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The $T(5)$ property of packings of squares *

T. Bisztriczky †

Department of Mathematics, U. of Calgary, Calgary, Alberta, Canada T2N 1N4

K. Böröczky ‡

Department of Geometry, Eötvös Loránd University, Pázmány Péter sétány 1/c, Budapest, 1117 Hungary

K.J. Böröczky §

Alfréd Rényi Institute of Mathematics, Reáltanoda u. 13-15, 1053 Budapest, Hungary

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Abstract

According to a classical theorem of Gruenbaum, if any five of a family of pairwise disjoint translates of a square has a transversal line (the family satisfies T(5)), then the whole family has a transversal line (satisfies T). First we show that this result is optimal in the sense that the ”T(5) implies T” property does not necessarily hold anymore if only the slightly shrinked versions of the squares are pairwise disjoint. Next we prove the ”T(5) implies T” property for a family of translates of squares if the interiors are pairwise disjoint and there exist two translates meeting at a common vertex.

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E-mail addresses: tbisztri@ucalgary.ca (T. Bisztriczky), kar.boroczky@gmail.com (K. Böröczky), boroczky.karoly.j@renyi.hu (K.J. Böröczky)

1 Introduction

A family $\mathcal{F}$ of *ovals* (compact convex sets with non-empty interior) in the Euclidean plane has the property T if there is a line (*transversal*) that intersects every member of $\mathcal{F}$. If each k-element subfamily has a transversal then $\mathcal{F}$ has the property $T(k)$.

The history of the study of the conditions under which "$T(k)$ implies T" is extensive, and we refer to Holmsen [8], Jeronimo-Castro, Roldan-Pensado [10] and Holmsen, Wenger [9] for reviews.

Our interest here is the case $k = 5$. The main result, due to Tverberg in [15], is that it is sufficient that the ovals are pairwise disjoint translates. For specific ovals, earlier verifications of this are due to Danzer [3] in the case of disks, and Grünbaum [5] in the case of parallelograms.

We note that in [5], Grünbaum conjectured Tverberg's result and presented an example (see Example 1.1) that showed that disjointedness is possibly a necessary condition for translates of a square. With that example in mind, we show that it is not. We note that the problem is affine invariant; therefore, considering translates of a parallelogram or translates of a square are equivalent.

For any oval C and $k \geq 3$, Grünbaum [5] indicated the problem of determining the infimum $\mu(C, k)$ of $\mu > 0$ such that if the finite family $\{c_i + C\}$ satisfies $T(k)$ and the translates $\{c_i + \mu\, C\}$ do not overlap, then the family $\{c_i + C\}$ has a common transversal.

Here $c_i + \mu\, C$ and $c_j + \mu\, C$ do not overlap means that their interiors are disjoint. This property can be written in the form $\|x_i - x_j\|_{DC} \geq 2\mu$ in terms of the norm $\|\cdot\|_{DC}$ with respect to the difference body $DC = \frac{1}{2}(C - C)$ where for $p \in \mathbb{R}^2$, we have

$$\|p\|_{DC} = \min\{\lambda \geq 0 : \, p \in \lambda\, DC\}.$$

In particular, $C = DC$ if C is origin symmetric, $\|\cdot\|_C$ is the Euclidean norm if C is a unit disk, and $\|(x, y)\|_C = \max\{|x|, |y|\}$ if $C = [-1, 1] \times [-1, 1]$.

Concerning $\mu(C, 5)$, the main result of Tverberg [15] cited above proves that

$$\mu(C, 5) \leq 1 \text{ for any oval } C. \tag{1.1}$$

The main focus of this paper is families of translates of parallelograms. First we recall Grünbaum's example at the end of [5] on page 469. We consider a family $\mathcal{F}$ of translated squares S_i of edge length 20 with center c_i and edges parallel to the coordinates axes for $i = 1, \ldots, 6$ satisfying $T(5)$ but not T. We assume that (x, y) is the Cartesian coordinate system in $\mathbb{R}^2$.

Example 1.1 (Grünbaum)**.** Let $c_1 = (-22, 4)$, $c_2 = (0, 15)$, $c_3 = (12, 11)$, $c_4 = (22, 4)$, $c_5 = (12, -11)$ and $c_6 = (0, -15)$.

It is easy to see that there exists a line transversal t_i of $\mathcal{F} \backslash \{S_i\}$ for $i = 1, \ldots, 6$. We note the unique choice for t_3 is the line with equation $y = -\frac{x}{2}$, which is the only transversal of $\{S_2, S_4, S_6\}$ with negative slope, and unique choice for t_5 is the line with equation $y = \frac{x}{2}$, which is the only transversal of $\{S_1, S_2, S_6\}$ with positive slope. Thus, $\mathcal{F}$ has no transversal.

In particular, Grünbaum's Example 1.1 shows that $\mu(C, 5) \geq \frac{1}{2}$ if C is a parallelogram. Our first result improves on this bound.

Theorem 1.2. *If C is a parallelogram, then $\mu(C, 5) = 1$.*

It is a natural question whether in Grünbaum's result in [5], the disjointedness of the compact parallelograms is necessary, or it is enough to assume that the interiors of the translates are pairwise disjoint; namely, the translated parallelograms do not overlap.

Conjecture 1.3. *If a family $\mathcal{F}$ of non-overlapping translates of a parallelogram satisfies $T(5)$, then $\mathcal{F}$ has a common transversal.*

Actually, we even conjecture the following stronger statement about translates of a square by imposing a lower bound on the distance between distinct centers in terms of the Euclidean distance.

Conjecture 1.4. *Let $\mathcal{F}$ be a family of $n \geq 6$ translates, of a square of side length s, with the property that the Euclidean distance between distinct centres is at least s. Then $T(5)$ implies that $\mathcal{F}$ has a transversal.*

We prove a weaker version of Conjecture 1.3.

Theorem 1.5. *If a family $\mathcal{F}$ of non-overlapping translates of a parallelogram satisfies $T(5)$, and there exist two parallelograms in $\mathcal{F}$ that intersect in a common vertex, then $\mathcal{F}$ has a common transversal.*

Returning to $\mu(C, 5)$ for any oval, we verify the following bounds.

Theorem 1.6. *For any oval C, we have $\frac{2}{3} \leq \mu(C, 5) \leq 1$.*

We note that the paper Bisztriczky, Böröczky, Heppes [2] verifies that $\mu(C, 5) = 2/3$ if C is an ellipsoid, and Theorem 1.2 proves that $\mu(C, 5) = 1$ if C is a parallelogram. Therefore the bounds in Theorem 1.6 are optimal.

We recall that according to Santaló [12], if a family of parallelograms with parallel sides satisfies $T(6)$, then the family has a common transversal. Therefore $\mu(C, 6) = 0$ if C is a parallelogram.

Concerning notation for Theorem 1.2 and Theorem 1.5, we write h and v to denote the horizontal and the vertical axis, respectively, for the coordinate system (x, y) in $\mathbb{R}^2$, and write $c_i = (x_i, y_i)$ to denote the centers of the translated squares in the family $\mathcal{F}$. For different points $p, q \in \mathbb{R}^2$, their line is denoted by $\operatorname{aff}\{p, q\}$. For a line $\ell = \{(x, y) : y = Ax + B\}$, we set $A = \operatorname{slope} \ell$ and write $\ell^+ = \{(x, y) : y > Ax + B\}$ and $\ell^- = \{(x, y) : y < Ax + B\}$ to denote the open halfplane of points above and below, respectively, ℓ.

2 Proof of Theorem 1.2

We may assume that C is the square $[-1, 1] \times [-1, 1]$. It follows from (1.1) that $\mu(C, 5) \leq 1$, therefore it is sufficient to prove the following statement:

For any $\varepsilon \in (0, \frac{1}{3})$, there exist $c_1, \ldots, c_6 \in \mathbb{R}^2$ such that $c_1 + (1 - 2\varepsilon)C, \ldots, c_6 + (1 - 2\varepsilon)C$ do not overlap, the family $\mathcal{F} = \{S_1, \ldots, S_6\}$ satisfies $T(5)$ for $S_i = c_i + C$, $i = 1, \ldots, 6$, and $\mathcal{F}$ has no common transversal.

We define (see Figure 1)

$$\begin{aligned} c_1 &= (-2, 1-\varepsilon) \\ c_2 &= (0, 1+\varepsilon) \\ c_3 &= (2-\varepsilon, 1) \\ c_4 &= (4-4\varepsilon, 1-3\varepsilon) \\ c_5 &= (2-\varepsilon, -1) \\ c_6 &= (0, -1-\varepsilon). \end{aligned}$$

We also consider some vertices of the S_is:

$$\begin{aligned} a &= (-1, \varepsilon) = c_2 + (-1,-1) \in S_1 \cap S_2 \\ b &= (-1, -\varepsilon) = c_1 + (1,-1) = c_6 + (-1,1) \in S_1 \cap S_6 \\ u &= (1-\varepsilon, 0) = c_3 + (-1,-1) = c_5 + (-1,1) \in S_3 \cap S_5 \\ z &= (1, \varepsilon) = c_2 + (1,-1) \in S_2 \cap S_3 \\ w &= (1, -\varepsilon) = c_6 + (1,1) \in S_5 \cap S_6. \end{aligned}$$

For $i = 1, \ldots, 6$, we write

$$\begin{aligned} t_1 &= \operatorname{aff}\{u, z\} = \{(x,y) : y = x - 1 + \varepsilon\}, \\ t_2 &= \operatorname{aff}\{u, b\} = \left\{(x,y) : y = \frac{\varepsilon}{2-\varepsilon}\,(x - 1 + \varepsilon)\right\}, \\ t_3 &= \operatorname{aff}\{a, w\} = \{(x,y) : y = -\varepsilon\, x\}, \\ t_4 &= \operatorname{aff}\{u, w\} = \{(x,y) : y = -x + 1 - \varepsilon\}, \\ t_5 &= \operatorname{aff}\{b, z\} = \{(x,y) : y = \varepsilon\, x\}, \\ t_6 &= \operatorname{aff}\{u, a\} = \left\{(x,y) : y = \frac{-\varepsilon}{2-\varepsilon}\,(x - 1 + \varepsilon)\right\}. \end{aligned}$$

We claim that t_i, $i = 1, \ldots, 6$, is a line transversal to $\mathcal{F} \backslash \{S_i\}$. We note that

$$c_4 + (-1,1) \in t_1 \cap S_4 \text{ and } c_4 + (-1,-1) \in t_3^- \cap S_4$$

for any $\varepsilon > 0$, and $\varepsilon < \frac{1}{2}$ yields that

$$c_4 + (-1,-1) \in t_6^- \cap S_4,$$

and hence $t_i \cap S_j \neq \emptyset$ for $i \neq j$ easily follows.

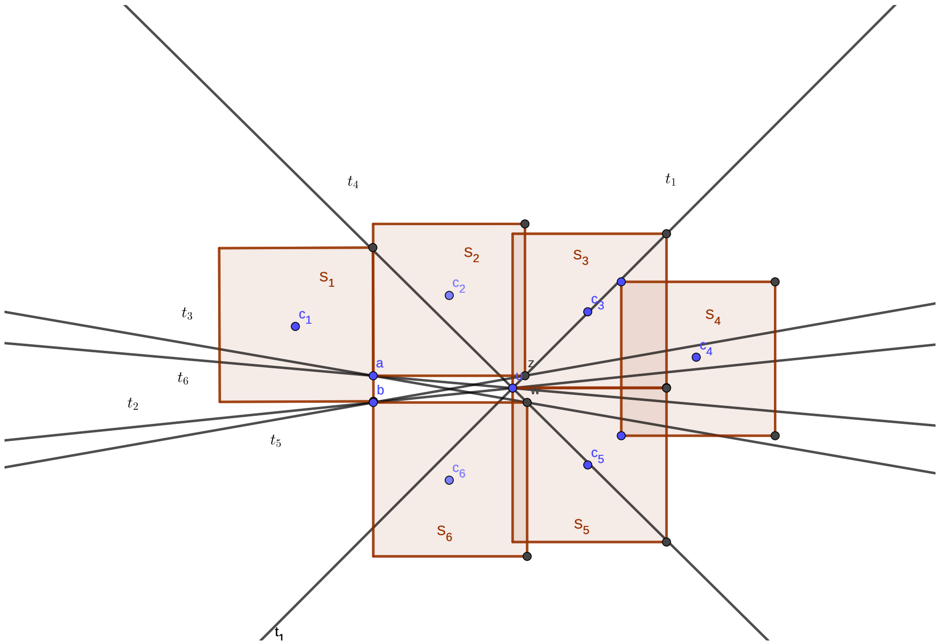

Figure 1: The transversals t_i of $\{S_1, \ldots, S_6\} \backslash \{S_i\}$

Finally, we observe that $S_2 \cap S_6 = \emptyset$ and both t_3 and t_5 are separating and supporting lines of S_2 and S_6. Thus if ℓ is a transversal of $\{S_2, S_6\}$, then

- either ℓ is parallel to v,
- or $\operatorname{slope} \ell \leq \operatorname{slope} t_3 = -\varepsilon$,
- or $\operatorname{slope} \ell \geq \operatorname{slope} t_5 = \varepsilon$.

In particular, if ℓ is parallel to v, then ℓ is disjoint from either S_1 or S_4.

Let $\operatorname{slope} \ell \leq \operatorname{slope} t_3 = -\varepsilon$, and we distinguish two cases. If $\operatorname{slope} \ell > \operatorname{slope} t_4 = -1$ and ℓ intersects S_6, then ℓ is disjoint from S_3. If $\operatorname{slope} \ell \leq \operatorname{slope} t_4 = -1$ and ℓ intersects S_6, then ℓ is disjoint from S_4.

Finally, let $\operatorname{slope} \ell \geq \operatorname{slope} t_5 = \varepsilon$, and we distinguish three cases. If $\operatorname{slope} \ell < \operatorname{slope} t_1 = 1$ and ℓ intersects S_2, then ℓ is disjoint from S_5. If $\operatorname{slope} \ell > \operatorname{slope} t_1 = 1$ and ℓ intersects S_2, then ℓ is disjoint from S_6. If $\operatorname{slope} \ell = \operatorname{slope} t_1 = 1$ and ℓ intersects S_2 and S_5, then $\ell = t_1$, and hence ℓ is disjoint from S_1.

Therefore, $\mathcal{F}$ has no transversal, proving $\mu(C, 5) = 1$. □

3 Proof of Theorem 1.5

Let $\mathcal{F} = \{S_1, \ldots, S_n\}$, $n \geq 6$, be a packing of n translates of the square $[-1, 1] \times [-1, 1]$ such that $\mathcal{F}$ satisfies $T(5)$ and two translates intersect in a common vertex. In addition, let $c_i = (x_i, y_i)$ be the center of S_i, $i = 1, \ldots, n$. We may assume that $c_1 = (-1, 1)$ and $c_2 = (1, -1)$.

We say that S_j and S_k are *split* if $|x_j - x_k| \geq 2$, $|y_j - y_k| \geq 2$ and $|x_j - x_k| + |y_j - y_k| > 4$. It is well known (see Grünbaum [5]) that if $\mathcal{F}$ satisfies $T(5)$ and contains a split pair of squares, then $\mathcal{F}$ has a transversal. Accordingly, we assume that

$$\mathcal{F} \text{ contains no split pair.} \tag{3.1}$$

Case 1 $S_k \cap (h \cup v) = \emptyset$ for some $k \in \{3, \ldots, n\}$

We may assume that $S_3 \cap (h \cup v) = \emptyset$ and $x_3, y_3 > 0$, and hence $x_3, y_3 > 1$. Since S_3 is disjoint and is not split from S_i, $i = 1, 2$, we deduce that

$$1 < x_3 < 3 \text{ and } 1 < y_3 < 3. \tag{3.2}$$

We claim that

$$S_m \cap (h \cup v) \neq \emptyset \text{ for } m \in \{4, \ldots, n\}. \tag{3.3}$$

We suppose that $S_m \cap (h \cup v) = \emptyset$ for an $m \in \{4, \ldots, n\}$, and hence $|x_m|, |y_m| > 1$, and seek a contradiction. If $x_m > 0$ and $y_m > 0$, then as S_3 and S_m do not overlap, (3.2) yields that either $x_m \geq x_3 + 2 > 3$ or $y_m \geq y_3 + 2 > 3$, thus S_m is split for either S_2 or S_1, respectively. If $x_m < 0$ and $y_m < 0$, then S_m is split from S_3, and if $x_m > 0$ and $y_m < 0$, then S_m is split from S_1; futhermore, if $x_m < 0$ and $y_m > 0$, then S_m is split from S_2. In turn, we conclude (3.3). It follows from (3.3) that possibly after interchanging h and v, and a reflection to keep S_3 in the first quadrant, we may assume that $S_4 \cap v \neq \emptyset$ and $S_5 \cap v \neq \emptyset$.

If v is a transversal of $\mathcal{F}$, then Theorem 1.5 has been proved. Therefore we may assume that $S_6 \cap v = \emptyset$, and hence $S_6 \cap h \neq \emptyset$ by (3.3).

As $S_6 \cap v = \emptyset$ and $S_6 \cap h \neq \emptyset$, we have $|x_6| > 1$ and $|y_6| \leq 1$. Since S_6 does not overlap S_1, and is not split from either one of S_2 and S_3, we deduce that

$$\text{if } x_6 < 0, \text{ then } x_6 \leq -3 \text{ and } -1 < y_6 < 1. \tag{3.4}$$

On the other hand, since S_6 does not overlap S_2 and S_3, and is not split from S_1, we deduce that

$$\text{if } x_6 > 0, \text{ then } x_6 \geq 3 \text{ and } -1 < y_6 \leq 1. \tag{3.5}$$

Turning to S_4 and S_5, we may assume that $y_4 \leq y_5$, and if $y_4 = y_5$, then $x_4 = -1$ and $x_5 = 1$.

Case 1.1 $x_4 < 0$

Since S_4 does not overlap S_1 and S_2, and is not split from S_3, we have $y_4 \leq -3$. Thus $-1 \leq x_4 \leq 1$, (3.4) and (3.5) yield that S_4 and S_6 are split, contradicting (3.1).

Case 1.2 $x_4 > 0$

Since S_4 does not overlap S_1, S_2 and S_3, we have

$$y_4 \geq 3 \text{ and } -1 \leq x_4 \leq 1, \text{ and if } y_4 = 3, \text{ then even } x_4 \leq x_3 - 2 < 1. \tag{3.6}$$

Comparing (3.6) to (3.4) and (3.5) shows that S_4 and S_6 are split, contradicting again (3.1).

Case 2 $S_k \cap (h \cup v) \neq \emptyset$ for any $k \in \{3, \ldots, n\}$

We may assume that neither h nor v is a transversal of $\mathcal{F}$, thus we may assume that $|y_3| > 1$ and $|x_4| > 1$. In addition, we may assume that S_3 is farthest from h, S_4 is farthest from v, and S_3 is closer to h than S_4 to v; or in other words, $1 < y_3 \le |x_4|$, $y_3 \ge |y_i|$ for $i \ge 3$ and $|x_4| \ge |x_i|$ for $i \ge 3$. It follows from $S_3 \cap v \ne \emptyset$ and $S_4 \cap h \ne \emptyset$ that

$$-1 \le x_3 \le 1 \text{ and } -1 \le y_4 \le 1.$$

Since S_3 and S_4 are not split, we have $y_3 \le 3$ and if $y_3 = 3$, then either $c_3 = (1,3)$ and $c_4 = (3,1)$, or $c_3 = (-1,3)$ and $c_4 = (-3,1)$. However, if $c_3 = (-1,3)$, then S_2 and S_3 are split, thus if $y_3 = 3$, then $c_3 = (1,3)$ and $c_4 = (3,1)$.

Case 2.1 $y_3 = 3$, and hence $c_3 = (1,3)$ and $c_4 = (3,1)$

In this case, the only common transversals of $\{S_1, S_2, S_3, S_4\}$ are $\ell_1 = \{(x,y) : y = x\}$ and $\ell_2 = \{(x,y) : y = 1 - x\}$. Let us assume that ℓ_2 is not a transversal of $\mathcal{F}$, thus we may assume that $S_5 \cap \ell_2 = \emptyset$ and ℓ_1 is a common transversal of $\{S_1, S_2, S_3, S_4, S_5\}$. In addition, we may assume that $|x_5| \le |y_5|$.

As $S_5 \cap (h \cup v) \ne \emptyset$, $S_5 \cap \ell_1 \ne \emptyset$ and $S_5 \cap \ell_2 = \emptyset$, we deduce that $S_5 \subset l_2^-$. Therefore combining the conditions $|x_5| \le |y_5|$, $S_5 \cap (h \cup v) \ne \emptyset$, $S_5 \cap \ell_1 \ne \emptyset$ and S_5 does not overlap S_1 and S_2 implies that $x_5 = -1$ and $-3 \le y_5 \le -1$. In particular, S_3 and S_5 are split, contradicting (3.1).

Case 2.2 $y_3 < 3$

Since S_3 does not overlap S_1 and S_2, we have $x_3 = 1$ and

$$x_3 = 1 \text{ and } 1 < y_3 < 3. \tag{3.7}$$

We claim that

$$y_3 - 2 \le y_i \le 1 \text{ and } |x_i| \ge 3 \text{ for } i = 4, \dots, n. \tag{3.8}$$

We suppose that there exists $j \in \{4, \dots, n\}$ with $y_j < y_3 - 2$, and seek a contradiction. As $1 > y_j \ge -|y_3| > -3$ and S_j does not overlap with S_1, S_2, S_3, we deduce that $|x_j| \ge 3$. Therefore $y_j < y_3 - 2$ and (3.7) imply that S_j and S_3 are split, contradicting (3.1), and verifying that $y_i \ge y_3 - 2$ for $i = 4, \dots, n$. For any $i = 4, \dots, n$, we have $-1 < y_3 - 2 \le y_i \le y_3 < 3$, S_i does not overlap S_1, S_2, S_3 and $S_i \cap (h \cup v) \ne \emptyset$, therefore $|x_i| \ge 3$ and $y_i \le 1$, as in (3.8).

We set $\ell_1 = \operatorname{aff}\{(0, y_3 - 1), (2, 0)\}$ and $\ell_2 = \operatorname{aff}\{(2, y_3 - 1), (0, 0)\}$, and note that they are separating and supporting lines of S_2 and S_3 with $\operatorname{slope} \ell_1 < 0$ and $\operatorname{slope} \ell_2 > 0$. We may assume that ℓ_1 is not a transversal of $\mathcal{F}$, and hence there exists $m \in \{4, \dots, n\}$ such that $\ell_1 \cap S_m = \emptyset$. In particular, either $x_m \ge 3$ and $S_m \subset \ell_1^+$, or $x_m \le -3$ and $S_m \subset \ell_1^-$.

We observe that if ℓ is a transversal of S_2 and S_3 with $\operatorname{slope} \ell < 0$, then

$$\{(x,y) \in \ell_1^+ : x \ge 2\} \subset \ell^+ \text{ and } \{(x,y) \in \ell_1^- : x \le -2\} \subset \ell^-. \tag{3.9}$$

We claim that

$$S_i \cap \ell_2 \ne \emptyset \text{ for } i = 1, \dots, n. \tag{3.10}$$

Let ℓ be a transversal of S_1, S_2, S_3, S_m, S_i, and hence (3.9) yields that $\operatorname{slope} \ell > 0$. Since ℓ is a transversal of S_1 and S_2, it contains the origin $(0,0)$. As $S_i \cap h \ne \emptyset$ and ℓ_2 has minimal slope among transversals of S_2 and S_3, we deduce that $S_i \cap \ell_2 \ne \emptyset$. In turn, we conclude from (3.10) that ℓ_2 is a transversal of $\mathcal{F}$. $\square$

4 Proof of Theorem 1.6

For references about Minkowski Geometry and properties of ovals in this section, see Schneider [13] and Thompson [14]. For an oval C, we say that a polygon P is circumscribed around C (inscribed into C) if each side of P touches P (each vertex of P lies on the boundary ∂P of P), respectively. We say that a polygon P is an affine regular hexagon if it is the image of a regular hexagon by a linear transformation. The proof of Theorem 1.6 rests on the following statement.

Proposition 4.1. *If C is an origin symmetric oval that is not a parallelogram, then there exists an affine regular hexagon H circumscribed around C such that no vertex of H lies in C.*

Since the proof of Proposition 4.1 is rather technical and uses ideas very different from the ones used in the rest of the paper, we present the argument in the Appendix (Section 5).

The following observation due to Tverberg in [15] shows that it is sufficient to consider origin symmetric ovals in our study.

Lemma 4.2 (Tverberg)**.** *For any oval C and $x_1, \ldots, x_k \in \mathbb{R}^2$, $x_1 + C, \ldots, x_k + C$ has a transversal if, and only if, $x_1 + \frac{1}{2}(C - C), \ldots, x_k + \frac{1}{2}(C - C)$ has a parallel transversal.*

Proof. We fix a line ℓ passing through the origin, and search for transversals parallel to ℓ. Let u be a unit vector orthogonal to ℓ, and let $b > a$ be defined by the property that $\ell + tu$ intersects C if, and only if, $a \leq t \leq b$, and hence $\ell + tu$ intersects $x + \frac{1}{2}(C - C)$ if, and only if, $\frac{a-b}{2} \leq t \leq \frac{b-a}{2}$. We write $u \cdot v$ to denote the scalar product of the vectors u and v. For an $x \in \mathbb{R}^2$ and $t, s \in \mathbb{R}$, it follows that $\ell + tu$ intersects $x + C$ if, and only if, $a + x \cdot u \leq t \leq b + x \cdot u$; moreover, $\ell + su$ intersects $x + \frac{1}{2}(C - C)$ if, and only if, $\frac{a-b}{2} + x \cdot u \leq s \leq \frac{b-a}{2} + x \cdot u$, which is in turn equivalent to saying that $\ell + (s + \frac{a+b}{2})u$ intersects $x + C$. We conclude that a line $\ell + su$ parallel to ℓ is a transversal of $x_1 + \frac{1}{2}(C - C), \ldots, x_k + \frac{1}{2}(C - C)$ if, and only if, $\ell + (s + \frac{a+b}{2})u$ is a transversal of $x_1 + C, \ldots, x_k + C$. □

Proof of Theorem 1.6: It follows from Tverberg [15] (see also (1.1)) that $\mu(C, 5) \leq 1$ for any oval C.

Let us turn to the proof of $\mu(C, 5) \geq \frac{2}{3}$ for any oval C. Since $\frac{1}{2}(C - C)$ is a parallelogram if, and only if, C is a parallelogram, we may assume that C is origin symmetric according to Lemma 4.2.

If the origin symmetric oval C is a parallelogram, then Theorem 1.2 verifies $\mu(C, 5) = 1$. Therefore we assume that C is an origin symmetric oval that is not a parallelogram, and hence Proposition 4.1 yields a circumscribed (origin symmetric) affine regular hexagon H such that no vertex of H is contained in ∂C.

Let $H_0 = \frac{2}{3}H$, and let $H_1, \ldots, H_6$ be the six non-overlapping translates of H_0 in a way such that $H_0 \cap H_i$ is a common side for $i = 1, \ldots, 6$, and $H_1, \ldots, H_6$ are situated around H_0 in counterclockwise order. We write c_i to denote the center of H_i, and hence $c_1 + \frac{2}{3}C, \ldots, c_6 + \frac{2}{3}C$ do not overlap.

Let us consider the family $\mathcal{F} = \{c_1 + C, \ldots, c_6 + C\}$. We observe that $c_1 + H, c_3 + H, c_5 + H$ enclose a triangle T_{135}. For $i = 1, 3, 5$, T_{135} has a common side with $c_i + H$ which touches $c_i + C$, and let ℓ_i be the line containing this side. We observe that ℓ_i, $i = 1, 3, 5$, touches $c_1 + C, c_3 + C, c_5 + C$, it is a common transversal to $\mathcal{F} \backslash \{c_j + C\}$ where $j \in \{1, \ldots, 6\}$ and $|j - i| = 3$, and $\ell_i \cap (c_j + C) = \emptyset$.

Similarly, $c_2 + H, c_4 + H, c_6 + H$ enclose a triangle T_{246}. For $i = 2, 4, 6$, T_{246} has a common side with $c_i + H$ which touches $c_i + C$, and let ℓ_i be the line containing this side. We observe that ℓ_i, $i = 2, 4, 6$, touches $c_2 + C, c_4 + C, c_6 + C$, it is a common transversal to $\mathcal{F}\backslash\{c_j + C\}$ where $j \in \{1, \dots, 6\}$ and $|j - i| = 3$, and $\ell_i \cap (c_j + C) = \emptyset$.

So far we have verified that $c_1 + \frac{2}{3}\,C, \dots, c_6 + \frac{2}{3}\,C$ do not overlap, $\mathcal{F}$ satisfies $T(5)$, and the fact that $\mathcal{F}$ has no transversal provided for any transversal ℓ of $c_1 + C, c_3 + C, c_5 + C$, we have

$$\ell \in \{\ell_1, \ell_3, \ell_5\}. \tag{4.1}$$

Since each ℓ_i, $i = 1, 3, 5$, separates two of $c_1 + C, c_3 + C, c_5 + C$, we may assume that ℓ is not parallel to ℓ_1, ℓ_3, ℓ_5. In this case, there exists a vertex v of T_{135} and a line ℓ' parallel to ℓ such that ℓ' passes through v and intersects $\operatorname{int} T_{135}$. We may assume that $\{v\} = (c_1 + H) \cap (c_3 + H)$. As ℓ' strictly separates $(c_1 + H)\backslash\{v\}$ and $(c_3 + H)\backslash\{v\}$ and $v \notin (c_i + C)$ for $i = 1, 3$, we deduce that ℓ' strictly separates $c_1 + C$ and $c_3 + C$. This contradicts that ℓ intersects both $c_1 + C$ and $c_3 + C$, and proves (4.1). In turn, we conclude Theorem 1.6. □

5 Appendix - proof of Proposition 4.1

We prove in fact Proposition 5.1 (the equivalent form of Proposition 4.1 via polarity) through a series of simple statements Lemma 5.2, Lemma 5.3 and Lemma 5.4.

If C is an oval with $o \in \operatorname{int} C$, then its polar is the oval

$$C^* = \{p \in \mathbb{R}^2 : \langle p, q\rangle \le 1\ \forall q \in C\}.$$

We note that $(C^*)^* = C^*$, and assuming that $C \subset K$ for an oval K, we have $K^* \subset C^*$. If C is a polygon, then so is C^*, and there exists a bijective correspondence between the vertices of P and the sides of P^*; namely, if v is a vertex of P, then $\{p \in C^* : \langle p, v\rangle = 1\}$ is the corresponding side of C^*. Since if A is a linear transformation and C is any oval, then $(AC)^* = A^{-t}C^*$ where A^{-t} is the inverse of the transpose of A, we have that P^* is an affine regular hexagon for any affine regular hexagon P centered at the origin, and P^* is a parallelogram for any parallelogram P centred at the origin.

Polarity shows that Proposition 4.1 is equivalent to Proposition 5.1.

Proposition 5.1. *If C is an origin symmetric oval that is not a parallelogram, then there exists an affine regular hexagon H inscribed into C such that no side of H lies in ∂C.*

Any origin symmetric oval C induces a Minkowski geometry where the length of a segment $[p, q]$ with endpoints p and q is $\|p - q\|_C$. For a polygon P, its correponding Minkowski perimeter $M_C(P)$ is the sum of the lengths of its sides with respect to $\|\cdot\|_C$. This notion of Minkowski perimeter can be extended to any oval K by approximation where $M_C(K_1) \le M_C(K_2)$ holds for ovals K_1 and K_2 satisfying $K_1 \subset K_2$. The following statement is well known, see Lemma 4.1.1 in Thompson [14] or Martini, Swanepoel, Weiss [11], or Asplund and Grünbaum [1] for related results.

Lemma 5.2. *If C is an origin symmetric oval, then for any $p \in \partial C$, there exists an origin symmetric affine regular hexagon H inscribed into C such that p is a vertex of H.*

Actually Lemma 4.1.1 in Thompson [14] states that there exists a $q \in \partial C$ in Lemma 5.2 such that $q - p \in \partial C$, and therefore $\pm p, \pm q, \pm(q - p)$ are vertices of an inscribed affine regular hexagon. We observe that if H is an origin symmetric affine regular hexagon inscribed into an origin symmetric oval C, then each side of H is of length 1 with respect to both $\|\cdot\|_H$ and $\|\cdot\|_C$. The self-perimeter of any origin symmetric oval is between 6 and 8 according to Golab [4]. For the sake of the reader, we present the simple argument.

Lemma 5.3 (Golab)**.** *If C is an origin symmetric oval, then $6 \leq M_C(C) \leq 8$.*

Remark We have $M_C(C) = 6$ if C is an affine regular hexagon, and $M_C(C) = 8$ if C is a parallelogram.

Proof Let H be an affine regular hexagon inscribed into C, and let P be a parallelogram of minimal area containing C. Since the midpoints of P lie in ∂C, we have

$$6 = M_C(H) \leq M_C(C) \leq M_C(P) = 8. \quad \square.$$

We note that Golab [4] defined a notion of self perimeter for any (not necessarily centrally symmetric) oval. For this generalized notion of self perimeter, Grünbaum [6] verified that it is at least 6 (with equality for affine regular hexagons) and at most 9 (with equality for triangles) for any oval.

Lemma 5.4. *If C is an origin symmetric oval that is not a parallellogram, then then there exists a $p \in \partial C$ not lying on any segment contained in ∂C of length longer than 1 with respect to $\|\cdot\|_C$.*

Proof We suppose that ∂C is the union of segments of length longer than 1 with respect to $\|\cdot\|_C$, and seek a contradiction. Since C is origin symmetric, we deduce from Lemma 5.3 that C is a hexagon. Let p_1, p_2, p_3 be vertices of C such that p_2 and p_3 are neighbors of p_1. Let P be the parallelogram such that $\pm p_1$ are opposite vertices and p_2 and p_3 lie on sides of P emanating from p_1. We may assume that P coincides with $[-1, 1] \times [-1, 1]$ in a way such that $p_1 = (1, 1)$, $p_2 = (1 - t, 1)$ and $p_3 = (1, 1 - s)$ where $0 < s, t < 2$. We may also assume that $s \leq t$.

We claim that

$$s > 1. \tag{5.1}$$

We suppose that $s \leq 1$, and seek a contradiction. Since $\|p_3 - p_1\|_C > 1$, it follows that the point $q = (0, -s)$ lies outside of C; therefore, there exists a line ℓ disjoint from C passing through q. Since $(1, 1 - s) \in C$ and $(-1, -1) \in C$, we deduce that $0 < \text{slope}\,\ell < 1$, and hence there exits $w = (-1 + r, -1) \in \ell$ with $0 < r \leq s$. However $-p_2 = (-1 + t, -1)$ lies on ∂C with $t \geq s$, thus $w \in [-p_1, -p_2] \subset \partial C$. This fact contradicts $\ell \cap C = \emptyset$, and in turn proves (5.1).

We deduce from $t \geq s > 1$ that $(1, 0), (0, 1) \in \partial C$, and in turn $p_3 - (-p_2) = (2-t, 2-s) \in \text{int}\, C$, and hence the length of the side $[-p_2, p_3]$ of C is $\|p_3-(-p_2)\|_C < 1$. This contradicts the conditions on C, and completes the proof of Lemma 5.4. $\square$

Proof of Proposition 4.1 In fact, we prove the equivalent Proposition 5.1. Let C be an origin symmetric oval that is not a parallelogram. It follows from Lemma 5.3, that ∂C contains at most 8 maximal segments of length at least 1 with respect to $\|\cdot\|_C$.

According to Lemma 5.4, there exists a $p \in \partial C$ not lying on any segment contained in ∂C of length longer than 1 with respect to $\|\cdot\|_C$. Possibly varying p, we may also assume that

(i) if p is contained in a segment s with $s \subset \partial C$ (thus the length of s is at most one), then p lies in the relative interior of s,

(ii) the line op is not parallel to any segment contained in ∂C of length at least 1 with respect to $\|\cdot\|_C$.

Let H be an affine regular hexagon inscribed into C such that p is a vertex of H. It follows that each side of H is of length 1 with respect to $\|\cdot\|_C$. The two sides of H parallel to p are not contained in ∂C by (ii). If a side s_0 of H containing p or $-p$ is part of ∂C then s_0 is a proper subset of the segment s of length at most 1 by (i); and that is a *reductio ad absurdum*. This completes the proof that no side of H is a subset of ∂C. $\square$

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