infty(\mathbb{R}^n) \times \mathrm{Val}^\infty(\mathbb{R}^n) \to \mathrm{Val}^\infty(\mathbb{R}^n)$ through the work by Alesker, Bernig and Fu (see [13]). For the Alesker product $Z \cdot \widetilde{Z} \in \mathrm{Val}_{i+j}^\infty(\mathbb{R}^n)$ for $Z \in \mathrm{Val}_i^\infty(\mathbb{R}^n)$ and $\widetilde{Z} \in \mathrm{Val}_j^\infty(\mathbb{R}^n)$ where $i + j \leqslant n$, the product induces a perfect pairing $\mathrm{Val}_i^\infty(\mathbb{R}^n) \times \mathrm{Val}_{n-i}^\infty(\mathbb{R}^n) \to \mathrm{Val}_n^\infty(\mathbb{R}^n) = \mathbb{R} \cdot V_n$. In addition, it satisfies the Hard Lefschetz theorem stating that $Z \mapsto Z \cdot V_1^{n-2i}$ is an isomorphism $\mathrm{Val}_i^\infty(\mathbb{R}^n) \to \mathrm{Val}_{n-i}^\infty(\mathbb{R}^n)$ for $0 \leqslant i < n/2$. The convolution $Z * \widetilde{Z}$ goes the other way, and the two products are connected by the so-called Fourier transform $\mathbb{F} : \mathrm{Val}^\infty(\mathbb{R}^n) \to \mathrm{Val}^\infty(\mathbb{R}^n)$ mapping $\mathrm{Val}_i^\infty(\mathbb{R}^n)$ into $\mathrm{Val}_{n-i}^\infty(\mathbb{R}^n)$ and satisfying $\mathbb{F} \circ \mathbb{F}(Z)(K) = Z(-K)$ and $\mathbb{F}(Z \cdot \widetilde{Z}) = \mathbb{F}(Z) * \mathbb{F}(\widetilde{Z})$.

Concerning some applications of the theory of valuations on compact convex sets, Hadwiger's Theorem 8.D.2 directly yields the principal kinematic formula, and

Bernig, Hug [83] prove very general kinematic formulas based on results on valuation. Haberl, Schuster [376] applies results on Minkowski valuations to obtain new Sobolev type inequalities. In addition, Aleksandrov-Fenchel-type inequalities are verified by Alesker [14] and Kotrbatý, Wannerer [484, 485] based on Kotrbatý's Hodge-Riemann relations on valuations in [482] (see also Bernig, Kotrbatý, Wannerer [84]). A surprising development that an extension of Firey's results on bodies of axial rotational concerning the Christoffel-Minkowski problem (a Monge-Ampère partial differential equation on the sphere, see Section 9.5.1) has been substantially generalized by Brauner, Hofstätter, Ortega-Moreno [156] using the theory of valuations.

We note that other important families of valuations that are defined on polytopes (see Alesker [13] and Böröczky, Ludwig [139] for classical results, and for example Haberl, Parapatits [375], Jochemko, Sanyal [449, 450] for recent advances), on polyhedra (see e.g. Barvinok [75]), on lattice polytopes (see Böröczky, Ludwig [139]), on fans of polyhedra (see Backman, Manecke, Sanyal [48]), on hyperplanes (see Gates, Hug, Schneider [324]), on function spaces (see Ludwig [542, 543] or Colesanti, Ludwig, Mussnig [211]), and even on manifolds (see Alesker [12] and Alesker, Bernig [15]).

Chapter 9

The Minkowski Problem, the L_p-Minkowski problem, and the L_p-Brunn-Minkowski inequality/conjecture

This chapter discusses some recent versions of the more than a century old Minkowski problem that are Monge-Àmpere equations on the sphere; namely, non-linear elliptic partial differential equations. One of the most beautiful aspects of the theory is that the uniqueness of the solution of certain Monge-Àmpere equations is intimately tied to the validity of fundamental Brunn-Minkowski-type inequalities (see also Böröczky [120] for a survey). In addition, many notions and inequalities discussed in this book have a significant role in the theory, like surface area measure, affine surface area, the Brunn-Minkowski inequality and the Blaschke-Santaló inequality.

9.1 Monge-Ampère equations in $\mathbb{R}^d$

In this section, we collect some properties of convex functions that are related to Monge-Ampère equations. Our main references are Figalli [272] and Trudinger, Wang [704] (see also Section 10.6). For the whole section, let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, be a convex, bounded, open set, and let $\varphi : \Omega \to \mathbb{R}$ be a convex function. We consider three key notions:

Subdifferential: $\partial\varphi(x) = \{z \in \mathbb{R}^d : \varphi(y) \geqslant \varphi(x) + \langle z, y - x\rangle \text{ for all } y \in \Omega\}$ for $x \in \Omega$, where $\partial\varphi(x)$ is a compact convex set.

Monge-Ampère measure μ_φ on Ω: If $\omega \subset \Omega$ is a Borel set, then

$$\mu_\varphi(\omega) = \mathcal{H}^d\left(\bigcup_{x\in\omega} \partial\varphi(x)\right). \tag{9.1}$$

If φ is C^2, then

$$\mu_\varphi(\omega) = \int_\omega \det D^2\varphi \, d\mathcal{H}^d.$$

Monge-Ampère equation: For a finite Borel measure μ on Ω, find a convex function φ on Ω such that

$$\mu_\varphi = \mu; \tag{9.2}$$

$$\det D^2\varphi = f \quad \text{if } d\mu = f\, d\mathcal{H}^n \text{ for "smooth enough" } f : \Omega \to [0,\infty) \tag{9.3}$$

where the φ in (9.2) is called the solution in Aleksandrov's sense, or in the sense of measure.

The regularity of the solution of the Minkowski problem (9.3) was intensively investigated by Nirenberg [605], Cheng, Yau [198] and Pogorelov [624] in the middle of the 20th century, and finally, Caffarelli [169,170] settled this issue around 1990 (see also Chapter 4 in Figalli [272]).

Definition 9.1.1 (Hölder continuity). For $\alpha \in (0,1]$ and $Z \subset \mathbb{R}^d$, a function $f : Z \to \mathbb{R}$ is $C^{0,\alpha}$, if there exists $c > 0$ such that $|f(x) - f(y)| \leqslant c \cdot \|x - y\|^\alpha$ for $x, y \in Z$, and f is locally $C^{0,\alpha}$, if each $z \in Z$ has a neighborhood where f is $C^{0,\alpha}$ (and the implied constant factor depends on the neighborhood).

In addition, for open $U \subset \mathbb{R}^d$ and integer $m \geqslant 1$, a function $h : U \to \mathbb{R}$ is $C^{m,\alpha}$ if h is C^m, and each partial derivative of h of order m is $C^{0,\alpha}$.

Remark. The $C^{0,1}$ functions are the Lipschitz functions.

We recall that for a convex, compact $K \subset \mathbb{R}^n$, an $x \in K$ is called an extreme point if $x = (1-\lambda)y + \lambda z$ for $y, z \in K$ and $\lambda \in (0,1)$ implies $y = z = x$, and hence $x \in \mathrm{relbd} K$ in this case (cf. Definition 1.6.4 and Lemma 1.6.5).

Theorem 9.1.2 (Caffarelli [169]). *For $\lambda_2 > \lambda_1 > 0$ and convex open bounded $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, let the convex function $\varphi : \Omega \to \mathbb{R}$ satisfy*

$$\lambda_1 \leqslant \det D^2\varphi \leqslant \lambda_2 \tag{9.4}$$

in the sense of measure; namely, $\lambda_1 \mathcal{H}^d(\omega) \leqslant \mu_\varphi(\omega) \leqslant \lambda_2 \mathcal{H}^d(\omega)$ for any Borel $\omega \subset \Omega$.

(i) *If $\varphi(y) \geqslant 0$ for all $y \in \Omega$ and $S = \{y \in \Omega : \varphi(y) = 0\}$ is non-empty and not a point, then S (more precisely, $\mathrm{cl}\, S$) has no extreme point in Ω.*

(ii) *If φ is strictly convex, then there exists an $\alpha \in (0,1)$ such that φ is locally $C^{1,\alpha}$.*

Caffarelli [170] (cf. Chapter 4 in Figalli [272]) deals with the case when the Monge-Ampère measure has a Hölder continuous density function.

Theorem 9.1.3 (Caffarelli [170]). *For an open bounded convex $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, let the functions φ, f on Ω satisfy that φ is strictly convex, $\lambda^{-1} \leqslant f \leqslant \lambda$ for a constant $\lambda > 1$, and*

$$\det D^2\varphi = f$$

in the sense of measure; namely, $d\mu_\varphi = f\, d\mathcal{H}^d$.

(i) *If f is continuous, then φ is locally $C^{1,\alpha}$ for any $\alpha \in (0,1)$; and in general, if f is C^k for an integer $k \geqslant 0$, then φ is locally $C^{k+1,\alpha}$ for any $\alpha \in (0,1)$.*

(ii) *If f is locally $C^{k,\alpha}$ in Ω for some $\alpha \in (0,1)$ and integer $k \geqslant 0$, then φ is locally $C^{k+2,\alpha}$.*

Remarks.

- Instead of (i) in Theorem 9.1.3, Caffarelli [170] actually proves that if f is continuous, then for any open ball B whose closure is in Ω, φ is in the Sobolev space $W^{2,l}(B)$ for any $l > n$. Since $W^{2,l}(B) \subset C^{1,\alpha}(B)$ if $\frac{n}{l} = 1 - \alpha$ according the Sobolev Embedding Theorem (cf. Demengel, Demengel [232]), φ is locally $C^{1,\alpha}$ for any $\alpha \in (0, 1)$.
- We note that (i) in Theorem 9.1.2 does not rule out the possibility that S is an open segment whose endpoints are in $\partial\Omega$. For $n \geqslant 3$, Pogorelov (see Section 3.2 in Figalli [272]) constructed a convex function $\varphi : \operatorname{int} B^n \to [0, \infty)$ such that $\{\varphi = 0\}$ is an open segment that connects two opposite points of S^{n-1}, and φ is not $C^{1,\alpha}$ for $\alpha > 1 - \frac{2}{n}$, but $\det D^2\phi$ is positive and analytic.

9.2 The Minkowski Problem

This section discusses the classical Minkowski problem dating back to around 1900. We use several notions introduced in Section 8.1 and Section 8.2, like the curvature function f_K and the surface area measure S_K on S^{n-1}, etc, for a convex body $K \subset \mathbb{R}^n$.

Remark 9.2.1 (Minkowski Problem). Characterize a finite non-trivial Borel measure μ on S^{n-1} such that

$$\mu = S_K \tag{9.5}$$

for a convex body $K \subset \mathbb{R}^n$, and characterize the case when $S_K = S_C$ for another convex body $C \subset \mathbb{R}^n$. If $dS_K = f_K \, d\mathcal{H}^{n-1}$, then determine how smooth K is under some smoothness assumptions on f_K.

Let us discusses the results in Section 8.1 and Section 8.2 from the point of view of the Minkowski problem. For a convex body $K \subset \mathbb{R}^n$ and some Borel set $\omega \subset S^{n-1}$, the surface area measure of ω is (cf. Definition 2.5.2 and Lemma 1.6.6)

$$S_K(\omega) = \mathcal{H}^{n-1}\big(\cup_{u\in\omega} F(K, u)\big) = \mathcal{H}^{n-1}\big(\cup_{u\in\omega} \partial h_K(u)\big);$$

therefore, S_K is the "Monge-Ampère measure" for the restriction $h = h_K|_{S^{n-1}}$. On the one hand, S_K is a discrete measure on $\{u_1, \ldots, u_m\}$ with $S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i)$ if K is an n-polytope with facets $F_1, \ldots, F_m$ and corresponding exterior unit normals $u_1, \ldots, u_m$. On the other hand, for any C^2 function $h : S^{n-1} \to \mathbb{R}$, there exists a convex body $K \subset \mathbb{R}^n$ (or a point) such that $h = h_K|_{S^{n-1}}$ if and only if $\nabla^2 h + hI_{n-1}$ is positive semi-definite where ∇ (or ∇^2) is the spherical gradient (spherical Hessian) on S^{n-1}. Here $o \in K$ ($o \in \operatorname{int} K$) if and only if $h \geqslant 0$ ($h > 0$) on S^{n-1}. Now, if ∂K is C^2_+ for a convex body $K \subset \mathbb{R}^n$, $x \in \partial K$, $u = \nu_K(x)$, and $h = h_K|_{S^{n-1}}$, then

- $x = Dh_K(u) = \nabla h(u) + h(u)\, u$ (cf. Lemma 1.6.6 and (8.12));
- $dS_K = \det\left(\nabla^2 h + hI_{n-1}\right) d\mathcal{H}^{n-1}$ (cf. (8.18));

- the eigenvalues of $\nabla^2 h(u) + h(u) I_{n-1}$ are $\frac{1}{\kappa_1(x)}, \dots, \frac{1}{\kappa_{n-1}(x)}$ where $\kappa_1(x), \dots, \kappa_{n-1}(x)$ are the principal curvatures at $x \in \partial K$, and $\det(\nabla^2 h(u) + h(u) I_{n-1}) = f_K(u) = \frac{1}{\kappa(x)}$ for the Gaussian curvature $\kappa(x) = \prod_{i=1}^{n-1} \kappa_i(x)$ and curvature function f_K (cf. Theorem 8.1.5 and Definition 8.1.6).

Minkowski Problem as a Monge-Ampère equation on S^{n-1}: Given an integrable function $f : S^{n-1} \to [0, \infty)$, find $h : S^{n-1} \to \mathbb{R}$ such that

$$\det(\nabla^2 h + h I_{n-1}) = f, \tag{9.6}$$

that is, the Minkowski problem (9.5) for the absolute continuous measure $d\mu = f \, d\mathcal{H}^{n-1}$.
Remark. We search for an h that is the restriction of a convex 1-homogeneous function on $\mathbb{R}^n$.

Our proof of Theorem 9.2.3 is based on the variational method, and an essential tool is the Aleksandrov lemma Theorem 7.5.2 that we now recall.

Lemma 9.2.2 (Aleksandrov Lemma for Wulff shapes). *Let $\Omega \subset S^{n-1}$ be a closed set not contained in a closed hemisphere, let $\varphi : \Omega \times (-t_0, t_0) \to (0, \infty)$, $t_0 > 0$, be continuous such that*

$$\lim_{t \to 0} \frac{\varphi(u,t) - \varphi(u,0)}{t} = \partial_t \varphi(u,0)$$

exists uniformly in $u \in \Omega$ where $\partial_t \varphi(u,0)$ is continuous on Ω. For $t \in (-t_0, t_0)$, the Wulff shape $K_t = \{x \in \mathbb{R}^n : \langle x, u \rangle \leqslant \varphi(u,t) \text{ for } u \in \Omega\}$ satisfies that

$$\lim_{t \to 0} \frac{|K_t| - |K_0|}{t} = \int_{S^{n-1}} \partial_t \varphi(u,0) \, dS_{K_0}(u).$$

In the solution Theorem 9.2.3 of the Minkowski problem, we use the notion of a spherical cap; namely, for $u \in S^{n-1}$ and $\delta \in [0, 1)$, an open spherical cap centered at u is

$$\Omega(u, \delta) = \{v \in S^{n-1} : \langle v, u \rangle > \delta\} \tag{9.7}$$

where $\Omega(u, 0)$ is an open hemisphere. It was Minkowski [590,591] himself who solved the Minkowski problem around 1900 if the measure μ in (9.5) is discrete (the case of polytopes) or absolutely continuous with positive C^∞ density function (cf. (9.6)), and the solution in the general case is due to Aleksandrov [6, 11] in 1938 (see also the nice variational argument by Carlier [180]). Still, the statement is typically is known as Minkowski's theorem.

Theorem 9.2.3 (Minkowski's theorem). *For a finite Borel measure μ on S^{n-1}, there exists a convex body K with $\mu = S_K$ if and only if*

(a) $\operatorname{supp} \mu$ *intersects each open hemisphere of S^{n-1};*

(b) $\int_{S^{n-1}} u\, d\mu(u) = o \in \mathbb{R}^n$.

In addition, $S_K = S_C$ if and only if K and C are translates.

Remark. Given (b), (a) is equivalent to the condition that $\operatorname{supp}\mu$ is not contained in a great subsphere of S^{n-1}.

Proof. The necessity of the conditions (a) and (b) have been proved in Lemma 2.5.6 and Lemma 2.5.7. The uniqueness of the surface area measure has been characterized in Theorem 2.5.11.

For the most involved part of the proof, for the sufficiency of the conditions in Theorem 9.2.3, let μ be a finite Borel measure on S^{n-1} satisfying (a) and (b). The main idea comes from the fact that if $S_M = \mu$ for a convex body M, then the form (2.31) or (7.28) of the Minkowski inequality says that $\int_{S^{n-1}} h_C\, d\mu \geqslant \int_{S^{n-1}} h_M\, d\mu$ for any convex body C with $|C| = |M|$; or in other words, $C = M$ minimizes the integral $\int_{S^{n-1}} h_C\, d\mu$ over all convex bodies C with $|C| = |M|$.

For the family $\mathcal{F}$ of convex bodies C with $o \in C$ and $|C| = 1$, we consider $\inf \int_{S^{n-1}} h_C\, d\mu$ for $C \in \mathcal{F}$. Let $C_m \in \mathcal{F}$ such that

$$\lim_{m\to\infty} \int_{S^{n-1}} h_{C_m}\, d\mu = \inf\left\{ \int_{S^{n-1}} h_C\, d\mu : C \in \mathcal{F} \right\}.$$

We claim that $\{C_m\}$ is bounded. There exists a $\Theta > 0$ such that $\int_{S^{n-1}} h_{C_m}\, d\mu \leqslant \Theta$ for each m, and the condition (a) on μ yields the existence of $\delta \in (0, 1)$ and $q > 0$ such that $\mu(\Omega(u, \delta)) \geqslant q$ for any $u \in S^{n-1}$. For each C_m, consider $R_m = \max\{\|x\| : x \in C_m\}$, and a $u_m \in S^{n-1}$ such that $R_m u_m \in C_m$, and hence

$$\Theta \geqslant \int_{S^{n-1}} h_{C_m}\, d\mu \geqslant \int_{\Omega(u_m,\delta)} \langle R_m u_m, v\rangle\, d\mu(v) \geqslant R_m \delta \cdot q.$$

Therefore, $R_m \leqslant \Theta/(\delta q)$, proving the claim that $\{C_m\}$ is bounded.

Since $\{C_m\}$ is bounded, we may assume that C_m tends to a convex body K with $o \in K$ and $|K| = 1$ by the Blaschke Selection Theorem 1.7.4 the continuity of volume (cf. Lemma 1.7.5). Therefore, $K \in \mathcal{F}$ and

$$\int_{S^{n-1}} h_K\, d\mu = \min\left\{ \int_{S^{n-1}} h_C\, d\mu : C \in \mathcal{F} \right\}. \tag{9.8}$$

We deduce from the condition (b) that $\int_{S^{n-1}} h_K\, d\mu = \int_{S^{n-1}} h_{K-z}\, d\mu$ for $z \in \operatorname{int} K$ (cf. Lemma 1.6.2), and hence we may assume that $o \in \operatorname{int} K$.

We claim that

$$\mu = \lambda \cdot S_K \quad \text{for } \lambda = \frac{1}{n}\int_{S^{n-1}} h_K\, d\mu, \tag{9.9}$$

where (9.9) is equivalent with saying that

$$\int_{S^{n-1}} g\, d\mu = \lambda \cdot \int_{S^{n-1}} g\, dS_K \tag{9.10}$$

for any continuous function $g : S^{n-1} \to \mathbb{R}$. Assuming that $|t|$ is small, we consider the Wulff-shape

$$K_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_K(u) + tg(u)\ \forall u \in S^{n-1}\},$$

thus $K_0 = K$ and $o \in \operatorname{int} K_t$; moreover, Aleksandrov's Lemma 9.2.2 yields that

$$\frac{d}{dt}\,|K_t|\Big|_{t=0} = \int_{S^{n-1}} g\, dS_K. \tag{9.11}$$

Since $|K_t|^{\frac{-1}{n}} \cdot K_t \in \mathcal{F}$, and we deduce from $h_{K_t} \leqslant h_K(u) + tg(u)$ and from (9.8) that

$$f(t) = \log \int_{S^{n-1}} (h_K + tg)\, d\mu - \frac{1}{n} \log |K_t| \geqslant \log \frac{\int_{S^{n-1}} h_{K_t}\, d\mu}{|K_t|^{\frac{1}{n}}} \geqslant f(0).$$

In particular, the differentiable function f has a minimum at $t = 0$, and hence (9.11) implies that

$$0 = f'(0) = \lambda^{-1} \int_{S^{n-1}} g\, d\mu - \frac{1}{n} \int_{S^{n-1}} g\, dS_K,$$

proving (9.10), and in turn (9.9). Therefore, $\mu = S_M$ for $M = \lambda^{\frac{1}{n-1}} K$. ∎

Remark 9.2.4 discusses why understanding the equality case in the Minkowski inequality $\int_{S^{n-1}} h_C\, dS_K \geqslant \int_{S^{n-1}} h_K\, dS_K$ for convex bodies $K, C \subset \mathbb{R}^n$ (cf. (2.31)) is equivalent with characterization of the uniqueness of the solution of the Minkowski Problem (9.5) up to translation.

Remark 9.2.4 (The Minkowski inequality and uniqueness in the Minkowski Problem). On the one hand, the uniqueness of the surface area measure up to translation was proved in Theorem 2.5.11 based on the equality case of the Minkowski inequality.

On the other hand, let us assume that the uniqueness of the solution of the Minkowski Problem $\mu = S_K$ (cf.(9.5)) for a convex body $K \subset \mathbb{R}^n$ is known to hold up to translation. The proof of Theorem 9.2.3 taking $\mu = S_K$ for a convex body $K \subset \mathbb{R}^n$ shows that $\int_{S^{n-1}} h_C\, dS_K$ among convex bodies C with $|C| = |K|$ attains its minimum at a $C = \widetilde{C}$ and $S_{\widetilde{C}} = S_K$ (the latter follows by the variational argument). Now the uniqueness of the solution of the Minkowski Problem (9.5) implies that $\widetilde{C} = K + z$ for a $z \in \mathbb{R}^n$; therefore,

$$\int_{S^{n-1}} h_C\, dS_K \geqslant \int_{S^{n-1}} h_{\widetilde{C}}\, dS_K = \int_{S^{n-1}} h_K\, dS_K$$

for any convex body $C \subset \mathbb{R}^n$ with $|C| = |K|$.

In order to apply the results discussed in Section 9.1 to the regularity of the solution of the Minkowski problem on S^{n-1}, we need to transfer the local equation into an equation on a convex set in $\mathbb{R}^{n-1}$.

Remark 9.2.5 (Transferring Monge-Ampère equation on S^{n-1} to Monge-Ampère in $\mathbb{R}^{n-1}$). Let $e \in S^{n-1}$, and let $h = h_K|_{S^{n-1}}$ be a solution of the Monge-Ampère equation (9.6) for a convex body $K \subset \mathbb{R}^n$. Setting $\varphi(y) = h_K(y+e)$ for $y \in e^\perp$, (8.8) implies that

$$\det D^2\varphi(y) = g(y) \quad \text{for } y \in e^\perp \tag{9.12}$$

in the sense of measure where for $y \in e^\perp$, we have

$$g(y) = \left(1 + \|y\|^2\right)^{-\frac{n+1}{2}} \cdot f\left(\frac{e+y}{\sqrt{1+\|y\|^2}}\right).$$

Combining (9.12) with the results in Section 9.1 mostly due to Caffarelli [169,170] imply the differentiability of the solution of the Minkowski problem in the case of Hölder continuous density function (cf. Definition 9.1.1).

Theorem 9.2.6 (Caffarelli). *If $\int_{S^{n-1}} u f(u)\, du = o$ holds for a $C^{0,\alpha}$, $\alpha \in (0,1)$, function $f : S^{n-1} \to (0,\infty)$, then any convex body K with $dS_K(u) = f(u)\, du$ has $C^{2,\alpha}_+$ boundary, and satisfies that h_K is $C^{2,\alpha}$ on $\mathbb{R}^n \backslash \{o\}$, and $f = f_K$ is the curvature function.*

If, in addition, f is $C^{m,\alpha}$ for an integer $m \geqslant 1$, then ∂K and h_K are $C^{m+2,\alpha}$.

Proof. We may assume that $o \in \operatorname{int} K$.

First, we show that ∂K is C^1. The argument is indirect, we suppose that there exists a $z \in \partial K$ such that $\dim N_K(z) \geqslant 2$ for the normal cone $N_K(z)$ at z, and seek a contradiction (cf. Section 1.5). Let $w \in \mathbb{R}^n$ such that $z - w \in \operatorname{int} K$. It follows that $(w + w^\perp) \cap N_K(z)$ is a compact convex set of dimension at least 1, and let $\tilde{e}$ be an extreme point of this set (cf. Lemma 1.6.5). It follows that $e = \tilde{e}/\|\tilde{e}\| \in S^{n-1}$ is an extreme point of the compact, convex set $S' = N_K(z) \cap (e + e^\perp) \cap (e + B^n)$ of dimension at least 1, and hence o is an extreme point of $S = S' - e$ satisfying $\dim S \geqslant 1$. We deduce from Remark 9.2.5 that $\tilde{\varphi}(y) = h_K(y+e)$ for $y \in e^\perp$ satisfies the Monge-Ampère equation

$$\det D^2\tilde{\varphi}(y) = g(y)$$

for a positive continuous function g on $e^\perp$. It follows from the definition of a support function that $\tilde{\varphi}(y) = h_K(y+e) \geqslant \langle y+e, z\rangle$ for $y \in e^\perp$, and $\tilde{\varphi}(y) = \langle y+e, z\rangle$ if and only if $y + e \in N_K(z)$. We deduce that $\varphi(y) = \tilde{\varphi}(y) - \langle y+e, z\rangle \geqslant 0$ is convex on $\Omega = e^\perp \cap \operatorname{int} B^n$, satisfies $S = \operatorname{cl}\{y \in \Omega : \varphi(y) = 0\}$, and there exists a function g on Ω such that $\lambda^{-1} < g(y) < \lambda$ for a constant $\lambda > 1$ and

$$\det D^2\varphi(y) = g(y). \tag{9.13}$$

This contradicts Theorem 9.1.2 as o is an extreme point of S satisfying $\dim S \geqslant 1$; therefore, ∂K is C^1.

Next, we show that h_K is $C^{0,\alpha}$ near any $e \in S^{n-1}$. As ∂K is C^1, we deduce from Lemma 1.9.6 that the function $\psi(y) = h_K(y+e)$ for $y \in e^\perp$ is strictly convex, and it satisfies (9.13) (with $\psi = \varphi$) for a function g on $\Omega = e^\perp \cap \operatorname{int} B^n$ that extends to a positive $C^{0,\alpha}$ function on $e^\perp \cap B^n$ according to Remark 9.2.5. Therefore, Theorem 9.1.3 (with $\psi = \varphi$) yields that ψ on Ω, and in turn h_K on $e + \operatorname{int} B^n$ is locally $C^{2,\alpha}$.

Finally, the case when f is $C^{m,\alpha}$ for $m \geqslant 1$ follows via Theorem 9.1.3. ∎

9.3 The L_p-Minkowski problem

In this section, we summarize some major results and conjectures concerning the L_p-Minkowski problem initiated by Erwin Lutwak in the 1990's (starting with the case $p > 1$ in Lutwak [550], who solved the even L_p-Minkowski problem for $p > 1$) that extends the classical Minkowski problem ($p = 1$, cf. Section 9.2), and by Firey [287] who posed the Logarithmic (L_0) Minkowski problem in 1974 about the cone volume measure. In the case $p \geqslant 0$, more detailed discussions about these topics are provided in Sections 9.A, 9.B and 9.C because the uniqueness of the solution of a Monge-Ampère equation is connected to Brunn-Minkowski type inequalities in this case. Since convex bodies in the rest of the chapter contain the origin, let $\mathcal{K}^n_o$ and $\mathcal{K}^n_{(o)}$ denote the family of convex bodies $K \subset \mathbb{R}^n$ such that $o \in K$ or $o \in \operatorname{int} K$, respectively.

Definition 9.3.1 (L_p-surface area measure $S_{K,p}$). For $p \in \mathbb{R}$, and a convex body $K \in \mathcal{K}^n_o$, the L_p-surface area measure $S_{K,p}$ is the Borel measure on S^{n-1} defined as

$$dS_{K,p} = h_K^{1-p}\, dS_K.$$

If $p > 1$ and $o \in \partial K$, then we assume that $S_K(\{u \in S^{n-1} : h_K(u) = 0\}) = 0$; or equivalently, $S_K(N_K(o) \cap S^{n-1}) = 0$, which condition we abbreviate as $S_K(\{h_K = 0\}) = 0$.

Let us list some basic properties that directly follow from the definition where $p \in \mathbb{R}$ and $K \in \mathcal{K}_o$, and we also assume $S_K(\{h_K = 0\}) = 0$ if $p > 1$ and $o \in \partial K$.

- $\Phi_* S_{K,p} = S_{\Phi K,p}$ for $\Phi \in O(n)$; namely, $S_{K,p}$ is equivariant under orthogonal transformations.
- If $g : S^{n-1} \to \mathbb{R}$ Borel, then (cf. (2.21))

$$\int_{S^{n-1}} g\, dS_{K,p} = \int_{\partial' K} g(\nu_K(x)) \cdot \langle \nu_K(x), x\rangle^{1-p}\, d\mathcal{H}^{n-1}(x) \tag{9.14}$$

  where the integral makes sense even if $p > 1$ and $o \in \partial K$ as in this case, the conditions $S_K(\{h_K = 0\}) = 0$ and $\mathcal{H}^{n-1}(\{x \in \partial' K : \langle \nu_K(x), x\rangle = 0\}) = 0$ are equivalent.

- $S_{K,p}$ is a finite measure if $o \in \operatorname{int} K$ or $p \leqslant 1$. In the case $p > 1$ and $o \in \partial K$, $S_{K,p}(S^{n-1})$ might be infinite even if $S_K(\{h_K = 0\}) = 0$.
- If $\lambda > 0$, then
$$S_{\lambda K,p} = \lambda^{n-p} \cdot S_{K,p}. \tag{9.15}$$
- $S_{K,p}$ is weakly continuous for $K \in \mathcal{K}_{(o)}$, and even for $K \in \mathcal{K}_o$ if $p \leqslant 1$.

For the last property, if K_m tends to K for $K_m, K \in \mathcal{K}_o$, then S_{K_m} tends weakly to S_K according to Proposition 2.6.13 (see also Proposition 8.4.1). On the other hand, $h_{K_m}^{1-p}$ tends uniformly to h_K^{1-p} either if $p \leqslant 1$, or if $p > 1$ and $K_m, K \in \mathcal{K}_{(o)}$ (see Lemma 9.A.1 for a related result in the case $p > 1$, $K_m \in \mathcal{K}_{(o)}$ and $o \in \partial K$).

Example 9.3.2 (Some fundamental L_p surface area measures).

$p = 1$. $S_{K,1} = S_K$ is the surface area measure (cf. Sections 2.5 and 8.2) that satisfies
$$S_{K+z} = S_K \text{ for } z \in \mathbb{R}^n.$$

$p = 0$. $S_{K,0} = nV_K$ where V_K is the cone volume measure (cf. Section 2.6).
In particular, $S_{K,0}$ is "$\mathrm{SL}(n)$ equivariant"; namely, if $\omega \subset S^{n-1}$ is a Borel set and $\Phi \in \mathrm{GL}(n)$ with $\det \Phi = \pm 1$, then
$$S_{\Phi K,0}(\omega) = S_{K,0}\left(\left\{\frac{\Phi^t u}{\|\Phi^t u\|} : u \in \omega\right\}\right).$$

$p = -n$. If ∂K is C_+^2, and $\nu_K(x(u)) = u$ for $u \in S^{n-1}$ and $x(u) \in \partial K$, then
$$dS_{K,-n}(u) = \kappa_0(K, x(u))^{-1}\, d\mathcal{H}^{n-1}(u)$$
where $\kappa_0(K,x) = \frac{\kappa(x)}{\langle x, \nu_K(x)\rangle^{n+1}} = \kappa_0(\Phi K, \Phi x)$ if $\Phi \in \mathrm{GL}(n)$ with $\det \Phi = \pm 1$ (cf. Section 8.9.2). In particular, if ∂K is C_+^2, then characterizing $S_{K,-n}$ is equivalent with characterizing the $\mathrm{SL}(n)$ invariant centro-affine curvature $\kappa_0(K, x(u))$.

The L_p-surface area pops up in various related problems; for example, in the definition of the L_p-affine surface area (cf. Theorem 8.9.12) proposed by Lutwak [552], or in the characterisation of $\mathrm{GL}(n)$ contravariant measure valued valuations by Haberl, Parapatits [373]. The main question addressed by this section is the following version of the Minkowski problem proposed by Lutwak [550] in 1993.

Remark 9.3.3 (L_p-Minkowski Problem). Let $p \in \mathbb{R}$ and $n \geqslant 2$.

Monge-Ampère equation on S^{n-1}: Given integrable $f : S^{n-1} \to [0, \infty)$, find suitably differentiable $h : S^{n-1} \to [0, \infty)$ such that
$$\det(\nabla^2 h + h I_{n-1}) = h^{p-1} f \qquad \text{if } p \geqslant 1; \tag{9.16}$$
$$h^{1-p} \det(\nabla^2 h + h I_{n-1}) = f \qquad \text{if } p \leqslant 1. \tag{9.17}$$

In the sense of measure:

- $h = h_K|_{S^{n-1}}$ for $K \in \mathcal{K}_o^n$ is an *Aleksandrov solution* of (9.16) or (9.17) if

$$dS_{K,p} = f\, d\mathcal{H}^{n-1} \tag{9.18}$$

  where $S_K(\{h_K = 0\}) = 0$ if $p > 1$.
- For $p \in \mathbb{R}$, characterize L_p-surface area measure $S_{K,p}$ of a $K \in \mathcal{K}_o^n$ as a Borel measure on S^{n-1} where $S_K(\{h_K = 0\}) = 0$ if $p > 1$.

In particular, the L_p-Minkowski Problem is just the classical Minkowski problem if $p = 1$ (cf. Section 9.2), and the Logarithmic Minkowski problem about the cone volume measure posed by Firey [287] in 1974 if $p = 0$.

Let us list the known not too technical existence results about the solution of the L_p-Minkowski Problem without symmetry assumption. If $p = n$, then no scaling is possible ($S_{K,p} = S_{\lambda K,p}$ for $\lambda > 0$, cf. (9.15)); therefore, the solution is only known up to homothety (cf. Hug, Lutwak, Yang, Zhang [427]). Out of the three fundamental cases; namely, $p = 1, 0, -n$, we exclude the classical $p = 1$ case because it is discussed in Section 9.2 in detail.

Remark 9.3.4 (Some known results about the L_p-Minkowski Problem).

$p > 0$, $p \neq 1, n$: If μ is a non-trivial finite Borel measure on S^{n-1} not concentrated on any closed hemisphere, then $\mu = S_{K,p}$ for convex body $K \in \mathcal{K}_o^n$ where $S_K(\{h_K = 0\}) = 0$ if $p > 1$ (cf. Theorem 9.C.1).

This result is due to Chou, Wang [201] and Hug, Lutwak, Yang, Zhang [427] if $p > 1$, and to Chen, Li, Zhu [194] if $p \in (0, 1)$ (cf. Section 9.C).

$p = 0$ (Logarithmic Minkowski Problem): If μ is a non-trivial finite Borel measure on S^{n-1} such that

$$\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1}) \quad \text{for any proper linear subspace } L \subset \mathbb{R}^n, \tag{9.19}$$

then $\mu = S_{K,0} = nV_K$ for a convex body $K \in \mathcal{K}_o^n$ (e.g. any absolutely continuous measure is a cone volume measure). This result, proved as Theorem 9.C.1, is due to Chen, Li, Zhu [195].

On the other hand, there are some non-trivial obstructions for a finite Borel measure on S^{n-1} to be a cone volume measure. For example, for any convex body $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$, Böröczky, Hegedűs [130] (cf. (2.40)) prove that

$$V_K(u) + V_K(-u) + 2(n-1)\sqrt{V_K(u)V_K(-u)} \leqslant V_K(S^{n-1})(= |K|). \tag{9.20}$$

$-n < p < 0$: If $d\mu = f\, d\mathcal{H}^{n-1}$ is a non-trivial measure for a non-negative function $f \in L^{\frac{n}{n+p}}(S^{n-1}, \mathcal{H}^{n-1})$, then $\mu = S_{K,p}$ for a convex body $K \in \mathcal{K}_o^n$, see Bianchi, Böröczky, Colesanti, Yang [91].

$p = -n$: The $p = -n$ case of the L_p-Minkowski problem is the critical case because its link to the $\mathrm{SL}(n)$ invariant century old notion of centro-affine curvature. If $K \in \mathcal{K}^n_{(o)}$ has C^2_+ boundary, then $dS_{K,-n}(u) = \kappa_0(x(u))^{-1}\, du$ where $\nu_K(x(u)) = u$ for $u \in S^{n-1}$ and $x(u) \in \partial K$, and $\kappa_0(K, x(u)) = \kappa_K(x(u))/h_K(u)^{n+1}$ is the $\mathrm{SL}(n)$ invariant centro-affine curvature (cf. Proposition 8.9.3). This case is far the most mysterious, hardly anything is known.

If the f in (9.17) is unconditional and satisfies certain additional technical conditions, then Jian, Lu, Zhu [448] verify the existence of a solution. Moreover, the paper Guang, Li, Wang [367] solves a variant of the centro-affine Minkowki problem. Stancu [689] links the centro-affine Minkowki problem to the logarithmic Minkowski problem (when $p = 0$).

On the other hand, Du, Liu, Lu [238] construct the first explicit example of a positive $C^{0,\alpha}$, $\alpha \in (0, 1)$, function f on S^{n-1} such that (9.17) has no solution when $p = -n$, and Chou, Wang [201] proved an implicit condition on possible functions f in (9.17) (see also [91]).

$p < -n$: In this super-critical case, if μ is an absolutely continuous measure on S^{n-1} with positive continuous density, then $\mu = S_{K,p}$ for a $K \in \mathcal{K}^n_{(o)}$ according to Li, Guang, Wang [365] where $h_K|_{S^{n-1}}$ is $C^{1,\alpha}$.

On the other hand, for $p < -n$, Du [237] constructs a non-negative $C^{0,\alpha}$, $\alpha \in (0, 1)$, function f on S^1 that is positive everywhere but a fixed pair of antipodal points and the L_p-Minkowski problem (9.17) has no solution, not even in the Aleksandrov sense (see Du [237] for the case $n = 2$).

Both the variational and the flow methods - the two main methods leading to the results above - are reviewed in Remark 9.3.7.

Concerning the super-critical case $p < -n$ open for many decades in spite of intense research, it is not surprising that the flow method is the one succeeding, as E. Milman [578] points out the limitations of the variational argument in this case. The paper Guang, Li, Wang [365] solves the problem by introducing a whole new approach, mixing the flow method with homology theory to find the right initial condition. When $n = 2$, the case $p \leqslant -2 = -n$ is further investigated by Yang, Liu, Fang [732].

Let μ be a finite Borel measure on S^{n-1} not concentrated on any closed hemisphere. If $K_{\mu,p} \in \mathcal{K}^n_o$ satisfies $\mu = S_{K_p,p}$ for $p > n$, then Zou [740] proves that

$$\lim_{p\to\infty} K_{\mu,p} = K_\mu = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant 1,\ \forall u \in \operatorname{supp}\mu\};$$

for example, $h_{K_p}|_{S^{n-1}}$ tends uniformly to the constant 1 function as p tends to infinity if μ has a positive density function f in (9.16).

A general sufficiency condition for all $p \leqslant 0$ (see Theorem 9.C.2) is due to Zhu [738]: if μ is a discrete measure on S^{n-1} such that $\operatorname{supp}\mu$ is not contained in a closed hemisphere and any n elements of $\operatorname{supp}\mu$ are independent, then there exists a polytope

$Q \in \mathcal{K}^n_{(o)}$ such that $\mu = S_{Q,p}$. However, this result does seem to be useful to construct solutions for more general measures if $p < 0$ because no method is known how to control the diameter of the solution in the case of weak approximation by such discrete measures.

Remark 9.3.5 (Open problems).

$p \leqslant 0$: In the case of any $p \leqslant 0$, not even a conjecture is known what properties characterize an L_p surface area measure. Probably, the most challenging case is when $p = -n$.

$p \in (0,1)$ *and* $\operatorname{supp} S_{K,p}$ *is lower dimensional:* Let $p \in (0,1)$, and for a non-trivial finite Borel measure μ on S^{n-1}, let $L = \operatorname{lin}\operatorname{supp}\mu$. If $n = 2$, then Böröczky, Trinh [149] and Chen, Li, Zhu [194] prove that $\mu = S_{K,p}$ for $K \in \mathcal{K}^2_o$ if and only if $\operatorname{supp}\mu$ is not a pair of antipodal points; or in other words, $\dim L \neq 1$. However, a full characterization of the L_p surface area measure under the condition $\dim L \leqslant n-1$ is still not known. So let $\dim L \leqslant n-1$. If $\operatorname{supp}\mu \subset L$ is contained in a closed hemisphere centered at a point of $L \cap S^{n-1}$, then μ is an L_p-surface area measure according to Bianchi, Böröczky, Colesanti, Yang [91]. On the other hand, Saroglou [649] proved that if $\mu(\omega)$ is the $\dim L$-dimensional Lebesgue measure of $\omega \cap L$ for any Borel $\omega \subset S^{n-1}$, then μ is not an L_p-surface area measure.

We shortly discuss the L_0 (Logarithmic) Minkowski Problem in more details because this has been very intensely investigated in the last decade. The even case has been completely characterized by Böröczky, Lutwak, Yang, Zhang [141] (cf. Theorem 9.B.4).

Theorem 9.3.6. *For a non-trivial finite even Borel measure μ on S^{n-1}, there exists an o-symmetric convex body $K \subset \mathbb{R}^n$ with $\mu = S_{K,0} = nV_K$ if and only if*

(i) $\mu(L \cap S^{n-1}) \leqslant \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *for any proper linear subspace $L \subset \mathbb{R}^n$;*

(ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *in (i) is equivalent with the existence of a complementary linear subspace $L' \subset \mathbb{R}^n$ with* $\operatorname{supp}\mu \subset L \cup L'$, *and in this case, $K = C + C'$ for o-symmetric compact convex sets $C \subset L^{\perp}$ and $C' \subset L'^{\perp}$.*

Remarks.

- For any non-trivial finite Borel measure μ on S^{n-1} satisfying (i) and (ii), Chen, Li, Zhu [195] prove that $\mu = S_{K,0} = nV_K$ for a $K \in \mathcal{K}^n_o$.
- The cone volume measure of quadrilaterals is characterized by Yude Liu, Xinbao Lu, Qiang Sun, Ge Xiong [527], and Baumbach, Henk [77] provides many characteristic properties of cone volume measures of polytopes.
- The cone volume measure $V_K = \mu$ of any centered convex body $K \subset \mathbb{R}^n$ (the centroid $\sigma_K = o$) satisfies the properties (i) and (ii) according to Böröczky, Henk [132]. In addition, Böröczky, Henk [133] prove a stability version of (ii), and Freyer, Henk, Kipp [306] prove an analoguos property of centered convex poly-

topes with respect to affine subspaces, which result has been extended to any convex body by Eller, Freyer [246].

- A Borel probability measure μ on S^{n-1} satisfies the conditions (i) and (ii) if and only if there exists an isotropic linear image $\widetilde{\Phi}_*\mu$ for a $\Phi \in \mathrm{GL}(n)$ according to Böröczky, Lutwak, Yang, Zhang [142] where a Borel measure ν on S^{n-1} is isotropic if $n\int_{S^{n-1}} u \otimes u \, dv(u) = I_n$, and for any Borel set $\omega \subset S^{n-1}$, we have

$$\widetilde{\Phi}_*\mu(\omega) = \mu\left(\left\{\frac{\Phi^{-1}(u)}{\|\Phi^{-1}(u)\|} : u \in \omega\right\}\right).$$

Let us review the main approaches to find a solution of the L_p-Minkowski problem. One approach based on weak approximation by discrete measures if $p \geqslant 0$ is discussed in more detail in Section 9.C because that approach is the most elementary.

Remark 9.3.7 (How to find a solution of the L_p-Minkowski problem).
Overall strategy: For $p \in \mathbb{R}$, a fundamental tool is the entropy of a $K \in \mathcal{K}_o^n$ with respect to a Borel probability measure μ on S^{n-1}. If $p > 1$, then the entropy is

$$\mathcal{E}_{\mu,p}(K) = \frac{1}{p}\int_{S^{n-1}} h_K^p \, d\mu,$$

and if $p < 1$ and $\xi \in \operatorname{int} K$, then the corresponding entropy is

$$\mathcal{E}_{\mu,p}(K,\xi) = \begin{cases} \frac{1}{p}\int_{S^{n-1}} h_{K-\xi}^p \, d\mu & \text{if} \quad p \neq 0 \\ \int_{S^{n-1}} \log h_{K-\xi} \, d\mu & \text{if} \quad p = 0. \end{cases} \tag{9.21}$$

To solve the L_p-Minkowski Problem (cf. Remark 9.3.3), first one considers special measures, either discrete ones, or ones that are absolutely continuous with a positive continuous density function, and finds a solution with suitably bounded entropy using either the variational method or the flow method. Then the more general measures in Remark 9.3.4 are weakly approximated by the special type of measures, and the entropy bound ensures that the solutions for the approximating special type of measures are bounded for $p > -n$ (cf. Proposition 9.A.2), and hence a subsequence converges to a solution of the original problem.

The flow method for $p < 1$: Here we mostly follow Böröczky, Guan [136] (see Bryan, Ivaki, Scheuer [165] in the even case, and Andrews, Chow, Guenther, Langford [32] for thorough introduction into the theory of geometric flows). Given $p \in (-n, 1)$, and a positive C^∞ probability density function f on S^{n-1}, let $\alpha = \frac{1}{1-p} > \frac{1}{n+1}$, and our aim is to solve the Monge-Ampère equation

$$h^{1-p}\det(\nabla^2 h + hI_{n-1}) = h^{1/\alpha}\det(\nabla^2 h + hI_{n-1}) = f. \tag{9.22}$$

For a convex body $M_0 \subset \mathbb{R}^n$ with C^∞_+ boundary and $|M_0| = 1$, the theory of anisotropic flows (see [32] or [136] for references) provides us a family of convex bodies M_t of C^∞_+ boundary for $t \in [0, T)$ that satisfy the evolution equation

$$\frac{\partial}{\partial t} X(x,t) = -f(\nu)^\alpha \kappa(x,t)^\alpha \, \nu(x,t) \tag{9.23}$$

where for $t \in [0,T)$ and $x \in \partial M_t$, we have $X(x,t) = x$, $\nu = \nu(x,t) = \nu_{M_t}(x)$, and $\kappa(x,t)$ is the Gaussian curvature at $x \in \partial M_t$, and in addition, M_t tends to a point $q \in \operatorname{int} M_0$ as t tends to T. Assuming $q = o$, the dilate $\widetilde{M}_t$ of M_t with $|M_t| = 1$ corresponds to the normalized flow satisfying the evolution equation

$$\frac{\partial}{\partial t} \widetilde{X}(x,t) = -\frac{f(\tilde{\nu})^\alpha \tilde{\kappa}(x,t)^\alpha}{\int_{S^{n-1}} f(u)^\alpha \bar{\kappa}(u,t)^{\alpha-1} \, du} \cdot \tilde{\nu}(x,t) + \widetilde{X}(x,t) \tag{9.24}$$

where for $t \in [0,T)$ and $x \in \partial \widetilde{M}_t$, we have $\widetilde{X}(x,t) = x$, $\tilde{\nu} = \tilde{\nu}(x,t) = \nu_{\widetilde{M}_t}(x)$, and $\tilde{\kappa}(x,t)$ is the Gaussian curvature at $x \in \partial \widetilde{M}_t$; moreover, $\bar{\kappa}(\tilde{\nu}(x,t),t) = \tilde{\kappa}(x,t)$. It can be proved that the supremum $\sup_{\xi \in \operatorname{int} \widetilde{M}_t} \mathcal{E}_{\mu,p}(\widetilde{M}_t, \xi)$ of the entropy for the probability measure $d\mu = f \, d\mathcal{H}^{n-1}$ is non-increasing in t. Therefore, as Guan, Ni [361] verify, diameter bounds in terms of the entropy for $p > -n$ (cf. Proposition 9.A.2) yield that $\widetilde{M}_t$ converges to a convex body $K \in \mathcal{K}_o$ as $t \to T$, and $h_K|_{S^{n-1}}$ satisfies (9.22).

When $p = -n$, the argument breaks down because the entropy does not bound the diameter anymore - consider for example centered ellipsoids, whose L_{-n}-surface area is a constant multiple of the Lebesgue measure - and $\widetilde{M}_t$ can actually be unbounded (see Andrews [28]). When $p < -n$, Li, Guang, Wang [365] use homology theory to find a suitable initial condition M_0 to tackle this issue.

Historically, Firey [287] indicated in 1974 that the surface of the worn stone can be modeled by the differential equation (9.23) when f is a suitable constant function and $\alpha = 1$ (and hence $p = 0$), and the support function of the limit shape of the normalized equation (9.24) satisfies (9.22).

The variational method for $p > -n$: When $p \geqslant 0$, one can find a solution of the L_p-Minkowski problem for a rather general measure μ on S^{n-1} by first using the variational method to find a polytopal solution in the case when μ is a discrete measure (see Section 9.C). The discrete case is due to Hug, Lutwak, Yang, Zhang [427] if $p > 1$, to Zhu [737] if $p \in (0,1)$, and to Böröczky, Hegedűs, Zhu [131] if $p = 0$.

If $p \in (-n, 1)$ and f is a positive continuous function on S^{n-1}, then the variational method can be also used to solve the Monge-Ampère equation

$$h^{1-p} \det(\nabla^2 h + h I_{n-1}) = f, \tag{9.25}$$

as it is done by Chou, Wang [201] and Bianchi, Böröczky, Colesanti, Yang [91]. We sketch the main steps. Let $d\mu = f \, d\mathcal{H}^{n-1}$, and let $\mathcal{F}$ be the family of convex bodies

$C \in \mathcal{K}_o^n$ with $|C| = 1$. For any $C \in \mathcal{F}$, there exists a unique $\xi_C \in C$ (depending also on p and μ) such that the entropy $\mathcal{E}_{\mu,p}(K,\xi)$ for $\xi \in C$ is maximized at $\xi = \xi_C$. Using the diameter bounds in Proposition 9.A.2 in terms of entropy (see (9.21) for the definition of entropy), we obtain a $\widetilde{C} \in \mathcal{F}$ such that

$$\mathcal{E}_{\mu,p}(\widetilde{C}, \xi_{\widetilde{C}}) = \min_{C \in \mathcal{F}} \mathcal{E}_{\mu,p}(C, \xi_C), \tag{9.26}$$

such that $h = h_{\widetilde{C} - \xi_{\widetilde{C}}}$ satisfies

$$\det(\nabla^2 h + h I_{n-1}) = h^{p-1} f \tag{9.27}$$

in the sense of measure. The path from (9.26) to (9.27) is a variational argument (similar to the one in Theorem 9.2.3 in the case of the classical Minkowski-problem) if $\xi_{\widetilde{C}} \in \operatorname{int} \widetilde{C}$. However, $\xi_{\widetilde{C}}$ may lie in $\partial \widetilde{C}$ if $p > 2 - n$ (see Example 9.3.9); therefore, we need a twist in the argument (see Chou, Wang [201] or Bianchi, Böröczky, Colesanti, Yang [91]). For small $\varepsilon > 0$, we replace the function $t \mapsto \frac{1}{p} t^p$ (or $t \mapsto \log t$ if $p = 0$) in the definition of the entropy by a C^1 strictly concave function $\varphi_\varepsilon(t)$ that coincides with the original function if $t \geqslant 3\varepsilon$ and $\varphi_\varepsilon(t) = \frac{-1}{|p|} t^{-n}$ (or $\varphi_\varepsilon(t) = -t^{-n}$ if $p = 0$) if $t \in (0, \varepsilon]$. Now, the $\widetilde{C}_\varepsilon$ minimizing this modified entropy satisfies that the corresponding "center" (the analogue of $\xi_{\widetilde{C}}$) lies in $\operatorname{int} \widetilde{C}_\varepsilon$, and hence assuming the corresponding "center" is the origin, a variational argument yields that

$$\det(\nabla^2 h + h I_{n-1}) = \varphi_\varepsilon'(h) f$$

holds for the support function. Finally, letting ε tending to zero, we obtain a common solution of (9.26) and (9.27).

Starting with Haberl, Lutwak, Yang, Zhang [371], Orlicz versions of the L_p-Minkowski problem have been intensively investigated; namely, the function $t \mapsto t^{1-p}$ in the Monge-Ampère equations (9.16) and (9.17) corresponding to the L_p-Minkowski problem is replaced by certain $\varphi : (0,\infty) \to (0,\infty)$, and hence the new Monge-Ampère equations is of the form

$$\varphi(h) \det(\nabla^2 h + h\, I_{n-1}) = f, \tag{9.28}$$

where f is a given non-negative measurable function on S^{n-1}. Typically, the solution is only up to a constant factor; namely, there exists some $c > 0$ such that $\varphi(h) \det(\nabla^2 h + h\, I_{n-1}) = c \cdot f$. The known existence results about the L_p-Minkowski problem for $p > -n$ have been generalized to the Orlicz L_p-Minkowski problem where $\varphi(t)$ replaces t^{1-p} by Huang, He [417] if $p > 1$ (see also Xie [725], and uniqueness if f is constant in (9.28) is clarified by Ivaki [440]), by Jian, Lu [447] if $p \in (0,1)$, and by Bianchi, Böröczky, Colesanti [89] if $p \in (-n, 0)$.

9.3.1 Positiveness and smoothness in the L_p-Minkowski problem

According to Böröczky, Fodor [127] (correcting a formula in Chou, Wang [201]), the L_p-Minkowski problem - that is a Monge-Ampère equation on S^{n-1} (cf. (9.16) and (9.17)) - can be locally written as a Monge-Ampère equation on $\mathbb{R}^{n-1}$ in a rather natural way (see (9.29) and (9.30)) using the homogeneity of the support function (cf. (8.8)).

Remark 9.3.8 (Transfering local equation on S^{n-1} to Monge-Ampère in $\mathbb{R}^{n-1}$). Let $p \in \mathbb{R}$ and $e \in S^{n-1}$. Setting $\varphi(y) = h_K(y+e)$ for $y \in e^\perp$, a solution $h = h_K|_{S^{n-1}}$, $K \subset \mathbb{R}^n$ convex body, of the Monge-Ampère equations (9.32) and (9.33), (8.8) implies that

$$\det(D^2\varphi) = \varphi^{p-1} g \qquad \text{if } p \geqslant 1; \tag{9.29}$$

$$\varphi^{1-p} \det(D^2\varphi) = g \qquad \text{if } p \leqslant 1. \tag{9.30}$$

in the sense of measure where for $y \in e^\perp$, we have

$$g(y) = \left(1 + \|y\|^2\right)^{-\frac{n+p}{2}} \cdot f\left(\frac{e+y}{\sqrt{1+\|y\|^2}}\right).$$

As it was observed by Hug, Lutwak, Yang, Zhang [427] if $1 < p < n$ and by Bianchi, Böröczky, Colesanti [90] if $2 - n < p < 1$, the solution of the L_p-Minkowski problem may not be positive even if the f in (9.16) and (9.17) is positive and $C^{0,\alpha}$.

Example 9.3.9. For $p \in (2-n, n)$, we show that there exists a convex body $K \in \mathcal{K}_o^n$ such that ∂K and $h = h_K|_{S^{n-1}}$ are $C^{1,\alpha}$ for some $\alpha > 0$, $o \in \partial K$ (and hence $h(e) = 0$ for the exterior unit normal e at o), and

$$dS_{K,p} = f \, d\mathcal{H}^{n-1}$$

on S^{n-1} for a positive and $C^{0,\alpha}$ function f.

We fix $e \in S^{n-1}$, and write $x + te = (x,t)$ for $x \in e^\perp$ and $t \in \mathbb{R}$. For

$$q = \frac{2(n-1)}{n+p-2} > 1 \ \text{ satisfying } \ \frac{q}{q-1} = \frac{2n-2}{n-p} > 1,$$

we choose the convex body $K \subset \mathbb{R}^n$ in a way such that $(x, -\|x\|^q) \in \partial K$ if $x \in e^\perp \cap B^n$ and $\partial K \backslash \{o\}$ is C^2_+. For $\psi(x) = \|x\|^q$, $x \in e^\perp \cap \operatorname{int} B^n$, we observe that $D\psi(x) = q\|x\|^{q-2}x$ if $x \neq o$ and $D\psi(o) = o$; moreover, one exterior normal to K at $(x, -\|x\|^q) \in \partial K$ is $\tilde{\nu}(x) = (D\psi(x), 1)$ and

$$h_K(\tilde{\nu}(x)) = (q-1)\|x\|^q.$$

In particular, writing θ_1, θ_2 to denote positive constants that depend on p and n, if $y = D\psi(x) = q\|x\|^{q-2}x$, then

$$\varphi(y) = h_K(y,1) = \theta_1 \cdot \|y\|^{\frac{q}{q-1}} = \theta_1 \cdot \|y\|^{\frac{2(n-1)}{n-p}}$$
$$D^2\varphi(y) = \theta_2 \cdot \|y\|^{(n-1)(\frac{q}{q-1}-2)} = \theta_2 \cdot \|y\|^{(n-1)\frac{2(p-1)}{n-p}}.$$

We deduce that φ solves the Monge-Ampère equation

$$\varphi^{1-p} D^2\varphi(y) = \theta_1^{1-p}\theta_2$$

on $e^\perp \cap B^n$, and hence the statement in Example 9.3.9 follows from Remark 9.3.8.

Next we show, following Chou, Wang [201], that Example 9.3.9 is optimal in the sense that if $p \geqslant n$ or $p \leqslant 2-n$ and the f in the L_p-Minkowski problem (9.18) is bounded and bounded away from zero, then the solution $h = h_K|_{S^{n-1}}$ for $K \in \mathcal{K}_o^n$ is positive; namely, $o \in \operatorname{int} K$.

Lemma 9.3.10. *For $f : S^{n-1} \to (\lambda^{-1}, \lambda)$, $\lambda > 1$, if $K \in \mathcal{K}_o^n$ is a solution of the L_p-Minkowski problem (9.18) where*

- *either $p \geqslant n$ and $S_K(\{h_K = 0\}) = 0$;*
- *or $p \leqslant 2-n$;*

then $o \in \operatorname{int} K$, and hence $h > 0$ for $h = h_K|_{S^{n-1}}$.

Proof. We suppose that $o \in \partial K$, and seek a contradiction. According to Lemma 1.2.11, there exist $t, r \in (0,1)$ and an exterior unit normal $e \in S^{n-1}$ to K at o such that

$$-te + rB^n \subset \operatorname{int} K. \tag{9.31}$$

First let $p \geqslant n$ and $S_K(\{h_K = 0\}) = \mathcal{H}^{n-1}(\{x \in \partial' K : \langle \nu_K(x), x\rangle = 0\}) = 0$. As $h_K(\nu_K(x)) = \langle \nu_K(x), x\rangle \leqslant \|x\|$ for $x \in \partial' K$, (9.14) and (9.31) yield

$$\begin{aligned}\lambda \cdot \mathcal{H}^{n-1}(S^{n-1}) &\geqslant \int_{S^{n-1}} f = \int_{S^{n-1}} h_K^{1-p}\, dS_K = \int_{\partial' K} \langle \nu_K(x), x\rangle^{1-p}\, d\mathcal{H}^{n-1}(x)\\ &\geqslant \int_{\partial' K} \|x\|^{1-p}\, d\mathcal{H}^{n-1}(x) \geqslant \int_{e^\perp \cap rB^n} \|y\|^{1-p}\, d\mathcal{H}^{n-1}(y)\\ &= (n-1)\omega_{n-1}\int_{s=0}^{r} s^{n-1-p}\, ds = \infty,\end{aligned}$$

which is absurd, verifying Lemma 9.3.10 if $p \geqslant n$.

Assuming $p \leqslant 2-n$, we consider the spherical cap $\Omega(e,\varrho) = \{u \in S^{n-1} : \langle u, e\rangle > \cos\varrho\}$ for $\varrho \in (0, \frac{\pi}{2})$, which satisfies $\mathcal{H}^{n-1}(\Omega(e,\varrho)) > \omega_{n-1}(\sin\varrho)^{n-1}$. If $D = \operatorname{diam} K$, $x \in K$ and $u \in \Omega(e,\varrho)$, then writing x in the form $y - se$ for $s \geqslant 0$ and $y \in e^\perp$,

we have $\|y\| \leqslant \|x\| \leqslant D$, and there exists an orthonormal basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ with $e = v_1$ and $\|y\| v_2 = y$, and hence

$$\langle x, u\rangle \leqslant \langle y, u\rangle \leqslant D\langle v_2, u\rangle \leqslant D\sqrt{1 - \langle v_1, u\rangle^2} \leqslant D \sin \varrho.$$

Therefore $h_K(u) \leqslant D \sin \varrho$ for any $u \in \Omega(e, \varrho)$ and $\varrho \in (0, , \frac{\pi}{2})$, thus

$$\begin{aligned}\lambda^{-1} &\leqslant \frac{1}{\mathcal{H}^{n-1}\left(\Omega(e,\varrho)\backslash\{e\}\right)} \int_{\Omega(e,\varrho)\backslash\{e\}} f = \frac{1}{\mathcal{H}^{n-1}\left(\Omega(e,\varrho)\backslash\{e\}\right)} \int_{\Omega(e,\varrho)\backslash\{e\}} h_K^{1-p}\, dS_K \\ &\leqslant \frac{D^{1-p}}{\omega_{n-1}} \cdot (\sin\varrho)^{2-n-p} \cdot S_K\left(\Omega(e,\varrho)\backslash\{e\}\right) \leqslant \frac{D^{1-p}}{\omega_{n-1}} \cdot S_K\left(\Omega(e,\varrho)\backslash\{e\}\right).\end{aligned}$$

Since $\bigcap_{\varrho>0}\left(\Omega(e,\varrho)\backslash\{e\}\right) = \varnothing$ and $f \leqslant \lambda$ yield $\lim_{\varrho\to 0^+} S_K\left(\Omega(e,\varrho)\backslash\{e\}\right) = 0$, we arrive at a contradiction, which in turn proves Lemma 9.3.10. ∎

Concerning the smoothness of the solution of the L_p-Minkowski problem (9.16) and (9.17), Chou, Wang [201] explained that the argument verifying Theorem 9.2.6 based on Theorem 9.1.2 and Theorem 9.1.3 due to Cafarelli [169, 170] yields the following.

Theorem 9.3.11 (Caffarelli). *If $p \in \mathbb{R}$ and the f in (9.16) and (9.17) is positive and $C^{0,\alpha}$, and in addition, the solution h is positive (or equivalently, $o \in \operatorname{int} K$ for the corresponding convex body K), then h is $C^{2,\alpha}$ (and ∂K is $C^{2,\alpha}_+$).*

Remark. If f is positive and $C^{0,\alpha}$, then the solution h in (9.16) or (9.17) is positive

- if $p \geqslant n$ or $p \leqslant 2 - n$ (cf. Lemma 9.3.10),
- or if f and h are even.

Additional results about the smoothness of the solution are provided by Bianchi, Böröczky, Colesanti [90] in the case $2 - n < p < 1$. For example, if $p < \min\{1, 4 - n\}$, and the f in (9.17) satisfies $\lambda^{-1} < f < \lambda$ for some $\lambda > 1$, then h_K is strictly convex and ∂K is C^1 for the corresponding convex body K even if $o \in \partial K$.

9.4 Uniqueness in the L_p-Minkowski problem

Uniqueness of the solution of a differential equation has always been a central problem. In the case of the L_p-Minkowski problem for $p \geqslant 0$, the role of uniqueness is even more significant as it is intimately connected to Brunn-Minkowski type inequalities, as it is explained by Remark 9.4.7. In addition, Firey's celebrated uniqueness problem is concerned with the case when the f in (9.33) is constant (see Remark 9.4.2).

We recall (cf. (9.16) and (9.17)) that for $p \in \mathbb{R}$ and measurable $f : S^{n-1} \to [0, \infty)$, the L_p-Minkowski Problem is the Monge-Ampère equation

$$\det(\nabla^2 h + hI_{n-1}) = h^{p-1} f \qquad \text{if } p \geqslant 1; \tag{9.32}$$

$$h^{1-p} \det(\nabla^2 h + hI_{n-1}) = f \qquad \text{if } p \leqslant 1. \tag{9.33}$$

Remark 9.4.1 (Uniqueness of the solution of the L_p-Minkowski problem).

Unique solution for $p > 1$: See Proposition 7.6.8 due to Hug, Lutwak, Yang, Zhang [427] in the case (9.32) where we have uniqueness even in the Aleksandrov sense.

No uniqueness if $0 \leqslant p < 1$ in general, and not even for even measures if $p < 0$: See Chen, Li, Zhu [194, 195] if $p \in [0, 1)$, and Li, Liu, Lu [513] if $p < 0$ (in the letter case, we may assume that the f and h in (9.33) are even and have axial rotational symmetry according to [513]). In addition, E. Milman [578] shows that for any o-symmetric convex body $C \subset \mathbb{R}^n$ with C^2_+ boundary, one finds a $q \in [-n, 0)$ where $q = -n$ if and only if C is a centered ellipsoid with the property that for any $p < q$, there exist multiple even solutions of the L_p-Minkowski problem (9.33); or in other words, there exists o-symmetric convex body $K \neq C$ with C^2_+ boundary with $S_{K,p} = S_{C,p}$.

One of the useful tools to verify non-uniqueness is that if $p \in (-n, 1)$ and f is a positive continuous function on S^{n-1}, then there always exists a solution of (9.33) minimizing the entropy (cf. (9.26) and (9.27)); therefore, one just needs to produce a solution with higher entropy.

According Remark 9.4.1, the uniqueness question concerning (9.33) is still an intriguing one in the even case if $p \in [0, 1)$, but before discussing this problem, we survey results about Firey's classical question whether the solution of (9.33) is unique if f is a constant function.

Remark 9.4.2 (Firey's isotropic L_p-Minkowski problem). In his seminal work about the "shapes of worn stones" in 1974, Firey [287] proved that if $p = 0$, then there is a unique even solution - that is the constant 1 function corresponding to the unit ball B^n - of the Monge-Ampère equation

$$h^{1-p} \det(\nabla^2 h + hI_{n-1}) = 1 \text{ on } S^{n-1}, \tag{9.34}$$

which is now called the "isotropic L_p-Minkowski problem". After a long sequence of partial results like Andrews And99, Andrews, Guan, Ni [30], finally, Brendle, Choi, Daskalopoulos [158] verified that the constant 1 function is the unique solution of (9.34) without the evenness condition if $-n < p < 1$. On the other hand, the $\mathrm{SL}(n)$ invariance Proposition 8.9.3 of the centro-affine curvature yields that any centered ellipsoid of volume $|B^n|$ is a solution of (9.34), and the highly non-trivial statement that these centered ellipsoids are the only solutions was proved, for example, by Calabi

[176], Pogolerov [622] and Cheng, Yau [199] in the 20th century, and more recently by Crasta, Fragalá [226], Ivaki, E. Milman [441] and Saroglou [648]. Concerning the uniqueness of the solution of (9.34) assuming $p < -n$, if $n = 2$, then it was clarified by Andrews [29]; namely, (9.34) has the unique constant 1 solution if $-7 \leqslant p < -2$, and the solution is not unique if $p < -7$. If $n > 2$, then Du [238] proves that (9.34) has multiple solutions if $p < -2n - 3$.

As a variation of the isotropic L_p-Minkowski problem, uniqueness of the solution of (9.33) if $p \in (-1, 1)$ and f is $C^{0,\alpha}$ close to 1 (without the evenness assumption) is proved by Böröczky, Chen, Liu, Saroglou [122] (see Böröczky, Saroglou [147], Chen, Feng, Liu [190], and Hu, Ivaki [414] for related results). In addition, for any $p > -n$ (with evenness condition for certain range of p), Ivaki [439] proved that if the f in (9.33) is $C^{0,\alpha}$ close to be a constant, then any solution has to be close to be a ball.

The following fundamental conjecture concerning the uniqueness of the solution in the even L_p-Minkowski problem was proposed by Lutwak [550] if $p = 0$ in 1993, and by Böröczky, Lutwak, Yang, Zhang [140] if $p \in (0, 1)$.

Conjecture 9.4.3 (Even L_p-Minkowski conjecture for $p \in [0, 1)$ and smooth data). *If $p \in [0, 1)$ and f is a positive even C^∞ function in (9.33), then (9.33) has a unique even solution.*

Remark. Kolesnikov, E. Milman [480] prove that knowing the L_p-Minkowski conjecture for some $p \in [0, 1)$ yields the L_q-Minkowski inequality when $q \in (p, 1)$ (see also Remark 9.4.7).

Conjecture 9.4.3 has been verified, for example, in the following cases:

- $n = 2$*:* Proved by Böröczky, Lutwak, Yang, Zhang [140].
- *f and h are unconditional in $\mathbb{R}^n$:* Proved by Saroglou [646].
- *$p \in (p_n, 1)$ where $0 < p_n < 1 - \frac{c}{n \log n}$ for an absolute constant $c > 0$:* Proved by Chen, Huang, Li, Liu [192] combining the local result by Kolesnikov, E. Milman [480] and the local to global approach in PDE (see Puttermann [629] for an Aleksandrov-type argument for the local to global approach).

In general, the following two conjectures are due to Böröczky, Lutwak, Yang, Zhang [140] if $p \in [0, 1)$ and the convex bodies are o-symmetric.

Conjecture 9.4.4 (Even L_p-Minkowski Conjecture for $p \in (0, 1)$). *If $p \in (0, 1)$ and $K, C \subset \mathbb{R}^n$ are o-symmetric convex bodies with $S_{K,p} = S_{C,p}$, then $K = C$.*

The o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ have *dilated summands* if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for $m \geqslant 1$ and o-symmetric compact convex sets $C_i, K_i \subset \mathbb{R}^n$ of dimension at least 1 such that $\sum_{i=1}^m \dim K_i = n$ and $C_i = \lambda_i K_i$ for

$\lambda_i > 0, i = 1, \dots, m$. If in addition, we have $|K| = |C|$, then Lemma 9.B.5 yields that $V_K = V_C$.

Conjecture 9.4.5 (Even log-(L_0-)Minkowski Conjecture). *For o-symmetric convex bodies $K, C \subset \mathbb{R}^n$, we have $S_{K,0} = S_{C,0}$ (or equivalently, $V_K = V_C$) if and only if $|K| = |C|$ and K and C have dilated summands.*

Again, Conjecture 9.4.4 and Conjecture 9.4.5 have been verified in the planar $n = 2$ case by Böröczky, Lutwak, Yang, Zhang [140], and if both K and C are unconditional in any dimension by Saroglou [646].

Given a non-trivial finite even Borel measure μ on S^{n-1}, while solving the even L_p-Minkowski problem for $p \in (0, 1)$ by Haberl, Lutwak, Yang, Zhang [371], and for $p = 0$ by Böröczky, Lutwak, Yang, Zhang [140], the authors considered the entropy

$$\mathcal{E}_{\mu,p}(C) = \begin{cases} \frac{1}{p} \int_{S^{n-1}} h_C^p \, d\mu & \text{if} \quad p \in (0, 1) \\ \int_{S^{n-1}} \log h_C \, d\mu & \text{if} \quad p = 0 \end{cases}$$

of an o-symmetric convex body $C \subset \mathbb{R}^n$, and proved the following (the argument is presented for Proposition 9.B.1).

Proposition 9.4.6. *Let $p \in [0, 1)$, and let μ be a finite even Borel measure μ on S^{n-1} such that any open hemisphere has positive measure, and if $p = 0$, then also*

$$\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1}) \text{ for any non-trivial linear subspace } L \subset \mathbb{R}^n.$$

(i) $\mathcal{E}_{\mu,p}(C)$ attains its minimum for o-symmetric convex bodies $C \subset \mathbb{R}^n$ with $|C| = 1$.

(ii) If an o-symmetric convex body $\widetilde{C} \subset \mathbb{R}^n$ with $|\widetilde{C}| = 1$ minimizes $\mathcal{E}_{\mu,p}(C)$ among o-symmetric convex bodies $C \subset \mathbb{R}^n$ with $|C| = 1$, then there exists $\lambda > 0$ with $\mu = S_{\lambda \widetilde{C},p}$.

Remark 9.4.7 (Uniqueness for the Monge-Ampère equation L_p-Minkowski problem versus L_p-Minkowski inequality). More precisely, we consider the even L_p-Minkowski problem and the conjectured L_p-Minkowski inequality for o-symmetric convex bodies and $p \in [0, 1)$ (cf. Conjecture 8.8.3 and Conjecture 8.7.3). Let $K \subset \mathbb{R}^n$ be an o-symmetric convex body such that if $p = 0$, then K is not of the form $K = K_0 + K_1$ for the at least one dimensional compact convex sets $K_0, K_1 \subset \mathbb{R}^n$ with $\dim K_0 + \dim K_1 = n$, and hence $S_{K,0}(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot S_{K,0}(S^{n-1})$ for any non-trivial linear subspace $L \subset \mathbb{R}^n$ by Theorem 9.3.6. According to Proposition 9.4.6, given $p \in [0, 1)$ and such an o-symmetric convex body $K \subset \mathbb{R}^n$, $\mathcal{E}_{S_{K,p},p}(C)$ attains its minimum among o-symmetric convex bodies $C \subset \mathbb{R}^n$ with $|C| = |K|$, and any minimizer $\widetilde{C}$ satisfies $S_{K,p} = S_{\lambda \widetilde{C},p}$ for a $\lambda > 0$. Now if Conjecture 9.4.4 and Conjecture 9.4.5 about the uniqueness of the solution of the even L_p-Minkowski problem hold, then $\widetilde{C} = K$ for

any minimizer; therefore, any o-symmetric convex body $C \subset \mathbb{R}^n$ with $|C| = |K|$ satisfies

$$\begin{aligned} \int_{S^{n-1}} h_C^p \, dS_{K,p} &\geqslant \int_{S^{n-1}} h_K^p \, dS_{K,p} \qquad \text{if} \quad p \in (0,1) \\ \int_{S^{n-1}} \log h_C \, dS_{K,0} &\geqslant \int_{S^{n-1}} \log h_K \, dS_{K,0} \qquad \text{if} \quad p = 0. \end{aligned}$$

As $dS_{K,p} = nh_K^{-p} \, dV_K$ for the cone volume measure $dV_K = \frac{1}{n} h_K \, dS_K$ satisfying $V_K(S^{n-1}) = |K|$ (see Section 2.6), we deduce that Conjecture 9.4.4 and Conjecture 9.4.5 about the uniqueness of the solution of the Monge-Ampère equation even L_p-Minkowski problem for $p \in [0,1)$ are equivalent to saying that if $K, C \subset \mathbb{R}^n$ are o-symmetric convex bodies, then

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K \geqslant |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}} \text{ provided } p \in (0,1), \tag{9.35}$$

with equality if and only if K and C are dilates, and

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \, dV_K \geqslant \frac{|K|}{n} \log \frac{|C|}{|K|} \tag{9.36}$$

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for o-symmetric compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one where K_i and C_i are dilates, $i = 1, \ldots, m$, and $\sum_{i=1}^m \dim K_i = n$.

According to Lemma 8.8.4, (9.35) is equivalent with the L_p-Brunn-Minkowski Conjecture for o-symmetric convex bodies and $p \in (0,1)$, and according to Lemma 8.7.4, (9.36) is equivalent to the L_0 or Logarithmic Brunn-Minkowski Conjecture for o-symmetric convex bodies. Conjecture 9.4.3 is equivalent to the cases of (9.35) and (9.36) when ∂K and ∂C are C_+^∞. On the other hand, assuming that Conjecture 9.4.3 holds for $p = q$ for some $q \in [0,1)$, then the inequalities in (9.35) and (9.36) hold for $p \in [q,1)$, and equality holds in (9.35) provided $p \in (q,1)$ and ∂K is C^1 if and only if $K = C$ according to Remark 8.8.5. For more details, see Section 7.6 when $p > 1$, Section 9.2 when $p = 1$, Section 8.8 when $p \in (0,1)$ and Section 8.7 when $p = 0$.

9.5 Comments to Chapter 9

In this section, we collect some recently well-investigated analogues of the L_p-Minkowski problem. For Minkowski-type problems for log-concave functions, see Section 10.9.2. Recall that $\mathcal{K}^n_{(o)}$ ($\mathcal{K}^n_o$) is the set of convex bodies $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ ($o \in K$).

9.5.1 The Christoffel-Minkowski problem about the radii of curvatures

Given a positive continuous function f on S^{n-1}, the Christoffel problem - posed by Christoffel [202] in 1865, and hence preceeding the Minkowski problem - considers

the existence of a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary such that the sum of the principal radii of curvature at $x \in \partial K$ is $f(\nu_K(x))$.

From the point of view of measures, for any convex body $K \subset \mathbb{R}^n$ and $i = 1, \dots, n-1$, we consider the finite Borel *ith area measure* $S_i(K, \cdot)$ on S^{n-1} introduced by Aleksandrov [4] and Fenchel, Jessen [268] (cf. (8.39)), which coincides with the mixed area measure

$$S_i(K, \cdot) = S_{\underbrace{B^n, \dots, B^n}_{n-1-i}, \underbrace{K, \dots, K}_{i}}(\cdot).$$

Its main properties are discussed in Proposition 8.4.4. For example, $S_{n-1}(K, \cdot) = S_K$, and if ∂K is C^2_+ and $i = 1, \dots, n-1$, then $S_i(K, \cdot)$ is absolutely continuous whose density function is the ith symmetric function of the radii of curvature as a function of the exterior unit normal. The corresponding non-linear elliptic partial differential equation on S^{n-1} is the *Christoffel-Minkowski problem* (cf. Proposition 8.4.4)

$$\sigma_i \left(\nabla^2 h + h I_{n-1} \right) = f \tag{9.37}$$

where a C^2 function h on S^{n-1} is thought such that $\nabla^2 h + h I_{n-1}$ is positive definite everywhere (and hence $h = h_K|_{S^{n-1}}$ for a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary by Lemma 8.1.8) and $\sigma_i(A)$ is the ith symmetric function of the eigenvalues of a positive definite matrix A. Since

$$\int_{S^{n-1}} u \, dS_i(K, u) = o,$$

holds for any convex body $K \subset \mathbb{R}^n$ according to Proposition 8.4.4, the function f in the Christoffel-Minkowski problem (9.37) should satisfy

$$\int_{S^{n-1}} u \, f(u) \, du = o. \tag{9.38}$$

The original *Christoffel problem* posed by Christoffel [202] in 1865 is the case $i = 1$ of (9.37); namely, it is about the sum of the radii of curvatures. After various attempts to characterize the possible f in (9.37) for $i = 1$, among others by Hilbert and Hurwitz around 1900, it was Firey [285] in 1967 first providing a full but rather complicated classification (similar results are due to Berg [80] and Goodey, Yaskin, Yaskina [328]). A more accessible classification is provided by Li, Wan, Wang [516], who link the Christoffel problem on S^{n-1} to the Laplace equation $\Delta h_K = \mathring{f}$ on $\mathbb{R}^n \backslash \{o\}$ where $\mathring{f}(\lambda u) = \lambda^{-1} f(u)$ for $\lambda > 0$ and $u \in S^{n-1}$. Schneider [656] solved the discrete Christoffel problem on S^{n-1}; namely, characterized $S_1(P, \cdot)$ for n-polytopes $P \subset \mathbb{R}^n$.

The Christoffel-Minkowski problem (9.37) when $1 < i < n-1$ is even more mysterious, see Guan, Ma [357], Guan, Lin, Ma [359], Guan, Ma, Zhou [360] up to 2006, and the recent paper Bryan, Ivaki, Scheuer [167] (see Guan [353] for a survey of the methods of the earlier papers). The case in which a measure with smooth density has axial rotational symmetry has been characterized by Firey [286], which results has

been recently extended by Brauner, Hofstätter, Ortega-Moreno [156] and Mussnig, Ulivelli [594]

For L_p versions of the Christoffel-Minkowski problem, see, for example, Guan, Xia [363], Ivaki [438], Chen [189], Sheng, Yi [681], Li, Ju, Liu [509], Guan [354], Zhang [736], Hu, Ivaki [413] and Patsalos, Saroglou [614].

9.5.2 The Minkowski problem for complete non-compact convex hypersurfaces in $\mathbb{R}^n$

The Minkowski problem for complete but not compact convex hypersurfaces in $\mathbb{R}^n$ was initiated by Aleksandrov [10] in his fundamental monograph from 1950, which was followed by decades of intensive research, see for example the monographs Pogorelov [621] and Bakelman [49], or the paper Chou, Wang [200]. The connection of the problem to algebraic geometry is discussed by Khovanskii, Timorin [457].

For a line free unbounded closed convex set $K \subset \mathbb{R}^n$ with non-empty interior, its recession cone C is pointed, and hence $\operatorname{int} C^* \neq \varnothing$ (see (1.54) in Section 1.B). We note that ∂K is differentiable at $\mathcal{H}^{n-1}$ a.e $x \in \partial K$, $\partial' K$ denotes the set of these points x, and $\nu_K(x)$ denotes the exterior unit normal at an $x \in \partial' K$ (cf. Definition 1.5.1).

According to Corollary 1.B.4, the convex, lower semi-continuous and homogeneous support function

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle$$

determines K uniquely, is finite on $\operatorname{int} C^*$, and is infinity on $\mathbb{R}^n \backslash C^*$. Actually, knowing h_K as a finite convex homogeneous function on $\operatorname{int} C^*$ is sufficient, because the h_K on $\mathbb{R}^n$ is the convex closure of the function on $\operatorname{int} C^*$ (see Definition 1.A.4).

The *surface area measure* S_K associated to K (or the complete non-compact convex line-free hypersurface $X = \partial K$) is defined on $\Omega = S^{n-1} \cap \operatorname{int} C^*$, and for any Borel set $\omega \subset \Omega$, we have (cf. Proposition 1.B.3)

$$\begin{aligned} S_K(\omega) &= \mathcal{H}^{n-1}\left(\bigcup \{\partial h_K(u) : u \in \omega\}\right) \\ &= \mathcal{H}^{n-1}\left(\{z \in \partial' K : \nu_K(z) \in \omega\right). \end{aligned} \tag{9.39}$$

In particular, S_K is the Monge-Ampére measure on Ω corresponding to $h_K|_\Omega$. It follows from Proposition 1.B.3 that

$$S_K \text{ is locally finite} \tag{9.40}$$

(any compact subset of Ω has finite measure).

Given a locally finite Borel measure μ on $\Omega = S^{n-1} \cap \operatorname{int} C^*$, the Minkowski problem for unbounded line-free closed convex sets with non-empty interior associated to a pointed closed convex (recession) cone C asks for a finite convex homogeneous

function h on intC^* such that for the corresponding closed convex set K (cf. Corollary 1.B.4), we have

$$S_K = \mu. \tag{9.41}$$

Similarly to the classical Minkowski problem (cf. (9.6)), if $d\mu = f d\mathcal{H}^{n-1}$ for a measurable function $f : \Omega \to [0, \infty)$, then (9.41) is equivalent to providing a possible weak solution of the Monge-Ampére equation

$$\det(\nabla^2 h + hI_{n-1}) = f. \tag{9.42}$$

As in the case of classical Minkowski problem, if $h = h_K|_\Omega$ and ∂K is C^2_+, then

$$\det(\nabla^2 h(\nu_K(x) + h(\nu_K(x)I_{n-1}) = \frac{1}{\kappa(x)}.$$

While we have formulated the Minkowski problem in terms of closed convex set $K \subset \mathbb{R}^n$, it is frequently formulated in terms of the complete line-free non-compact convex hypersurface $X = \partial K$ as

$$h_K(u) = \sup_{x \in X} \langle x, u \rangle \quad \text{for any } u \in \mathbb{R}^n. \tag{9.43}$$

Next, we discuss how the Minkowski problem for complete non-compact line-free hypersurfaces can be formulated as a classical Monge-Ampère equation on a bounded open convex subset of $\mathbb{R}^{n-1}$. Using the notation as above, Lemma 1.3.6 yields the existence of a $w \in S^{n-1} \cap \operatorname{relint} C$ such that $-w \in \operatorname{int} C^*$ (remember that $\operatorname{int} C^* \neq \varnothing$ as K and C are line-free). We consider the bounded (relatively) open convex subset $\widetilde{\Omega}$ of $w^\perp \equiv \mathbb{R}^{n-1}$ obtained from the radial projection of Ω into $w^\perp - w$ such that $\widetilde{\Omega} - w = (w^\perp - w) \cap \operatorname{int} C^*$. For $\theta(y) = h_K(y - w)$ for $y \in \widetilde{\Omega}$, we obtain similarly to (9.2.5) that the Monge-Ampére equation on $\widetilde{\Omega}$ equivalent to (9.42) is

$$\det D^2\theta(y) = g(y) \quad \text{for } y \in \widetilde{\Omega} \tag{9.44}$$

for a finite convex function φ on $\widetilde{\Omega}$ where if $y \in \widetilde{\Omega}$, then

$$g(y) = \left(1 + \|y\|^2\right)^{-\frac{n+1}{2}} \cdot f\left(\frac{y - w}{\sqrt{1 + \|y\|^2}}\right).$$

As $K \subset \mathbb{R}^n$ is a closed convex line free set with non-empty interior and its recession cone C contains w, K is the epigraph of the lower semi-continuous convex function $\varphi : w^\perp \to (-\infty, \infty]$ (cf. Section 1.A)

$$\varphi(x) = \sup\{t \in \mathbb{R} : x + tw \notin K\}.$$

It follows that (cf. Proposition 1.B.3) if $y \in \widetilde{\Omega}$, then (cf. Proposition 1.B.3 (ii) and (9.43))

$$\theta(y) = h_K(y-w) = \sup_{x \in w^\perp} \langle x + \theta(x)w, y - w\rangle = \sup_{x \in w^\perp} \langle x, y\rangle - \theta(x),$$

and hence $\theta = \mathcal{L}\varphi$ is the Legendre transform of φ. We deduce from the self-duality of the Legendre transform (cf. Lemma 1.A.7) that we can recover K from the solution θ of the Monge-Ampère equation (9.44) as the epigraph of $\varphi = \mathcal{L}\theta$. We observe that under the additional hypothesis that

$$\nu_K(x) \notin w^\perp \text{ for any } x \in \partial' K, \tag{9.45}$$

the complete line free non-compact convex hypersurface ∂K is the graph of $\mathcal{L}\theta$. Here (9.45) readily holds, and hence ∂K is the graph of $\mathcal{L}\theta$, if

$$\operatorname{int} C \neq \varnothing. \tag{9.46}$$

Let us discuss some known results about the Minkowski problem for non-compact line-free complete convex hypersurfaces. If C^* is a half-space; or equivalently, $C = [0,\infty)w$ or $\widetilde{\Omega} = w^\perp \equiv \mathbb{R}^{n-1}$, then Chou, Wang [200] solves the "non-compact" Minkowski problem (9.44) (or equivalently, (9.41)) under the assumptions of (9.45) and that the Gaussian curvature of K is positive and $C^{0,\alpha}$; or equivalently, the μ in (9.41) is absolutely continuous with a positive and $C^{0,\alpha}$ density function.

Next we consider the "C-asymptotic" Minkowski problem. Let us fix a convex pointed cone $C \subset \mathbb{R}^n$ with $\operatorname{int} C \neq \varnothing$ for the rest of the section. As initiated by Aleksandrov [10], here we only discuss the case of C-asymptotic complete convex hypersurfaces. Here $X = \partial K$ is C-asymptotic for an unbounded closed convex set $K \subset \mathbb{R}^n$ if $K \subset C$, and if $x_m \in X$ satisfies $\lim_{m\to\infty} \|x_m\| = \infty$, then $\lim_{m\to\infty} d(x_m, \partial C) = 0$ where $d(x_m, \partial C)$ is the distance of x_m from ∂C. In terms of the convex, homogeneous support function h_K on the spherically convex open set $\Omega = S^{n-1} \cap \operatorname{int} C^*$ whose closure is contained in an open hemi-sphere, $X = \partial K$ is C-asymptotic for an unbounded closed convex set $K \subset \mathbb{R}^n$ according to Schneider [666] (see also the recent preprint Semenov, Zhao [676]) if and only if (cf. Corollary 1.B.4)

$$\begin{array}{rcll} h_K(u) & \leqslant & 0 & \text{for } u \in \operatorname{cl}\Omega; \\ h_K(u) & = & 0 & \text{for } u \in \partial\Omega = S^{n-1} \cap \partial C^* \\ h_K & \text{is} & \text{continuous} & \text{on } \operatorname{cl}\Omega, \end{array} \tag{9.47}$$

and $h_K(u) = \infty$ for $u \notin S^{n-1}\backslash C^*$. According to Corollary 1.B.4, there exists a unique C-asymptotic closed convex set $K \subset C$ such that $h = h_K$ for any solution $h : \operatorname{cl}\Omega \to$

$(-\infty, 0]$ of (9.42) under the Dirichlet boundary condition in (9.47). Using the formulation (9.44), the Minkowski problem for C-asymptotic complete convex hypersurfaces is equivalent with the following classical Monge-Ampére equation with Dirichlet boundary condition:

$$\det D^2\theta(y) = g(y) \quad \text{for } y \in \widetilde{\Omega}; \tag{9.48}$$

$$\theta(y) = 0 \quad \text{for } y \in \partial\widetilde{\Omega} \tag{9.49}$$

for a convex continuous θ on $\operatorname{cl}\widetilde{\Omega}$.

The Minkowski problem for C-asymptotic sets has been revitalized by Schneider [663, 666]. It follows via the theory of Monge-Ampére equations (cf. Figalli [272]) and Corollary 1.B.4 that any finite Borel measure on Ω is the surface area measure of a unique C-asymptotic closed convex set (see also Schneider [665]). The case of non-finite Borel measures on Ω is much more intriguing: If $n = 2$, then Semenov, Zhao [676] prove that a locally finite Borel measure μ on $\Omega = S^1 \cap \operatorname{int} C^*$ is a surface area measure of a C-asymptotic closed convex set if and only if

$$\int_\Omega d(x, \partial\Omega)\, d\mu(x) < \infty$$

where $d(x, \partial\Omega)$ is the spherical distance of $x \in \Omega$ from $\partial\Omega$ (naturally, one can consider the Euclidean distance in $\mathbb{R}^2$, as well). However, the problem is still open when $n \geqslant 3$, and the necessary growth condition near $\partial\Omega$ provided by Schneider [665], and the sufficient ones proved either by Schneider [667] or by Bakelman [49] Theorem 24.1 and Chou, Wang [200] Theorem A are far apart.

We note that Schneider [667] also discusses the logarithmic (L_0) Minkowski problem for C-asymptotic closed convex sets, which is generalized to the corresponding L_p-Minkowski problem by Yang, Ye, Zhu [731] and to the corresponding dual-Minkowski problem by Li, Ye, Zhu [512].

9.5.3 Minkowski problem on the sphere and in the hyperbolic space

The Minkowski problem on the sphere is solved by Guang, Li, Wang [366] (see [366] for related references, as well), and in the hyperbolic space, partial results - also about the hyperbolic Christoffel-Minkowski problem - are obtained by Gerhardt [319]. The even discrete L_p-Minkowski problem is solved in the hyperbolic space by Li, Wan, Xu [511].

9.5.4 Aleksandrov's problem on the integral curvature and some of its versions (prescribed curvature measures, L_p Aleksandrov problem)

For $K \in \mathcal{K}^n_{(o)}$, we parametrize the boundary of K using the radial image as

$$r_K(u) = \varrho_K(u)\, u \in \partial K$$

for $u \in S^{n-1}$ and the radial function $\varrho_K(u) > 0$. Then Aleksandrov [9] defined the *integral curvature measure* of a Borel set $\omega \subset S^{n-1}$ as

$$J_K(\omega) = \mathcal{H}^{n-1}\left(\left\{v \in S^{n-1} : \exists u \in \omega \text{ such that } v \in N_K(r_K(u))\right\}\right), \tag{9.50}$$

where $v \in N_K(r_K(u))$ says that v is an exterior normal at $r_K(u) \in \partial K$. It follows that

$$J_K(S^{n-1}) = \mathcal{H}^{n-1}(S^{n-1}) = n\omega_n.$$

If ∂K is C^2 (and hence ϱ_K is C^2, as well), then (cf. Proposition 2.2.5)

$$J_K(\omega) = \int_{r_K(\omega)} \kappa\left(r_K(u)\right)\, d\mathcal{H}^{n-1}(u) \tag{9.51}$$

$$= \int_{\omega} \kappa\left(r_K(u)\right) \varrho_K(u)^{n-2} \sqrt{\varrho_K(u)^2 + \|\nabla \varrho_K(u)\|^2}\, d\mathcal{H}^{n-1}(u). \tag{9.52}$$

The formula (9.51) explains the name *integral curvature measure*.

Aleksandrov [9] characterized integral curvature measures, and hence solved what later was called Aleksandrov problem.

Theorem 9.5.1 (Aleksandrov). *For a Borel measure μ on S^{n-1} with $\mu(S^{n-1}) = \mathcal{H}^{n-1}(S^{n-1})$, $\mu = J_K$ for a $K \in \mathcal{K}^n_{(o)}$ if and only if*

$$\mu(\omega) < \mathcal{H}^{n-1}\left(\left\{u \in S^{n-1} : \exists v \in \omega,\ \langle u, v\rangle > 0\right\}\right) \tag{9.53}$$

for any Borel set ω contained in a closed hemisphere, and K is unique up to dilation.

Remark. According to Remark 4.A.1, $\omega \subset S^{n-1}$ is a (non-empty) spherically convex compact set if $\omega = C \cap S^{n-1}$ for a non-trivial closed convex cone $C \subset \mathbb{R}^n$ (cf. Definition 1.2.8). The condition (9.53) can be expressed in terms of the polar

$$\omega^* = \{u \in S^{n-1} : \langle u, v\rangle \leqslant 0\ \forall v \in \omega\}$$

of a spherically convex compact set $\omega \subset S^{n-1}$, which is spherically convex and compact, as well, and satisfies $\omega^{**} = \omega$. Now (9.53) is equivalent to saying that if $\omega \subset S^{n-1}$ is a spherically convex compact set, then

$$\mu(\omega) + \mathcal{H}^{n-1}(\omega^*) < \mathcal{H}^{n-1}(S^{n-1}). \tag{9.54}$$

Many papers cite (9.54) instead of (9.53) as Aleksandrov's condition.

Regularity of the solution of the Aleksandrov problem is investigated by Pogorelov [623], Oliker [607] and Guan, Li [356].

Remark 9.5.2 (Known proofs of Theorem 9.5.1).

- Aleksandrov [9,11]: First proving for polytopes (discrete measures), and then using topology to handle the case of general convex bodies.
- Oliker [608] and Bertrand [86, 87]: Using optimal transport.
- Böröczky, Lutwak, Yang, Zhang, Zhao [144]: Variational argument.

As (9.52) indicates (cf. Guan, Li, Li [355]), given a positive continuous function f on S^{n-1}, finding a convex body $K \in \mathcal{K}^n_{(o)}$ with $dJ_K = f d\mathcal{H}^{n-1}$ leads to a fully non-linear elliptic partial differential equation S^{n-1}. On the other hand, Lemma 9.5.3 below shows that the Aleksandrov problem for the polar (dual) body is a Monge-Ampère equation, a familiar setting in this book.

First, in line with (2.7), the *reverse radial Gauss image* of a Borel set $\omega \subset S^{n-1}$ is the set of $u \in S^{n-1}$ such that some $v \in \omega$ is an exterior normal at $\varrho_K(u)u$; namely,

$$\alpha^*_K(\omega) = \left\{u \in S^{n-1} : N_K(r_K(u)) \cap \omega \neq \varnothing\right\}, \tag{9.55}$$

and hence (9.50) and Proposition 1.9.2 yield that

$$J_{K*}(\omega) = \mathcal{H}^{n-1}\left(\alpha^*_K(\omega)\right). \tag{9.56}$$

In order to provide an integral formula based on (9.56), let $\pi(x) = x/\|x\|$ be the radial projection $\mathbb{R}^n\backslash\{o\} \to S^{n-1}$, and for $K \in \mathcal{K}^n_{(o)}$, let $\aleph_K = (\pi\partial' K) \cap (\pi\partial' K^*)$ be the set of points $u \in S^{n-1}$ such that

$$\alpha_K(u) = \nu_K(r_K(u)) = \nu_K(\varrho_K(u) \cdot u) \tag{9.57}$$

is well defined for $u \in \aleph_K$, and $\alpha_K(u)$ is an exterior normal only at $r_K(u) \in \partial K$ and

$$\alpha^*_K(\alpha_K(u)) = u. \tag{9.58}$$

Here we used that for $z = r_{K*}(\alpha_K(u)) \in \partial' K^*$, we have $\pi(z) = u$ and $\alpha^*_K(\pi(z)) = \nu_{K^*}(z)$. Since $\mathcal{H}^{n-1}$ a.e. boundary point of a convex body in $\mathbb{R}^n$ is regular, and π is locally Lipschitz, we deduce that

$$\mathcal{H}^{n-1}\left(S^{n-1}\backslash\aleph_K\right) = 0. \tag{9.59}$$

Using the function $f(v) = \varrho_K\left(\alpha^*_K(v)\right)^{-n}$ for $v \in \aleph_K$ in Proposition 2.6.9, and the formulas (9.56), (9.58) and (9.59), we deduce from Lemma 2.2.6 that

$$J_{K*}(\omega) = \mathcal{H}^{n-1}\left(\alpha^*_K(\omega)\right) = n\int_{\omega\cap\aleph_K} \varrho_K^{-n}\, dV_K = \int_\omega h\left(\|\nabla h\| + h^2\right)^{\frac{-n}{2}}\, dS_K \tag{9.60}$$

for the Lipschitz function $h = h_K|_{S^{n-1}}$ and any Borel set $\omega \subset S^{n-1}$. Thus (8.18) yields the Monge-Ampère equation for h if J_{K*} is absolutely continuous.

Lemma 9.5.3. *If $K \in \mathcal{K}^n_{(o)}$ is a convex body with C^2_+ boundary, then $dJ_{K*} = f\, d\mathcal{H}^{n-1}$ where the C^2 function $h = h_K|_{S^{n-1}}$ satisfies*

$$h\left(\|\nabla h\| + h^2\right)^{\frac{-n}{2}} \det\left(\nabla^2 h + hI_{n-1}\right) = f. \tag{9.61}$$

Let us briefly discuss two generalizations of the Aleksandrov problem. The first is the characterization of curvature measures introduced by Federer [260] originally for sets of positive reach, and discussed in depth by Schneider [661], Chapter 4 in the case of convex bodies. Here we only consider the case of a convex body $K \in \mathcal{K}^n_{(o)}$ with C^2_+ boundary, when the problem becomes prescribing the mth symmetric function of the principal curvatures at the points of ∂K for $m = 1, \dots, n-1$. Now, for $i = 0, \dots, n-1$ and Borel $\eta \subset \partial K$, the Federer's ith curvature measure $C_i(K, \eta)$ satisfies (cf. Proposition 2.2.5 for the second step)

$$\begin{aligned} C_i(K,\eta) &= \tfrac{1}{(n-i)\omega_{n-i}} \int_\eta \sigma_{n-1-i}(\kappa_1(x), \dots, \kappa_{n-1}(x))\, dx \\ &= \tfrac{1}{(n-i)\omega_{n-i}} \int_{r_K^{-1}(\eta)} \sigma_{n-1-i}(\kappa_1, \dots, \kappa_{n-1}) \circ r_K \cdot \varrho_K^{n-2} \sqrt{\varrho_K^2 + \|\nabla \varrho_K\|^2}\, d\mathcal{H}^{n-1} \end{aligned}$$

where $\sigma_j(\kappa_1, \dots, \kappa_{n-1})$ at $x \in \partial K$ stands for the jth symmetric function of the principal curvatures $\kappa_1(x), \dots, \kappa_{n-1}$ at x. Here the normalization is chosen in a way such that $C_i(K, S^{n-1}) = V_i(K)$ (cf. (8.32)). According to Guan, Li, Li [355], characterizing the measure $C_i(K, r_K(\omega))$ of Borel sets $\omega \subset S^{n-1}$ leads to a fully non-linear elliptic partial differential equation S^{n-1}, which coincides with the Aleksandrov problem if $i = 0$ by (9.52).

For $p \in \mathbb{R}$, the L_p version of the Aleksandrov problem posed by the fundamental paper Huang, Lutwak, Yang, Zhang [420], asks for characterization of the L_p *Aleksandrov integral curvature measure*

$$dJ_{K,p} = \varrho_K^p dJ_K \tag{9.62}$$

for $K \in \mathcal{K}^n_{(o)}$ where the classical Aleksandrov problem is the case $p = 0$. It follows from Lemma 9.5.3 that the Monge-Ampère equation for the L_p Aleksandrov integral curvature measure $J_{K*,p}$ of the polar body is

$$\det(\nabla^2 h + h\, I_{n-1}) = h^{p-1}(\|\nabla h\|^2 + h^2)^{\frac{n}{2}} \cdot f.$$

For related results, see, for example, Zhao [735], Feng, He [269], Feng, Hu, Liu [270], Li, Sheng, Ye, Yi [514], Wu, Wu, Xiang [720] and Mui [593].

In general, for L_p versions of the characterization problem for the curvature measures, see, for example, Guan, Ren, Wang [362], Bryan, Ivaki, Scheuer [166], Lee [499] and Hu, Ivaki [413] where the current state of art is surveyed in [413].

9.5.5 L_p Dual curvature measures

In order to define the dual curvature measures for $q \in \mathbb{R}$, first let $K \in \mathcal{K}^n_{(o)}$. For a Borel set $\omega \subset S^{n-1}$, using the reverse radial Gauss image $\alpha^*_K(\omega)$ in (9.55), the fundamental paper Huang, Lutwak, Yang, Zhang [419]) introducing a whole new perspective within the theory of Minkowski type problems defines the *qth dual curvature measure* by the formula

$$\widetilde{C}_{K,q}(\omega) = \int_{\alpha^*_K(\omega)} \varrho_K^q \, d\mathcal{H}^{n-1}.$$

If $q > 0$, then this definition makes sense even if $K \in \mathcal{K}^n_o$, and hence the origin might lie on the boundary in that case. We deduce from Corollary 2.6.7 (cf. Huang, Lutwak, Yang, Zhang [419]) that if either $K \in \mathcal{K}^n_{(o)}$, or $q > 0$ and $K \in \mathcal{K}^n_o$, then

$$\widetilde{C}_{K,q}(\omega) = \int_{\nu_K^{-1}(\omega)\cap\partial' K} \|x\|^{q-n} \langle \nu_K(x), x\rangle \, d\mathcal{H}^{n-1}(x).$$

For example, if h_K is differentiable at each point of ω (and hence $\nu_K^{-1}(\omega)$ contains no segment), then (cf. (9.60) and Corollary 2.6.7)

$$\widetilde{C}_{K,q}(\omega) = n \int_\omega \varrho_K(\alpha^*_K(u))^{q-n} \, dV_K(u) \tag{9.63}$$

$$= \int_\omega h \left(\|\nabla h\|^2 + h^2\right)^{\frac{q-n}{2}} \, dS_K \tag{9.64}$$

for $h = h_K|_{S^{n-1}}$. We note that (9.63) and (9.64) make sense even if $q > 0$ and $o \in \partial K$. This follows because if $o \in \partial K$, then $V_K(S^{n-1} \cap N_K(o)) = 0$ as $h_k(u) = 0$ for $u \in N_K(o)$, while $\varrho_K(\alpha^*_K(u)) > 0$ if $u \in S^{n-1}\backslash N_K(o)$.

Examples for the dual curvature measures that we have already encountered are

- $\widetilde{C}_{K,n} = nV_K$ (cf. Section 2.6);
- $\widetilde{C}_{K,0} = J_{K*}$ (cf. Section 9.5.4, especially (9.60)).

For $q \in \mathbb{R}$, Lutwak [550] defined the *qth dual volume* of a $K \in \mathcal{K}_{(o)}$ as

$$\widetilde{V}_q(K) = \frac{1}{n} \int_{S^{n-1}} \varrho_K^q \, d\mathcal{H}^{n-1},$$

and the same formula works if $q > 0$ and $K \in \mathcal{K}^n_o$ (so possibly $o \in \partial K$ in this case). Readily, $\widetilde{V}_n(K) = |K|$.

Remark 9.5.4 (Continuity of $\widetilde{V}_q$ and weak continuity of $\widetilde{C}_{K,q}$). If $q \in \mathbb{R}$ and $K_m \in \mathcal{K}_{(o)}$ tends to $K \in \mathcal{K}_{(o)}$, then readily $\widetilde{V}_q(K_m)$ tends to $\widetilde{V}_q(K)$, and (9.63) and the weak continuity of V_K yields that $\widetilde{C}_{K_m,q}$ tends weakly to $\widetilde{C}_{K,q}$.

If $q > 0$, then the same holds even if $K \in \mathcal{K}_o$ according to Böröczky, Fodor [127].

Generalizing Aleksandrov's Lemma 9.2.2 about the Wulff-shape in the case $\Omega = S^{n-1}$, Huang, Lutwak, Yang, Zhang [419] established the characteristic property of the dual curvature measure that it encodes the first variation of the dual volume. Here, (i) in Theorem 9.5.5 about the derivative of the radial function yields (ii), and (ii) in the case $q = n$ is Aleksandrov's Lemma 9.2.2 in the case $\Omega = S^{n-1}$.

Theorem 9.5.5. *Let $\varphi : S^{n-1} \times (-t_0, t_0) \to (0, \infty)$, $t_0 > 0$, be continuous such that*

$$\lim_{t\to 0} \frac{\varphi(u,t) - \varphi(u,0)}{t} = \partial_t \varphi(u,0)$$

exists uniformly in $u \in S^{n-1}$ where $\partial_t \varphi(u,0)$ is continuous on S^{n-1}, and let

$$K_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant \varphi(u,t),\ \forall u \in S^{n-1}\} \in \mathcal{K}^n_{(o)}$$

be the corresponding Wulff shape for $t \in (-t_0, t_0)$ where $K = K_0$.

(i) For $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$, we have (cf. (9.57) for α_K)

$$\lim_{t\to 0} \frac{\varrho_{K_t}(u) - \varrho_K(u)}{t} = \varrho_K(u) \cdot \frac{\partial_t \varphi(\alpha_K(u), 0)}{h_K(\alpha_K(u))},$$

and there exists $M > 0$ such that for $0 \neq t \in (-t_0/2, t_0/2)$ and $u \in S^{n-1}$, we have

$$\frac{\varrho_{K_t}(u) - \varrho_K(u)}{t} \leqslant M.$$

(ii) If $q \in \mathbb{R}\backslash\{0\}$, then

$$\lim_{t\to 0} \frac{\widetilde{V}_q(K_t) - \widetilde{V}_q(K)}{t} = \frac{q}{n} \int_{S^{n-1}} \frac{\partial_t \varphi(u,0)}{h_K(u)}\, d\widetilde{C}_{K,q}(u).$$

We observe that the cone volume measure V_K (cf. Section 2.6, Section 9.3 and Section 9.4) lies at the cross road: nV_K is the L_0-surface area measure (cf. Section 9.3) on the one hand, and the nth dual curvature measure on the other hand.

Concerning the dual Minkowski Problem for $q \in \mathbb{R}$, it follows from (9.64) that the corresponding Monge-Ampère equation is

$$\left(\|\nabla h\|^2 + h^2\right)^{\frac{q-n}{2}} \cdot h \det(\nabla^2 h + h\, I_{n-1}) = f. \tag{9.65}$$

The case $q = n$ about the cone volume measure is discussed in Sections 9.3 and 9.4, and the case $q = 0$ about the Aleksandrov problem is covered in Section 9.5.4. If $q \neq 0, n$, then the following results are known:

- If $q < 0$, then any Borel measure on S^{n-1} not concentrated on a closed hemisphere is a qth dual Minkowski curvature measure according to Zhao [734] and Li, Sheng, Wang [515].

- If $q > 0$ and $n = 2$, then (9.65) has a solution for any measurable f provided $\frac{1}{c} < f < c$ for a $c > 1$ according to Chen, Li [193].
- If $0 < q < n$, then a finite even Borel measure μ on S^{n-1} is a qth dual Minkowski curvature measure if and only if

$$\mu(L \cap S^{n-1}) < \tfrac{\dim L}{q} \cdot \mu(S^{n-1})$$

for any proper linear subspace $L \subset \mathbb{R}^n$ according to Böröczky, Lutwak, Yang, Zhang, Zhao [143] where one needs to add that μ is not concentrated onto a great subsphere if $q < 1$. These results have been partially extended to centered convex bodies by Eller, Henk [247].

- If $q \geqslant n + 1$ and $K \in \mathcal{K}^n_{(o)}$ is origin symmetric, then Henk, Pollehn [386] prove the necessary condition

$$\widetilde{C}_{K,q}(L \cap S^{n-1}) < \tfrac{q-n+\dim L}{q} \cdot \widetilde{C}_{K,q}(S^{n-1})$$

for any proper linear subspace $L \subset \mathbb{R}^n$.

Remark. In particular, it is an intriguing open problem to characterize a qth dual Minkowski curvature measure on S^{n-1} if $q > 0$, or similarly, to characterize an even qth dual Minkowski curvature measure on S^{n-1} if $q > n$.

Uniqueness of the solution of the dual Minkowski problem (9.65) does not hold in general, for example, in the case $q > 2n$ and $f \equiv 1$ (the isotropic case), even if assuming that the solution h is even (see Chen, Chen, Li [187]). On the other hand, if q is close to n, then uniqueness holds in the near isotropic qth dual Minkowski problem according to Böröczky, Chen, Liu, Saroglou [122]. Concerning uniqueness in the even qth dual Minkowski problem on S^{n-1}, Ivaki, E. Milman [442] verify uniqueness in the isotropic case if $0 < q \leqslant n$, which result is extended to the even near isotropic case by Böröczky, Chen, Liu, Saroglou [122] (see also Hu [404] if $\max\{0, n-4\} < q \leqslant n$).

If $0 < q < n$, then Sadovsky, Zhang [641] have recently proved the even qth dual Brunn-Minkowski inequality

$$\widetilde{V}_q(\alpha K + \beta C)^{\frac{1}{q}} \geqslant \alpha\, \widetilde{V}_q(K)^{\frac{1}{q}} + \beta\, \widetilde{V}_q(C)^{\frac{1}{q}} \tag{9.66}$$

for $\alpha, \beta \geqslant 0$ and o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ where equality holds if K and C are homothetic. However, equality in (9.66) is not characterized yet. It is also not clear whether (9.66) holds if K and C are not o-symmetric.

Next we discuss the L_p dual Minkowski problem introduced by Lutwak, Yang, Zhang [553] that is a common generalization of the L_p-Minkowski problem and the L_p-Aleksandrov problem. For $p, q \in \mathbb{R}$, Lutwak, Yang, Zhang [553] defines the L_p

qth dual curvature measure on S^{n-1} by

$$d\widetilde{C}_{K,p,q} = h_K^{-p}\, d\widetilde{C}_{K,q},$$

and hence Corollary 2.6.7 yields that if either $K \in \mathcal{K}^n_{(o)}$, or $q > 0$, $p < 1$ and $K \in \mathcal{K}^n_o$, then

$$\widetilde{C}_{K,p,q}(\omega) = \int_{\nu_K^{-1}(\omega)\cap\partial' K} \|x\|^{q-n}\langle \nu_K(x), x\rangle^{1-p}\, d\mathcal{H}^{n-1}(x).$$

As well-known examples, we have

$$\begin{aligned}\widetilde{C}_{K,p,n} &= S_{K,p};\\ \widetilde{C}_{K,0,q} &= \widetilde{C}_{K,q};\\ \widetilde{C}_{K,p,0} &= J_{K^*,p}.\end{aligned}$$

Given a Borel measure μ on S^{n-1}, the simplest version of the L_p qth dual Minkowski problem asks for a $K \in \mathcal{K}^n_{(o)}$ with $\mu = \widetilde{C}_{K,p,q}$, and the correspondig Monge-Ampère equation is (see (9.65), and note that $p = 0$ for the qth dual Minkowski problem)

$$h^{1-p}\det(\nabla^2 h + h\, I_{n-1}) = (\|\nabla h\|^2 + h^2)^{\frac{n-q}{2}} \cdot f. \tag{9.67}$$

Improving on Böröczky, Fodor [127] and Huang, Zhao [422], Chen, Li [188] and Lu, Pu [534] prove that if $p > 0$ and $q \neq p, 0$, then any finite Borel measure not concentrated on a closed hemisphere is a L_p qth dual Minkowski curvature measure (more precisely, if $p \leqslant q$, then some modification of the Monge-Ampère equation might be needed). Huang, Zhao [422] proved the same for $p, q < 0$ and $p \neq q$ within the category of even measures. See also Guang, Li, Wang [364] for a flow approach when $p < 0$ and $q > n$ under regularity assumptions.

Concerning uniqueness of the solution of the L_p qth dual Minkowski problem (9.67), if $p > q$, then the solution is unique provided f is positive and $C^{0,\alpha}$ according to Huang, Zhao [422], and provided the L_p qth dual curvature measure $\widetilde{C}_{K,p,q}$ is discrete by Lutwak, Yang, Zhang [553]. Therefore, in the following, only results about unique solution in the case $p \leqslant q$ are reviewed. We note that according to Li, Liu, Lu [513], uniqueness of the solution of (9.67) fails if $0 < q \leqslant 1$ and $p < q - 1$.

Like in the case of L_p-Minkowski problem (cf. Section 9.4), there is a very nice interplay between a possible L_p qth dual Brunn-Minkowski-type inequality and uniqueness of the solution of the corresponding Monge-Ampère equation (9.67). For example, (9.66) and the Jensen's inequality (10.6) yield that if $p > 1$ and $0 < q < n$, then

$$\widetilde{V}_q(\alpha \cdot K +_p \beta \cdot C)^{\frac{p}{q}} \geqslant \alpha\, \widetilde{V}_q(K)^{\frac{p}{q}} + \beta\, \widetilde{V}_q(C)^{\frac{p}{q}} \tag{9.68}$$

holds for o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with equality if and only if K and C are homothetic (see Section 7.6 for analogous results when $q = n$). In turn, we deduce

like in the proof of Lemma 8.8.4, only using Theorem 9.5.5 (ii) for the Wulff shape instead of Lemma 7.5.1, that if $p > 1$, $0 < q < n$ and $K, C \subset \mathbb{R}^n$ are o-symmetric convex bodies, then

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, d\widetilde{C}_{K,q} \geqslant |K|^{\frac{q-p}{q}} |C|^{\frac{p}{q}} \tag{9.69}$$

with equality if and only if K and C are dilates. Since the variational method produces a solution for $p > 1$ and $q > 0$ according to Böröczky, Fodor [127], analogously as in Remark 9.4.7 in the case $q = n$, we conclude the following statement.

Proposition 9.5.6. *If $1 < p \leqslant q < n$, then the even L_p qth dual Minkowski problem (9.67) on S^{n-1} has a unique even solution (also in the sense of measure).*

Concerning uniqueness in the isotropic setting, when the f on S^{n-1} in (9.67) is a constant function, the case when $n = 2$ has been completely clarified by Li, Wan [510]. In higher dimensions, uniqueness in the isotropic setting has been verified if $-n \leqslant p \leqslant q < n$ in the even case by Ivaki, E. Milman [441] (see Chen, Huang, Zhao [185] for earlier results), and uniqueness in the isotropic case is extended not necessarily even solutions by Hu, Ivaki [412] if $-n < p < -1$ and $n - 1 < q < n$. The paper Böröczky, Chen, Liu, Saroglou [122] verified uniqueness of the solution of (9.65) in the near isotropic case (when the C^α norm of $f - 1$ is small) if $-1 < p < 1$ and q is close to n (see some additional results in the even case by Hu [406]). On the other hand, Chen, Chen, Li [187] provide cases when no uniqueness of the solution of the isotropic L_p dual Minkowski problem (9.65) holds in general, for example, when $p \geqslant 0$ and $q > p + 2n$, even if assuming that the solution h is even.

Orlicz versions of these Monge-Ampère equations have been considered by Li, Sheng, Ye, Yi [514], Feng, Hu, Liu [270] and Hu, Liu, Ma [409] in the case of the Aleksandrov problem, by Xing, Ye [726] in the case of the dual Minkowski problem, and Gardner, Hug, Weil, Xing, Ye [316, 317], Xing, Ye, Zhu [727] and Liu, Lu [526] in the case of the L_p dual Minkowski problem in general.

Another important related variant of the dual Minkowski problem is the so-called "Chord Minkowski Problem" (cf. Lutwak, Xi, Yang, Zhang [554]) and its L_p version either for $p > 0$ (cf. Xi, Yang, Zhang, Zhao [724]), or for $p < 0$ (cf. Li [517, 518]). In addition, see Guo, Xi, Zhao [368] for partial results on the critical interval $p \in (0, 1)$. A variant of the Chord Minkowski Problem, the "Affine dual Minkowski problem" is proposed by Cai, Leng, Wu, Xi [175].

9.5.6 Gaussian Minkowski problem

For the Gaussian density $d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\|x\|^2}{2}}\, dx$ in $\mathbb{R}^n$ and a convex body $K \in \mathcal{K}^n_{(0)}$, in line with the more general set up in Livshyts [528] (cf. (2.50) and (2.51)), Huang,

Xi, Zhao [421] defined the *Gaussian surface area measure* on S^{n-1} by the formula

$$S_{(\gamma_n,K)}(\omega) = \int_{\nu_K^{-1}(\omega)} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\|x\|^2}{2}} d\mathcal{H}^{n-1}(x)$$

for any Borel $\omega \subset S^{n-1}$, and hence if $C \in \mathcal{K}^n_{(o)}$, then (cf. (5.55))

$$\lim_{t\to 0^+} \frac{\gamma_n(K+tC)-\gamma_n(K)}{t} = \int_{S^{n-1}} h_C \, dS_{(\gamma_n,K)}.$$

In particular, if K is strictly convex (no segment on the boundary), thus α_K^* (cf. (9.55)) is a continuous function $S^{n-1} \to S^{n-1}$, then

$$dS_{(\gamma_n,K)}(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{\varrho_K(\alpha_K^*(u))^2}{2}\right) dS_K.$$

We deduce from Lemma 2.2.6 that the *Gaussian Minkowski problem* leads to the Monge-Ampère equation

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \cdot e^{-\frac{\|\nabla h\|^2+h^2}{2}} \cdot \det(\nabla^2 h + h\, I_{n-1}) = f.$$

These notions are due to Huang, Xi, Zhao [421], who obtain significant results about the even Gaussian Minkowski problem. Their results are extended to the not necessarily even case by Feng, Liu, Xu [267] and Chen, Hu, Liu, Zhao [191]. Uniqueness of the solution is discussed in the works above and in Ivaki, E. Milman [441]. Various additional properties of the Gaussian surface area measure is discussed in depth by Fradelizi, Langharst, Madiman, Zvavitch [301].

Remark 9.5.7. Uniqueness can't be expected in the Gaussian Minkowski Problem in general as, for example, a small ball and suitable large ball have the same Gaussian surface area measure. For $K, C \in \mathcal{K}^n_{(0)}$ with $\gamma_n(K), \gamma_n(C) \geqslant \frac{1}{2}$, Huang, Xi, Zhao [421] prove that $S_{(\gamma_n,K)} = S_{(\gamma_n,C)}$ implies $K = C$. Therefore, the uniqueness question is really interesting under the assumptions $\gamma_n(K), \gamma_n(C) < \frac{1}{2}$ (cf. Chen, Hu, Liu, Zhao [191] and Ivaki, E. Milman [441]).

Concerning sufficient conditions for the Gaussian Minkowski Problem, let us quote the following result from Feng, Liu, Xu [267].

Theorem 9.5.8. *If μ is a Borel measure on S^{n-1} with $\mu(S^{n-1}) < \frac{1}{\sqrt{2\pi}}$, then $\mu = S_{(\gamma_n,K)}$ for a $K \in \mathcal{K}^n_{(o)}$ with $\gamma_n(K) > \frac{1}{2}$.*

The L_p-Gaussian Minkowski problem is considered, for example, by Liu [524], Feng, Hu, Xu [266] and Sheng, Xue [682]. Orlicz versions of the Gaussian Minkowski problem are discussed by Gardner, *et al* [316], Li, Sheng, Ye, Yi [514] and Hu [402, 403].

9.5.7 Capacity, Torsional rigidity and the first eigenvalue of the Laplacian

This section discusses theories analogues to the Brunn-Minkowski theory that have been built for the classical notion of electrostatic (Newtonian) capacity, the torsional rigidity and the first eigenvalue of the Laplacian (see, for example, Colesanti, Hug [210] and Langharst, Ulivelli [493] for surveys).

The *electrostatic capacity* $C_2(X)$ (see Colesanti, *et al* [213] for a background) of a compact set $X \subset \mathbb{R}^n$, $n \geqslant 3$, is defined by

$$C_2(X) = \inf \left\{ \int_{\mathbb{R}^n} \|D\varphi\|^2 : \varphi \in C_c^\infty(\mathbb{R}^n),\ \varphi(x) \geqslant 1 \text{ for } x \in X \right\},$$

and if $\Omega \subset \mathbb{R}^n$ is a bounded open convex set, then

$$C_2(\Omega) = \sup \{C_2(X) : X \subset \Omega \text{ compact}\},$$

which satisfies $C_2(s\,\Omega) = s^{n-2} C_2(\Omega)$ for $s > 0$. For $n \geqslant 3$, $s \in (0,1)$ and bounded open convex sets $\Omega_0, \Omega_1 \subset \mathbb{R}^n$, Borell [113] proved the Brunn-Minkowski type inequality

$$C_2((1-s)\Omega_0 + s\,\Omega_1)^{\frac{1}{n-2}} \geqslant (1-s) C_2(\Omega_0)^{\frac{1}{n-2}} + s\, C_2(\Omega_1)^{\frac{1}{n-2}}, \tag{9.70}$$

where equality holds if and only if Ω_0 and Ω_1 are homothetic according to Caffarelli, Jerison, Lieb [173].

For a bounded open convex set $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, the equilibrium potential $U = U_\Omega$ associated to Ω is a continuous real function on $\mathbb{R}^n \backslash \Omega$ that is the unique solution of the boundary value problem

$$\left\{ \begin{array}{rcll} \Delta U & = & 0 & \text{on } \mathbb{R}^n \backslash (\text{cl}\,\Omega) \\ U(x) & = & 1 & \text{for } x \in \partial\Omega \\ \lim_{\|x\| \to \infty} U(x) & = & 0 & \end{array} \right. \tag{9.71}$$

where Δ is the Laplace operator. Now, if $\Omega = \text{int}\, K$ for the convex convex body $K = \text{cl}\,\Omega$, then the corresponding finite Borel measure $\mu_{2,\Omega}$ on S^{n-1} - the so-called electrostatic capacitary measure - is defined by the formula

$$\mu_{2,\Omega}(\omega) = \int_{\nu_K^{-1}(\omega)} \|DU_\Omega\|^2\, d\mathcal{H}^{n-1} \tag{9.72}$$

for Borel $\omega \subset S^{n-1}$ where the integral on $\partial K = \partial\Omega$ makes sense and is finite according to Dahlberg [228]. According to the so-called Poincaré formula, we have

$$C_2(\Omega) = \frac{1}{n-2} \int_{S^{n-1}} h_K\, d\mu_{2,\Omega}. \tag{9.73}$$

For a continuous $g : S^{n-1} \to \mathbb{R}$ and the open Wulff shape $\Omega_t = \{x \in \mathbb{R}^n : \langle x, u\rangle < h_K(u) + tg(u),\ \forall u \in S^{n-1}\}$, the electrostatic capacitary measure satisfies the variational formula (cf. Jerison [446])

$$\frac{d}{dt} C_2(\Omega_t)\Big|_{t=0} = \int_{S^{n-1}} g \, d\mu_{2,\Omega} \tag{9.74}$$

which actually yields (9.73) by taking $g = h_K$.

The Minkowski problem for capacity and a finite Borel measure μ on S^{n-1}, $n \geqslant 3$, has been solved by Jerison [445], proving that $\mu = \mu_{2,\Omega}$ for a bounded open convex set $\Omega \subset \mathbb{R}^n$ if and only if μ satisfies the conditions (a) and (b) in Minkowki's Theorem 9.2.3; namely,

(a) $\int_{S^{n-1}} u \, d\mu(u) = o \in \mathbb{R}^n$;

(b) μ is not concentrated on any closed hemisphere.

Uniqueness of the solution up to translation was clarified by Caffarelli, Jerison, Lieb [173] using their result on (9.70).

If $n = 2$, then the notion of capacity is replaced by the notion of so-called *transfinite diameter*, and the corresponding Minkowski problem is solved again by Jerison [446].

For $n \geqslant 3$, $p \in (1, n)$ and a bounded open convex set $\Omega \subset \mathbb{R}^n$, the p-capacity of Ω is $C_p(\Omega) = \sup\{C_p(X) : X \subset \Omega \text{ compact}\}$, where

$$C_p(X) = \inf\left\{\int_{\mathbb{R}^n} \|D\varphi\|^p : \varphi \in C_c^\infty(\mathbb{R}^n),\ \varphi(x) \geqslant 1 \text{ for } x \in X\right\}$$

for any compact $X \subset \mathbb{R}^n$. We note that $C_p(s\,\Omega) = s^{n-p} C_p(\Omega)$ for $s > 0$. Colesanti, *et al* [213] associates a so-called p-capacitary measure - a finite Borel measure - $\mu_{p,\Omega}$ on S^{n-1} to Ω, and proves the variational formula

$$\frac{d}{dt} C_p(\Omega_t)\Big|_{t=0} = (p-1)\int_{S^{n-1}} g \, d\mu_{p,\Omega}$$

using the notion of (9.74). It follows that

$$C_p(\Omega) = \frac{p-1}{n-p}\int_{S^{n-1}} h_K \, d\mu_{p,\Omega}$$

where $K = \operatorname{cl}\Omega$. Here the case $p = 2$ is the classical case. Actually, the p-capacitary measure $\mu_{p,\Omega}$ can be defined analogously to (9.72), only the Laplace operator should be replaced by the p-Laplace operator in (9.71) (cf. Colesanti, *et al* [213]).

The analogue of the Brunn-Minkowski type inequality (9.70) was proved by Colesanti, Salani [214] for $n \geqslant 3$ and $p \in (1, n)$; namely,

$$C_p((1-s)\Omega_0 + s\,\Omega_1)^{\frac{1}{n-p}} \geqslant (1-s) C_p(\Omega_0)^{\frac{1}{n-p}} + s\, C_p(\Omega_1)^{\frac{1}{n-p}}$$

holds for $s \in (0,1)$ and bounded open convex sets $\Omega_0, \Omega_1 \subset \mathbb{R}^n$, where equality holds if and only if Ω_0 and Ω_1 are homothetic. Next, Colesanti, *et al* [213]) propose the Minkowski problem for p-capacity for $p \in (1,n)$, characterize uniqueness for $p \in (1,n)$, and characterize the solution for $p \in (1,2)$. The characterization of the solution in the case $p \in (2,n)$ is due to Akman, *et al* [2]. For some additional related papers, see, for example, Hong, Ye [397], Hong, Ye, Zhang [398], Xiong, Xiong, Xu [730], Zou, Xiong [741], Xiong, Xiong [729], Hu [405] and Liu, Sheng [525], Akman, *et al* [1] and the extensive survey in Akman, *et al* [2].

Next, we turn to the torsional rigidity $\tau(\Omega)$ of a bounded open convex set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, (cf. Colesanti, Fimiani [208] or Langharst, Ulivelli [493]), namely,

$$\tau(\Omega) = \int_\Omega \|DU\|^2$$

where U is the unique solution of the boundary value problem

$$\begin{cases} -\Delta U &= 2 \quad \text{on } \Omega, \\ U(x) &= 0 \quad \text{for } x \in \partial\Omega, \end{cases} \tag{9.75}$$

and hence $\tau(s\,\Omega) = s^{n+2}\tau(\Omega)$ for $s > 0$. Using the U in (9.75) and the convex body $K = \operatorname{cl}\Omega$, the corresponding finite Borel measure $\mu_{\tau,\Omega}$ on S^{n-1} is defined by

$$\mu_{\tau,\Omega}(\omega) = \int_{\nu_K^{-1}(\omega)} \|DU\|^2\, d\mathcal{H}^{n-1}$$

for Borel $\omega \subset S^{n-1}$ where the integral on $\partial K = \partial\Omega$ makes sense and is finite according to Dahlberg [228].

Borell [113] proved that torsional rigidity satisfies the Brunn-Minkowski type inequality

$$\tau((1-s)\Omega_0 + s\,\Omega_1)^{\frac{1}{n+2}} \geqslant (1-s)\tau(\Omega_0)^{\frac{1}{n+2}} + s\,\tau(\Omega_1)^{\frac{1}{n+2}} \tag{9.76}$$

for $n \geqslant 2$, $s \in (0,1)$ and bounded open convex sets $\Omega_0, \Omega_1 \subset \mathbb{R}^n$, where equality holds if and only if Ω_0 and Ω_1 are homothetic according to Colesanti [206]. In addition, Colesanti, Fimiani [208] handled the first variation of torsional rigidity at a bounded open convex set $\Omega \subset \mathbb{R}^n$ with $K = \operatorname{cl}\Omega$. For a continuous $g : S^{n-1} \to \mathbb{R}$ and the open Wulff shape $\Omega_t = \{x \in \mathbb{R}^n : \langle x,u\rangle < h_K(u) + tg(u),\ \forall u \in S^{n-1}\}$, [208] proves that

$$\frac{d}{dt}\tau(\Omega_t)\bigg|_{t=0} = \int_{S^{n-1}} g\, d\mu_{\tau,\Omega},$$

and hence

$$\tau(\Omega) = \frac{1}{n+2}\int_{S^{n-1}} h_K\, d\mu_{\tau,\Omega}.$$

Concerning the Minkowski problem for torsional rigity, the equality condition in (9.76) yields uniqueness of the solution up to translation whenever it exists (cf. Colesanti [206]). However, the Minkowski problem for torsional rigity is still open, the characteristic properties of the measure $\mu_{\tau,\Omega}$ are not known. As a partial result, Colesanti, Fimiani [208] prove that if a finite Borel measure μ on S^{n-1} satisfies Minkowski's conditions (a) and (b) in Theorem 9.2.3 (these conditions are also available just after (9.74)), then $\mu = \mu_{\tau,\Omega}$ holds for a bounded open convex set $\Omega \subset \mathbb{R}^n$.

The L_p torsional rigidity Minkowski problem was posed by Chen, Dai [197] to characterize the Borel measure $h_K^{1-p}\, d\mu_{\tau,\Omega}$ on S^{n-1} where $K = \operatorname{cl}\Omega$ for the bounded open convex set $\Omega \subset \mathbb{R}^n$, and [197] proves existence and uniqueness results when $p > 1$. Further related existence results have been provided by Hu, Liu [407] if $p \in (0,1)$, and by Hu [401] if $p = 0$, where in the latter case, "subspace concentration conditions" like in (9.19) play a crucial role (see Langharst, Ulivelli [493] for the even case). Additional versions of the Minkowski problem for torsional rigidity have been considered by Fang, Hu, Zhao [255], Hu [400], Hu, Liu, Ma [408], Hu, Zhang [410], Hu, Li [416] and Hu [406].

Another related problem is about the first eigenvalue of the Laplacian. For any open bounded convex set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, let $\lambda(\Omega)$ be the smallest positive (the principal) eigenvalue of $-\Delta$ on Ω with Dirichlet boundary condition; namely, the smallest positive number such that there exists a continuous function $V = V_\Omega$ on $\operatorname{cl}\Omega$, which is C^∞ on Ω, and satisfies

$$\begin{cases} -\Delta V &= \lambda(\Omega) V & \text{on } \Omega, \\ V(x) &= 0 & \text{for } x \in \partial\Omega \end{cases}$$

(cf. Jerison [446]) where - using that $\lambda(\Omega)$ is the principal eigenvalue - we may also assume that V_Ω is positive on Ω and $\int_\Omega V^2 = 1$ in order to make it unique (cf. Proposition 10.7.1). Here $\lambda(s\,\Omega) = s^{-2}\lambda(\Omega)$ for $s > 0$. For a continuous $g : S^{n-1} \to \mathbb{R}$ and the open Wulff shape $\Omega_t = \{x \in \mathbb{R}^n : \langle x, u\rangle < h_K(u) + t g(u),\ \forall u \in S^{n-1}\}$, the first eigenvalue of the Laplacian satisfies the variational formula (cf. Jerison [446])

$$\frac{d}{dt}\lambda(\Omega_t)\bigg|_{t=0} = \int_{\partial K} g(\nu_K(x))\|V_\Omega(x)\|^2\, d\mathcal{H}^{n-1} \tag{9.77}$$

for the convex body $K = \operatorname{cl}\Omega$ where the integral on $\partial K = \partial\Omega$ makes sense and finite according to Dahlberg [228]. In turn, one deduces from (9.77) taking $g = h_K$ that

$$\lambda(\Omega) = \frac{1}{2}\int_{\partial K} h_K(\nu_K(x))\|V_\Omega(x)\|^2\, d\mathcal{H}^{n-1}.$$

Brascamp, Lieb [155] proved that the first eigenvalue of the Laplacian satisfies the Brunn-Minkowski type inequality

$$\lambda((1-s)\Omega_0 + s\,\Omega_1)^{\frac{-1}{2}} \geqslant (1-s)\lambda(\Omega_0)^{\frac{-1}{2}} + s\,\tau(\Omega_1)^{\frac{-1}{2}} \tag{9.78}$$

for $n \geqslant 2$, $s \in (0,1)$ and bounded open convex sets $\Omega_0, \Omega_1 \subset \mathbb{R}^n$, where equality holds if and only if Ω_0 and Ω_1 are homothetic according to Colesanti [206].

Concerning the Minkowski problem on S^{n-1} for the first eigenvalue of the Laplacian, the equality condition in (9.78) yields uniqueness of the solution if exists (cf. Colesanti [206]). In the even case, Langharst, Ulivelli [493] handles the sufficiency part under the "subspace concentration conditions" as in (9.19) (see also Hu [406]).

Jerison [446] placed the notions of capacity, torsion rigidity and the first eigenvalue of the Laplacian into a more general framework where the homogeneity and the formula for the first variation play crucial role, see Colesanti [206] for the corresponding Brunn-Minkowski type inequalities, and Crasta, Fragalà [227] for the corresponding Firey-type evolution equations, and Langharst, Ulivelli [493] for the sufficiency concerning the even Minkowski problem under the "subspace concentration conditions" (9.19).

9.A Supplement: Weak continuity of the L_p surface area measures and diameter bounds in terms of the entropy

In this section, we discuss two fundamental technical properties related to the L_p surface area measure for $p \in \mathbb{R}$. The first is the weak continuity of the L_p surface area measure, and the second property is a bound of the diameter of a convex body in terms of the "L_p entropy".

Lemma 9.A.1. *Let $p \in \mathbb{R}$, and let K_m tend to K for $K_m, K \in \mathcal{K}_o$. Then $S_{K_m,p}$ tends weakly to $S_{K,p}$ under the following conditions.*

(i) $p \leqslant 1$;

(ii) $p > 1$ and $K_m, K \in \mathcal{K}_{(o)}$;

(iii) $p > 1$, $K_m \in \mathcal{K}_{(o)}$ and the sequence $\{S_{K_m,p}(S^{n-1})\}$ is bounded. In this case, $S_K(\{h_K = 0\}) = 0$ also follows, and hence $S_{K,p}$ is well-defined.

Proof. We deduce (i) and (ii) from the facts that S_{K_m} tends weakly to S_K according to Proposition 2.6.13 (see also Proposition 8.4.1), and $g \cdot h_{K_m}^{1-p}$ tends uniformly to $g \cdot h_K^{1-p}$ for any continuous $g : S^{n-1} \to \mathbb{R}$ as h_{K_m} tends uniformly to h_K (cf. Definition 1.7.1).

For (iii), we may assume that $o \in \partial K$ by (ii), and let $R > 0$ such that $S_{K_m,p}(S^{n-1}) < R$ for every m. First we show that $S_{K,p}$ is well-defined; namely, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$S_K(\Xi_\delta) \leqslant \varepsilon \ \text{ for } \ \Xi_\delta = \left\{u \in S^{n-1} : \ h_K(u) < \delta\right\}. \tag{9.79}$$

We choose $\delta > 0$ in a way such that $(2\delta)^{1-p} \cdot \frac{\varepsilon}{2} > R$. If m is large, then $h_{K_m}(u) < 2\delta$ for $u \in \Xi_\delta$, and as Ξ_δ is a non-empty open subset of S^{n-1}, we have $S_{K_m}(\Xi_\delta) > \frac{1}{2} \cdot S_K(\Xi_\delta)$

by the weak convergence of S_{K_m} (cf. Remark 10.1.1). We deduce that

$$(2\delta)^{1-p}\cdot\frac{S_K\left(\Xi_\delta\right)}{2}\leqslant(2\delta)^{1-p}\cdot S_{K_m}\left(\left\{u\in S^{n-1}:\,h_{K_m}(u)\leqslant 2\delta\right\}\right)\leqslant\int_{\{u\in S^{n-1}:\,h_{K_m}(u)\leqslant 2\delta\}}h_{K_m}^{1-p}\,dS_{K_m}\leqslant R,$$

yielding (9.79), and in turn the well-definedness of $S_{K,p}$.

Given a continuous $g:S^{n-1}\to\mathbb{R}$, $g\cdot h_{K_m}^{1-p}$ tends uniformly to $g\cdot h_K^{1-p}$ on $S^{n-1}\backslash\Xi_\delta$ for any small $\delta>0$. Therefore, $\lim_{m\to\infty}\int_{S^{n-1}}g\,dS_{K_m,p}=\int_{S^{n-1}}g\,dS_{K,p}$ follows from (9.79). ∎

Diameter bounds in terms of the entropy - that we present in Proposition 9.A.2 -, are useful tools when handling the L_p-Minkowski problem. The point is that (see Remark 9.3.7) both in the cases of the flow method and the variational method, the convex body solving the L_p-Minkowski problem is sought as the limit of a sequence of special type of convex bodies, and the diameter bounds in Proposition 9.A.2 ensure that the sequence of approximating convex bodies is bounded, and hence contains a convergent subsequence by the Blaschke Selection Theorem.

Let μ be a Borel probability measure on S^{n-1}, and let $C\in\mathcal{K}_o$. If $p>1$, then we use the entropy function

$$\mathcal{E}_{\mu,p}(C)=\frac{1}{p}\int_{S^{n-1}}h_C^p\,d\mu \tag{9.80}$$

of a C, which is continuous in C. For $p<1$ and $\xi\in\operatorname{int}C$, we consider the entropy function

$$\mathcal{E}_{\mu,p}(C,\xi)=\begin{cases}\frac{1}{p}\int_{S^{n-1}}h_{C-\xi}^p\,d\mu & \text{if}\quad p\neq 0,\\ \int_{S^{n-1}}\log h_{C-\xi}\,d\mu & \text{if}\quad p=0,\end{cases} \tag{9.81}$$

which is continuous in C and in $\xi\in\operatorname{int}C$.

We introduce some additional notions which help to describe how much a measure on S^{n-1} is concentrated to great sub-spheres. Let $\delta\in[0,1)$. For a linear i-subspace L of $\mathbb{R}^n$ with $1\leqslant\dim L\leqslant n-1$, we consider the collar

$$\Psi(L\cap S^{n-1},\delta)=\{x\in S^{n-1}:\langle x,y\rangle\leqslant\delta\text{ for }y\in L^\perp\cap S^{n-1}\}.$$

In addition, for a $u\in S^{n-1}$, an open spherical cap centered at u is

$$\Omega(u,\delta)=\{v\in S^{n-1}:\langle v,u\rangle>\delta\}$$

where $\Omega(u,0)$ is an open hemisphere (cf. (9.7)).

While only the cases (i) and (ii) (when $p\geqslant 0$) of Proposition 9.A.2 are used in this book (see Section 9.B and Section 9.C), we also discuss the case $p\in(-n,0)$, used in

some equivalent forms, for example, by Bianchi, Böröczky, Colesanti, Yang [91] and Böröczky, Guan [136]. We recall that σ_K is the centroid of a convex body $K \subset \mathbb{R}^n$ (cf. Definition 1.11.1).

Proposition 9.A.2. *Let μ be a Borel probability measure on S^{n-1}, let $K \in \mathcal{K}^n_o$ with* $\operatorname{diam} K = D$, *and let $\delta, \tau \in (0,1)$.*

(i) *If $\mu\left(\Omega(u,\delta)\right) > \tau$ for any $u \in S^{n-1}$, then*

$$\frac{\tau\,\delta^p}{2^p}\cdot D^p \leqslant \begin{cases} p\,\mathcal{E}_{\mu,p}(K) & \text{provided} \quad p > 1 \\ p\,\mathcal{E}_{\mu,p}(K,\sigma_K) & \text{provided} \quad p \in (0,1). \end{cases}$$

(ii) *If $p = 0$, $|K| = 1$, and*

$$\mu\left(\Psi(L \cap S^{n-1},\delta)\right) < \frac{(1-\tau)\cdot i}{n} \tag{9.82}$$

for any linear i-subspace $L \subset \mathbb{R}^n$, $i = 1,\dots,n-1$, then

$$\mathcal{E}_{\mu,0}(K,\sigma_K) \geqslant \tau \log D + \log\delta - 6\log n.$$

(iii) *If $-n < p < 0$, and $d\mu = f\,\mathcal{H}^{n-1}$ for $f \in L^{\frac{n}{n+p}}(S^{n-1},\mathcal{H}^{n-1})$ where*

$$\int_{\Psi(u^\perp\cap S^{n-1},\delta)} f^{\frac{n}{n+p}} \leqslant (n\omega_n^2)^{\frac{p}{n+p}}\cdot\tau^{\frac{n}{n+p}}$$

for any $u \in S^{n-1}$, and in addition, $\tau \leqslant \frac{p}{2}\,\mathcal{E}_{\mu,p}(K,\sigma_K)$, then

$$\text{either } D \leqslant 4n^2/\delta^2, \quad \text{or} \quad D \leqslant \left(\frac{p}{2}\,\mathcal{E}_{\mu,p}(K,\sigma_K)\right)^{\frac{2}{p}}.$$

Remark. In the applications, the K in (iii) is obtained via minimizing the entropy; namely, it satisfies $|K| = 1$ and $\mathcal{E}_{\mu,p}(K,\sigma_K) \leqslant \sup_{\xi\in\operatorname{int}\widetilde{B}} \mathcal{E}_{\mu,p}(\widetilde{B},\xi)$ for the centered ball $\widetilde{B}$ of volume 1 and of radius $\omega_n^{-1/n}$. For such a K, it is sufficient to choose τ in (iii) in a way such that (remember, $p < 0$)

$$\tau \leqslant 2^{p-1}\omega_n^{-p/n}\mu(S^{n-1})\left(\leqslant \inf_{\xi\in\operatorname{int}\widetilde{B}} \frac{1}{2}\int_{S^{n-1}} h^p_{\widetilde{B}-\xi}\,d\mu\right). \tag{9.83}$$

In particular, the τ in (iii) can be chosen to satisfy (9.83), thus independently of K.

Proof. Let $R = \max_{x\in K}\|x\|$, and hence $D \leqslant 2R$, and let $x_0 \in K$ and $u_0 \in S^{n-1}$ with $u_0 \in S^{n-1}$ and $x_0 = R\,u_0$.

If $p < 1$, then we may assume that in addition $\sigma_K = o$, and hence the KLS ellipsoid E (cf. Lemma 1.11.5) satisfies that

$$E \subset K \subset n\,E, \tag{9.84}$$

which, in turn, also yields that

$$-x/n \in K \text{ for } x \in K. \tag{9.85}$$

For (i), we observe that $h_K(u) \geqslant \delta R$ for $u \in \Omega(u_0, \delta)$ and $\mu\left(\Omega(u_0, \delta)\right) \geqslant \tau$; therefore, $\int_{S^{n-1}} h_K^p \, d\mu \geqslant \tau \, (R\delta)^p \geqslant \tau \, (D\delta/2)^p$.

Let us turn to (ii), which case requires a more involved argument. Let $e_1, \ldots, e_n \in S^{n-1}$ be orthonormal basis of $\mathbb{R}^n$ forming the principal directions associated to the ellipsoid E in (9.84), and let $r_1, \ldots, r_n > 0$ be the half axes of E with $r_i e_i \in \partial E$ where we may assume that $r_1 \leqslant \ldots \leqslant r_n$. In particular, (9.84) yields that

$$\prod_{i=1}^{n} r_i = \frac{|nE|}{n^n \omega_n} \geqslant \frac{|K|}{n^n \omega_n} = \aleph_n \text{ for } \aleph_n = n^{-n} \omega_n^{-1} < 1. \tag{9.86}$$

We observe that for any $v \in S^{n-1}$, there exists e_i such that $|\langle v, e_i \rangle| \geqslant \frac{1}{\sqrt{n}} > \frac{\delta}{n}$. For $i = 1, \ldots, n$, we define

$$B_i = \left\{ v \in S^{n-1} : \, |\langle v, e_i \rangle| \geqslant \frac{\delta}{n} \text{ and } |\langle v, e_j \rangle| < \frac{\delta}{n} \text{ for } j > i \right\}.$$

In particular, the i-dimensional linear subspace $L_i = \operatorname{lin}\{e_1, \ldots, e_i\}$ satisfies

$$B_j \subset \Psi(L_i \cap S^{n-1}, \delta) \text{ for } i = 1, \ldots, n-1 \text{ and } j \leqslant i. \tag{9.87}$$

It follows that S^{n-1} is partitioned into the Borel sets $B_1, \ldots, B_n$, and (9.82) and (9.87) yield that

$$\mu(B_1) + \ldots + \mu(B_i) \leqslant \frac{i(1-\tau)}{n} \text{ for } i = 1, \ldots, n-1 \tag{9.88}$$

$$\mu(B_1) + \ldots + \mu(B_n) = 1. \tag{9.89}$$

For $\zeta = \frac{1-\tau}{n}$, we have $0 < \zeta < \frac{1}{n}$, and we define

$$\beta_i = \mu(B_i) - \zeta \text{ for } i = 1, \ldots, n-1 \tag{9.90}$$

$$\beta_n = \mu(B_n) - \zeta - \tau \tag{9.91}$$

where (9.88) and (9.89) yield

$$\beta_1 + \ldots + \beta_i \leqslant 0 \text{ for } i = 1, \ldots, n-1 \tag{9.92}$$

$$\beta_1 + \ldots + \beta_n = 0. \tag{9.93}$$

As $r_i e_i \in K$, it follows from the definition of B_i that $h_K(u) \geqslant \langle u, r_i e_i \rangle \geqslant r_i \cdot \frac{\delta}{n}$ for $u \in B_i$, $i = 1, \ldots, n$. We deduce from the definition of $B_1, \ldots, B_n$, applying (9.89),

and the definition of β_i in (9.90) and (9.91) that

$$\begin{aligned}\int_{S^{n-1}} \log h_K \, d\mu &= \sum_{i=1}^{n} \int_{B_i} \log h_K \, d\mu \\ &\geqslant \sum_{i=1}^{n} \mu(B_i) \log r_i + \sum_{i=1}^{n} \mu(B_i) \log \frac{\delta}{n} = \sum_{i=1}^{n} \mu(B_i) \log r_i + \log \frac{\delta}{n} \\ &= \sum_{i=1}^{n} \beta_i \log r_i + \sum_{i=1}^{n} \zeta \log r_i + \tau \log r_n + \log \frac{\delta}{n}.\end{aligned}$$

Here, applying first (9.86), and then (9.92), (9.93), $r_1 \leqslant \ldots \leqslant r_{n+1}$ and $\zeta < \frac{1}{n}$ lead to

$$\begin{aligned}\int_{S^{n-1}} \log h_K \, d\mu &\geqslant \sum_{i=1}^{n} \beta_i \log r_i + \zeta \log \aleph_n + \tau \log r_n + \log \frac{\delta}{n} \\ &= (\beta_1 + \ldots + \beta_n) \log r_n + \sum_{i=1}^{n-1} (\beta_1 + \ldots + \beta_i)(\log r_i - \log r_{i+1}) \\ &\quad + \zeta \log \aleph_n + \tau \log r_n + \log \frac{\delta}{n} \\ &\geqslant \tau \log r_n + \log \delta + \log \aleph_n^{1/n} - \log n.\end{aligned}$$

Now, $D \leqslant n \operatorname{diam} E = 2nr_n \leqslant n^2 r_n$ and $\tau < 1$, and hence $\tau \log r_n \geqslant \tau \log D - 2 \log n$. In addition, $\aleph_n^{1/n} = \frac{\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}{n\sqrt{\pi}} > (2e\pi n)^{\frac{-1}{2}} > n^{-3}$; therefore,

$$\tau \log r_n + \log \delta + \log \aleph_n^{1/n} - \log n \geqslant \tau \log D + \log \delta - 6 \log n.$$

For (iii), when $-n < p < 0$, we may assume that

$$D \geqslant 4n^2/\delta^2,$$

and we consider

$$\begin{aligned}\Phi_0 &= \left\{ u \in S^{n-1} : \, h_K(u) > \sqrt{2R} \right\}; \\ \Phi_1 &= \left\{ u \in S^{n-1} : \, h_K(u) \leqslant \sqrt{2R} \right\}.\end{aligned}$$

Concerning Φ_0, we have

$$\int_{\Phi_0} f \cdot h_K^p \leqslant (2R)^{p/2} \int_{\Phi_0} f \leqslant D^{\frac{p}{2}}. \tag{9.94}$$

On the other hand, we have $\pm \frac{R}{n} u_0 \in K$ by (9.85). Thus, any $u \in \Phi_1$ satisfies

$$\sqrt{2R} \geqslant h_K(u) \geqslant \left| \left\langle u, \frac{R}{n} u_0 \right\rangle \right|,$$

and hence $|\langle u, u_0\rangle| \leqslant n\sqrt{\frac{2}{R}} \leqslant \frac{2n}{\sqrt{D}} \leqslant \delta$; or in other words,

$$\Phi_1 \subset \Psi(u_0^{\perp} \cap S^{n-1}, \delta).$$

It follows from $|K| = 1$ and the form (6.32) of the Blaschke-Santaló inequality that

$$\int_{S^{n-1}} h_K^{-n} \leqslant n\omega_n^2.$$

For $p \in (-n, 0)$, Hölder's inequality and $\int_{\Phi_1} f^{\frac{n}{n+p}} < (n\omega_n^2)^{\frac{p}{n+p}} \cdot \tau^{\frac{n}{n+p}}$ yield

$$\int_{\Phi_1} f \cdot h_K^p \leqslant \left(\int_{\Phi_1} f^{\frac{n}{n+p}}\right)^{\frac{n+p}{n}} \left(\int_{\Phi_1} h_K^{-n}\right)^{\frac{|p|}{n}} \leqslant \tau.$$

Finally, adding the last estimate to (9.94) yields

$$p\,\mathcal{E}_{\mu,p}(K, \sigma_K) = \int_{S^{n-1}} f \cdot h_K^p \leqslant D^{\frac{p}{2}} + \tau,$$

and hence the condition $\tau \leqslant \frac{p}{2}\,\mathcal{E}_{\mu,p}(K)$ on τ implies (iii). ∎

9.B Supplement: The even L_p-Minkowski problem for $p \in [0, 1)$

This section about the even L_p-Minkowski surface area measures discusses work by Haberl, Lutwak, Yang, Zhang [371] if $p \in (0, 1)$ (cf. Corollary 9.B.2), and by Böröczky, Lutwak, Yang, Zhang [141] if $p = 0$, and proves Proposition 9.4.6 (cf. Proposition 9.B.1), and Theorem 9.3.6 (cf. Theorem 9.B.4). The key tool is the notion of entropy defined in (9.81); namely, if $C \subset \mathbb{R}^n$ is an o-symmetric convex body, then

$$\mathcal{E}_{\mu,p}(C, o) = \begin{cases} \frac{1}{p}\int_{S^{n-1}} h_C^p \, d\mu & \text{if} \quad p \in (0, 1), \\ \int_{S^{n-1}} \log h_C \, d\mu & \text{if} \quad p = 0, \end{cases} \tag{9.95}$$

which is continuous in C. For the reader's convenience, we restate Proposition 9.4.6.

Proposition 9.B.1. *For $p \in [0, 1)$, let μ be an even probability Borel measure on S^{n-1} such that any open hemisphere has positive measure, and, if $p = 0$, then, in addition,*

$$\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1}) \tag{9.96}$$

holds for any non-trivial linear subspace $L \subset \mathbb{R}^n$.

(a) $\mathcal{E}_{\mu,p}(C, o)$ attains its minimum among o-symmetric convex bodies $C \subset \mathbb{R}^n$ satisfying $|C| = 1$.

(b) Moreover, if an o-symmetric convex body $K \subset \mathbb{R}^n$ with $|K| = 1$ minimizes $\mathcal{E}_{\mu,p}(C,o)$ among o-symmetric convex bodies $C \subset \mathbb{R}^n$ with $|C| = 1$, then $\mu = S_{\theta K,p}$ for $\theta = \left(\frac{1}{n}\int_{S^{n-1}} h_K^p \, d\mu\right)^{\frac{1}{n-p}}$.

Proof. Let $\mathcal{C}$ be the set of o-symmetric convex bodies C with $|C| = 1$. As for any $i = 1, \dots, n-1$, the Grassmannian space of all i-dimensional linear subspaces is compact, the conditions on μ imply that there exist $\delta, \tau \in (0,1)$ such that

$$\begin{aligned} \mu\left(\Omega(u,\delta)\right) &> \tau && \text{if } p \in (0,1) \text{ and } u \in S^{n-1}; \\ \mu\left(\Psi(L \cap S^{n-1}, \delta)\right) &< \tfrac{(1-\tau)i}{n} && \text{if } p = 0 \text{ and } L \subset \mathbb{R}^n \text{ linear } i\text{-subspace}, \\ & && i = 1, \dots, n-1; \end{aligned} \tag{9.97}$$

namely, μ satisfies the conditions in Proposition 9.A.2.

The argument is easier presented using another notion of entropy that is equivalent to the one defined (9.95) for our purposes: if $C \subset \mathbb{R}^n$ is an o-symmetric convex body, then let

$$\overline{\mathcal{E}}_{\mu,p}(C,o) = \begin{cases} \frac{1}{p}\log \int_{S^{n-1}} h_C^p \, d\mu - \frac{1}{n}\log|C| & \text{if} \quad p \in (0,1), \\ \int_{S^{n-1}} \log h_C \, d\mu - \frac{1}{n}\log|C| & \text{if} \quad p = 0, \end{cases}$$

which notion is invariant under rescaling; namely, it satisfies

$$\begin{aligned} \overline{\mathcal{E}}_{\mu,p}(\lambda C, o) &= \overline{\mathcal{E}}_{\mu,p}(C,o) \quad \text{for } \lambda > 0, \\ \overline{\mathcal{E}}_{\mu,p}(C,o) &= \mathcal{E}_{\mu,p}(C,o) \quad \text{if } |C| = 1. \end{aligned} \tag{9.98}$$

Let $\widetilde{B}^n$ be the o-symmetric Euclidean ball with $|\widetilde{B}^n| = 1$, and let $C_m \in \mathcal{C}$ satisfy that $\overline{\mathcal{E}}_{\mu,p}(C_m,o) \leqslant \overline{\mathcal{E}}_{\mu,p}(\widetilde{B}^n,o)$ and $\overline{\mathcal{E}}_{\mu,p}(C_m,o)$ tends to $\inf\left\{\overline{\mathcal{E}}_{\mu,p}(C,o) : C \in \mathcal{C}\right\}$. It follows from Proposition 9.A.2 that the sequence $\{C_m\}$ is bounded; therefore, we may assume that C_m tends to an o-symmetric convex compact set K by the Blaschke Selection Theorem 1.7.4. As each $|C_m| = 1$, we have $|K| = 1$ by the continuity of volume (cf. Lemma 1.7.5), thus $K \in \mathcal{C}$; therefore,

$$\overline{\mathcal{E}}_{\mu,p}(K,o) = \min\left\{\overline{\mathcal{E}}_{\mu,p}(C,o) : C \in \mathcal{C}\right\}. \tag{9.99}$$

We claim that

$$\mu = \lambda \cdot S_{K,p} \quad \text{for } \lambda = \frac{1}{n}\int_{S^{n-1}} h_K^p \, d\mu, \tag{9.100}$$

where (9.100) is equivalent with saying that

$$\int_{S^{n-1}} g \cdot h_K^{p-1} \, d\mu = \lambda \cdot \int_{S^{n-1}} g \, dS_K \tag{9.101}$$

for any continuous function $g : S^{n-1} \to \mathbb{R}$. For $t \geqslant 0$, we consider the Wulff-shape

$$K_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_K(u) + tg(u)\ \forall u \in S^{n-1}\},$$

and hence $K_0 = K$, and the Aleksandrov Lemma 9.2.2 yields that

$$\frac{d}{dt}\,|K_t|\Big|_{t=0} = \int_{S^{n-1}} g\, dS_K. \tag{9.102}$$

If $|t|$ is small, then we consider the differentiable function

$$f(t) = \begin{cases} \frac{1}{p}\log \int_{S^{n-1}} (h_K + tg)^p\, d\mu - \frac{1}{n}\log|K_t| & \text{if} \quad p \in (0,1), \\ \int_{S^{n-1}} \log(h_K + tg)\, d\mu - \frac{1}{n}\log|K_t| & \text{if} \quad p = 0, \end{cases}$$

which, according to $|K_t|^{\frac{-1}{n}} \cdot K_t \in \mathcal{C}$, $h_{K_t} \leqslant h_K(u) + tg(u)$, (9.98) and (9.99), satisfies that

$$f(t) \geqslant \overline{\mathcal{E}}_{\mu,p}(K_t, o) = \overline{\mathcal{E}}_{\mu,p}\left(|K_t|^{\frac{-1}{n}} \cdot K_t, o\right) \geqslant f(0).$$

In particular, f has a minimum at $t = 0$, and hence (9.102) implies that

$$0 = f'(0) = \frac{1}{n\lambda}\int_{S^{n-1}} g \cdot h_K^{p-1}\, d\mu - \frac{1}{n}\int_{S^{n-1}} g\, dS_K,$$

proving (9.101), and in turn (9.100). Therefore, $\mu = S_{M,p}$ for $M = \lambda^{\frac{1}{n-p}} K$. ∎

Proposition 9.B.1 directly implies the characterization of even L_p-surface area measures for $p \in (0,1)$ due to Haberl, Lutwak, Yang, Zhang [371].

Corollary 9.B.2. *For $p \in (0,1)$, and a finite even Borel measure μ on S^{n-1}, there exists an o-symmetric convex body $K \subset \mathbb{R}^n$ with $\mu = S_{K,p}$ if and only if the measure of any open hemisphere is positive.*

The characterization of even L_0-surface area measure (cone volume measure) due to Böröczky, Lutwak, Yang, Zhang [141] is more involved. We recall that according to (2.34) and (2.37), if $\omega \subset \mathbb{R}^n$ is a Borel set and $K \in \mathcal{K}_o^n$, then

$$V_K(\omega) = \left|\bigcup \{\mathrm{conv}\{o, x\} : x \in \partial K \text{ and } N_K(x) \cap \omega \neq \varnothing\}\right| \tag{9.103}$$

$$= \frac{1}{n}\int_{\nu_K^{-1}(\omega)} \langle \nu_K(x), x\rangle\, d\mathcal{H}^{n-1}(x). \tag{9.104}$$

We say that the linear subspaces $L_1, L_2 \subset \mathbb{R}^n$ of dimension at least 1 are complementary if $L_1 + L_2 = \mathbb{R}^n$ and $L_1 \cap L_2 = \{o\}$. The following statement is a direct consequence of (9.104).

Lemma 9.B.3. *Let $C_i \subset \mathbb{R}^n$ be compact convex set of dimension $d_i \geqslant 1$ and containing o for $i = 1,2$ such that $L_1 = \operatorname{lin} C_1$ and $L_2 = \operatorname{lin} C_2$ are complementary linear subspaces. In this case, writing $V_{C_i'}$ to denote the cone volume measure of $C_i' = C_i|L_j^{\perp}$ on $L_j^{\perp} \cap S^{n-1}$ for $\{i,j\} = \{1,2\}$, the convex body $K = C_1 + C_2 \subset \mathbb{R}^n$ satisfies that*

- $\operatorname{supp} V_K \subset L_1^{\perp} \cup L_2^{\perp}$,
- *if $\{i,j\} = \{1,2\}$ and $\omega \subset L_j^{\perp} \cap S^{n-1}$ is Borel, then*

$$V_K(\omega) = \frac{d_i}{n} \cdot \mathcal{H}^{d_j}(C_j) \cdot V_{C_i'}(\omega).$$

We arrived at the main result of this section.

Theorem 9.B.4. *For a non-trivial finite even Borel measure μ on S^{n-1}, there exists an o-symmetric convex body $K \subset \mathbb{R}^n$ with $\mu = V_K = \frac{1}{n} S_{K,0}$ if and only if*

(i) $\mu(L \cap S^{n-1}) \leqslant \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *for any proper linear subspace $L \subset \mathbb{R}^n$;*

(ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *in (i) is equivalent with the existence of a complementary linear subspace $L' \subset \mathbb{R}^n$ with $\operatorname{supp} \mu \subset L \cup L'$, and in this case, $K = C + C'$ for o-symmetric compact convex sets $C \subset L^{\perp}$ and $C' \subset L'^{\perp}$.*

Proof. Step 1. If a non-trivial even Borel measure μ on S^{n-1}, $n \geqslant 1$, satisfies (i) and (ii), then $\mu = V_K$ for an o-symmetric convex body $K \subset \mathbb{R}^n$.

We prove this by induction on $n \geqslant 1$. When $n = 1$, then the only condition on μ on S^0 is its evenness, and $\mu = V_K$ for the o-symmetric segment $K \subset \mathbb{R}$ with $|K| = \mu(S^0)$. Therefore, let $n \geqslant 2$.

If $\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1})$ for any non-trivial linear subspace $L \subset \mathbb{R}^n$, then Proposition 9.B.1 yields the existence of an o-symmetric convex body $K \subset \mathbb{R}^n$ with $\mu = V_K$.

Next, we assume that, in line with condition (ii), there exist complementary linear subspaces $L_1, L_2 \subset \mathbb{R}^n$ of dimensions $d_1, d_2 \geqslant 1$ such that $\operatorname{supp} \mu \subset L_1 \cup L_2$. For $i = 1,2$, now the condition (i) on μ yields that $\mu(L_i \cap S^{n-1}) = \frac{d_i}{n} \cdot \mu(S^{n-1})$, and if $L \subset L_i$ is a non-trivial linear subspace, then $\mu(L \cap S^{n-1}) \leqslant \frac{\dim L}{d_i} \cdot \mu(L_i \cap S^{n-1})$. Therefore, for $i = 1,2$, the induction hypothesis yields the existence of a d_i-dimensional o-symmetric compact convex set $\widetilde{C}_i' \subset L_i$ such that

$$\mu_{\llcorner}(L_i \cap S^{n-1}) = V_{\widetilde{C}_i'}.$$

For $\{i,j\} = \{1,2\}$, let $\widetilde{C}_i \subset L_j^{\perp}$ be the d_i-dimensional o-symmetric compact convex set such that $\widetilde{C}_i|L_i = \widetilde{C}_i'$. For $\widetilde{K} = \widetilde{C}_1 + \widetilde{C}_2$, we deduce from Lemma 9.B.3 (exchanging the role of L_i and $L_i^{\perp}$) that $V_{\widetilde{K}} = \lambda\mu$ for a $\lambda > 0$, and hence we may choose $K = \lambda^{-\frac{1}{n}} \widetilde{K}$ to complete Step 1.

Step 2. If $K \subset \mathbb{R}^n$ is an o-symmetric convex body, and $L \subset \mathbb{R}^n$ is an i-dimensional linear subspace for $i \in \{1, \dots, n-1\}$, then (i) and (ii) hold.

We consider the orthogonal projection $K' = \Pi_L K$. For any $x \in \operatorname{relbd} K'$, let $M(x) = K \cap (x + L^\perp)$, and hence $-x \in \operatorname{relbd} K'$ and $M(-x) = -M(x)$. If $z = tx$ for $t \in [0,1]$, then $z = (1-\lambda)x + \lambda(-x)$ for $\lambda = \frac{1-t}{2} \in [0, \frac{1}{2}]$, and the convexity of K, $M(-x) = -M(x)$ and the Brunn-Minkowski inequality Theorem 1.12.3 imply

$$\mathcal{H}^{n-i}\left(K \cap (z + L^\perp)\right) \geqslant \mathcal{H}^{n-i}\left((1-\lambda)M(x) + \lambda\, M(-x))\right) \geqslant \mathcal{H}^{n-i}\left(M(x)\right). \tag{9.105}$$

We deduce from the Fubini theorem and using polar coordinates in L (cf. (10.13)) that

$$\begin{aligned} |K| &= \int_{K'} \mathcal{H}^{n-i}\left(K \cap (z + L^\perp)\right)\, d\mathcal{H}^i(z) \\ &= \int_{S^{n-1}\cap L} \int_0^{\varrho_{K'}(u)} \mathcal{H}^{n-i}\left(K \cap (ru + L^\perp)\right) r^{i-1}\, dr\, d\mathcal{H}^{i-1}(u) \\ &\geqslant \int_{S^{n-1}\cap L} \int_0^{\varrho_{K'}(u)} \mathcal{H}^{n-i}\left(M\left(\varrho_{K'}(u)u\right)\right) r^{i-1}\, dr\, d\mathcal{H}^{i-1}(u) \\ &= \frac{1}{i} \int_{S^{n-1}\cap L} \varrho_{K'}(u)^i \mathcal{H}^{n-i}\left(M\left(\varrho_{K'}(u)u\right)\right)\, d\mathcal{H}^{i-1}(u) \end{aligned} \tag{9.106}$$

where $\varrho_{K'}(u)u \in \operatorname{relbd} K'$ for $u \in S^{n-1} \cap L$ and the radial function $\varrho_{K'}(u) > 0$. On the other hand, (9.103), the Fubini theorem and using polar coordinates in L (cf. (10.13)) yield that

$$\begin{aligned} V_K(L \cap S^{n-1}) &= \left|\left\{t\, y : t \in [0,1] \text{ and } y \in (L^\perp + \operatorname{relbd} K')\right\}\right| \\ &= \int_{S^{n-1}\cap L} \int_0^{\varrho_{K'}(u)} \frac{r^{n-i} \cdot \mathcal{H}^{n-i}\left(M\left(\varrho_{K'}(u)u\right)\right)}{\varrho_{K'}(u)^{n-i}} \cdot r^{i-1}\, dr\, d\mathcal{H}^{i-1}(u) \\ &= \frac{1}{n} \int_{S^{n-1}\cap L} \varrho_{K'}(u)^i \mathcal{H}^{n-i}\left(M\left(\varrho_{K'}(u)u\right)\right)\, d\mathcal{H}^{i-1}(u). \end{aligned} \tag{9.107}$$

We conclude (i) from comparing (9.106) and (9.107).

Let us assume that that equality holds in (i), and hence equality holds in (9.105) for any $x \in \operatorname{relbd} K'$, $\lambda \in [0, \frac{1}{2}]$ and $z = (1-\lambda)x + \lambda(-x)$. We deduce from the equality conditions in the Brunn-Minkowski inequality Theorem 1.12.3 that $M(x)$ and $M(-x) = -M(x)$ are translates. In particular, equality in (9.105) yields that $K \cap (z + L^\perp)$ is a translate of C for any $z \in K'$ and the $(n-i)$-dimensional o-symmetric compact convex set $C = K \cap L^\perp$. In other words, there exists an even function $\varphi : K' \to L^\perp$ such that $K \cap (z + L^\perp) = C + \varphi(z)$ for any $z \in K'$. Now if $z_1, z_2 \in K'$ and $\alpha \in [0,1]$, then $\varphi((1-\alpha)z_1 + \alpha z_2) = (1-\alpha)\varphi(z_1) + \alpha\,\varphi(z_2)$ follows from the convexity of K, and hence φ is a linear function. We deduce that $K = C + C'$ for the i-dimensional o-symmetric compact convex set $C' = \{z + \varphi(z) : z \in K'\}$, and in turn $\operatorname{supp} V_K \subset L \cup L'$ where $L' = (\operatorname{lin} C')^\perp$. ∎

We deduce via the $\mathrm{SL}(n)$ equivariance of the cone volume measure (cf. Proposition 2.6.15) the following ambiguity concerning recovering the o-symmetric convex body from its cone volume measure, where the Logarithmic Minkowski Conjecture 9.4.5 states that this is the only ambiguity in the even case.

Lemma 9.B.5. *If $\sum_{i=1}^m C_i$ is a convex body in $\mathbb{R}^n$ for compact convex sets $C_1, \dots, C_m \subset \mathbb{R}^n$ of dimension $d_i \geqslant 1$ and containing o such that $\sum_{i=1}^m \dim C_i = n$, and $\lambda_1, \dots, \lambda_m > 0$ satisfy that $\prod_{i=1}^m \lambda_i^{d_i} = 1$, or equivalently, $|\sum_{i=1}^m C_i| = |\sum_{i=1}^m \lambda_i C_i|$, then*

$$V_{\sum_{i=1}^m C_i} = V_{\sum_{i=1}^m \lambda_i C_i}.$$

9.C Supplement: About the L_p-Minkowski problem in general for $p \geqslant 0$

For $p \geqslant 0$ with $p \neq 1, n$, and for non-negative $f \in L^1(S^{n-1})$ with $\mathcal{H}^{n-1}(\{f = 0\}) = 0$, the main goal of this section is to show the existence of the solution of the Monge-Ampère equation

$$\det(\nabla^2 h + hI_{n-1}) = h^{p-1} f \qquad \text{if } p > 1; \tag{9.108}$$

$$h^{1-p} \det(\nabla^2 h + hI_{n-1}) = f \qquad \text{if } p < 1 \tag{9.109}$$

in the sense of measure. As defined in Section 9.3, for a convex body $K \in \mathcal{K}_o^n$ and $p \in \mathbb{R}$, the L_p-surface area measure $S_{K,p}$ is the Borel measure

$$dS_{K,p} = h_K^{1-p}\, dS_K$$

on S^{n-1} where in the case when $p > 1$ and $o \in \partial K$, we assume that $S_K(\{h_K = 0\}) = 0$. In particular, if $p \in \mathbb{R}$, $K \in \mathcal{K}_o^n$ and $\lambda > 0$, then

$$S_{\lambda K,p} = \lambda^{n-p} \cdot S_{K,p}. \tag{9.110}$$

For a $p \in \mathbb{R}$ and finite non-trivial Borel μ on S^{n-1}, the L_p-Minkowski problem (cf. Section 9.3) asks whether there exists a convex body $K \in \mathcal{K}_o^n$ such that

$$\mu = S_{K,p}. \tag{9.111}$$

In this section, we prove the following sufficiency conditions concerning the L_p-Minkowski problem (9.111) for $p \geqslant 0$ using the variational method where the case $p > 1$ is due to Hug, Lutwak, Yang, Zhang [427] and Chou, Wang [201], and the case $p \in [0, 1)$ is due to Chen, Li, Zhu [194, 195].

Theorem 9.C.1. *Let μ be a finite Borel measure on S^{n-1}.*

$p > 0$, $p \neq 1, n$: *If μ is not concentrated on any closed hemisphere, then $\mu = S_{K,p}$ for convex body $K \in \mathcal{K}_o^n$ (where $S_K(\{h_K = 0\}) = 0$ if $p > 1$ and $o \in \partial K$).*

$p = 0$: *If $\mu(L \cap S^{n-1}) < \frac{i}{n} \cdot \mu(S^{n-1})$ for any linear i-space $L \subset \mathbb{R}^n$, $i = 1, \dots, n-1$, then $\mu = S_{K,0} = nV_K$ for a convex body $K \in \mathcal{K}_o^n$.*

Remarks.

- In particular, any non-trivial absolutely continuous measure on S^{n-1} is a cone volume measure.
- If $p = n$, what known is (cf. Hug, Lutwak, Yang, Zhang [427]) - and what our method yields, as well - that if μ is a finite Borel measure on S^{n-1} not concentrated on any closed hemisphere, then $\mu = \lambda S_{K,p}$ for a convex body $K \in \mathcal{K}_o^n$ and $\lambda > 0$. In this case, rescaling does not help (cf. (9.110)).
- Here we do not handle the case $p \in (-n, 0)$ even if a variational argument does exist (cf. Bianchi, Böröczky, Colesanti, Yang [91]) because the diameter bound in terms of entropy in Proposition 9.A.2 works only for certain absolutely continuous measures, however, the proof of Theorem 9.C.1 uses discrete measures for weak approximation instead absolutely continuous ones.
- Let μ be a finite Borel measure on S^{n-1} not concentrated on any closed hemisphere. If $K_p \in \mathcal{K}_o^n$ satisfies $\mu = S_{K_p,p}$ for $p > n$, then Zou [740] proves that

$$\lim_{p \to \infty} K_p = \{x \in \mathbb{R}^n : \langle x, u \rangle \leqslant 1, \ \forall u \in \operatorname{supp} \mu\}.$$

  In particular, if $f \in L^1(S^{n-1})$ is non-negative and $\mathcal{H}^{n-1}(\{f = 0\}) = 0$, then the solution h_p of the L_p Monge-Ampère equation (9.108) on S^{n-1} for $p > 1$ tends uniformly to the constant 1 function as p tends to infinity.

The idea to prove Theorem 9.C.1 is to prove it first for discrete measures whose support is in general position (cf. Theorem 9.C.2), and for the more general measures in Theorem 9.C.1, to use weak approximation by discrete measures. The conditions in Theorem 9.C.1 ensure that the convex bodies that we use in the case of weak approximation are of bounded diameter (cf. Proposition 9.A.2).

Theorem 9.C.2 (The L_p-Minkowski problem for "general" discrete measures). *If $p \in \mathbb{R}$, $p \neq 1, n$, and μ is a discrete measure on S^{n-1} such that* supp μ *is not contained in a closed hemisphere, and any n vectors in* supp μ *are independent, then there exists a $Q \in \mathcal{K}_{(o)}^n$ such that $\mu = S_{Q,p}$.*

Remark. The measures in Theorem 9.C.2 are very special, but any finite measure can be weakly approximated by them, and we have a result for essentially all $p \in \mathbb{R}$. Theorem 9.C.2 is due to Hug, Lutwak, Yang, Zhang [427] if $p > 1$, and to Zhu [737, 738] if $p < 1$.

In order to handle discrete measures whose support is in general position for $p \in \mathbb{R}^n$, let $\mathcal{U}^n$ be the family of all finite subsets $U \subset S^{n-1}$ that are not contained in a closed hemisphere and any n elements of U are independent. For a $U \in \mathcal{U}^n$, let $\mathcal{P}(U) \subset \mathcal{K}_o^n$ be the family of n-dimensional polytopes whose facets' exterior unit normals are from U (possibly a proper subset). This idea of considering polytopes whose exterior unit normals are in general position is due to Zhu [738]. To prove Theorem 9.C.2, we borrow ideas from Hug, Lutwak, Yang, Zhang [427] if $p > 1$, from Zhu [737] if $0 < p < 1$, and from Zhu [738] if $p \leqslant 0$.

Lemma 9.C.3. *Let $U \in \mathcal{U}^n$.*

(i) There exists $D_U > 0$ such that $\operatorname{diam} Q \leqslant D_U |Q|^{\frac{1}{n}}$ *for $Q \in \mathcal{P}(U)$.*

(ii) Any sequence $Q_m \in \mathcal{P}(U)$ with $|Q_m| = 1$ has a convergent subsequence whose limit lies in $\mathcal{P}(U)$.

Proof. Let $\widetilde{D}_U$ be the maximum of the diameters of any simplices of the form $\{x \in \mathbb{R}^n : \langle x, u_i \rangle \leqslant 1,\ i = 1, \dots, n+1\}$ where $u_1, \dots, u_{n+1} \in U$ run through all $n+1$ element subsets of U that are not contained in any closed hemisphere.

For a $Q \in \mathcal{P}(U)$, let $z + rB^n \subset Q$, $r > 0$ be a ball of maximal radius contained in Q. As any n elements of U are independent, there exist $u_1, \dots, u_{n+1} \in U$ not contained in any closed hemisphere such that $\langle x - z, u_i \rangle \leqslant r$ holds for $x \in Q$ and $i = 1, \dots, n+1$, and hence $\operatorname{diam} Q \leqslant \widetilde{D}_U r$. We conclude (i), and in turn (ii) follows from the Blaschke Selection Theorem 1.7.4 and the continuity of the volume (cf. Lemma 1.7.5). ∎

The next auxiliary statement follows from the Aleksandrov Lemma Theorem 7.5.2 for the Wulff shapes. We note that for any $U \in \mathcal{U}^n$ and $Q \in \mathcal{P}(U)$, there exists $z_0 \neq o$ with $z_0 \in \operatorname{int} Q$, and hence there also exists a $u \in U$ such that $h_Q(u) > 0$.

Lemma 9.C.4. *For $U \in \mathcal{U}^n$, $Q \in \mathcal{P}(U)$, and a function $g : U \to \mathbb{R}$, we consider the n-polytope $Q_t = \{x \in \mathbb{R}^n : \langle x, u \rangle \leqslant h_Q(u) + t \cdot g(u),\ \forall u \in U\}$ if $|t|$ is small.*

(i) If $o \in \operatorname{int} Q$, then $Q_t \in \mathcal{P}(U)$ if $|t|$ is small, and

$$\lim_{t \to 0} \frac{|Q_t| - |Q|}{t} = \int_{S^{n-1}} g \, dS_Q. \tag{9.112}$$

(ii) If $o \in \partial Q$, and hence $U_0 = \{u \in U :\ h_Q(u) = 0\} \neq \varnothing$, and $g(u) \geqslant 0$ for $u \in U_0$, then $Q_t \in \mathcal{P}(U)$ if $t > 0$ is small, and

$$\lim_{t \to 0^+} \frac{|Q_t| - |Q|}{t} = \int_{S^{n-1}} g \, dS_Q. \tag{9.113}$$

Let μ be a Borel probability measure on S^{n-1}, and let $C \in \mathcal{K}_o$. The main tool in proving Theorem 9.C.2 is to consider an entropy function that is similar to the one

also used in Section 9.B. If $p > 1$, then the entropy function is

$$\overline{\mathcal{E}}_{\mu,p}(C) = \frac{1}{p}\log\int_{S^{n-1}} h_C^p \, d\mu - \frac{1}{n}\log|C| \tag{9.114}$$

of a convex body $C \in \mathcal{K}_o^n$, which is continuous in C and zero homogeneous; namely, if $\lambda > 0$, then

$$\overline{\mathcal{E}}_{\mu,p}(\lambda C) = \overline{\mathcal{E}}_{\mu,p}(C). \tag{9.115}$$

For $p < 1$ and $\xi \in \operatorname{int} C$, the entropy function is

$$\overline{\mathcal{E}}_{\mu,p}(C,\xi) = \begin{cases} \frac{1}{p}\log\int_{S^{n-1}} h_{C-\xi}^p \, d\mu - \frac{1}{n}\log|C| & \text{if} \quad p \neq 0 \\ \int_{S^{n-1}} \log h_{C-\xi} \, d\mu - \frac{1}{n}\log|C| & \text{if} \quad p = 0, \end{cases} \tag{9.116}$$

which is again zero homogeneous; namely, if $\lambda > 0$, then

$$\overline{\mathcal{E}}_{\mu,p}(\lambda C, \lambda\xi) = \overline{\mathcal{E}}_{\mu,p}(C,\xi) \tag{9.117}$$

(here we use that $\mu(S^{n-1}) = 1$ if $p = 0$).

We note that according to Lemma 9.C.6, if $p < 1$, and μ is a discrete probability measure on S^{n-1} such that $\operatorname{supp}\mu \in \mathcal{U}^n$, then for $Q \in \mathcal{P}(\operatorname{supp}\mu)$, there exists a unique $\xi_Q \in \operatorname{int} Q$ (that depends on p and μ, as well) such that

$$\overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q) = \max_{\xi\in\operatorname{int} Q} \overline{\mathcal{E}}_{\mu,p}(Q,\xi).$$

When proving Theorem 9.C.2, we will actually show that there exists a solution that has bounded entropy.

Proposition 9.C.5. *Let $p \in \mathbb{R}$, $p \neq 1, n$, and let μ be a discrete probability measure on S^{n-1} such that* $\operatorname{supp}\mu$ *is not contained in a closed hemisphere, and any n vectors in* $\operatorname{supp}\mu$ *are independent. Given a $\breve{Q} \in \mathcal{P}(\operatorname{supp}\mu)$ with $o \in \operatorname{int}\breve{Q}$, there exists $Q \in \mathcal{K}_{(o)}^n$ such that $\mu = S_{Q,p}$, we have $\xi_Q = o$ in the case $p < 1$, and*

$$\log\operatorname{diam}\breve{Q} - \frac{\log|\breve{Q}|}{n} \geqslant \begin{cases} \overline{\mathcal{E}}_{\mu,p}(Q) & \text{if} \quad p > 1, \ p \neq n, \\ \overline{\mathcal{E}}_{\mu,p}(Q,o) & \text{if} \quad p < 1; \end{cases} \tag{9.118}$$

moreover,

$$n|Q|^{\frac{n-p}{n}} = \begin{cases} \exp\left(p\cdot\overline{\mathcal{E}}_{\mu,p}(Q)\right) & \text{if} \quad p > 1, \ p \neq n, \\ \exp\left(p\cdot\overline{\mathcal{E}}_{\mu,p}(Q,o)\right) & \text{if} \quad p < 1, \ p \neq 0, \\ \overline{\mathcal{E}}_{\mu,0}(Q,o) & \text{if} \quad p = 0. \end{cases} \tag{9.119}$$

For any probability measure μ as in Theorem 9.C.1, the idea, due to Hug, Lutwak, Yang, Zhang [427] if $p > 1$, and Chen, Li, Zhu [194, 195] if $p \in (0,1)$ and $p = 0$, is

using weak convergence as follows. We approximate the measure μ in Theorem 9.C.1 by a sequence $\{\mu_m\}$ of discrete probability measures that also satisfy the same conditions as μ and whose support is in general position. For each μ_m, let $Q_m \in \mathcal{K}^n_{(0)}$ be a solution of the L_p-Minkowski problem provided by Proposition 9.C.5 such that the entropy $\overline{\mathcal{E}}_{\mu_m,p}(Q_m, \sigma_{Q_m})$ stays bounded. We deduce via Proposition 9.A.2 that the sequence $\{Q_m\}$ is bounded and $|Q_m|$ stays bounded away from zero, and hence a subsequence of $\{Q_m\}$ tends to a convex body K with $S_{K,p} = \mu$.

9.C.1 Proof of Theorem 9.C.2 and Proposition 9.C.5 if $p > 1$

First we prove Theorem 9.C.2 when $p > 1$ and $p \neq n$ because the argument is much easier in that case.

Proof of Theorem 9.C.2 when $p > 1$. Let $p > 1$, and let μ be a discrete measure on S^{n-1} with $\operatorname{supp}\mu \in \mathcal{U}^n$. We deduce from Lemma 9.C.3 that there exists a $\widetilde{Q} \in \mathcal{P}(\operatorname{supp}\mu)$ such that $|\widetilde{Q}| = 1$ and (cf. (9.115))

$$\begin{aligned}\overline{\mathcal{E}}_{\mu,p}(\widetilde{Q}) &= \min\left\{\overline{\mathcal{E}}_{\mu,p}(Q) : Q \in \mathcal{P}(\operatorname{supp}\mu) \text{ and } |Q| = 1\right\}\\ &= \min\left\{\overline{\mathcal{E}}_{\mu,p}(Q) : Q \in \mathcal{P}(\operatorname{supp}\mu)\right\}. \end{aligned} \tag{9.120}$$

Step 1. We claim that

$$o \in \operatorname{int}\widetilde{Q}. \tag{9.121}$$

Otherwise, there exists $u_0 \in \operatorname{supp} S_{\widetilde{Q}}$ such that $h_{\widetilde{Q}}(u_0) = 0$, and we seek a contradiction. Let $U_0 \subset U$ be the set of all $u \in \operatorname{supp}\mu$ such that $h_{\widetilde{Q}}(u) = 0$, and let

$$\widetilde{Q}_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_{\widetilde{Q}}(u) + t \cdot \tilde{g}(u),\ \forall u \in \operatorname{supp}\mu\}$$

where

$$\tilde{g}(u) = \begin{cases} 1 & \text{if} \quad u \in U_0,\\ 0 & \text{if} \quad u \in (\operatorname{supp}\mu)\backslash U_0. \end{cases}$$

As $u_0 \in U_0 \cap \operatorname{supp} S_{\widetilde{Q}}$, we deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t}\left|\widetilde{Q}_t\right|\Big|_{t=0+} > 0. \tag{9.122}$$

On the other hand, if $t \geqslant 0$, then $h_{\widetilde{Q}}(u) = 0 \leqslant h_{\widetilde{Q}_t}(u) \leqslant t$ for $u \in U_0$ and $h_{\widetilde{Q}}(u) = h_{\widetilde{Q}_t}(u)$ for $u \in (\operatorname{supp}\mu)\backslash U_0$, and hence there exists $\tilde{\theta} > 0$ depending on μ such that

$$\int_{S^{n-1}} h^p_{\widetilde{Q}}\, d\mu \leqslant \int_{S^{n-1}} h^p_{\widetilde{Q}_t}\, d\mu \leqslant \int_{S^{n-1}} h^p_{\widetilde{Q}}\, d\mu + \tilde{\theta}\cdot t^p.$$

Therefore, combining these estimates with $p > 1$, and using the definition (9.114) of entropy and (9.122) yield

$$\frac{\partial}{\partial t}\,\overline{\mathcal{E}}_{\mu,p}(\widetilde{Q}_t)\,\Big|_{t=0^+} = -\frac{1}{n}\cdot\frac{\partial}{\partial t}\,\Big|\widetilde{Q}_t\Big|\,\Big|_{t=0^+} < 0.$$

Since $\widetilde{Q}_t \in \mathcal{P}(\operatorname{supp}\mu)$ for small $t \geqslant 0$, and $\widetilde{Q}_0 = \widetilde{Q}$, the last estimate contradicts the minimality property (9.120), and in turn proves (9.121).

Step 2. We claim that

$$\operatorname{supp} S_{\widetilde{Q}} = \operatorname{supp}\mu. \tag{9.123}$$

Otherwise, there exists $\bar{u} \in (\operatorname{supp}\mu)\backslash(\operatorname{supp} S_{\widetilde{Q}})$, and let

$$\overline{Q}_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_{\widetilde{Q}}(u) + t\cdot\bar{g}(u),\ \forall u \in \operatorname{supp}\mu\}$$

where for $u \in \operatorname{supp}\mu$, we have

$$\bar{g}(u) = \begin{cases} -1 & \text{if} \quad u = \bar{u},\\ 0 & \text{if} \quad u \neq \bar{u}. \end{cases}$$

Here $o \in \operatorname{int}\widetilde{Q}$ by (9.121) in Step 1. As $\bar{u} \notin S_{\widetilde{Q}}$, we deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t}\,\Big|\overline{Q}_t\Big|\,\Big|_{t=0} = 0. \tag{9.124}$$

On the other hand, if $t > 0$ is small, then $h_{\overline{Q}_t}(\bar{u}) = h_{\widetilde{Q}}(\bar{u}) - t$ and $h_{\overline{Q}_t}(u) \leqslant h_{\widetilde{Q}}(u)$ hold for any $u \neq \bar{u}$, and hence

$$\frac{1}{p}\log\int_{S^{n-1}} h^p_{\overline{Q}_t}\,d\mu < \frac{1}{p}\log\int_{S^{n-1}} h^p_{\widetilde{Q}}\,d\mu - \bar{\theta}\cdot t$$

for a $\bar{\theta} > 0$ depending on $\widetilde{Q}$, $\bar{u}$ and μ. Therefore, combining this estimate with (9.124) yields

$$\overline{\mathcal{E}}_{\mu,p}(\overline{Q}_t) < \overline{\mathcal{E}}_{\mu,p}(\widetilde{Q})$$

for small $t > 0$. Since $\overline{Q}_t \in \mathcal{P}(\operatorname{supp}\mu)$ for small $t > 0$, the last estimate contradicts the minimality property (9.120), and in turn proves (9.123).

Step 3. We claim that

$$\mu = \frac{\lambda}{n}\cdot S_{\widetilde{Q},p} \quad \text{for} \quad \lambda = \int_{S^{n-1}} h^p_{\widetilde{Q}}\,d\mu. \tag{9.125}$$

Since $h_{\widetilde{Q}}(u) > 0$ for $u \in \operatorname{supp}\mu$ by (9.121) in Step 1 and (9.123) in Step 2, it is equivalent to saying that for any $g : \operatorname{supp}\mu \to \mathbb{R}$, we have

$$\int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1}\, d\mu = \frac{\lambda}{n} \cdot \int_{S^{n-1}} g\, dS_{\widetilde{Q}}. \tag{9.126}$$

Let us consider

$$Q_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_{\widetilde{Q}}(u) + g(u)\cdot t,\ \forall u \in \operatorname{supp}\mu\}.$$

We deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t}\, |Q_t| \,\Big|_{t=0} = \int_{S^{n-1}} g\, dS_{\widetilde{Q}}. \tag{9.127}$$

If $|t|$ is small, then $o \in \operatorname{int} Q_t$ by (9.121) in Step 1, and $h_{Q_t}(u) = h_{\widetilde{Q}}(u) + g(u)\cdot t$ for $u \in \operatorname{supp}\mu$ as $\operatorname{supp} S_{\widetilde{Q}} = \operatorname{supp}\mu$ by (9.123) in Step 2. In turn, we conclude that

$$\frac{\partial}{\partial t}\, \frac{1}{p} \log \int_{S^{n-1}} h_{Q_t}^p\, d\mu \,\Big|_{t=0} = \lambda^{-1} \int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1}\, d\mu.$$

Since $Q_t \in \mathcal{P}(\operatorname{supp}\mu)$ if t is small, and $Q_0 = \widetilde{Q}$, combining the last formula with (9.127) and the minimality property (9.120) yields that

$$0 = \frac{\partial}{\partial t}\, \overline{\mathcal{E}}_{\mu,p}(Q_t) \,\Big|_{t=0} = \lambda^{-1} \int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1}\, d\mu - \frac{1}{n} \int_{S^{n-1}} g\, dS_{\widetilde{Q}}.$$

We conclude (9.126), and in turn (9.125).

Finally, it follows from (9.125) and (9.110) that $\mu = S_{Q,p}$ for $Q = (\lambda/n)^{\frac{1}{n-p}} \widetilde{Q}$. ∎

Proof of Proposition 9.C.5 if $p > 1$. The proof of Theorem 9.C.2 if $p > 1$ above provides a $Q \in \mathcal{P}(\operatorname{supp}\mu)$ such that $\mu = S_{Q,p}$ and $\overline{\mathcal{E}}_{\mu,p}(Q) \leqslant \overline{\mathcal{E}}_{\mu,p}(\check{Q})$ for the $\check{Q}$ in Proposition 9.C.5. Since μ is a probability measure, and $h_{\check{Q}}(u) \leqslant \operatorname{diam}\check{Q}$ for any $u \in \operatorname{supp}\mu$, we have $\overline{\mathcal{E}}_{\mu,p}(Q) \leqslant \log\operatorname{diam}\check{Q} - \frac{1}{n}\log|\check{Q}|$, verifying (9.118).

To prove (9.119), $|\widetilde{Q}| = 1$ for the $\widetilde{Q}$ in (9.125) yields that $|Q| = (\lambda/n)^{\frac{n}{n-p}}$. ∎

9.C.2 Proof of Theorem 9.C.2 and Proposition 9.C.5 if $p < 1$

Let $p < 1$, and let μ be a discrete probability measure on S^{n-1} such that $\operatorname{supp}\mu \in \mathcal{U}^n$. We recall that if $Q \in \mathcal{P}(\operatorname{supp}\mu)$ and $\xi \in \operatorname{int} Q$, then the entropy function is

$$\overline{\mathcal{E}}_{\mu,p}(Q,\xi) = \begin{cases} \frac{1}{p}\log\int_{S^{n-1}} h_{Q-\xi}^p\, d\mu - \frac{1}{n}\log|Q| & \text{if} \quad p \neq 0 \\ \int_{S^{n-1}} \log h_{Q-\xi}\, d\mu - \frac{1}{n}\log|Q| & \text{if} \quad p = 0. \end{cases}$$

We note that if $u \in S^{n-1}$ and $\xi \in Q$, then

$$h_{Q-\xi}(u) = h_Q(u) - \langle \xi, u \rangle. \tag{9.128}$$

We deduce from (9.128) that $\overline{\mathcal{E}}_{\mu,p}(Q,\xi)$ is a continuous function of $Q \in \mathcal{P}(\mathrm{supp}\mu)$ and $\xi \in \mathrm{int}\, Q$, and for fixed $Q \in \mathcal{P}(\mathrm{supp}\mu)$, the function $\xi \mapsto \overline{\mathcal{E}}_{\mu,p}(Q,\xi)$ is C^1 on $\mathrm{int}\, Q$. Let us verify some useful properties of our entropy function.

Lemma 9.C.6. *Let $p < 1$, and let μ be a discrete probability measure on S^{n-1} such that $\mathrm{supp}\,\mu \in \mathcal{U}^n$.*

(i) For $Q \in \mathcal{P}(\mathrm{supp}\,\mu)$, the function $\xi \mapsto \overline{\mathcal{E}}_{\mu,p}(Q,\xi)$ over all $\xi \in \mathrm{int}\, Q$ attains its maximum at a unique $\xi_Q \in \mathrm{int}\, Q$ (ξ_Q naturally also depends on p and μ), and

$$\int_{S^{n-1}} u \cdot h_{Q-\xi_Q}(u)^{p-1}\, d\mu(u) = o. \tag{9.129}$$

(ii) The functions ξ_Q and $\overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q)$ of $Q \in \mathcal{P}(\mathrm{supp}\,\mu)$ are continuous.

(iii) If $g : \mathrm{supp}\,\mu \to \mathbb{R}$, $Q \in \mathcal{P}(\mathrm{supp}\,\mu)$ satisfies that $\mathrm{supp}\, S_Q = \mathrm{supp}\,\mu$, and $\mathrm{supp}\, S_{Q_t} = \mathrm{supp}\,\mu$ and $o \in Q_t$ hold for $t \in (-t_0, t_0)$, $t_0 > 0$, and

$$Q_t = \{x \in \mathbb{R}^n : \langle x, u \rangle \leqslant h_Q(u) + t\, g(u),\ \forall u \in \mathrm{supp}\,\mu\},$$

then ξ_{Q_t} is a differentiable function of t.

Proof. If $p \leqslant 0$, then the existence of $\xi_Q \in \mathrm{int}\, Q$ in (i) follows from the observations that the function $\xi \mapsto \log(h_Q(u) - \langle \xi, u \rangle)$ is strictly concave, and the function $(h_Q(u) - \langle \xi, u \rangle)^p$ is strictly convex for $\xi \in \mathrm{int}\, Q$ where $u \in \mathrm{supp}\,\mu$; moreover, if $\xi_m \in \mathrm{int}\, Q$ tends to a $\tilde{\xi} \in \partial Q$, then there exists a $u \in \mathrm{supp}\,\mu$ such that $\lim_{m\to\infty} h_Q(u) - \langle \xi_m, u \rangle = 0$, and hence $\lim_{m\to\infty} \overline{\mathcal{E}}_{\mu,p}(Q,\xi_m) = -\infty$.

To consider (i) if $p \in (0,1)$, we note that the function $\xi \mapsto (h_Q(u) - \langle \xi, u \rangle)^p$ is strictly concave even on Q for any $u \in \mathrm{supp}\,\mu$, therefore, the function $\xi \mapsto \overline{\mathcal{E}}_{\mu,p}(Q,\xi)$ attains its maximum at a unique $\xi_Q \in Q$.

We suppose that $\xi_Q \in \partial Q$, and seek a contradiction. We choose a $w \in S^{n-1}$ and $t_0 > 0$ such that $\xi(t) = \xi_Q - tw \in \mathrm{int}\, Q$ if $t \in (0, t_0)$, and $\varrho > 0$ such that $h_{Q-\xi_Q}(u) > \varrho$ if $u \in \mathrm{supp}\,\mu$ and $h_{Q-\xi_Q}(u) > 0$. For the non-empty subset $U_0 \subset \mathrm{supp}\,\mu$ such that $h_{Q-\xi_Q}(u) = 0$ for $u \in U_0$ and $\theta = \min_{u\in U_0} |\langle u, w \rangle| > 0$, we have

$$\begin{array}{lll} h_{Q-\xi(t)}(u)^p \ \geqslant\ \theta^p t^p = h_{Q-\xi_Q}(u)^p + \theta^p t^p & \text{if} & u \in U_0, \\ h_{Q-\xi(t)}(u)^p \ \leqslant\ h_{Q-\xi_Q}(u)^p + p\varrho^{p-1} \cdot t & \text{if} & u \in (\mathrm{supp}\,\mu)\backslash U_0. \end{array}$$

We deduce from $p \in (0,1)$ that $\overline{\mathcal{E}}_{\mu,p}(Q,\xi(t)) < \overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q)$ for small $t > 0$, which is a contradiction proving that $\xi_Q \in \mathrm{int}\, Q$ also if $p \in (0,1)$.

Now (9.129) follows from the maximality property of $\xi_Q \in \mathrm{int}\, Q$, and (ii) follows from the uniqueness of ξ_Q.

For (iii), (9.129) and (ii) yield that for $\xi(t) = \xi_{Q_t}$ and

$$F(t,\xi) = \int_{S^{n-1}} u \cdot (h_{Q_t}(u) - \langle \xi, u\rangle)^{p-1}\, d\mu(u),$$

there exists $\varrho > 0$ such $F(t,\xi)$ is C^1 on the set $|t| < \varrho$ and $\|\xi - \xi_Q\| < \varrho$, $\xi(t)$ is a continuous function on $(-\varrho, \varrho)$, and $\xi(t)$ is the solution of the functional equation $F(t,\xi(t)) = o$. Since the derivative $D_\xi F$ of F with respect to ξ at $t = 0$ and $\xi = \xi_Q \in \operatorname{int} Q$ is

$$D_\xi F(0,\xi_Q) = (p-1)\int_{S^{n-1}} (u \otimes u) \cdot (h_Q(u) - \langle \xi_Q, u\rangle)^{p-2}\, d\mu(u)$$

(here $u \otimes u = u\,u^t$), which is negative definite as $u \otimes u$ is positive semi-definite for any $u \in \operatorname{supp}\mu$, and $\operatorname{supp}\mu$ contains n independent vectors. Therefore, the implicit function theorem yields (iii). ∎

Proof of Theorem 9.C.2 *if* $p < 1$. Let $p < 1$, and let μ be a discrete measure on S^{n-1} with $\operatorname{supp}\mu \in \mathcal{U}^n$.

We deduce from the diameter bound Lemma 9.C.3 and the continuity of $\mathcal{E}_{\mu,p}(Q,\xi_Q)$ and ξ_Q (cf. Lemma 9.C.6) that there exists a $\widetilde{Q} \in \mathcal{P}(\operatorname{supp}\mu)$ such that $|\widetilde{Q}| = 1$ and (cf. (9.117) for (9.130))

$$\begin{aligned}\overline{\mathcal{E}}_{\mu,p}(\widetilde{Q},\xi_{\widetilde{Q}}) &= \min\left\{\overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q) : Q \in \mathcal{P}(\operatorname{supp}\mu) \text{ and } |Q| = 1\right\}\\ &= \min\left\{\overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q) : Q \in \mathcal{P}(\operatorname{supp}\mu)\right\}. \end{aligned} \tag{9.130}$$

In the estimates below, we also use the expression

$$\widetilde{\mathcal{E}}_{\mu,p}(Q,\xi) = \begin{cases} \frac{1}{p}\log\int_{S^{n-1}} h^p_{Q-\xi}\, d\mu & \text{if} \quad p \neq 0\\ \int_{S^{n-1}} \log h_{Q-\xi}\, d\mu & \text{if} \quad p = 0 \end{cases}$$

for $Q \in \mathcal{P}(\operatorname{supp}\mu)$ and $\xi \in \operatorname{int} Q$, which then satisfies that if $\xi \in \operatorname{int} Q$, then

$$\widetilde{\mathcal{E}}_{\mu,p}(Q,\xi) \leqslant \widetilde{\mathcal{E}}_{\mu,p}(Q,\xi_Q). \tag{9.131}$$

Step 1. We claim that

$$\operatorname{supp} S_{\widetilde{Q}} = \operatorname{supp}\mu. \tag{9.132}$$

Otherwise, there exists $\bar{u} \in (\operatorname{supp}\mu)\backslash\operatorname{supp} S_{\widetilde{Q}}$, and let

$$\overline{Q}_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leqslant h_{\widetilde{Q}}(u) + \bar{g}(u)\cdot t,\ \forall u \in \operatorname{supp}\mu\}$$

where if $u \in \operatorname{supp}\mu$

$$\bar{g}(u) = \begin{cases} -1 & \text{if} \quad u = \bar{u}\\ 0 & \text{if} \quad u \neq \bar{u}. \end{cases}$$

As $\bar{u} \notin S_{\widetilde{Q}}$, we deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t} \left| \overline{Q}_t \right| \Big|_{t=0} = 0. \tag{9.133}$$

On the other hand, let $\xi(t) = \xi_{\overline{Q}_t}$. We deduce from the continuity of $\xi(t)$ (cf. Lemma 9.C.6) that there exists $\varrho, t_0 > 0$ such that if $t \in (0, t_0)$, then $\xi(t) \in \xi_{\widetilde{Q}} + \varrho B^n, \xi_{\widetilde{Q}} + 2\varrho B^n \subset \overline{Q}_t$ and $\overline{Q}_t \in \mathcal{P}(\operatorname{supp}\mu)$. Since $h_{\overline{Q}_t - \xi(t)}(\bar{u}) = h_{\widetilde{Q} - \xi(t)}(\bar{u}) - t$ and $h_{\overline{Q}_t - \xi(t)}(u) \leqslant h_{\widetilde{Q} - \xi(t)}(u)$ for $u \neq \bar{u}$ for small $t > 0$, it follows from (9.131) that

$$\widetilde{\mathcal{E}}_{\mu,p}(\overline{Q}_t, \xi(t)) < \widetilde{\mathcal{E}}_{\mu,p}(\widetilde{Q}, \xi(t)) - \bar{\theta} \cdot t \leqslant \widetilde{\mathcal{E}}_{\mu,p}(\widetilde{Q}, \xi_{\widetilde{Q}}) - \bar{\theta} \cdot t$$

for some $\bar{\theta} > 0$ depending on μ, $\widetilde{Q}$, $\bar{u}$ and ϱ. Therefore, combining this estimate with (9.133) yields

$$\overline{\mathcal{E}}_{\mu,p}(\overline{Q}_t, \xi(t)) < \overline{\mathcal{E}}_{\mu,p}(\widetilde{Q}, \xi_{\widetilde{Q}})$$

for small $t > 0$. This last estimate contradicts the minimality property (9.130), and in turn proves (9.132).

Step 2. Assuming that $\xi_{\widetilde{Q}} = o$, we claim that

$$\mu = \frac{\lambda}{n} \cdot S_{\widetilde{Q},p} \quad \text{for} \quad \lambda = \int_{S^{n-1}} h_{\widetilde{Q}}^p \, d\mu. \tag{9.134}$$

Since $h_{\widetilde{Q}}(u) > 0$ for $u \in \operatorname{supp}\mu$ by $o = \xi_{\widetilde{Q}} \in \operatorname{int}\widetilde{Q}$, it is equivalent to saying that for any $g : \operatorname{supp}\mu \to \mathbb{R}$, we have

$$\int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1} \, d\mu = \frac{\lambda}{n} \cdot \int_{S^{n-1}} g \, dS_{\widetilde{Q}}. \tag{9.135}$$

If $|t|$ is small, then let us consider

$$Q_t = \{x \in \mathbb{R}^n : \langle x, u \rangle \leqslant h_{\widetilde{Q}}(u) + g(u) \cdot t, \ \forall u \in \operatorname{supp}\mu\},$$

and let $\xi(t) = \xi_{\overline{Q}_t}$. We deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t} \, |Q_t| \Big|_{t=0} = \int_{S^{n-1}} g \, dS_{\widetilde{Q}}. \tag{9.136}$$

We note that if $|t|$ is small, then $\operatorname{supp}\mu = \operatorname{supp} S_{\widetilde{Q}}$ (cf. (9.132)) yields that $\operatorname{supp}\mu = \operatorname{supp} S_{Q_t}$, and hence

$$\widetilde{\mathcal{E}}_{\mu,p}(Q_t, \xi(t)) = \frac{1}{p} \cdot \log \int_{S^{n-1}} \Big(h_{\widetilde{Q}}(u) + g(u) \cdot t - \langle \xi(t), u \rangle \Big)^p \, d\mu(u).$$

It follows from the differentiability of $\xi(t)$ (cf. Lemma 9.C.6) and $\xi(o) = o$ that

$$\frac{\partial}{\partial t}\,\widetilde{\mathcal{E}}_{\mu,p}(Q_t,\xi(t))\,\Big|_{t=0} = \lambda^{-1}\cdot\int_{S^{n-1}} h_{\widetilde{Q}}^{p-1}(u)(g(u)-\langle\xi'(o),u\rangle)\,d\mu(u).$$

Here $\int_{S^{n-1}} h_{\widetilde{Q}}^{p-1}(u)\cdot\langle\xi'(o),u\rangle\,d\mu(u) = 0$ by (9.129); therefore,

$$\frac{\partial}{\partial t}\,\widetilde{\mathcal{E}}_{\mu,p}(Q_t,\xi(t))\,\Big|_{t=0} = \lambda^{-1}\cdot\int_{S^{n-1}} h_{\widetilde{Q}}^{p-1}\cdot g\,d\mu.$$

Combining the last formula with (9.136) and the minimality property (9.130) yields that

$$0 = \frac{\partial}{\partial t}\,\overline{\mathcal{E}}_{\mu,p}(Q_t)\,\Big|_{t=0} = \lambda^{-1}\int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1}\,d\mu - \frac{1}{n}\int_{S^{n-1}} g\,dS_{\widetilde{Q}}.$$

We conclude (9.135), and in turn (9.134).

Finally, to prove Theorem 9.C.2 if $p<1$, we may assume that $\mu(S^{n-1}) = 1$ by the homogeneity (9.110) of $S_{K,p}$, and we also assume that $\xi_{\widetilde{Q}} = o$ as in Step 2. It follows from (9.134) and (9.110) that $\mu = S_{Q,p}$ for $Q = (\lambda/n)^{\frac{1}{n-p}}\widetilde{Q}$. ∎

Proof of Proposition 9.C.5 *if* $p<1$. The proof of Theorem 9.C.2 if $p<1$ above provides a $Q\in\mathcal{P}(\operatorname{supp}\mu)$ such that $\mu = S_{Q,p}$, $\xi_Q = o$ and $\overline{\mathcal{E}}_{\mu,p}(Q,o)\leqslant\overline{\mathcal{E}}_{\mu,p}(\check{Q},\xi_{\check{Q}})$. Since $h_{\check{Q}-\xi_{\check{Q}}}(u)\leqslant\operatorname{diam}\check{Q}$ for any $u\in\operatorname{supp}\mu$, and μ is a probability measure, we have

$$\overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q)\leqslant\overline{\mathcal{E}}_{\mu,p}(\check{Q},\xi_{\check{Q}})\leqslant\log\operatorname{diam}\check{Q}-\frac{1}{n}\log|\check{Q}|,$$

verifying (9.118). To prove (9.119), $|\widetilde{Q}| = 1$ for the $\widetilde{Q}$ in (9.134) yields that $|Q| = (\lambda/n)^{\frac{n}{n-p}}$. ∎

9.C.3 Proof of Theorem 9.C.1 for $p\geqslant 0$, $p\neq 1, n$

For $p\geqslant 0$, $p\neq 1,n$, we may assume that the finite Borel measure μ on S^{n-1} satisfying the conditions in Theorem 9.C.1 is a probability measure (cf. (9.110)). In particular, the measure of any open hemisphere is positive, and if $p=0$, then μ also satisfies the condition $\mu(L\cap S^{n-1})<\frac{i}{n}$ for any linear i-space $L\subset\mathbb{R}^n$, $i=1,\dots,n-1$. Using the compactness of the space of the i-dimensional linear subspaces of $\mathbb{R}^n$, we deduce the existence of $\delta,\tau\in(0,\frac{1}{2})$ such that if $p>0$, then

$$\mu\left(\Omega(u,2\delta)\right)>2\tau \tag{9.137}$$

for any $u \in S^{n-1}$ where $\Omega(u,\delta) = \{v \in S^{n-1} : \langle v,u\rangle > \delta\}$, and if $p = 0$ and $L \subset \mathbb{R}^n$ is any non-trivial linear subspace, then

$$\mu\left(\Psi(L \cap S^{n-1}, 2\delta)\right) < \frac{(1-2\tau)i}{n} \tag{9.138}$$

where $\Psi(L \cap S^{n-1},\delta) = \{x \in S^{n-1} : \langle x,y\rangle \leqslant \delta \text{ for } y \in L^{\perp} \cap S^{n-1}\}$.

We approximate weakly the measure μ by a sequence $\{\mu_m\}$ of discrete probability measures. For large integer m, let Ξ_m be a finite $\frac{1}{m}$-net on S^{n-1} such that any n points of Ξ_m are independent as vectors of $\mathbb{R}^n$, and hence for any $u \in S^{n-1}$, there exists $v \in \Xi_m$ such that $\|u-v\| \leqslant \frac{1}{m}$. We write $N(m)$ to denote the cardinality of Ξ_m, and for any $v \in \Xi_m$, let

$$\widetilde{\mathcal{D}}_{v,m} = \left\{u \in S^{n-1} : \|u-v\| \leqslant \|u-w\| \text{ for any } w \in \Xi_m\right\}$$

be the Dirichlet-Voronoi cell of v that is a closed spherically convex set of diameter at most $\frac{2}{m}$. Enumerating the elements of Ξ_m as v_i, $i = 1,\dots,N(m)$, let $\mathcal{D}_{v_1,m} = \widetilde{\mathcal{D}}_{v_1,m}$, and for $i \geqslant 2$, let

$$\mathcal{D}_{v_i,m} = \widetilde{\mathcal{D}}_{v_i,m} \setminus \bigcup_{j=1}^{i-1} \widetilde{\mathcal{D}}_{v_j,m},$$

and hence $\mathcal{D}_{v,m}$, $v \in \Xi_m$, are Borel measurable pairwise disjoint subsets of S^{n-1} of diameter at most $\frac{2}{m}$, form a partition of S^{n-1}, and $v \in \mathcal{D}_{v,m}$ for $v \in \Xi_m$.

Now for large m, let μ_m be the probability discrete measure on S^{n-1} such that $\operatorname{supp}\mu_m = \Xi_m$, and for $v \in \Xi_m$,

$$\mu_m(v) = \left(1 - \frac{1}{m}\right)\mu\left(\mathcal{D}_{v,m}\right) + \frac{1}{m \cdot N(m)}.$$

It follows that $\{\mu_m\}$ tends weakly to μ, and as the diameter of $\mathcal{D}_{v,m}$ for $v \in \Xi_m$ is at most $\frac{2}{m}$, we deduce that if $p > 0$, then

$$\mu_m\left(\Omega(u,\delta)\right) > \tau \tag{9.139}$$

for any $u \in S^{n-1}$, and if $p = 0$ and $L \subset \mathbb{R}^n$ is any non-trivial linear subspace, then

$$\mu_m\left(\Psi(L \cap S^{n-1},\delta)\right) < \frac{(1-\tau)i}{n}. \tag{9.140}$$

In order to apply Proposition 9.C.5 to μ_m, we take $\check{Q}_m$ to be the polytope whose facets touch B^n in the points of Ξ_m; therefore,

$$\operatorname{diam}\check{Q}_m < 3 \ \text{ and } \ |\check{Q}_m| > \omega_n \tag{9.141}$$

hold for large m. Applying Proposition 9.C.5 to μ_m, we have $\mu_m = S_{\theta_m Q_m,p}$ where $\theta_m > 0$ and $Q_m \in \mathcal{K}^n_{(o)}$ is a polytope with $|Q_m| = 1$ and $\xi_{Q_m} = o$, and $\theta_m Q_m$ satisfies

(9.118) and (9.119). We claim that there exists $\eta_1 > 1$ depending on p, n, δ, τ such that

$$\begin{array}{rcccccl} \eta_1^{-1} & \leqslant & \overline{\mathcal{E}}_{\mu,p}(Q_m) & \leqslant & \eta_1 & & \text{if } p > 1, \\ \eta_1^{-1} & \leqslant & \overline{\mathcal{E}}_{\mu,p}(Q_m, \sigma_{Q_m}) & \leqslant & \overline{\mathcal{E}}_{\mu,p}(Q_m, o) & \leqslant & \eta_1 \quad \text{if } p \in [0,1). \end{array} \tag{9.142}$$

The upper bound in (9.142) directly follows from (9.141) and from (9.118) in Proposition 9.C.5. For the lower bound in (9.142), we note that the two notions of entropy in Proposition 9.A.2 and in Proposition 9.C.5 are equivalent for our purposes; for example, $\overline{\mathcal{E}}_{\mu,p}(Q_m) = \frac{1}{p} \log p\mathcal{E}_{\mu,p}(Q_m)$ if $p > 1$. Since $\operatorname{diam} Q_m$ is at least the diameter of a ball in $\mathbb{R}^n$ of volume one according to the Isodiametric Inequality Theorem 1.10.5, the lower bound in (9.142) follows from Proposition 9.A.2. In turn, (9.142), $|\theta_m Q_m| = \theta_m^n$ and (9.119) in Proposition 9.C.5 yield that there exists $\eta_2 > 1$ depending on p, n, δ, τ such that

$$\eta_2^{-1} \leqslant \theta_m \leqslant \eta_2. \tag{9.143}$$

We also deduce from (9.142) and Proposition 9.A.2 that there exists $\eta_3 > 1$ depending on p, n, δ, τ such that

$$\operatorname{diam} Q_m \leqslant \eta_3. \tag{9.144}$$

Since $|\theta_m Q_m| \geqslant \eta_2^{-n}$ and $\operatorname{diam}(\theta_m Q_m) \leqslant \eta_2\eta_3$, a subsequence of $\{\theta_m Q_m\}$ tends to a convex body $K \in \mathcal{K}_o^n$, and the weak continuity of the L_p surface area measure (cf. Lemma 9.A.1) yields that $S_{K,p} = \mu$.

Chapter 10

Appendix: Background from Analysis and Algebra

10.1 Weak convergence, Regular measures and the Lebesgue measure

In this section, first, we consider general measures, and then we focus on the Lebesgue measure. We need the following properties of the notion of weak convergence of measures.

Remark 10.1.1 (Weak convergence of measures). For a compact metric space X, let μ_m and μ be finite Borel measures on X. Then the following properties are equivalent, and characterize the weak convergence of $\{\mu_m\}$ to μ.

- $\lim_{m\to\infty} \int_X g \, d\mu_m = \int_X g \, d\mu$ for any continuous function $g : S^{n-1} \to \mathbb{R}$.
- $\liminf_{m\to\infty} \mu_m(U) \geqslant \mu(U)$ for any open $U \subset X$.
- $\limsup_{m\to\infty} \mu_m(C) \leqslant \mu(C)$ for any closed $C \subset X$.

Next we discuss the notion of a regular or a Radon measure.

Definition 10.1.2 (Regularity of Borel measures). A Borel measure μ on a topological space X is regular or a Radon measure if for any measurable $A \subset X$, we have

- $\mu(A) = \inf\{\mu(U) : U \supset A \text{ open}\}$;
- $\mu(A) = \sup\{\mu(C) : C \subset A \text{ compact}\}$;
- μ is finite on compact subsets.

For such a measures, the term *regular Borel measure* appears for example in Rudin [640], and the term *Radon measure* occurs in Ambrosio, Fusco, Pallara [25] and Maggi [557]. For properties of regular measures, see for example Rudin [640], especially Theorem 2.18.

Theorem 10.1.3 (Regularity of some Borel measures). *If X is a locally compact Hausdorff space where every open set is the union of countable many compact subsets, then any Borel measure μ on X that is finite on compact subsets is regular.*

Remark. In particular, the Lebesgue measure on $\mathbb{R}^n$, or any finite Borel measure on S^n are regular.

A popular version of Theorem 10.1.3 is that any finite Borel measure on a separable complete metric space (Polish space) is regular (see Aliprantis, Border [17], Theorem 12.7).

Given a topological space X and a Borel measure μ on X, as it is common, we write $L^1(X,\mu)$ to denote the Banach space of μ-measurable functions $f : X \to \mathbb{R}$ such that $\|f\|_1 = \int_X |f|\, d\mu < \infty$.

Theorem 10.1.4 (Lebesgue's Dominated Convergence theorem). *Given a topological space X and a Borel measure μ on X, if $f, g, f_m : X \to \mathbb{R}$, $m \geqslant 1$, are μ-measurable functions, $|f_m(x)| \leqslant g(x)$ and $\lim_m f_m(x) = f(x)$ for μ a.e. $x \in X$, and $g \in L^1(X,\mu)$, then $f_m, f \in L^1(X,\mu)$ and $\lim_m \int_X f_m\, d\mu = \int_X f\, d\mu$.*

One of the cornerstones of analysis are the versions of the Riesz Representation Theorem; namely, various types of well behaved linear functionals can be represented by integrals with respect to certain measures. For a locally compact Hausdorff space X, we write $C_c(X)$ to denote the family of continuous functions with compact support, and we call a linear operator $L : C_c(X) \to \mathbb{R}$ positive if $L(f) \geqslant 0$ whenever $f \geqslant 0$.

The first relevant version of Riesz' theorem is in Cohn [205].

Theorem 10.1.5 (Riesz Representation Theorem for Positive functionals). *If X is a locally compact Hausdorff space, and L is a positive linear functional on $C_c(X)$, then there exists a unique regular Borel measure (i.e. Radon measure) on X such that*

$$L(f) = \int_X f\, d\mu$$

holds for every $f \in C_c(X)$.

The second version of Riesz' theorem in Ambrosio, Fusco, Pallara [25] and Maggi [557] is about linear functionals on $C_c(\mathbb{R}^n;\mathbb{R}^n)$; namely, the space of vector valued continuous functions with compact support. For an ($\mathbb{R}$ valued) linear functional L on $C_c(\mathbb{R}^n;\mathbb{R}^n)$, the total variation $|L|$ is an outer measure on $\mathbb{R}^n$ such that if $A \subset \mathbb{R}^n$ is open, then

$$|L|(A) = \sup\{L(\varphi) : \varphi \in C_c(A;\mathbb{R}^n) \text{ and } \|\varphi\|_\infty \leqslant 1\},$$

and for any $E \subset \mathbb{R}^n$,

$$|L|(E) = \inf\{|L|(A) : A \subset \mathbb{R}^n \text{ open and } E \subset A\}.$$

Theorem 10.1.6 (Riesz Representation Theorem for Vector fields). *If L is a bounded linear functional on $C_c(\mathbb{R}^n;\mathbb{R}^n)$, then its total variation $|L|$ is a Radon measure on $\mathbb{R}^n$, and there exist vector valued measure μ on $\mathbb{R}^n$ and an $|L|$-measurable function $\nu : \mathbb{R}^n \to \mathbb{R}^n$ such that $\|\nu\| = 1$ holds $|L|$ a.e. on $\mathbb{R}^n$, $d\mu = \nu\, d|L|$ and*

$$L(\varphi) = \langle \varphi, d\mu \rangle = \int_{\mathbb{R}^n} \langle \nu, \varphi \rangle\, d|L|$$

holds for every $\varphi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$.

Remark. We frequently use the notation $|\mu| = |L|$ for the total variation measure.

Next, we turn to properties of the Lebesgue measure. In this book, an $X \subset \mathbb{R}^n$ is called measurable if it is Lebesgue measurable. Let us list some related notions:

- If $X \subset \mathbb{R}^n$ measurable, then $|X|$ is the Lebesgue measure of X where
$$|X| = \sup\{|Y| : Y \subset X \text{ compact}\} = \inf\{|Z| : Z \supset X \text{ open}\}.$$
- If $A \subset \mathbb{R}^n$, then $|A|^* = \text{outer measure} = \min\{|X| : X \supset A \text{ measurable}\}$.
- If $A \subset \mathbb{R}^n$, then $|A|_* = \text{inner measure} = \max\{|X| : X \subset A \text{ measurable}\}$.

Remark 10.1.7 (Volume of the Unit ball). For the Euclidean unit ball $B^n = \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}$ centered at the origin, we have

$$|B^n| = \omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \sim \left(\frac{2e\pi}{n}\right)^{\frac{n}{2}} (n\pi)^{-\frac{1}{2}} \tag{10.1}$$

as n tends to infinity. Here Γ is Euler's Gamma function satisfying $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$, $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, and the Stirling formula says that

$$(t/e)^t\sqrt{2\pi t} < \Gamma(t+1) < (t/e)^t\sqrt{2\pi(t+1)} \tag{10.2}$$

for $t \geqslant 1$ (cf. Artin [35]). For $n \geqslant 2$, we have

$$\sqrt{\frac{n}{2\pi}} < \frac{\omega_{n-1}}{\omega_n} < \sqrt{\frac{n+1}{2\pi}} \tag{10.3}$$

as $\frac{\omega_m}{\omega_{m-1}} = \int_{-1}^{1}(1-t^2)^{\frac{m-1}{2}}\,dt > \frac{\omega_{m+1}}{\omega_m}$ and $\frac{\omega_m}{\omega_{m-2}} = \frac{2\pi}{m}$ for $m \geqslant 2$.

We set $\omega_\alpha = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(\frac{\alpha}{2}+1\right)}$ for any $\alpha \geqslant 0$.

Remark 10.1.8 (Inner Density points). For $A \subset \mathbb{R}^n$, we define
$A_* = \left\{x \in \mathbb{R}^n : \lim_{r\to 0^+} \frac{|A\cap(x+rB^n)|_*}{|rB^n|} = 1\right\}$.

- A_* is Borel (as for a fixed $r > 0$, $x \mapsto |A \cap (x + rB^n)|_*$ is continuous);
- $|A_*| = |A|_*$;
- if A is measurable, then $|A_* \cap A| = |A|$.

10.2 Hölder's, Jensen's and Minkowski's inequalities and the L^p function spaces

For a topological space X, and non-trivial Borel measure μ on X, we write $L^1(X,\mu)$ the space of μ measurable real functions f such that $\int_X |f|\,d\mu < \infty$. We discuss the

probably three most basic inequalities in analysis (see, for example, Rudin [640]); namely, Hölder's, Jensen's and Minkowski's inequalities. The most fundamental is Hölder's inequality, which we state in both forms how we use it.

Theorem 10.2.1 (Hölder inequality I). *Let X be a topological space, and μ be a non-trivial Borel measure on X. If $p, q \geqslant 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ (where $p = 1$ if and only if $q = \infty$), then*

$$\int_X |fg|\, d\mu \leqslant \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q\, d\mu\right)^{\frac{1}{q}}. \tag{10.4}$$

Theorem 10.2.2 (Hölder inequality II). *Let X be a topological space, and μ be a non-trivial Borel measure on X. For non-negative $f_1, \ldots, f_k \in L^1(X, \mu)$ and $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$, $k \geqslant 2$, we have*

$$\int_X \left(\prod_{i=1}^k |f_i|^{\lambda_i}\right) d\mu \leqslant \prod_{i=1}^k \left(\int_X |f_i|\, d\mu\right)^{\lambda_i}. \tag{10.5}$$

Assuming that every $\int |f_i| > 0$, equality holds if and only if there exist $a_2, \ldots, a_k > 0$ such that $|f_i(x)| = a_i |f_1(x)|$ for μ a.e. $x \in X$, $i = 2, \ldots, k$.

Next, for the Jensen inequality about p means for $p \in \mathbb{R}$, let X be a topological space, and let μ be a probability Borel measure on X; namely, $\mu(X) = 1$. If $p > 0$ and $f \geqslant 0$ is μ measurable on X with $\int_X f^p\, d\mu < \infty$, then its p-mean is

$$M_\mu^p(f) = \left(\int_X f^p\, d\mu\right)^{\frac{1}{p}}.$$

In addition, let $p \in \mathbb{R}$, and let $f > 0$ be μ measurable on X with $\int_X f^p\, d\mu < \infty$ if $p \neq 0$, and $\int_X |\log f|\, d\mu < \infty$ if $p = 0$. In this case,

$$M_\mu^p(f) = \begin{cases} \left(\int_X f^p\, d\mu\right)^{\frac{1}{p}} & \text{if} \quad p \neq 0; \\ \exp\left(\int_X \log f\, d\mu\right) & \text{if} \quad p = 0. \end{cases}$$

We provide the simple proof of the Jensen inequality based on the Hölder inequality because the Jensen inequality exists in many forms in the literature, and here we state it in the forms we need in this book.

Theorem 10.2.3 (Jensen inequality). *For a topological space X, a Borel probability measure μ on X and $p, q \in \mathbb{R}$, if $f \geqslant 0$ is a μ measurable function on X such that*

(i) either $q > p > 0$ and $\int_X f^q\, d\mu < \infty$,

(ii) or $q > p$ and there exists $R > 3$ such that $R^{-1} < f < R$,

then

$$M_\mu^p(f) \leqslant M_\mu^q(f), \tag{10.6}$$

with equality if and only if there exists $\lambda \geqslant 0$ such that $f(x) = \lambda$ for μ a.e. $x \in X$. Moreover, if there exists $R > 3$ such that $R^{-1} < f < R$, then

$$\lim_{p\to 0} M_\mu^p(f) = M_\mu^0(f). \tag{10.7}$$

Remark. Strengthening (ii), if $p < q$ for $p, q \in \mathbb{R}$, and $\int_X f^q \, d\mu < \infty$ and $\int_X f^p \, d\mu < \infty$ hold for μ measurable $f : X \to (0, \infty)$, then

$$M_\mu^p(f) \leqslant M_\mu^q(f), \tag{10.8}$$

where (10.8) follows by (ii) and by approximating f by functions of the form $f_m = \min\left\{m, \max\{\frac{1}{m}, f\}\right\}$ for $m \geqslant 4$.

Proof. In the case of (i), we apply the Hölder inequality (10.5) with $f_1 = f^q$, $f_2 \equiv 1$ and $\lambda_1 = \frac{p}{q}$. In the case of (ii) and assuming $0 > q > p$, we apply the Hölder inequality (10.5) with $f_1 = f^p$, $f_2 \equiv 1$ and $\lambda_1 = \frac{q}{p}$.

Therefore, all we need is to prove (10.7). For this, we note that if $|t| \leqslant \frac{1}{2}$, then $e^t = 1 + t + O(t^2)$ and $\log(1 + t) = t + O(t^2)$, and hence

$$\begin{aligned}\lim_{p\to 0} \log\left(\int_X f^p \, d\mu\right)^{\frac{1}{p}} &= \lim_{p\to 0} \frac{1}{p} \log \int_X e^{p \log f} \, d\mu \\ &= \lim_{p\to 0} \frac{1}{p} \log \int_X 1 + p \log f + O\left(p^2 (\log R)^2\right) \, d\mu = \int_X \log f \, d\mu,\end{aligned}$$

verifying (10.7). ∎

We also need the Jensen inequality using a concave function to take the average.

Lemma 10.2.4 (Jensen inequality with concave functions). *For a topological space X, a Borel probability measure μ on X and a concave function φ on a real interval, if the range of an $f \in L^1(X, \mu)$ is contained in the domain of φ, then*

$$\int_X \varphi(f) \, d\mu \leqslant \varphi\left(\int_X f \, d\mu\right). \tag{10.9}$$

Proof. Let $t_0 = \int_X f \, d\mu$, which is in the domain of φ. As φ concave, there exists $a, b \in \mathbb{R}$ such that $\varphi(t) \leqslant a + bt$ and $\varphi(t_0) = a + bt_0$; therefore,

$$\int_X \varphi(f) \, d\mu \leqslant a + \int_X bf \, d\mu = a + bt_0 = \varphi(t_0).$$

∎

Finally, we prove the Minkowski inequality for integrals.

Theorem 10.2.5 (Minkowski inequality). *Let X be a topological space, and μ be a non-trivial Borel measure on X. If $p > 1$ and $|f|^p, |g|^p \in L^1(X, \mu)$, then*

$$\left(\int_X |f+g|^p \, d\mu\right)^{\frac{1}{p}} \leqslant \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \, d\mu\right)^{\frac{1}{p}}. \tag{10.10}$$

Proof. For the $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we deduce from the Hölder inequality (10.4) that

$$\begin{aligned}\int_X |f+g|^p \, d\mu &= \int_X |f+g| \cdot |f+g|^{p-1} \, d\mu \\ &\leqslant \int_X |f| \cdot |f+g|^{p-1} \, d\mu + \int_X |g| \cdot |f+g|^{p-1} \, d\mu \\ &\leqslant \left[\left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \, d\mu\right)^{\frac{1}{p}}\right] \cdot \left(\int_X |f+g|^{q(p-1)} \, d\mu\right)^{\frac{1}{q}}.\end{aligned}$$

We conclude (10.10) by $q(p-1) = p$ and $1 - \frac{1}{q} = \frac{1}{p}$. ∎

Definition 10.2.6 (L^p spaces). For $1 \leqslant p \leqslant \infty$, a topological space X, non-trivial Borel measure μ on X and μ measurable $f : X \to \mathbb{R}$, we have

$$f \in L^p(X) \text{ if and only if } \begin{cases} f \text{ is bounded} & \text{provided } p = \infty, \\ \int_X |f|^p \, d\mu < \infty & \text{provided } 1 \leqslant p < \infty. \end{cases}$$

Then $L^p(X, \mu)$ is a Banach space with the corresponding L_p-norm (cf. the Minkowski inequality (10.10))

$$\|f\|_p = \begin{cases} \sup |f| & \text{if } p = \infty, \\ \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} & \text{if } 1 \leqslant p < \infty. \end{cases}$$

For example, the Hölder inequality (10.4) says that if $\frac{1}{p} + \frac{1}{q} = 1$ for $p, q > 1$, $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$, then

$$\|f \cdot g\|_1 \leqslant \|f\|_p \cdot \|g\|_q.$$

10.3 Lebesgue Integral, Convolution, Fourier transform

In this section, we review the basic properties of integrals that we use in this book (see, for example, Grafakos [333]).

- For (Lebesgue) measurable $f : \mathbb{R}^n \to [0,\infty)$, $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f(x)\,dx$ is the *Lebesgue integral*. In order to simplify notation, we simply write $L^1(\mathbb{R}^n)$ if the corresponding measure is the Lebesgue measure.
- For $f : \mathbb{R}^n \to [0,\infty)$, the *outer Lebesgue integral* is

$$\int_{\mathbb{R}^n}^{*} f = \min\left\{ \int_{\mathbb{R}^n} g : g \geqslant f \text{ and } g \text{ measurable} \right\}.$$

- For $f : \mathbb{R}^n \to [0,\infty)$, the *inner Lebesgue integral* is

$$\int_{*,\mathbb{R}^n} f = \max\left\{ \int_{\mathbb{R}^n} g : g \leqslant f \text{ and } g \text{ measurable} \right\}.$$

  Actually there exists a measurable $\tilde{f}$ ("a witness") such that $\tilde{f} \leqslant f$ and the set $\{f > \tilde{f}\}$ contains no subset of positive Lebesgue measure; therefore, $\int_{*,\mathbb{R}^n} f = \int_{\mathbb{R}^n} \tilde{f}$.
- $f \in L^1(\mathbb{R}^n)$ if and only if f is measurable and $\int_{\mathbb{R}^n} |f| < \infty$.

Convolutions are tools for us to make functions smoother.

Definition 10.3.1 (Convolution). For $f, g \in L^1(\mathbb{R}^n)$, their convolution is $f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$.

We sketch the proof of the statement about the differentiability of convolutions.

Lemma 10.3.2. *Let $f, g \in L^1(\mathbb{R}^n)$.*

(i) $f * g \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f * g = \left(\int_{\mathbb{R}^n} f\right)\left(\int_{\mathbb{R}^n} g\right)$.

(ii) If either g is C^k for $k \geqslant 1$ and $\operatorname{supp} g$ compact, or g is C^{k+1} and each partial derivative of g of order at most $k+1$ is bounded, then $f * g \in C^k(\mathbb{R}^n)$.

Proof. (i) is just consequence of the Fubini Theorem, and (ii) follows from the fact that if g is C^1 and $\|Dg(y) - Dg(x)\|$ stays bounded under the condition $\|y - x\| \leqslant 1$, then Lebesgue's Dominated Convergence Theorem 10.1.4 and Taylor's formula imply that $f * g \in C^1(\mathbb{R}^n)$. ∎

Lemma 10.3.3 (Approximate Identity). *Let $k_\varepsilon(x) = \varepsilon^{-n} k(\frac{x}{\varepsilon})$ for $\varepsilon \in (0,1)$ where $k : \mathbb{R}^n \to [0,\infty)$ is a C^∞ function satisfying that $\int_{\mathbb{R}^n} k = 1$, moreover, k and all of its partial derivatives are bounded.*

If $f \in L^1(\mathbb{R}^n)$, then the C^∞ function $k_\varepsilon * f$ (cf. Lemma 10.3.2) satisfies

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} |f - k_\varepsilon * f| = 0,$$

and $\lim_{\varepsilon \to 0^+} k_\varepsilon * f(x) = f(x)$ for any density point x of f.

Remark. If $\operatorname{supp} k \subset B^n$ and $U_0 \subset U \subset U_1$ open with $\mathrm{cl}U_0 \subset U$ and $\mathrm{cl}U \subset U_1$, then $\mathbf{1}_{U_0} \leqslant k_\varepsilon * \mathbf{1}_U \leqslant \mathbf{1}_{U_1}$ for small $\varepsilon > 0$.

Let us list the main examples for approximate identity in this book.

Example 10.3.4 (Approximate Identity).

- $k(x) = e^{-\pi\|x\|}$.
- k is C^∞ with $\operatorname{supp} k \subset B^n$, for example, $k(x) = \gamma\varphi(1 - \|x\|^2)$ where $\gamma > 0$ constant, $\varphi(t) = 0$ if $t \leqslant 0$, and $\varphi(t) = e^{\frac{-1}{t^2}}$ if $t > 0$.

Fourier transform links the product and convolution of functions.

Definition 10.3.5 (Fourier transform). If $h \in L^1(\mathbb{R}^n)$, then

$$\hat{h}(z) = \int_{\mathbb{R}^n} h(x) e^{-2\pi i\langle x,z\rangle}\, dx$$

is the Fourier transform of h.

Remark. If $\psi(x) = e^{-\pi\|x\|^2}$, then $\hat{\psi} = \psi$.

Lemma 10.3.6. *Let $f, g \in L^1(\mathbb{R}^n)$.*

- $\widehat{f * g} = \hat{f} \cdot \hat{g}$.
- *If $\hat{f} = \hat{g}$, then $f(x) = g(x)$ for a.e. $x \in \mathbb{R}^n$.*

10.4 Hausdorff measure and Lipschitz functions

In this section, we discuss some fundamental properties of Lipschitz functions and the Hausdorff measure based on the monographs Falconer [254] and Federer [260].

We recall that for a metric space (Ξ_1, d_1) and (Ξ_2, d_2), a function $f : \Xi_1 \to \Xi_2$ is Lipschitz, if there exists $L > 0$ such that

$$d_2(f(x), f(y)) \leqslant L \cdot d_1(x, y)$$

for any $x, y \in \Xi_1$. In addition, f is locally Lipschitz if the restriction $f|_X$ is Lipschiz for any compact $X \subset \Xi_1$. One of the basic properties of Lipschitz functions is that they can be extended to a larger space in many situations (see Gutev [369]). In this book, we only need the special case of real valued Lipschitz functions.

Theorem 10.4.1 (McShane [572]). *For a metric space (Ξ, d) and non-empty $S \subset \Xi$, if $f : S \to \mathbb{R}$ and $L > 0$ satisfy that $\|f(x) - f(y)\| \leqslant L \cdot d(x, y)$ for any $x, y \in S$, then the $\varphi : \Xi \to \mathbb{R}$ defined by*

$$\varphi(x) = \sup\{f(z) - L \cdot d(z, x) : z \in S\}$$

satisfies that $\varphi|_S = f$, *and* $\|\varphi(x) - \varphi(y)\| \leqslant L \cdot d(x,y)$ *for any* $x, y \in \Xi$.

Remark. Fixing an $x_0 \in S$, we have $\varphi(x) \leqslant f(x_0) + L \cdot d(x, x_0)$ for any $x \in \Xi$.

Next we define the Hausdorff measure that was already considered for subsets of $\mathbb{R}^n$ in Section 1.D.

Definition 10.4.2 (Hausdorff measure $\mathcal{H}^s$). For a metric space (Ξ, d), $s \geqslant 0$, $\delta > 0$ and $X \subset \Xi$, let

$$\mathcal{H}^s_\delta(X) = \inf\left\{ \sum_{i=1}^{\infty} \omega_s \left(\frac{\operatorname{diam} Z_i}{2}\right)^s : X \subset \bigcup_{i=1}^{\infty} Z_i \text{ and } \forall \operatorname{diam} Z_i < \delta \right\}$$

where $\omega_s = \pi^{\frac{s}{2}}/\Gamma(\frac{s}{2}+1)$ and $\operatorname{diam} Z = \sup\{d(x,y) : x, y \in Z\}$. The Hausdorff outer measure is $\mathcal{H}^{*,s}(X) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(X)$, and let $\mathcal{H}^s_\Xi$ be the corresponding Borel measure (which naturally depends on the metric d on Ξ, as well).

Example 10.4.3. Let (Ξ, d) be a metric space.

(i) $\mathcal{H}^0_\Xi$ is the counting measure; namely, $\mathcal{H}^0_\Xi = \#X$ if X is finite, and $\mathcal{H}^0_\Xi = \infty$ if X is infinite.

(ii) The Hausdorff measure is normalized in a way such that if (Ξ, d) is either the Euclidean space $\mathbb{R}^n$, the spherical space S^n, or the hyperbolic space H^n, then $\mathcal{H}^n_\Xi(X) = |X|$ for a Borel set $X \subset \Xi$ where $|X|$ is the Haar measure corresponding to the transitive isometry group of Ξ (thus $|X|$ is the Lebesgue measure). This is proved in Theorem 1.D.5 if $\Xi = \mathbb{R}^n$, and the argument in the other two cases is similar, and the only essential difference is that concerning the volume of the ball $B(z,r) = \{x \in \Xi : d(x,z) \leqslant r\}$, $r > 0$, we have only $|B(z,r)| = \omega_n r^n + O(r^{n+2})$ as $r \to 0^+$ if $\Xi = S^n, H^n$ where the implied constant in $O(\cdot)$ depends only on n.

Remark 10.4.4 (Hausdorff dimension). For any metric space (Ξ, d) and $X \subset \Xi$, there exists a Hausdorff dimension $\alpha \geqslant 0$ of X such that $\mathcal{H}^s_\Xi(X) = \infty$ if $0 \leqslant s < \alpha$ and $\mathcal{H}^s_\Xi(X) = 0$ if $s > \alpha$. In particular, if $0 < \mathcal{H}^\alpha_\Xi(X) < \infty$, then α is the Hausdorff dimension.

Example 10.4.5. (i) The Hausdorff dimension of a finite set is 0.

(ii) The Hausdorff dimension of any measurable subset $X \subset \mathbb{R}^n$ with $0 < |X| \leqslant \infty$ is n according to Example 10.4.3 (ii).

Remark 10.4.6 (Lipschitz function). If $(\Xi, d), (Y, \tilde{d})$ are metric spaces, then $f : \Xi \to Y$ is Lipschitz if there exists $L > 0$ such that $\tilde{d}(f(x), f(y)) \leqslant L \cdot d(x,y)$ for $x, y \in \Xi$. For any $\mathcal{H}^s_\Xi$ measurable $X \subset \Xi$ and $s \geqslant 0$, the definition of the Hausdorff measure yields

$$\mathcal{H}^s_Y\Big(f(X)\Big) \leqslant L^s \cdot \mathcal{H}^s_\Xi(X).$$

In particular, $\mathcal{H}^s_Y(f(X)) = 0$ if $\mathcal{H}^s_\Xi(X) = 0$.

One of the most basic property of a Lipschitz function on $\mathbb{R}^n$ (or on a manifold, like S^n or H^n) is its almost everywhere differentiability (cf. Federer [260]).

Lemma 10.4.7 (Rademacher's theorem). *If $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$ is locally Lipschitz (Lipschitz on any compact $K \subset \Omega$), then f is differentiable at $\mathcal{H}^n$ a.e. $z \in \Omega$; namely, there exists $m \times n$ matrix $Df(z)$ at with*

$$f(x) = f(z) + Df(z)(x - z) + o(\|x - z\|);$$

or in other words, $\lim_{x\to z} \frac{f(x)-f(z)-Df(z)(x-z)}{\|x-z\|} = 0$.

For properties of Lipschitz manifolds, see Federer [260].

Remark 10.4.8 (locally Lipschitz map from a Lipschitz manifold). Let $\Xi \subset \mathbb{R}^q$ be an embedded Lipschitz n-manifold; namely, any point $x \in X$ has a neighbourhood bi-Lipschitz equivalent to $\mathbb{R}^n$, and the distance of $x, y \in \Xi$ is the infimum of $\text{length}(\gamma) = \int_0^1 \|D\gamma(t)\|\, dt$ for any Lipschitz curve $\gamma : [0, 1] \to \Xi$ with $\gamma(0) = x$ and $\gamma(1) = y$. In this case, the Hausdorff measure $\mathcal{H}^s_\Xi$ on Ξ coincides with the Hausdorff measure $\mathcal{H}^s$ with respect to $\mathbb{R}^n$.

(i) For $\mathcal{H}^n_\Xi$ a.e. $x \in X$, the tangent space $T_{\Xi,x}$ exists (and isomorphic to $\mathbb{R}^n$).

(ii) If $F : \Xi \to \mathbb{R}^m$ is locally Lipschitz, then there exists the differential (a linear map) $DF(x) : T_{\Xi,x} \to \mathbb{R}^m$ at $\mathcal{H}^n_\Xi$ a.e. $x \in X$ by Rademacher's theorem.

(iii) In the setting of (ii), if $m \leqslant n$ and there exists $T_{\Xi,x}$ and $DF(x)$ at $x \in \Xi$, then $J(F, x) = \sqrt{\det DF(x)\, DF(x)^t}$ is the Jacobian.
In particular, if $m = 1$, then $J(F, x) = \|DF(x)\|$.

We note that the spherical space $S^n \subset \mathbb{R}^{n+1}$ is an embedded Lipschitz (actually C^∞) manifold. The following statement follows from combining Theorem 3.2.12. in Federer [260] (the Euclidean case) and the paragraph 3.2.46. in Federer [260].

Theorem 10.4.9 (Coarea formula, Federer). *For $m \leqslant k < q$ and Lipschitz embedded k-manifold $\Xi \subset \mathbb{R}^q$, if $F : \Xi \to \mathbb{R}^m$ is locally Lipschitz and $\varphi : \Xi \to [0, \infty)$ is measurable, then*

$$\int_\Xi \varphi(x) \cdot J(F, x)\, d\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \int_{F^{-1}(y)} \varphi(x)\, d\mathcal{H}^{k-m}(x)\, d\mathcal{H}^m(y). \tag{10.11}$$

Remarks.

- In both Theorem 10.4.9 and Corollary 10.4.10, the integral on the left is finite if and only if the integral on the right hand side is finite.

- When $m = 1$ and $\Xi = \mathbb{R}^k$, (10.11) reads as

$$\int_{\mathbb{R}^k} \varphi \cdot \|DF\| \, d\mathcal{H}^k = \int_{\mathbb{R}} \int_{F^{-1}(t)} \varphi \, d\mathcal{H}^{k-1} \, dt. \tag{10.12}$$

- Applying (10.12) to $F(x) = \|x\|$ for $x \in \mathbb{R}^k$ where $\|DF(x)\| = 1$ for $x \in \mathbb{R}^k \backslash \{o\}$ yields the formula for integration in polar coordinates; namely, if $\varphi : \mathbb{R}^k \to [0, \infty)$ is measurable or $\varphi \in L^1(\mathbb{R}^n, \mathcal{H}^n)$, then

$$\begin{aligned} \int_{\mathbb{R}^k} \varphi \, d\mathcal{H}^k &= \int_0^\infty \int_{\partial(rB^n)} \varphi \, d\mathcal{H}^{k-1} \, dr \\ &= \int_0^\infty \int_{S^{n-1}} \varphi(ru) r^{n-1} \, d\mathcal{H}^{k-1}(u) \, dr. \end{aligned} \tag{10.13}$$

Corollary 10.4.10 follows from Theorem 10.4.9 by taking $\varphi = \psi \circ F$.

Corollary 10.4.10 (Coarea formula #2). *For $m \leqslant k < q$ and embedded Lipschitz k-manifold $X \subset \mathbb{R}^q$, if $F : X \to \mathbb{R}^m$ is locally Lipschitz and $\psi : \mathbb{R}^m \to [0, \infty)$ measurable, then*

$$\int_X \psi(F(x)) \cdot J(F, x) \, d\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \psi(y) \mathcal{H}^{k-m} \left(F^{-1}(y) \right) \, d\mathcal{H}^m(y).$$

We need the basic properties of compact sets with Lipschitz boundary in the n-dimensional Spherical and Hyperbolic space that are n-dimensional Riemannian manifolds. We recall that a compact $X \subset \mathbb{R}^n$ has Lipschitz boundary (cf. Definition 4.1.4) if X is the closure of its non-empty interior, and for any $x \in \partial X$, there exists $r > 0$ (depending on x, X) such that $\partial X \cap \operatorname{int}(x + rB^n)$ is the graph of a Lipschitz function.

Remark 10.4.11 (Lipschitz boundary on Riemannian manifolds). Let (Ξ, d) be an n-dimensional Riemannian manifold. A non-empty compact $X \subset \Xi$ has Lipschitz boundary if $X = \operatorname{cl} \operatorname{int} X$, and for any $x \in \partial X$, there exist an open neighbourhood $U \subset \Xi$ of x, an open $U' \subset \mathbb{R}^n$ and a C^2 diffeomorphism $\varphi : U \to U'$ such that $\varphi(U \cap \partial X)$ is the graph of a Lipschitz function. In this case, $0 < \mathcal{H}^{n-1}_\Xi(\partial X) < \infty$; moreover, it can be proved based on paragraph 3.2.46. in Federer [260] and on (5.56) about weighted outer Minkowski content in the Euclidean setting that the parallel domain $X^{(\varrho)} = \{z \in \Xi : \exists x \in X \text{ with } d(x, z) \leqslant \varrho\}$ for $\varrho > 0$ satisfies

$$\mathcal{H}^{n-1}_\Xi(\partial X) = \lim_{\varrho \to 0^+} \frac{\mathcal{H}^n_\Xi(X^{(\varrho)}) - \mathcal{H}^n_\Xi(X)}{\varrho}. \tag{10.14}$$

Next we discuss the relation between the perimeter and the lower Minkowski content of the typical level set of a Lipschitz function on $\mathbb{R}^n$ where the lower Minkowski

content of a compact $X \subset \mathbb{R}^n$ is

$$\mathcal{M}_-(X) = \liminf_{\varrho \to 0^+} \frac{\left|X^{(\varrho)}\right| - |X|}{\varrho}$$

for $X^{(\varrho)} = X + \varrho B^n$.

Theorem 10.4.12 (Ambrosio, Di Marino, Gigli [24]). *If $f : \mathbb{R}^n \to [0, \infty)$ is Lipschitz such that any non-empty level set $\{f > t\}$ for $t > 0$ is bounded, then*

$$\mathcal{H}^{n-1}\left(f^{-1}(t)\right) = \mathcal{M}_-\left(\{f \geqslant t\}\right) \tag{10.15}$$

for a.e. $t > 0$ in the image of f.

Remark. Proposition 4.2 in Ambrosio, Di Marino, Gigli [24] actually states Theorem 10.4.12 under the assumption that $\{f > 0\}$ is bounded. However, applying the original result to $f_k = \max\{f - \frac{1}{k}, 0\}$ for any large k leads to the statement of Theorem 10.4.12.

10.5 The Stone-Weierstrass theorem

We need the Stone-Weierstrass theorem for compact Hausdorff spaces (see Rudin [640]). For any compact Hausdorff space X, we write $C(X, \mathbb{R})$ to denote the Banach space of real continuous functions equipped with $\|\cdot\|_\infty$ norm (maximum norm).

Theorem 10.5.1 (Stone-Weierstrass). *Let X be a compact Hausdorff space, and let $\mathcal{A} \subset C(X, \mathbb{R})$ contain all constant functions, and satisfy that $f \cdot g \in \mathcal{A}$ and $f + g \in \mathcal{A}$ for $f, g \in \mathcal{A}$. If for any different $x, y \in X$, there exists an $f \in \mathcal{A}$ with $f(x) \neq f(y)$, then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ with respect to the $\|\cdot\|_\infty$ norm.*

Corollary 10.5.2 (Stone-Weierstrass on S^{n-1}). *If $\mathcal{A}$ is the family of restrictions of polynomials in $\mathbb{R}^n$ to S^{n-1}, then $\mathcal{A}$ is dense in $C(S^{n-1}, \mathbb{R})$ with respect to the $\|\cdot\|_\infty$ metric.*

10.6 Differentiability of Convex Functions

In this section, we collect some basic properties of convex functions (see Rockafellar [633] for an in depth study).

Definition 10.6.1 (Convex functions). If $\Omega \subset \mathbb{R}^n$ is convex, then $\varphi : \Omega \to \mathbb{R}$ is convex, if $\varphi((1-t)x + ty) \leqslant (1-t)\varphi(x) + t\varphi(y)$ for any $x, y \in \Omega$ and $t \in (0, 1)$.

Similarly, $\psi : \Omega \to \mathbb{R}$ is concave, if $-\psi$ is convex; or in other words, if $\psi((1-t)x+ty) \geqslant (1-t)\psi(x)+t\psi(y)$ for $x, y \in \Omega$ and $t \in (0,1)$.

Remark. For fixed $t \in (0,1)$, if $\Omega \subset \mathbb{R}^n$ is open convex, $\varphi : \Omega \to \mathbb{R}$ is measurable and $\varphi((1-t)x+ty) \leqslant (1-t)\varphi(x)+t\varphi(y)$ for any $x, y \in \Omega$, then φ is convex (cf. Theorem 10.9.12).

We say that a real function φ is locally Lipschitz on an open set $\Omega \subset \mathbb{R}^n$ if it is Lipschitz on any compact subset of Ω.

Theorem 10.6.2 (The differentiability of a convex function). *Let $\varphi : \Omega \to \mathbb{R}$ be convex for a convex open set $\Omega \subset \mathbb{R}^n$.*

(i) Rademacher's theorem: *φ is locally Lipschitz (and hence continuous), and differentiable $\mathcal{H}^n$ a.e. in Ω.*
φ is C^1 if and only if it is differentiable at each $x \in \Omega$.

(ii) Aleksandrov's theorem, [8, 11]: *φ is twice differentiable $\mathcal{H}^n$ a.e. in Ω in the following sense: For $\mathcal{H}^n$ a.e. $z \in \Omega$, there exist $D\varphi(z)$ and a positive semi-definite quadratic form Q_z such that*

$$\varphi(x) = \varphi(z) + \langle D\varphi(z), x-z\rangle + \frac{1}{2}\cdot Q_z(x-z) + o(\|x-z\|^2);$$

namely, $\lim_{x\to z} \frac{\varphi(x)-\varphi(z)-\langle D\varphi(z),x-z\rangle-\frac{1}{2}\cdot Q_z(x-z)}{\|x-z\|^2} = 0.$
We write $D^2\varphi(z)$ to denote the $n \times n$ symmetric positive semi-definite matrix associated to Q_z. In particular, $D^2\varphi(z)$ is the derivative of $z \mapsto D\varphi(z)$ for $\mathcal{H}^n$ a.e $z \in \Omega$, and $\det D^2\varphi$ *is a measurable function on Ω.*

Definition 10.6.3 (Subdifferential of a convex function). If $\Omega \subset \mathbb{R}^n$ is open and convex, and $\varphi : \Omega \to \mathbb{R}$ is convex, then for $z \in \Omega$, the subdifferential is

$$\partial\varphi(z) = \{u \in \mathbb{R}^n : \varphi(x) - \varphi(z) \geqslant \langle u, x-z\rangle \ \forall x \in \Omega\}.$$

We note that $\partial\varphi(z)$ is nonempty, convex and compact for each for $z \in \Omega$.

Remark. For an open and convex $\Omega \subset \mathbb{R}^n$, and a convex function $\varphi : \Omega \to \mathbb{R}$, the function φ is differentiable at $z \in \Omega$ with derivative $D\varphi(z)$ if and only if $\partial\varphi(z) = \{D\varphi(z)\}$; namely, if $\partial\varphi(z)$ has one element.

10.7 Self-adjoint Elliptic Linear Operators on the sphere S^{n-1}

For this section, see Evans [251], Chapter 6 and Gilbarg, Trudinger [325], Chapter 8.

For a C^2 function $h : S^{n-1} \to \mathbb{R}$, let $\tilde{h}(tu) = t \cdot h(u)$ and $\bar{h}(tu) = h(u)$ for $t \geqslant 0$ and $u \in S^{n-1}$, and hence $\tilde{h}, \bar{h} : \mathbb{R}^n \to \mathbb{R}$ are C^2 on $\mathbb{R}^n \backslash \{o\}$. Following Definition 8.1.6,

we write

$$\nabla h(u) = D\tilde{h}(u)|_{u^\perp} = D\bar{h}(u)|_{u^\perp} \tag{10.16}$$

where ∇h is the spherical gradient (see Schneider [661], Section 2.5). In addition, $\mathbb{R}u$ is an eigenspace (with eigenvalue zero) of $D^2\tilde{h}(u)$ and $D^2\bar{h}(u)$, and we define

$$\nabla^2 h(u) = D^2\bar{h}(u)|_{u^\perp} \tag{10.17}$$

$$\widetilde{D}^2 h(u) = \widetilde{D}^2\tilde{h}(u) = D^2\tilde{h}(u)|_{u^\perp}. \tag{10.18}$$

According to Schneider [661], Section 2.5, $\nabla^2 h$ is the spherical Hessian of h with respect to a moving orthogonal frame in the sense of Riemannian geometry, and $\tilde{h}(x) = \|x\| \cdot \bar{h}(x)$ implies that if $u \in S^{n-1}$, then

$$\widetilde{D}^2 h(u) = D^2\tilde{h}(u)|_{u^\perp} = \nabla^2 h(u) + h(u) I_{n-1} \text{ on the tangent space } u^\perp. \tag{10.19}$$

For an absolutely continuous measure μ on S^{n-1} with positive C^∞ density function, we consider the corresponding scalar product

$$(\varphi, \psi)_\mu = \int_{S^{n-1}} \varphi\psi \, d\mu$$

for $\varphi, \psi \in L^2(S^{n-1}, \mu)$.

First, we provide the traditional definition of a self-adjoint (symmetric) elliptic operators in terms of partial derivatives: Let ∂_i denote the ith partial derivative with respect to a moving frame on S^{n-1}, $i = 1, \ldots, n-1$, and hence $\nabla\varphi = (\partial_1\varphi, \ldots, \partial_{n-1}\varphi)$ for a C^∞ function φ on S^{n-1}. If e^w is the C^∞ density function of μ, then $\mathcal{E}$ is a self-adjoint (symmetric) elliptic linear operator on $C^\infty(S^{n-1})$ (extendable to $L^2(S^{n-1}, \mu)$) if there exists C^∞ functions $a_{ij}, i, j = 1, \ldots, n$, and c on S^{n-1} such that $a_{ij} = a_{ji}$ and

$$\mathcal{E}\varphi = e^{-w} \sum_i \partial_i \left(\sum_j e^w a_{ij} \partial_j \varphi \right) + c \cdot \varphi; \tag{10.20}$$

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j > 0 \text{ for } \xi = (\xi_1, \ldots, \xi_{n-1}) \in x^\perp \backslash \{o\} \tag{10.21}$$

for $x \in S^{n-1}$ where, as $\psi \sum_{i=1}^{n-1} \partial_i T_i = -\sum_{i=1}^{n-1} \partial_i\psi \cdot T_i$ for any C^1 tangent vector field $(T_1(x), \ldots, T_{n-1}(x)) \in x^\perp$ on S^{n-1} by the divergence theorem on S^{n-1}, we have

$$(\psi, \mathcal{E}\varphi)_\mu = -\int_{S^{n-1}} \sum_{ij} a_{ij} \partial_i\psi \partial_j\varphi \, e^w \, d\mathcal{H}^{n-1} = (\mathcal{E}\psi, \varphi)_\mu.$$

Let us describe the way how we meet self-adjoint elliptic linear operators in this book. We write $\mathcal{M}_d$ to denote the space of $d \times d$ symmetric matrices, and for $M \in \mathcal{M}_d$, we write $M \geq 0$ if M is positive semi-definite. For a real C^∞ function $F_2(M, x)$ of

$M \in \mathcal{M}_{n-1}$ and $x \in S^{n-1}$, and for a C^∞ function $F_1(v,t,x)$ of $x \in S^{n-1}$, $v \in x^\perp$ and $t \in \mathbb{R}$, we say that $\mathcal{E}$ is a *uniformly elliptic* (sometimes called strictly elliptic) operator of $\varphi \in C^\infty(S^{n-1})$ if

$$\begin{aligned} \mathcal{E}\varphi \quad = \quad & F_2(\nabla^2\varphi, \cdot) + F_1(\nabla\varphi, \varphi, \cdot) \text{ where} \\ & F_2(\cdot, x) \text{ and } F_1(\cdot, \cdot, x) \text{ are linear for fixed } x \in S^{n-1}, \text{ and} \\ & F_2(M,x) > 0 \text{ if } M \geq 0 \text{ and } M \neq 0 \text{ for } M \in \mathcal{M}_{n-1}. \end{aligned} \tag{10.22}$$

In particular, $F_2(M,x) = \sum_{ij} a_{ij}(x) m_{ij}$ for $M = [m_{ij}]$ where $a_{ij}(x)$ are suitable C^∞ functions satisfying the "positivity condition" in (10.22).

The uniformly elliptic operator $\mathcal{E}$ defined on $C^\infty(S^{n-1})$ is called *symmetric* with respect to $L^2(S^{n-1}, \mu)$; or in other words, has a self-adjoint extension to $L^2(S^{n-1}, \mu)$ if

$$(\mathcal{E}\varphi, \psi)_\mu = (\varphi, \mathcal{E}\psi)_\mu$$

for $\varphi, \psi \in C^\infty(S^{n-1})$. We note that for a differential operator, being symmetric and being self-adjoint is not equivalent in general, but these properties coincide in the case of uniformly elliptic operators. In particular, self-adjoint uniformly elliptic operators are the ones that can be written in the form of (10.20) where $a_{ij}(x)$ satisfies (10.21).

Concerning spectral properties of a self-adjoint elliptic linear operator $\mathcal{E}$, the theory of compact operators on Hilbert spaces and some classical results in PDE yield the existence of C^∞ eigenfunctions $\varphi_1, \varphi_2, \ldots$ on S^{n-1} that form an orthogonal basis of $L^2(S^{n-1}, \mu)$, and corresponding eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots$ with $\lim_{k\to\infty} \lambda_k = -\infty$ and $\mathcal{E}\varphi_i = \lambda_i \varphi_i$. It follows from the strong maximal principle that

$$\text{if } \varphi \text{ is an eigenfunction with } \mathcal{E}\varphi = \lambda_1\varphi, \text{ then } \varphi(x) \neq 0 \text{ for } x \in S^{n-1}. \tag{10.23}$$

According to the variational characterization of eigenvalues, if $j \geqslant 1$, and $(\varphi_i, \psi)_\mu = 0$ for $\psi \in C^2(S^{n-1})$ and $i = 1, \ldots, j$, then

$$(\mathcal{E}\psi, \psi)_\mu \leqslant \lambda_{j+1}(\psi, \psi)_\mu. \tag{10.24}$$

The following well-known properties, that explain why λ_1 is called the principal eigenvalue, are not discussed in many textbooks; therefore, we provide the simple arguments.

Proposition 10.7.1. *Using the notation as above, we have*

(i) $\lambda_i < \lambda_1$ for $i \geqslant 2$ (and hence λ_1 is a simple eigenvalue),

(ii) assuming that $\varphi_1(x) > 0$ for every $x \in S^{n-1}$ (cf. (10.23)),

$$\text{if } \varphi \geqslant 0 \text{ for an eigenfunction } \varphi \text{ of } \mathcal{E}, \text{ then } \varphi = r\varphi_1 \text{ for } r > 0. \tag{10.25}$$

Proof. First, let ψ be an eigenfucntion with $\mathcal{E}\psi = \lambda_1\psi$, and hence we may assume that $\psi > 0$ by (10.23). If $x_0 \in S^{n-1}$ and $r = \psi(x_0)/\varphi_1(x_0)$, then $\mathcal{E}(\psi - r\varphi_1) = \lambda_1(\psi - r\varphi_1)$

and $\psi - r\varphi_1$ has a zero; therefore, (10.23) yields that $\psi - r\varphi_1 \equiv 0$. In particular, λ_1 is a simple eigenvalue.

Finally, let $\varphi \geqslant 0$ for an eigenfunction φ. As $(\varphi, \varphi_1)_\mu > 0$, and eigenfunctions corresponding to different eigenvalues are orthogonal, the only possible eigenvalue for φ is λ_1. ∎

10.8 Matrices: The Hilbert-Schmidt norm, Hadamard's inequality and the Perron-Frobenius Theorem

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. For an $n \times n$ real matrix A; or equivalently, for a linear transform $A : \mathbb{R}^n \to \mathbb{R}^n$, its Hilbert-Schmidt norm is

$$\|A\| = \sqrt{\sum_{i=1}^{n} \|Ae_i\|^2};$$

that is, the Euclidean norm of the n^2-dimensional vector constructed from the entries of A. This second definition shows that $\|A^t\| = \|A\|$. The Hilbert-Schmidt norm of a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is independent of the orthonomal basis of $\mathbb{R}^n$ because if $f_1, \ldots, f_n$ is another orthonormal basis of $\mathbb{R}^n$, then

$$\|A\|^2 = \sum_{i=1}^{n} \|Ae_i\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle f_j, Ae_i\rangle^2 = \sum_{j=1}^{n}\sum_{i=1}^{n} \langle A^t f_j, e_i\rangle^2 = \sum_{j=1}^{n} \|A^t f_j\|^2 = \sum_{j=1}^{n} \|Af_j\|^2.$$

For any $n \times n$ matrix $A = [a_1, \ldots, a_n]$ with columns $a_1, \ldots, a_n \in \mathbb{R}^n$, Hadamard inequality states that

$$|\det A| \leqslant \prod_{i=1}^{n} \|a_i\|. \tag{10.26}$$

Our last topic is the Perron-Frobenius theorem. We provide the argument due to Pál Hegedűs to handle the case of symmetric matrices, see for example Bapat, Raghavan [62] for the general case.

Theorem 10.8.1 (Perron-Frobenius Theorem for positive symmetric matrices). *If A is a symmetric $d \times d$ matrix such that every entry is positive, and λ_1 is the largest eigenvalue, then*

- *$\lambda_1 > 0$ and λ_1 is a simple eigenvalue, and $|\lambda| < \lambda_1$ for any other eigenvalue λ;*
- *there exists an eigenvector x_1 whose coordinates are all positive and $Ax_1 = \lambda_1 x_1$;*
- *any eigenvector x of A whose coordinates are all non-negative satisfy $x = r\,x_1$ for $r > 0$.*

Proof. For $x = (x^{(1)}, \ldots, x^{(n)}) \in \mathbb{R}^n$ and $y = (y^{(1)}, \ldots, y^{(n)}) \in \mathbb{R}^n$, we write $x \succeq y$ if $x^{(i)} \geqslant y^{(i)}$ for $i = 1, \ldots, n$, and $x \succ y$ if $x^{(i)} > y^{(i)}$ for $i = 1, \ldots, n$. In particular, $x \succeq o$ is equivalent to saying that the coordinates of x are non-negative. Since every entry of A is positive, we deduce that $x \succeq y$ and $x \neq y$ yield

$$Ax \succ Ay \text{ and if in addition, } y \succeq o, \text{ then } \|y\| > \|x\|. \tag{10.27}$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A in a way such that $|\lambda_i| \leqslant |\lambda_1|$ for $i = 1, \ldots, n$. Considering A in an orthonormal basis of eigenvectors shows that

$$\|Ax\| \leqslant |\lambda_1| \cdot \|x\| \text{ holds for any } x \in \mathbb{R}^n. \tag{10.28}$$

We claim that if $x = (x^{(1)}, \ldots, x^{(n)}) \in \mathbb{R}^n \backslash \{o\}$ is an eigenvector with $Ax = \lambda_1 x$, then

$$\lambda_1 > 0 \text{ and either } x \succ o \text{ or } x \prec o. \tag{10.29}$$

We may assume that $x^{(j)} > 0$ for a $j \in \{1, \ldots, n\}$. First we verify that $x \succeq o$ using an indirect argument. We suppose that there exists an $x^{(i)} < 0$, and seek a contradiction. Let $\tilde{x} = (|x^{(1)}|, \ldots, |x^{(n)}|)$ Since every entry of A is positive and x has both negative and positive coordinate, the absolute value of each coordinate of $A\tilde{x}$ is larger than the corresponding coordinate of Ax, thus $\|A\tilde{x}\| > \|Ax\| = \|\lambda_1 x\| = |\lambda_1| \cdot \|\tilde{x}\|$. This contradicts (10.28), and hence yields $x \succeq o$. In turn, we deduce from (10.27) that $Ax \succ o$; therefore, $Ax = \lambda_1 x$ implies (10.29).

Let us fix an eigenvector $x_1 = (x_1^{(1)}, \ldots, x_1^{(n)})$ for λ_1, and hence $x_1 \succ o$ by (10.29). If $y = (y^{(1)}, \ldots, y^{(n)})$ is any other eigenvector for λ_1, then we may assume that $y \succ 0$ by (10.29). For $\alpha = y^{(1)}/x_1^{(1)} > 0$, $y - \alpha x_1$ has a zero coordinate. As $A(y - \alpha x_1) = \lambda_1(y - \alpha x_1)$, we deduce from (10.29) that $y = \alpha x_1$; therefore, λ_1 is a simple eigenvalue.

Finally, if $z \succeq 0$ is an eigenvector for A, then $\langle z, x_1 \rangle > 0$ and the fact that the eigenvectors corresponding to different eigenvalues are orthogonal yield that $Az = \lambda_1 z$, and hence $z = r\, x_1$ for some $r > 0$. ∎

Let $A = [a_{ij}]$ be a $d \times d$ matrix. We say that A is non-negative, if $a_{ij} \geqslant 0$ for any $i, j = 1, \ldots, d$, and that the off-diagonal entries of A are non-negative, if $e_{ij} \geqslant 0$ when $i \neq j$. Assuming that the off-diagonal entries of A are non-negative, we say that A is an *irreducible matrix with period* $p \geqslant 1$, if for any $i \neq j$, there exist $1 \leqslant k \leqslant p$ and pairwise different $i_0, \ldots, i_k \in \{1, \ldots, d\}$ such that $i_0 = i$, $i_k = j$, and $a_{i_{m-1}, i_m} > 0$ for $m = 1, \ldots, k$. Obviously, $p \leqslant n - 1$, and $p = 1$ if and only if each off-diagonal entry is positive. If the actual period is irrelevant, then we simply say that A is an *irreducible matrix*. Now the version of the Perron-Frobenius theorem we need states the following (see Bapat, Raghavan [62] for the general version):

Theorem 10.8.2 (Perron-Frobenius Theorem for irreducible symmetric matrices). *If A is symmetric non-negative irreducible $d \times d$ matrix with positive entries on the diagonal, and λ_1 is the largest eigenvalue, then*

- *$\lambda_1 > 0$ and λ_1 is a simple eigenvalue, and $|\lambda| < \lambda_1$ for any other eigenvalue λ;*
- *there exists an eigenvector x_1 whose coordinates are all positive and $Ax_1 = \lambda_1 x_1$;*
- *any eigenvector x of A whose coordinates are all non-negative satisfy $x = r\,x_1$ for $r > 0$.*

Proof. As A is non-negative and symmetric, the same holds for A^2. Let $A = [a_{ij}]$. As the entries of A on the diagonal are positive, if $a_{ij} > 0$, then the entry of A^2 in the ith row and jth column is positive. In addition, assuming that $a_{ij} > 0$ and $a_{jm} > 0$, then $a_{mj} > 0$ by symmetry; therefore, the entry of A^2 in the ith row and mth column is positive. It follows that if A has period $p \geqslant 2$, then A^2 has period at most $\lceil p/2 \rceil$ (the smallest integer not smaller than $p/2$). Since $p < n$ and $2^{\lceil \log_2 n \rceil} < 2n$, we deduce that each entry of the symmetric matrix A^{2n+1} is positive.

Let $x_1, \ldots, x_n$ be an orthonormal basis of eigenvectors of A with $Ax_i = \lambda_i x_i, \lambda_i \in \mathbb{R}$, where we assume that $|\lambda_i| \leqslant |\lambda_1|$ for $i = 1, \ldots, n$. Since, $x_1, \ldots, x_n$ are eigenvectors of A^{2n+1}, and $A^{2n+1} x_i = \lambda_i^{2n+1} x_i$, we conclude Theorem 10.8.2 from applying Theorem 10.8.1 to the positive symmetric matrix A^{2n+1}, which has λ_1^{2n+1} as the eigenvalue with maximal absolute value. ∎

10.9 Log-concave functions

Log-concave functions on $\mathbb{R}^n$ can be considered as functional analogues of convex bodies in $\mathbb{R}^n$. One of the core properties of log-concave functions is that they are essentially the extremizers in the Prékopa-Leindler inequality (cf. Section 3.4), which we recall in the following form: Given $\lambda \in (0,1)$, measurable $f, g, h : \mathbb{R}^n \to [0, \infty)$, $n \geqslant 1$, with $h((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} g(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} h \geqslant \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda}. \tag{10.30}$$

In this section, we do not discuss properties directly related to the Prékopa-Leindler inequality in detail because that is done in Chapter 3 (see also Section 6.7 and Section 8.7 for other geometrically inspired inequalities and conjectures for log-concave functions). We also do not discuss here properties related to isoperimetric type inequalities for log-concave measures (Cheeger constant, Kannan-Lovász-Simonovits conjecture) because that is done in Section 4.7.

Definition 10.9.1 (Log-concave functions). Given a convex set $\Omega \subset \mathbb{R}^n$, a function $f : \Omega \to [0, \infty)$ is log-concave if $f((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} f(y)^{\lambda}$ holds for $x, y \in \Omega$ and $\lambda \in (0,1)$.

The support of f is the closed convex set

$$\operatorname{supp} f = \operatorname{cl}\{f > 0\},$$

which is also the support of the corresponding log-concave measure $f\,d\mathcal{H}^n$ if the support is not contained in a hyperplane (cf. Definition 10.9.3).

Remarks.

- f is log-concave if and only if $f = e^{-\varphi}$ for a convex function $\varphi : \Omega \to (-\infty, \infty]$.
- For a log-concave function f, the level sets $\{f > t\}$ are convex for $t \in \mathbb{R}$.
- If f is log-concave and positive on an open convex $\Omega \subset \mathbb{R}^n$, then f is continuous on Ω.
- Typical examples for log-concave functions are $e^{-\pi\|x\|^2}$ (Gaussian density), or $f(x) = e^{-\|x\|_K}$, or $f(x) = e^{-\|x\|_K^2}$ for $\|x\|_K = \min\{t \geqslant 0 : x \in tK\}$ where $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$.
- $f = \mathbf{1}_X$ log-concave for $X \subset \mathbb{R}^n$ if and only if X is convex.

Theorem 10.9.2 (Operations preserving log-concavity, Prékopa [627]).

Affine invariance: If f is log-concave function on $\mathbb{R}^n$, $\Phi \in \mathrm{GL}(n)$, $w \in \mathbb{R}^n$ and $a > 0$, then $x \mapsto a \cdot f(\Phi x + w)$ is log-concave.

Product: If f, g are log-concave functions on $\mathbb{R}^n$, then $f \cdot g$ is log-concave.

Marginals: If $h(x, y)$ log-concave on $\mathbb{R}^{n+m}$ for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then $\varphi(x) = \int_{\mathbb{R}^m} h(x, y)\,dy$ is log-concave on $\mathbb{R}^n$.

Convolution: If f, g are log-concave functions on $\mathbb{R}^n$, then $f * g$ is log-concave.

Cumulative Distribution Function: If f is log-concave on $\mathbb{R}$, then $F(x) = \int_{-\infty}^{x} f$ is also log-concave.

Proof. Affine invariance and the log-concavity of a product directly follow from Definition 10.9.1.

For the log-concavity of marginals, let $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in (0, 1)$, and hence

$$h((1-\lambda)x_1 + \lambda x_2, (1-\lambda)y + \lambda z) \geqslant h(x_1, y)^{1-\lambda} h(x_2, z)^{\lambda}$$

for the three functions $h(x_1, y)$, $h(x_2, y)$ and $h((1-\lambda)x_1 + \lambda x_2, y)$ of $y \in \mathbb{R}^m$, thus the Prékopa-Leindler inequality (10.30) yields $\varphi((1-\lambda)x_1 + \lambda x_2) \geqslant \varphi(x_1)^{1-\lambda}\varphi(x_2)^{\lambda}$.

For convolution, we observe that $h(x, y) = f(y)g(x-y)$ is a log-concave function of $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ as it is the product of two log-concave functions; therefore, the convolution $f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy$ is a marginal, and hence log-concave.

The Prékopa-Leindler inequality (10.30) yields that the Cumulative Distribution Function associated to a log-concave probability density on $\mathbb{R}$ is also log-concave. ∎

We recall Theorem 10.1.3 saying that a Borel measure μ on $\mathbb{R}^n$ is regular (called also Radon) if and only if it is locally finite (any compact set has finite measure).

Definition 10.9.3 (Log-concave measures). A regular Borel measure μ on $\mathbb{R}^n$ is log-concave if whenever $\lambda \in (0,1)$ and $(1-\lambda)X + \lambda Y \subset Z$ for Borel sets $X, Y, Z \subset \mathbb{R}^n$, then

$$\mu(Z) \geqslant \mu(X)^{1-\lambda}\mu(Y)^{\lambda}.$$

According to the Brunn-Minkowski inequality (cf. Lemma 3.1.4), the Lebesgue measure is a log-concave measure. In general, any absolutely continuous finite Borel measure on $\mathbb{R}^n$ with log-concave density is a log-concave measure according to the Prékopa-Leindler inequality (10.30). The converse statement is due to Borell [112], Theorem 3.2.

Theorem 10.9.4 (Borell). *μ is a log-concave Borel measure on $\mathbb{R}^n$ with* supp μ *not contained in a hyperplane if and only if $d\mu = f\, d\mathcal{H}^n$ for a log-concave function f with $\int_{\mathbb{R}^n} f > 0$.*

The weak limit of log-concave measures is log-concave, see, for example, Dharmadhikari, Joag-Dev [234], Th. 2.10. For Radon (regular Borel) measures μ_k, μ, $k \in \mathbb{N}$, on $\mathbb{R}^n$, we note that $\{\mu_k\}$ tends weakly to μ if

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f\, d\mu_k = \int_{\mathbb{R}^n} f\, d\mu$$

for any continuous bounded $f : \mathbb{R}^n \to [0,\infty)$ (possibly $\int_{\mathbb{R}^n} f\, d\mu = \infty$).

Theorem 10.9.5. *If μ_k, μ are regular Borel measures on $\mathbb{R}^n$, each μ_k is log-concave and μ_k tends weakly to μ, then μ log-concave.*

A useful property of a log-concave function $f : \mathbb{R}^n \to [0,\infty)$ with $0 < \int_{\mathbb{R}^n} f < \infty$ is that it has exponentially small tail. For example, if $f(o) > 0$, then there exist some $a, R > 0$ such that $f(x) \leqslant f(o)e^{-aR}$ whenever $\|x\| = R$, which in turn yields that

$$f(x) \leqslant f(o)e^{-a\|x\|} \text{ whenever } \|x\| \geqslant R. \tag{10.31}$$

It follows from (10.31) that boundedness of the integral of a log-concave function can be nicely characterized (see, for example, Cordero-Erausquin, Klartag [218]).

Lemma 10.9.6. *For a log-concave function $f : \mathbb{R}^n \to [0,\infty)$, $0 < \int_{\mathbb{R}^n} f\, d\mathcal{H}^n < \infty$ holds if and only if* supp f *is not contained in a hyperplane and any non-empty level set $\{f > t\}$ for $t > 0$ is bounded (and hence $\lim_{x\to\infty} f(x) = 0$ and f is bounded).*

Recently, many notions associated to convex bodies have been generalized to log-concave functions. For example, even the John and Löwner ellipsoid (cf. Chapter 6) has been generalized to them by Alonso-Gutiérrez, González Merino, Jiménez, Villa [20] and Ivanov, Naszódi [443, 444], and V. Milman, Rotem [585] considered "mixed volumes" of log-concave functions and see Section 10.9.2 for associated measures, like surface area measures. We note that the "sum" and "dilation" operators for log-concave

functions defined in V. Milman, Rotem [585] are different from the one commonly used (see Section 10.9.1 the latter).

For additional similar properties of log-concave functions, see for example Saumard, Wellner [652] and Klartag, V. Milman [472].

10.9.1 Summation, dilation and polar of upper semi-continuous log-concave functions

For a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$, its Legendre transform (or convex conjugate, sometimes denoted as φ^*) is

$$\mathcal{L}\varphi(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - \varphi(y)\}, \tag{10.32}$$

which is a lower semi-continuous convex functions, and if φ is lower semi-continuous, then $\mathcal{L}\mathcal{L}(\varphi) = \varphi$ (see Section 1.A for a detailed discussion of the properties of the Legendre transform).

A log-concave function $f : \mathbb{R}^n \to [0, \infty)$ can be chosen rather arbitrarily on the boundary of the support $\operatorname{supp} f$; therefore, it is natural to consider the family LC_n of upper semi-continuous log-concave functions. This family has the property that if $f = e^{-\varphi} \in \mathrm{LC}_n$ for a convex function φ, then its polar is

$$f^\circ(x) = \inf_{y \in \mathbb{R}^n} \left\{ \frac{e^{-\langle x, y\rangle}}{f(y)} \right\} = e^{-\mathcal{L}\varphi(x)}$$

is also an upper semi-continuous log-concave function, and satisfies $(f^\circ)^\circ = f$ (see Section 6.7 for Santaló-type inequalities involving the polar).

For $\lambda > 0$, the "dilate" of an $f \in \mathrm{LC}_n$ is

$$\lambda \cdot f(x) = f\left(\frac{x}{\lambda}\right)^\lambda,$$

which is also an upper semi-continuous log-concave function, and the "sum" of $f, g \in \mathrm{LC}_n$ is the upper semi-continuous log-concave function (sup-convolution)

$$f \star g(x) = \sup_{y \in \mathbb{R}^n} g(y)\, f(x - y).$$

We observe that if $K, C \subset \mathbb{R}^n$ are convex bodies and $\lambda > 0$, then $\mathbf{1}_K, \mathbf{1}_C \in \mathrm{LC}_n$ satisfy $\lambda \cdot \mathbf{1}_K = \mathbf{1}_{\lambda K}$ and $\mathbf{1}_K \star \mathbf{1}_C = \mathbf{1}_{K+C}$. Using this notation as above, the Prékopa-Leindler inequality (10.30) for $f, g \in \mathrm{LC}_n$ and $\lambda \in (0, 1)$ reads as

$$\int_{\mathbb{R}^n} ((1-\lambda) \cdot f \star \lambda \cdot g) \geqslant \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^\lambda. \tag{10.33}$$

Riesz-type representation theorems using the notion above of linearity have been verified by Rotem [636]. Roysdon, Xing [639] defined the L_p linear combination of log-concave functions, and proved analogues of the Borell-Brascamp-Lieb inequalities (cf. (3.26)).

10.9.2 Surface area measures and relatives associated to log-concave functions

For an upper semi-continuous log-concave functions $f \in \mathrm{LC}_n$, its support function (see (10.32) for the Legendre transform) is the lower semi-continuous convex function

$$h_f = \mathcal{L}(-\log f)$$

on $\mathbb{R}^n$. The name is explained by the fact that if $f = \mathbf{1}_K$ for a convex body $K \subset \mathbb{R}^n$, then $h_f = h_K$ (the support function of K).

For an $f \in \mathrm{LC}_n$, we write $K_f = \operatorname{supp} f$ to denote the support. Extending the work of Cordero-Erausquin, Klartag [218] and Colesanti, Fragalá [209], Rotem [637] considers a pair of Borel measures μ_f on $\mathbb{R}^n$ and ν_f on S^{n-1} as *surface area measures of an* $f \in \mathrm{LC}_n$, which satisfy the following.

Theorem 10.9.7 (Rotem). *If $f, g \in \mathrm{LC}_n$ and $0 < \int_{\mathbb{R}^n} f < \infty$, then*

$$\lim_{t \to 0^+} \frac{\int_{\mathbb{R}^n} (f \star (t \cdot g)) - \int_{\mathbb{R}^n} f}{t} = \int_{\mathbb{R}^n} h_g \, d\mu_f + \int_{S^{n-1}} h_{K_g} \, d\nu_f. \tag{10.34}$$

In addition, $\mu_f = \mu_{\tilde{f}}$ and $\nu_f = \nu_{\tilde{f}}$ for $\tilde{f} \in \mathrm{LC}_n$ with $0 < \int_{\mathbb{R}^n} \tilde{f} < \infty$ if and only if there exists $x_0 \in \mathbb{R}^n$ such that $\tilde{f}(x) = f(x - x_0)$.

Remark. For the possibly unbounded closed convex set K_g, $h_{K_g}(u) = \sup_{z \in K_g} \langle u, z \rangle$ might be infinite in (10.34) for some $u \in S^{n-1}$, and hence both sides of (10.34) are infinite, for example, if $f = \mathbf{1}_K$ for a convex body $K \subset \mathbb{R}^n$ and $g(x) > 0$ for any $x \in \mathbb{R}^n$.

If $f \in \mathrm{LC}_n$ and $0 < \int_{\mathbb{R}^n} f < \infty$, then Rotem [637] proves that μ_f and ν_f are bounded measures with bounded first moments ($\int_{\mathbb{R}^n} |x| \, d\mu_f(x) < \infty$), and for any non-negative measurable functions α on $\mathbb{R}^n$ and β on S^{n-1}, we have

$$\int_{\mathbb{R}^n} \alpha \, d\mu_f = \int_{\mathbb{R}^n} \alpha(D\varphi(x)) f(x) \, d\mathcal{H}^n(x)$$
$$\int_{S^{n-1}} \beta \, d\nu_f = \int_{\partial K_f} \beta(\nu_{\partial K_f}(x)) f(x) \, d\mathcal{H}^{n-1}(x)$$

where $\nu_{\partial K_f}$ is continuous at the ($\mathcal{H}^{n-1}$ a.e.) regular points of ∂K_f (cf. Section 1.5). In particular, if $f = \mathbf{1}_K$ for a convex body $K \subset \mathbb{R}^n$, then $\mu_f = |K| \delta_o$ where δ_o is the Dirac delta at o, and $\nu_f = S_K$ - that is the surface area measure of K. In addition,

$\nu_f \equiv 0$ if and only if $f(x) = 0$ for $\mathcal{H}^{n-1}$ a.e. $x \in \partial K_f$ (which holds, for example, if $K_f = \mathbb{R}^n$) - in this case, Theorem 10.34 was verified earlier by Cordero-Erausquin, Klartag [218] using variational argument, and by Santambrogio [645] using optimal transport.

Using $g(x) = \langle x, p \rangle$ for any fixed $p \in \mathbb{R}^n$ in (10.34), we deduce that

$$\int_{\mathbb{R}^n} x \, d\mu_f(x) + \int_{S^{n-1}} u \, d\nu_f(u) = o$$

Cordero-Erausquin, Klartag [218] (see also the paper Santambrogio [645] using optimal transport) managed to characterize μ_f when $\nu_f \equiv 0$.

Theorem 10.9.8 (Cordero-Erausquin, Klartag). *For a Borel measure μ on $\mathbb{R}^n$, $\mu = \mu_f$ for an $f \in \mathrm{LC}_n$ such that $0 < \int_{\mathbb{R}^n} f < \infty$ and either $K_f = \mathbb{R}^n$, or $f(x) = 0$ for $\mathcal{H}^{n-1}$ a.e. $x \in \partial K_f$ if and only if μ is finite, is not concentrated on any hyperplane, and*

$$\int_{\mathbb{R}^n} x \, d\mu_f(x) = o.$$

Recently, Falah, Rotem [253] managed to solve the "Minkowski Problem for log-concave functions" in full generality.

Theorem 10.9.9 (Falah, Rotem). *For Borel measures μ on $\mathbb{R}^n$ and ν on S^{n-1}, we have $\mu = \mu_f$ and $\nu = \nu_f$ for an $f \in \mathrm{LC}_n$ with $0 < \int_{\mathbb{R}^n} f < \infty$ if and only if*

(i) μ is not the zero measure;

(ii) μ has finite first moment, and

$$\int_{\mathbb{R}^n} x \, d\mu_f(x) + \int_{S^{n-1}} x \, d\nu_f(x) = o;$$

(iii) μ and ν are not supported on a common hyperplane $H \subset \mathbb{R}^n$.

In this case, the function f is unique up to translations: If $\mu = \mu_g$ and $\nu = \nu_f$ for a $g \in \mathrm{LC}_n$ with $0 < \int_{\mathbb{R}^n} g < \infty$, then there exists a $v \in \mathbb{R}^n$ such that $g(x) = f(x + v)$.

The paper Falah, Rotem [253] also discusses the continuity of the pair (μ_f, ν_f).

In addition, Fang, Xing, Ye [256] considered the L_p-Minkowski problem for log-concave functions, and Huang, Liu, Xi, Zhao [418] and Fang, Ye, Zhang, Zhao [257] managed to extend the dual Minkowski problem to log-concave functions on $\mathbb{R}^n$ for $q > 0$.

10.9.3 Midpoint log-concave functions

What happens if we fix a $\lambda \in (0, 1)$, and we know the estimate $f((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} f(y)^{\lambda}$ in Definition 10.9.1 for all $x, y \in \mathbb{R}^n$ only for this fixed λ? When the

fixed λ is $\frac{1}{2}$, a function with this property is called *midpoint log-concave*. If the function is continuous, then midpoint log-concavity readily implies log-concavity.

Lemma 10.9.10 (Continuous "midpoint log-concave" functions). *Let $\lambda \in (0,1)$ and $X \subset \mathbb{R}^n$ convex. If continuous $f : X \to [0,\infty)$ satisfies $f((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} f(y)^{\lambda}$ for $x, y \in X$, then f is log-concave.*

Example 10.9.11 (A measurable "midpoint log-concave" function may not be log-concave). Define $\Xi_k \subset [0,1]$ by induction on $k = 0, 1, \ldots$ where $\{0,1\} = \Xi_0 \subset \Xi_1 \subset \ldots$, and $\Xi_{k+1} = \{\frac{1}{2}t + \frac{1}{2}s : t, s \in \Xi_k\}$. Let $\Xi = \cup_{k=0}^{\infty} \Xi_k$.

For $X = \Xi \times \{0\} \cup [0,1] \times (0,1] \subset \mathbb{R}^2$, $\mathbf{1}_X$ is "Midpoint log-concave" on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} \mathbf{1}_X > 0$, but $\mathbf{1}_X$ is not log-concave.

Remark. A classical result of Blumberg [99] and Sierpiński [684] is that if measurable $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfies $\varphi(\frac{1}{2}x + \frac{1}{2}y) \geqslant \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y)$ for $x, y \in \mathbb{R}^n$, then φ is convex. Note that in this case the log-concave $f = e^{-\varphi}$ is positive; namely, it does not take zero.

In this section we prove that a measurable midpoint log-concave function is "essentially" log-concave.

Theorem 10.9.12 (Measurable "Midpoint log-concave" functions). *Let $\lambda \in (0,1)$, and let measurable $f : \mathbb{R}^n \to [0,\infty)$ satisfy $\int_{\mathbb{R}^n} f > 0$ and $f((1-\lambda)x + \lambda y) \geqslant f(x)^{1-\lambda} f(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$. Then there exists a log-concave function φ on $\mathbb{R}^n$ and an open convex $\Omega \subset \mathbb{R}^n$ with $f(x) = \varphi(x) > 0$ if $x \in \Omega$ and $f(x) = \varphi(x) = 0$ if $x \notin \operatorname{cl}\Omega$, and hence $f(x) = \varphi(x)$ for a.e. $x \in \mathbb{R}^n$.*

Proof. Let D be the set of density points of the set $\{f > 0\}$.

Step 1. D is convex and $\Omega = \operatorname{int} D \neq \varnothing$

As $\int_D f = \int_{\mathbb{R}^n} f > 0$, D is not contained in a hyperplane. We define finite $\Xi_k \subset [0,1]$ by induction on $k = 0, 1, \ldots$ with the properties that $\{0,1\} = \Xi_0 \subset \Xi_1 \subset \ldots$, and if $t, s \in \Xi_k$, then $(1-\lambda)t + \lambda s \in \Xi_{k+1}$.

For $x_1, x_2 \in D$, we prove that

$$(1-s)x_1 + sx_2 \in D \quad \text{for any } s \in (0,1). \tag{10.35}$$

Let $\varepsilon \in (0,1)$. As x_1, x_2 are density points of D, there exists $\varrho > 0$ such that if $r \in (0, \varrho)$, then $|D \cap (x_i + rB^n)| > (1 - \frac{\varepsilon}{3})|r\,B^n|$ for $i = 1, 2$, thus induction on k, the "λ-log-concavity" of f and the Brunn-Minkowski inequality Theorem 3.2.1 for inner measure yield that if $t \in \Xi_k$, then $|D \cap ((1-t)x_1 + tx_2 + rB^n)| > (1 - \frac{\varepsilon}{2})|r\,B^n|$. Taking a sequence $t_k \in \Xi_k$ with $t_k \to s$ yields that $|D \cap ((1-s)x_1 + sx_2 + rB^n)| > (1-\varepsilon)|r\,B^n|$; therefore, $(1-s)x_1 + sx_2$ is a density point of D, which in turn yields (10.35) and the convexity of D.

As $|D| = |\{f > 0\}| > 0$, we have $\operatorname{int} D \neq \varnothing$.

Step 2. $\{f > 0\} \subset \operatorname{cl} D$.

Indirectly, we suppose that there exists an $x_0 \notin \operatorname{cl} D$ with $f(x_0) > 0$, and seek a contradiction. Let $y \in \partial D$ be closest to x_0, and hence there exists $z \in \operatorname{int} D$ with $y \in \operatorname{conv}\{z, x_0\}$ and $x = (1-\lambda)x_0 + \lambda z \notin \operatorname{cl} D$. Since the "$\lambda$ log-concavity" of f yield that x is a density point of $\{f > 0\}$, we have arrived at a contradiction.

Step 3. f is continuous on $\operatorname{int} D$

We may assume that $\lambda \leqslant \frac{1}{2}$. We observe that the "$\lambda$ log-concavity" of f yields that if $x, y, z \in \operatorname{int} D$ with $x = (1-\lambda)y + \lambda z$, then

$$f(y) \leqslant f(x) \cdot \left(\frac{f(x)}{f(z)}\right)^{\frac{\lambda}{1-\lambda}}. \tag{10.36}$$

The proof of Step 3 is also indirect, we suppose that f is not continuous at $z_0 \in \operatorname{int} D$. In this case, first we verify that

$$\text{there exists } x_m \to z_0, x_m \neq z_0 \text{ with } f(x_m) \to 0. \tag{10.37}$$

To prove (10.37), the indirect hypothesis yields the existence of a sequence $w_m \to z_0$, $w_m \neq z_0$ with $f(w_m) \to t \neq f(z_0)$. According to (10.36), we may assume that $t < f(z_0)$. For each w_m, we consider the sequence $\{u_k^{(m)}\}$ with $u_0^{(m)} = z_0$, $u_1^{(m)} = w_m$ and $u_k^{(m)} = (1-\lambda)u_{k+1}^{(m)} + \lambda u_{k-1}^{(m)}$ for $k \geqslant 1$. Now if $k \geqslant 2$, then

$$f(u_k^{(m)}) \leqslant f(z_0) \cdot \left(\frac{f(w_m)}{f(z_0)}\right)^{\frac{(k-1)\cdot\lambda}{1-\lambda}}$$

provided $u_k^{(m)} \in \operatorname{int} D$ by (10.36) where $\frac{f(w_m)}{f(z_0)}$ tends to $\frac{t}{f(z_0)}$. Choosing $k(m) \to \infty$ in a way such that $x_m = u_{k(m)}^{(m)} \in \operatorname{int} D$ satisfies $x_m \to z_0$, we conclude (10.37).

We may assume that

$$z_0 = o.$$

Choose $r > 0$ such that $\frac{1-\lambda}{\lambda} rB^n \subset \operatorname{int} D$ (remember, $\lambda \leqslant \frac{1}{2}$).

Next we claim that for any $y \in rB^n$,

$$\text{there exists a sequence } y_m \to y, y_m \neq y \text{ with } f(y_m) \to 0. \tag{10.38}$$

To prove (10.38), let $b = -\frac{1-\lambda}{\lambda} y$, and hence $o = (1-\lambda)y + \lambda b$. It follows from (10.36) and from using the sequence $\{x_m\}$ in (10.37) that $y_m = y + \frac{\lambda}{1-\lambda} x_m$ satisfies $y_m \to y$, $x_m = (1-\lambda)y_m + \lambda b$, $y_m \neq y$ and

$$f(y_m) \leqslant f(x_m) \cdot \left(\frac{f(x_m)}{f(b)}\right)^{\frac{\lambda}{1-\lambda}} \to 0.$$

In turn, we conclude (10.38).

Finally, we claim that if $\ell \geqslant 2$ and $y + 2\varrho B^n \subset rB^n$ for $y \in rB^n$, $\varrho > 0$, then

$$\left| \left\{ x \in y + \varrho B^n : f(x) < \frac{1}{\ell} \right\} \right| > \frac{\lambda^n}{\lambda^n + (1-\lambda)^n} \cdot |\varrho B^n|. \tag{10.39}$$

To prove (10.39), let $X = \left\{ x \in y + \varrho B^n : f(x) \geqslant \frac{1}{\ell} \right\}$, and for the sequence $\{y_m\}$ in (10.38), let $\widetilde{X}_m = \{ z \in \mathbb{R}^n : y_m = (1-\lambda)z + \lambda x \text{ for } x \in X \}$. For any $\varepsilon \in (0,1)$, we may assume by (10.38) and $\lambda \leqslant \frac{1}{2}$ that $f(y_m) < \frac{1}{\ell}$ and $\widetilde{X}_m \subset y + (\varrho + \varepsilon)B^n$. It follows from the "$\lambda$ log-concavity" of f that $X \cap \widetilde{X}_m = \varnothing$, and hence

$$\left(1 + \frac{\lambda^n}{(1-\lambda)^n} \right) \cdot |X| = |X| + \left| \widetilde{X}_m \right| \leqslant (\varrho + \varepsilon)^n |B^n|.$$

As the last estimates holds for any $\varepsilon \in (0,1)$, we conclude (10.39).

For $\ell \geqslant 2$ and $\Xi_\ell = \left\{ x \in rB^n : f(x) < \frac{1}{\ell} \right\}$, it follows from (10.39) that

$$|\Xi_\ell| \geqslant \frac{\lambda^n}{\lambda^n + (1-\lambda)^n} \cdot |rB^n|. \tag{10.40}$$

However, $\bigcap_{\ell=2}^{\infty} \Xi_\ell = \varnothing$ because $rB^n \subset D = \{f > 0\}$. Since f is measurable, we have $\lim_{\ell \to \infty} |\Xi_\ell| = 0$, which contradiction with (10.40) verifies Step 3.

Finally, combining Lemma 10.9.10 with Steps 1-3 yields the existence of a suitable log-concave function φ. ∎

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Index

L^p spaces, 458
L_0 linear combination, 334, 335
L_p Aleksandrov problem, 418
L_p dual Minkowski Problem, 421
L_p dual curvature measure, 421
L_p linear combination, 347
L_p linear combination of convex bodies, 276
L_p-affine surface area, 362
L_p-surface area measure, 3, 396
M-ellipsoid, 234
ε-net, 2, 27, 82
l_p ball, 22
1-Lipschitz function, 169

affine hull, 5
affine independence, 5
affine subspace, 5
affine surface area, 358
Aleksandrov lemma for Wulff shapes, 274
Aleksandrov problem, 416
Aleksandrov solution, 398
Aleksandrov-Fenchel inequality, 272, 289, 318
anisotropic coarea formula, 127
anisotropic isoperimetric inequality, 71, 126, 187, 271, 323
anisotropic perimeter, 70, 125, 182, 271, 323
apex of a cone, 12
approximate identity, 459
area measure, 317, 411
auxiliary cone volume measure, 78

Banach-Mazur distance, 85, 371
Beltrami-Cayley-Klein model, 161
Bonnesen inequality, 324
Borell-Brascamp-Lieb inequality, 113
Brunn-Minkowski inequality, 35, 42, 43, 93

capacity, 425
Carathédory theorem, 6
cardinality of a finite set, 4
Cauchy formula for surface area, 66
centered convex body, 38
centro-affine curvature, 359
centro-affine surface area, 358
centroid, 38
Cheeger constant, 144, 145, 216
Christoffel problem, 411
Christoffel-Minkowski problem, 411
circumradius, 32
circumscribed polytope, 206
closest point map, 2, 10
Coarea formula, 462
complementary linear subspaces, 436
cone volume measure, 3, 76, 77
convex body, 2, 22
convex closure of a convex function, 48
convex cone, 9, 11
convex conjugate, 49
convex function, 464
convex function's domain, 47
convex hull in H^n, $\mathbb{R}^n$, S^n, 161
convex set, 2, 5
convolution, 459
coordinatewise product of unconditional bodies, 105
covariance matrix, 148
cross-polytope, 17
cube, 17

curvature function, 303, 308, 309, 357
curvature measure, 418
cyclic polygon, 282

diameter, 6, 33, 45, 95
dilated summands, 337, 408
Divergence Theorem, 63, 89, 91, 175
dual cone, 11
dual curvature measure, 419
dual Minkowski Problem, 420
dual volume, 419

ellipsoid of inertia, 210
elliptic operator, 319, 339, 353, 467
embedded Lipschitz manifold, 329
epigraph of a convex function, 20, 47
equidistant curves, 166
exterior normal, 8
exterior unit normal, 3

face, 2, 8, 21
finite perimeter, 192
Fourier transform, 460

Gaussian Correlation inequality, 157
Gaussian curvature, 302, 303, 357
Gaussian Minkowski problem, 424
Gaussian surface area measure, 424

Hölder inequality, 456
Hadamard inequality, 468
Hausdorff distance, 2, 25, 107
Hausdorff measure, 1, 56, 119, 460
Hilbert-Schmidt norm, 468
homothetic convex bodies, 42
horosphere, 160
hyperbolic convex body, 160
Hyperbolic Law of Cosines for sides, 160
hyperbolic space H^n, 159

identity matrix, 3
inradius, 32
integral curvature measure, 416
intrinsic volume, 260, 266
irreducible matrix, 290, 469
isodiametric inequality, 33, 34, 44, 45, 95
isoperimetric inequality, 69, 121
isoperimetric problem in an open bounded convex set, 219
isotropic constant, 210
isotropic measure, 148, 246
isotropic position, 211

Jensen inequality, 456
John ellipsoid, 201, 202, 240

Kubota formula, 260, 315

Löwner ellipsoid, 201, 202
Layer Cake formula, 124
Lebesgue integral, 459
Lebesgue integral, inner, 99, 459
Lebesgue integral, outer, 459
Lebesgue measure, inner density points, 116, 455
Legendre ellipsoid, 210, 234
Legendre transform, 49, 226, 473
Lindelöf theorem, 281
Lipschitz boundary, 120–122, 125, 126, 131, 132, 136, 140, 144, 152, 156, 162, 463
Lipschitz function, 460
Lipschitz manifold, 20, 462
log-concave function, 4, 98, 144, 470
log-concave measure, 4, 472
lower semi-continuous function, 47, 129

Mahler Conjecture, 224
matrix of inertia, 210

mean curvature, 261
mean curvatures, 312
mean projections, 260, 315
mean width, 261, 297
median, 146, 169
midpoint log-concave function, 476
Minkowski content, 65, 121, 125
Minkowski inequality, 75, 257, 270, 316
Minkowski inequality for integrals, 458
Minkowski problem, 391, 392, 413
Minkowski sum, 1
Minkowski's second inequality, 271
mixed discriminant, 311, 319, 372
mixed volume, 265, 312

norm with respect to a convex body, 3, 23
normal cone, 2, 9, 18
normalized angle of a cone, 259

one-homogeneous function, 21, 51
operator norm, 149
origin, 1
orthogonal complement, 1
orthogonal projection, 1

parallel domain, 106, 121, 125, 131, 161
parallelepiped, 17, 206
Perron-Frobenius theorem, 468
pointed convex cone, 11
polar coordinates, 40
polar of a convex body, 2, 30
polyhedral cone, 17
polyhedral cone's angle, 17
polyhedron, 14
polytope, 15
positive eigenspace, 290, 319
positive hull, 12
projection body, 91
pushforward of a measure, 72

quermassintegral, 286

Rademacher's theorem, 20, 61–63, 68, 120, 462
radial function, 2, 31, 77
radial Gauss map, 79
radial projection, 78
Radon measure, 453
recession cone, 52
regular boundary point, 3, 19, 62
regular measure, 453
relative boundary, 1
relative interior, 1
reverse isoperimetric inequality, 205
reverse radial Gauss image, 417
Riesz representation theorem, 454

Santaló point, 220
Schwarz symmetrization, 36, 37
self-adjoint elliptic operator, 466
simple polytope, 27, 291
simplex, 17
space of compact convex sets, 2
sphere, 1
spherical cap, 392, 430
spherical convex body, 158
spherical gradient, 3, 466
spherical Hessian, 3, 466
Spherical Laws of Cosines, 158
spherical space S^n, 158
spherically convex set, 158
Steiner formula, 260
Steiner symmetrization, 32, 55, 108
Stone-Weierstrass theorem, 464
strictly convex body, 23
strongly isomorphic polytopes, 28, 291

subdifferential of a convex function, 3, 20, 49, 465
subgradient of a convex function, 49
support function, 2, 21
support function of unbounded closed convex sets, 51
supporting hyperplane, 8
Surface area, 65
surface area measure, 3, 71, 308
surface area measure of unbounded closed compact sets, 412
surface area measure, weighted, 90

tensor product of vectors, 4
torsional rigidity, 427
total variation measure, 454
transpose, 4
two-point symmetrization, 132

unconditional convex body, 33, 105, 345
unit ball, 1

valuations on convex compact sets, 381
volume product, 220
volume ratio, 208

weak convergence of measures, 453, 472
Wulff shape, 129, 189, 274, 334

zonotopes, zonoids, 22