On the variance of random polytopes

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Abstract

A random polytope is the convex hull of uniformly distributed random points in a convex body K. A general lower bound on the variance of the volume and f-vector of random polytopes is proved. Also an upper bound in the case when K is a polytope is given. For polytopes, as for smooth convex bodies, the upper and lower bounds are of the same order of magnitude. The results imply a law of large numbers for the volume and f-vector of random polytopes when K is a polytope.

1 The main results

Let $K \subset \mathbb{R}^d$ be a convex set of volume one. Assume $x_1, \dots, x_n$ is a random sample of n independent, uniform points from K. The *random polytope* K_n is just the convex hull of these points: $K_n = [x_1, \dots, x_n]$. It is one of the classical problems in stochastic geometry to investigate the asymptotic behaviour of K_n, see, e.g., the book of Kendall and Moran [14], and the recent book on stochastic geometry of Schneider and Weil [20]. Starting with Rényi and Sulanke [16] in 1963, there have been many results concerning the expectation of various functionals of K_n. For instance, the expectation of the volume $V(K_n)$, and of the number, $f_\ell(K_n)$, of ℓ-dimensional faces of K_n ($\ell = 0, \dots, d-1$) have been determined, see [23] for an extensive survey, and also [7] for more recent results.

Yet determining the variance is in general still an open problem. For *smooth* convex bodies this has been solved, up to order of magnitude, by Reitzner [17] and [19], extending an earlier upper bound, for the case of the unit ball, by Küfer [15] (and some other sporadic results in dimension 2). Recently Schreiber and Yukich [21] have determined the precise asymptotic behaviour of the variance of $f_0(K_n)$ when K is the unit ball, a significant breakthrough.

On the other hand for convex *polytopes* much less is known, and it seems that the situation there is much more delicate. In this case we denote the underlying polytope by P instead of K and the random polytope by P_n. In the planar case, variances and central limit theorems for $f_0(P_n)$ and $V(P_n)$ were proved by Groeneboom [12], and Cabo and Groeneboom [9], but it seems that the stated variances are incorrect (see the discussion in Buchta [8]). In this paper we determine the order of magnitude of the variance of the volume and the number of ℓ-dimensional faces of the random polytope when the mother body

P is a polytope in $\mathbb{R}^d$. Let $F(P)$ denote the number of flags of P. A *flag* is a sequence of faces $F_0, F_1, \ldots, F_{d-1}$ of P such that, for all i, $\dim F_i = i$ and $F_i \subset F_{i+1}$.

Theorem 1.1. *Assume P is a polytope of volume one. Let P_n be the random polytope inscribed in P. Then*

$$\begin{aligned} \mathrm{Var} V(P_n) &\ll F(P)^3 n^{-2} (\log n)^{d-1}, \\ \mathrm{Var} f_\ell(P_n) &\ll F(P)^3 (\log n)^{d-1}. \end{aligned}$$

Here (and throughout the paper) we use Vinogradov's $\ll$ notation, that is, we write $f(n) \ll g(n)$ if there are constants $C > 0$ and n_0, independent of n, such that $|f(n)| < Cg(n)$ for all $n \geq n_0$. The constants C and n_0 may, and usually do, depend on the dimension, but not on the convex polytope P or on the convex body K. Most likely, in both bounds the coefficient $F(P)^3$ can be replaced by $F(P)$.

From Theorem 1.1 we deduce a law of large numbers for the random variables $V(P_n)$ and $f_\ell(P_n)$. It is known by work of Bárány and Buchta [3] and Reitzner [18], that for P a polytope of volume one

$$\begin{aligned} 1 - \mathbb{E}V(P_n) &= \frac{F(P)}{(d+1)^{d-1}(d-1)!} n^{-1} (\log n)^{d-1} (1 + o(1)), \\ \mathbb{E} f_\ell(P_n) &= c(d,\ell) F(P) (\log n)^{d-1} (1 + o(1)), \end{aligned}$$

where $c(d,\ell) > 0$ is a constant depending on d and ℓ. Chebyshev's inequality, the above stated expectations and Theorem 1.1 immediately gives the following Corollary.

Corollary 1.2. *Assume P is a polytope of volume one. Let P_n be the random polytope inscribed in P. Then*

$$\begin{aligned} (1 - V(P_n))\, n (\log n)^{-(d-1)} &\to \frac{F(P)}{(d+1)^{d-1}(d-1)!}, \\ f_\ell(P_n)\, (\log n)^{-(d-1)} &\to c(d,\ell) F(P) \end{aligned}$$

in probability as $n \to \infty$.

It can be observed that the estimates for the variance in Theorem 1.1 and the corresponding results for smooth convex sets in [17] are closely related to the so-called *floating body* of K. To explain what the floating body is we first define the function $v : K \to \mathbb{R}$ via

$$v(z) = \min\{V(K \cap H) : \ H \text{ is a halfspace and } z \in H\}.$$

The *floating body* with parameter t is just the level set $K(v \geq t) = \{z \in K : v(z) \geq t\}$, which is clearly convex. The *wet part* is $K(v \leq t)$, that is, where

v is at most t. The name comes from the 3-dimensional picture when K is a container containing t units of water.

The volume of the wet part $V(K(v \leq t))$ is known when K is a smooth convex body and when it is a polytope. The case of polytopes is the main object of interest in this paper. It follows from results of Affentranger, Wieacker [1] and Bárány, Buchta [3], that for a polytope $P \subset \mathbb{R}^d$ of volume one, and for small enough $t > 0$

$$V(P(v \leq t)) = \frac{1}{(d+1)^{d-1}(d-1)!} F(P)\, t(\log 1/t)^{d-1}(1+o(1)). \tag{1.1}$$

Comparing Theorem 1.1 with (1.1) leads us to **conjecture** that for general convex bodies $K \subset \mathbb{R}^d$ the variance $\mathrm{Var} V(K_n)$ is - up to constants - always of order $n^{-1}V(K(v \leq n^{-1}))$, and the variance $\mathrm{Var} f_\ell(K_n)$ is always of order $nV(K(v \leq n^{-1}))$.

The second main result of this paper confirms this conjecture partially. We prove lower bounds for the variance of the random variables $V(K_n)$ and $f_\ell(K_n)$ for general convex sets K.

Theorem 1.3. *Assume K is a convex body of volume one. Then*

$$\begin{aligned} n^{-1}V(K(v \leq n^{-1})) &\ll \mathrm{Var} V(K_n) \\ nV(K(v \leq n^{-1})) &\ll \mathrm{Var} f_\ell(K_n). \end{aligned}$$

Thus for a polytope P in $\mathbb{R}^d$ of unit volume we have

$$\begin{aligned} F(P) n^{-2}(\log n)^{d-1} &\ll \mathrm{Var} V(P_n) \ll F(P)^3 n^{-2}(\log n)^{d-1}, \\ F(P)(\log n)^{d-1} &\ll \mathrm{Var} f_\ell(P_n) \ll F(P)^3 (\log n)^{d-1}. \end{aligned}$$

In Section 2 a second well-known notion of a random polytope, the Poisson polytope Π_n is introduced, and analogous lower bounds on the corresponding variances are stated there. In Sections 4 and 5 we give the detailed proof of the above results concerning the variance of the volume of K_n, resp. P_n. In section 6 we sketch the proofs for the variance for $f_\ell(K_n)$, resp. $f_\ell(P_n)$. Auxiliary definitions and results are given in Section 3.

Further distributional aspects of the volume and the number of faces will be discussed in a forthcoming paper [6], where we prove a central limit theorem for the volume and the f-vector of the Poisson random polytope Π_n (the definition is given in Section 2).

Both the upper and lower bounds on the variances in question build on the methods developed by Reitzner in [17] and [19] for smooth convex bodies. The main novelty in this paper is twofold. The first is the extension of the technique to give lower bound for general convex bodies (Theorem 1.3). This is achieved by using methods of convex geometry which was inspired by the philosophy of the cap covering theorem, see Theorem 3.2 below. The second main novelty is the upper bound on the variance for polytopes. The proof is based on the Efron-Stein jackknife inequality plus the cap covering theorem applied to convex polytopes. This application uses a subtle estimate of the volume of the visible part of $P(v \leq t)$, see Lemma 3.3 for details. Similar methods are used in [6] for the proof of the central limit theorem. Actually, the results of [6] and of this paper were reached simultaneously. We decided to separate the material by publishing the results in two (almost) self-contained papers in order to make them both shorter and also more accessible for the imaginary reader.

2 Poisson polytopes

As it turns out it is often more convenient, and perhaps more natural, to work with *Poisson polytopes*, see e.g. [6], [9], [12], [19]. To define the Poisson polytope Π_n inscribed in a convex body K, one first considers a Poisson point process $X(n)$ in $\mathbb{R}^d$ of intensity n and let Π_n be just $[K \cap X(n)]$, the convex hull of the points lying in K. This is the same as choosing first a random number N which is Poisson distributed with mean n, and then choosing N random, uniform independent points $x_1, \dots, x_N$ from K and let Π_n be the random polytope $K_N = [x_1, \dots, x_N]$.

As expected, the random polytope K_n and the Poisson polytope Π_n are very close to each other. The following result is a lower bound on the variances of these random variables and is analogous to Theorem 1.3.

Theorem 2.1. *If $K \subset \mathbb{R}^d$ is a convex body of volume one, then*

$$\begin{aligned} n^{-1} V(K(v \leq n^{-1})) &\ll \operatorname{Var} V(\Pi_n), \\ n V(K(v \leq n^{-1})) &\ll \operatorname{Var} f_\ell(\Pi_n) \end{aligned}$$

The proof of this result is almost identical to that of Theorem 1.3. It will be given in the end of Section 4. We mention further that the upper bounds of Theorem 1.1 are valid for $\operatorname{Var} V(\Pi_n)$ and $\operatorname{Var} f_\ell(\Pi_n)$ as well. We omit the straightforward proof.

3 Notation and background

To avoid some trivial complications we assume that the dimension d is at least 2. The unit sphere is S^{d-1}. As usual, $h_K(u)$ denotes the support function of K in direction $u \in S^{d-1}$:

$$h_K(u) = \max\{u \cdot x : \ x \in K\}.$$

A cap C of K is the intersection of K with a closed halfspace. This halfspace can be written as $\{x \in \mathbb{R}^d \mid u \cdot x \geq h_K(u) - \tau\}$ with $u \in S^{d-1}$. Thus

$$C = K \cap \{x \in \mathbb{R}^d \mid u \cdot x \geq h_K(u) - \tau\}.$$

The *bounding hyperplane* of C is the one with equation $u \cdot x = h_K(u) - \tau$. We define, for $\lambda > 0$, C^λ by

$$C^\lambda = K \cap \{x \in \mathbb{R}^d \,|\, u \cdot x \geq h_K(u) - \lambda\tau\}.$$

The *centre* of the cap $C = K \cap \{x \in \mathbb{R}^d :\ u \cdot x \geq h_K(u) - \tau\}$ is a point $x \in \partial K$ with $u \cdot x = h_K(u)$. The centre need not be unique, but this will cause no harm. Assuming that x is the centre of C, observe that for $\lambda \geq 1$, $C^\lambda \subset x + \lambda(C - x)$ implying that

$$V(C^\lambda) \leq \lambda^d V(C) \text{ holds for } \lambda \geq 1. \tag{3.1}$$

Recall that the function $v :\ K \to \mathbb{R}$ has been defined by

$$v(z) = \min\{V(K \cap H) :\ H \text{ is a halfspace and } z \in H\}.$$

The *minimal cap* of $z \in K$ is a cap $C(z) = C_K(z)$ containing z such that $v(z) = V(C(z))$. Again, it need not be unique.

The *Macbeath region, or M-region,* for short, with centre z and factor $\lambda > 0$ is

$$M(z, \lambda) = M_K(z, \lambda) = z + \lambda[(K - z) \cap (z - K)].$$

The M-region with $\lambda = 1$ is just the intersection of K and K reflected with respect to z. Thus $M(z, 1)$ is convex and centrally symmetric with centre z, and $M(z, \lambda)$ is a homothetic copy of $M(z, 1)$ with centre z and factor of homothety λ.

This definition is from [11], cf [5] as well. The following result is from [2]. We assume $K \subset \mathbb{R}^d$ is a convex body of volume one. Set

$$t_0 = (16d)^{-2d}. \tag{3.2}$$

Lemma 3.1. *Assume $t \leq t_0$. If the bounding hyperplane of a cap C is tangent to $K(v \geq t)$, then $t \leq V(C) \leq dt$.*

Let $K(v = t) = \partial K(v \geq t)$. Assume $t \leq t_0$ and choose a maximal system of points $Z = \{z_1, \ldots, z_m\}$ on $K(v = t)$ having pairwise disjoint Macbeath regions $M(z_i, \frac{1}{2})$. Such a system will be called *saturated*. Note that Z (and even m) is not defined uniquely. However, for each K (of volume one) and t (with $t \leq t_0$) we fix a saturated system Z. We write $Z(t)$ and $m(t) = |Z(t)|$ when we want to emphasize that our fixed saturated system comes from the level set $K(v = t)$. Evidently, $V(C(z_i)) = t$. Set

$$K_i'(t) = M(z_i, \frac{1}{2}) \cap C(z_i) \text{ and } K_i(t) = C^{16}(z_i),$$

where, of course, $C^{16}(z_i)$ is just $(C(z_i))^\lambda$ with $\lambda = 16$.

The sets $K_i'(t)$ and $K_i(t)$ for $i = 1, \ldots, m(t)$ form an *economic cap covering* in the following result, the so called *economic cap covering theorem*, that comes from Theorem 6 in [5] and Theorem 7 in [2].

Theorem 3.2. *Suppose $t \in (0, t_0]$, $K \subset \mathbb{R}^d$ is a convex body of volume one, and $Z = \{z_1, \dots, z_m\}$ is a saturated system on $K(v = t)$. Then, with $K_i(t)$ and $K_i'(t)$ as defined above, the following holds*

(i) $\bigcup_1^{m(t)} K_i'(t) \subset K(v \le t) \subset \bigcup_1^{m(t)} K_i(t)$,

(ii) $t \le V(K_i(t)) \le 16^d t$, *for* $i = 1, \dots, m(t)$,

(iii) $(6d)^{-d} t \le V(K_i'(t)) \le 2^{-d} t$, $i = 1, \dots, m(t)$,

(iv) every C with $V(C) \le t$ is contained in $K_i(t)$ for some i.

The sets $K_i'(t)$ are pairwise disjoint, all of them have volume $\ge (6d)^{-d} t$, and are all contained in $K(v \le t)$. This gives an upper bound for $m(t)$. Similarly, the sets $K_i(t)$ cover $K(v \le t)$, all of them have volume $\le 16^d t$. This gives a lower bound for $m(t)$. Summarizing, we have

$$\frac{1}{16^d t} V(K(v \le t)) \le m(t) \le \frac{(6d)^d}{t} V(K(v \le t)) \tag{3.3}$$

for $t \le t_0$. We will often use this in the form $V(K(v \le t))/t \ll m(t) \ll V(K(v \le t))/t$. So the inequality $f(t) \ll g(t)$ means that there are constants t_0 and C such that $|f(t)| \le C g(t)$ for all $t \in (0, t_0)$.

We need one more auxiliary result which follows from Lemma 4.1 of the companion paper [6].

Lemma 3.3. *Assume $P \subset \mathbb{R}^d$ is a polytope of volume one, $z \in P$ with $0 < 2v(z) \le t \le (16d)^{-d}$. Let $z_1, \dots, z_m$ (where $m = m(t)$) be a saturated system on $P(v = t)$ and let $K_i(t)$ be the caps from the cap covering theorem. Then the number of caps $K_i(t)$ containing z is at most*

$$\ll F(P) \left(\log \frac{t}{v(z)} \right)^{d-1}.$$

Here (and in the proof to come) the constant implied by $\ll$ depends only on d (and does not depend on $v(z)$). Note that the total number of caps, $m(t) \ll V(P(v \le t))/t \ll F(P) \log^{d-1} 1/t$ which is smaller than the bound given in the lemma when $1/t < t/v(z)$, that is, when $v(z) < t^2$.

Proof. The set of points in $P(v \le T)$ *visible from z* is, by definition,

$$S(z, T) = \{x \in P : \ [x, z] \cap P(v \ge T) = \emptyset\}.$$

Lemma 4.1 from [6] gives an upper bound on the volume of $S(z, T)$. Namely, assuming $0 < v(z) < 1/2$ and $2v(z) \le T$,

$$V(S(z, T)) \ll F(P)\, T \log^{d-1} \left(\frac{T}{v(z)} \right).$$

In our case $V(K_i(t)) \le 16^d t := T$. Thus $z \in K_i(t)$ implies $z_i \in S(z, T)$. Then the set $K_i'(t)$, which is half of the M-region $M(z_i, 1/2)$ cut off from $M(z_i, 1/2)$

by the hyperplane tangent to $P(v \geq t)$ at z_i, lies in $S(z,T)$. As the M-regions $M(z_i, 1/2)$ are pairwise disjoint, and each has volume $\gg t$, the number of caps $K_i(t)$ containing z is at most

$$\ll V(S(z,T))/t \ll F(P)\,\frac{T}{t}\log^{d-1}\left(\frac{T}{v(z)}\right) \ll F(P)\log^{d-1}\left(\frac{t}{v(z)}\right).$$

This finishes the proof. □

4 Lower bounds

Proof of Theorem 1.3 for $\mathrm{Var}V(P_n)$.

We start with some geometric preparations. Let $y \in K(v=t)$, and denote by $H(y)$ the bounding hyperplane of the minimal cap of y. Then $y \in H(y)$ and, as is well known, y is the center of gravity of $K \cap H(y)$. According to a classical result of Fritz John, the convex body $K \cap H(y)$ (in the hyperplane $H(y)$) is sandwiched between two concentric and homothetic ellipsoids with ratio of homothety $d-1$. We need a strengthening of this result where the common center of the ellipsoids coincides with the center of gravity, y, of the convex body $K \cap H(y)$. This is given by a recent result of Kannan, Lovász, and Simonovits, Theorem 4.1 in [13]: there is an ellipsoid $E \subset H(y)$ centered in y such that

$$y + \frac{1}{d-1}(E-y) \subset K \cap H(y) \subset E.$$

We choose a simplex $[x_1,\ldots,x_d] \subset y + \frac{1}{2(d-1)}(E-y)$ of maximal $(d-1)$-dimensional volume. The center of gravity of this simplex is clearly y. Let x be a boundary point of this simplex. We have $y + 2d^2(x-y) \notin E$ and thus $y + 2d^2(x-y)$ is not contained in K. Denote by y_0 the centre of the minimal cap $C(y)$. Then the halfline $y_0 + \lambda(y + 2d^2(x-y) - y_0)$, $\lambda \geq 0$, starting from y_0 meets $H(y)$ when $\lambda = 1$ in a point not contained in K, and thus is also outside K for all $\lambda > 1$. Put $x_0 = y + \frac{1}{2(3d^2-1)}(y_0 - y)$. The halfline $x_0 + \mu(x - x_0)$ $(\mu \geq 0)$ meets the line $y_0 + \lambda(y + 2d^2(x-y) - y_0)$ at $\lambda = \mu/(2d^2) = 3/2$, and thus $\{x_0 + \mu(x-x_0):\ \mu \geq 0\} \cap (K \setminus C^{1.5}(y))$ is empty for all x on the boundary of $[x_1,\ldots,x_d]$. We just proved the following claim.

Claim 4.1. *Suppose $x_0,\ldots,x_d$ are chosen as above and set $\triangle(y) = [x_0,\ldots,x_d]$. Then the simplex $\triangle(y)$ is contained in $M(y,\frac{1}{2}) \cap C(y)$. If for some $x \in K$ the segment $[x,x_0]$ is disjoint from $[x_1,\ldots,x_d]$, then $x \in C^{1.5}(y)$.*

Further observe that the volume of the simplex $\triangle = [x_0,\ldots,x_d]$ is precisely of order t (bounded independently of $y, x_0,\ldots,x_d$). Given $\delta > 0$, let $\triangle_i$ be small homothetic (and uniquely determined) copies of $\triangle$, with center of homothety x_i $(i = 0,1,\ldots,d)$ such that $V(\triangle_i) = \delta t$. By Claim 4.1 and by continuity the following strengthening of the last sentence of Claim 4.1 holds.

Claim 4.2. *There is a small $\delta > 0$, depending only on d and independent of K, y, and $\triangle$ such that, with $V(\triangle_i) = \delta t$, the conditions $z_i \in \triangle_i$ $(i = 0,1,\ldots,d)$, $z \in K$, and $[z,z_0] \cap [z_1,\ldots,z_d] = \emptyset$ imply $z \in C^2(y)$.*

The following observation is important as it connects the geometry of the simplices $\triangle_i$ with the variation in question. For fixed $z_i \in \triangle_i$, $i = 1, \dots, d$, and for randomly and uniformly chosen $Z \in \triangle_0$,

$$\mathrm{Var} V([Z, z_1, \dots, z_d]) \gg t^2. \tag{4.1}$$

The proof is elementary: the function $V([Z, z_1, \dots, z_d])$ is affine equivariant and homogeneous of the same degree as $V(\triangle)$. Thus $\mathrm{Var} V([Z, z_1, \dots, z_d])$ equals $V(\triangle)^2$ times the variance occurring if $\triangle$ is the regular simplex of volume one.

As the last step of the geometric preparations we choose a saturated system $Y(t) = \{y_1, \dots, y_m m\}$ from $K(v = t)$. For each y_j we construct the simplices $\triangle_i(y_j)$. For each j and for fixed $z_i \in \triangle_i(y_j)$ $(i = 1, \dots, d)$ we have $\mathrm{Var}_Z V([Z, z_1, \dots, z_d]) \gg t^2$ where Z varies uniformly in $\triangle_0(y_j)$. Also, inequality (3.3) implies that

$$m = m(t) \gg \frac{1}{t} V(K(v \le t)). \tag{4.2}$$

After these preparations we can now start proving the lower bound on the variance. Set $t = 1/n$ in the previous construction and consider the set $Y = \{y_1, \dots, y_m\} \subset K(v = 1/n)$ and the simplices $\triangle_i(y_j)$. Let $X_n = \{x_1, \dots, x_n\}$ be the random sample of n uniform independent points from K. For $j \in \{1, \dots, m\}$ let A_j be the event that exactly one point of X_n is contained in each set $\triangle_i(y_j)$ and no other point of X_n is in $C^2(y_j)$. Although the event A_j occurs only with small probability, we will show that this probability is bounded away from zero independently of n. Thus A_j will occur regularly, a fixed percentage of the configurations $\triangle_i(y_j)$ will satisfy A_j. This will in turn imply that a fixed percentage of the variance is determined by the variance given A_j. Using as a lower bound for $\mathrm{Var} V(P_n)$ only this conditional variance, the bound in (4.1) will suffice to produce the requested estimate.

Recall that $V(\triangle_i(y_j)) = \delta/n$ and $V(C^2(y_j)) \le 2^d V(C(y_j)) = 2^d/n$ follows from (3.1). Thus

$$\mathbb{P}(A_j) \ge \binom{n}{d+1} \left(\frac{\delta}{n}\right)^{d+1} \left(1 - \frac{2^d}{n}\right)^{n-d-1} \gg 1. \tag{4.3}$$

By (4.2)

$$\mathbb{E}\left(\sum_{j=1}^{m} I(A_j)\right) = \sum_{j=1}^{m} \mathbb{P}(A_j) \gg m \gg nV(K(v \le n^{-1})). \tag{4.4}$$

Assume next that A_j holds, and let $Z_j, z_1, \dots, z_d$, resp., be the points from X_n contained in $\triangle_0(y_j), \triangle_1(y_j), \dots, \triangle_d(y_j)$. Write H for the halfspace which contains Z_j and whose bounding hyperplane contains $z_1, \dots, z_d$. Now Claim 4.2 and condition A_j imply that

$$P_n \cap H = [Z_j, z_1, \dots, z_d]. \tag{4.5}$$

which means that, under A_j and conditioning on the points $z_1, \ldots, z_d$, $P_n \cap H$ depends only on Z_j and $P_n \setminus H$ is independent of Z_j. Further, since the sets $\triangle(y_j)$ are disjoint, Z_j and Z_i are independent for $i \neq j$ if $I(A_i) = I(A_j) = 1$.

Define $\mathcal{H}$ to be the σ-algebra that keeps track of everything except the locations of the points X_i in $\triangle_0(y_j)$ for which A_j occurs. More formally, let $\mathcal{J}$ denote the set of indices for which A_j occurs. Then $\mathcal{H}$ is the σ-algebra generated by $\mathcal{J}$ and

$$\{X_1, \ldots, X_n\} \cap \left(\bigcup_{j \in \mathcal{J}} \triangle_0(y_j) \right)^c.$$

We decompose the variance by conditioning on $\mathcal{H}$:

$$\begin{aligned} \mathrm{Var} V(P_n) &= \mathbb{E}\,\mathrm{Var}(V(P_n) \,|\, \mathcal{H}) + \mathrm{Var}\, \mathbb{E}(V(P_n) |\, \mathcal{H}) \\ &\geq \mathbb{E}\,\mathrm{Var}(V(P_n) \,|\, \mathcal{H}) \end{aligned}$$

Write P^* for the convex hull of the points from $X_n \cap K$ fixed by condition $\mathcal{H}$. Observe now that, under condition $\mathcal{H}$,

$$V(P_n) | \mathcal{H} = \sum_{I(A_j)=1} V([Z_j, z_1, \ldots, z_d]) + V(P^*). \tag{4.6}$$

Here in the summation the random variables are independent and the last term is constant. This implies that

$$\mathrm{Var}(V(P_n) \,|\, \mathcal{H}) = \sum_{I(A_j)=1} \mathrm{Var}_{Z_j} V(P_n)$$

where the variance is taken with respect to the random variable $Z_j \in \triangle_0(y_j)$, and we sum over all $j = 1, \ldots, m$ with $I(A_j) = 1$. Combining this with (4.1) and with (4.4) implies

$$\begin{aligned} \mathrm{Var} V(P_n) &\gg \mathbb{E} \left(\sum_{I(A_j)=1} n^{-2} \right) \gg n^{-2} \mathbb{E} \left(\sum_{j=1}^{m} I(A_j) \right) \\ &\gg n^{-1} V\left(P(v \leq n^{-1})\right) \end{aligned}$$

which is the first part of Theorem 1.3. □

Remark. Note that in (4.6) we made use of the fact that $\triangle_i(y_k) \cap \triangle_j(y_h) = \emptyset$ unless $k = h$ and $i = j$. This follows because system $y_1, \ldots, y_m$ is saturated and so the M-regions $M(y_k, 1/2)$ are pairwise disjoint.

Proof of Theorem 2.1 for $\mathrm{Var} V(\Pi_n)$. The previous proof works with the only change that this time for the estimate $\mathbb{P}(A_j) \gg 1$ one has to use the Poisson distribution: $\mathbb{P}(|\triangle_i(y_j) \cap X(n)| = 1) = nV(\triangle_i(y_j)) \exp\{-nV(\triangle_i(y_j))\} = \delta e^{-\delta}$ and the probability that $C^2(y_j)$ contains no further point of $X(n)$ is bounded from below by $\exp\{-nV(C^2(y_j)\}$. □

5 Proof of Theorem 1.1 for $V(P_n)$

The beginning of this proof works for all convex bodies. We start with a general convex body K of volume one, and change it to a polytope P when we have to.

Let T_n be the event that the floating body $K(v \geq (c \log n)/n)$ is contained in K_n. Here $c = c_d$ is a large constant to be specified soon. We write T_n^c for the complement of T_n. The main result of [4] says that there is a constant $\delta > 0$ depending only on d such that T_n^c occurs with probability $n^{-\delta c}$. For an alternative statement (and proof) see Van Vu's paper [22]

We use the jackknife inequality of Efron and Stein [10], which implies, in the form given by Reitzner [17], that

$$\begin{aligned} \mathrm{Var} V(K_n) \; &\leq \; (n+1) \cdot \mathbb{E}(V(K_{n+1}) - V(K_n))^2 \\ &= \; (n+1) \cdot \mathbb{E}[(V(K_{n+1}) - V(K_n))^2 \mathbf{1}(T_n)] \\ &+ \; (n+1) \cdot \mathbb{E}[(V(K_{n+1}) - V(K_n))^2 \mathbf{1}(T_n^c)]. \end{aligned}$$

The second term here is very small if the constant c is chosen large enough because $(V(K_{n+1}) - V(K_n))^2 \leq V(K_{n+1})^2 \leq V(K)^2$ which is a constant depending only on K, and $\mathbb{E}(\mathbf{1}(T_n^c)) \leq n^{-\delta c}$. We choose $c = c_d$ so large that the second term is smaller n^{-3}, say.

So we need to estimate the first term only. We use, quite naturally, a coupling argument since K_{n+1} is just the convex hull of K_n and x_{n+1}, the last point from the random sample consisting of $n+1$ points from K. For simpler notation we write y for x_{n+1}. Let $\mathcal{F}$ be the collection of those facets F of K_n for which y is not on the same side of the hyperplane $\mathrm{aff} F$ as K_n. Clearly $\mathcal{F} = \emptyset$ if $y \in K_n$. We write $[n]$ for the set $\{1, \ldots, n\}$. The difference $K_{n+1} \setminus K_n$ is the union of (internally disjoint) simplices $[F, y]$ with $F \in \mathcal{F}$. For a d-subset I of $[n]$ let F_I denote the convex hull of $\{x_i : i \in I\}$. Then, with $\sum_I$ denoting summation over all d-element subsets of $[n]$,

$$\begin{aligned} V(K_{n+1}) \; - \; V(K_n) &= \sum_{F \in \mathcal{F}} V([F, y]) \\ &= \sum_I \mathbf{1}\{F_I \in \mathcal{F}\} V([F_I, y]) \leq \sum_I \mathbf{1}\{F_I \in \mathcal{F}\} V(F_I), \end{aligned}$$

where $V(F_I)$ denotes the volume of the cap $C(F_I)$ containing y which is cut off by the hyperplane $\mathrm{aff} F_I$. (This is well defined if $F_I \in \mathcal{F}$, and irrelevant otherwise.) Now

$$(V(K_{n+1}) - V(K_n))^2 \leq \sum_I \sum_J \mathbf{1}\{F_I \in \mathcal{F}\} V(F_I) \mathbf{1}\{F_J \in \mathcal{F}\} V(F_J)$$

By symmetry we can assume $V(F_I) \geq V(F_J)$ (and a factor 2 appears). When integrating, we can assume, again by symmetry, that $I = [d]$, $I \cap J = [k]$ for some $k \in \{0, 1, \ldots, d\}$ and $J = [k] \cup \{d+1, \ldots, 2d-k\}$. Write $F = F_I$ and

$G = F_J$ with these I and J. So we have

$$\mathbb{E}((V(K_{n+1}) - V(K_n))^2 \mathbf{1}(T_n)) \le 2\sum_{k=0}^{d} \binom{n}{d}\binom{d}{k}\binom{n-d}{d-k} \int_K \cdots \int_K \mathbf{1}\{F \in \mathcal{F}\} \times$$
$$\times V(F)\mathbf{1}\{G \in \mathcal{F}\} V(G) \mathbf{1}\{V(F) \ge V(G)\}\mathbf{1}\{T_n\} dx_1 \dots dx_n dy.$$

Let Σ_k denote the above integral for a fixed k without the factor $2\binom{n}{d}\binom{d}{k}\binom{n-d}{d-k}$.

Since $F \in \mathcal{F}$, the variables $x_{2d-k+1}, \dots, x_n$ all lie in the complement of $C(F)$, their total contribution is at most $(1 - V(F))^{n-(2d-k)}$. Note that $\mathbf{1}\{F \in \mathcal{F}\}$ and $\mathbf{1}\{G \in \mathcal{F}\}$ imply $y \in C(F) \cap C(G)$. So we have

$$\Sigma_k \le \int_K \cdots \int_K (1 - V(F))^{n-2d+k} V(F)V(G)\mathbf{1}\{y \in C(F) \cap C(G)\} \times$$
$$\times \mathbf{1}\{V(G) \le V(F) \le \frac{c \log n}{n}\} dx_1 \dots dx_{2d-k} dy,$$

where the condition T_n has been replaced by $V(F) \le \frac{c \log n}{n}$.

We estimate this $2d - k + 1$-fold integral using the cap covering technique, which is based on Theorem 3.2. Let $\mathcal{M}_f = \{C_1^f, \dots, C_{m(2^{-f})}^f\}$ denote the set of caps from the cap covering for $K(v \le 2^{-f})$, (f is an integer) that is, $C_i^f = K_i(2^{-f})$. We assume that $f \ge f_0$ where f_0 is defined by $2^{-f_0} = (c \log n)/n$. Now we associate with every point $x_1, \dots, x_{2d-k}, y$ in the domain of integration two caps C_i^f and C_j^g as follows. By condition (iv) of the cap covering theorem there is a largest positive integer $f \ge f_0$ such that $C(F)$ is contained in some cap $C_i^f \in \mathcal{M}_f$, and, further, there is a largest positive integer g such that $C(G)$ is contained in a cap $C_j^g \in \mathcal{M}_g$. Here $g \ge f$ since $V(G) \le V(F)$. We integrate on these two caps in the sense that variables $x_{k+1}, \dots, x_d$ all lie in C_i^f so we integrate by them on C_i^f, variables $x_{d+1}, \dots, x_{2d-k}$ all lie in C_i^g so we integrate by them on C_j^g and the remaining variables $x_1, \dots, x_k, y$ lie in $C_i^f \cap C_j^g$ and we integrate by them on $C_i^f \cap C_j^g$. Then we sum these integrals for all $C_i^f \in \mathcal{M}_f$ and $C_j^g \in \mathcal{M}_g$, and then for all $g \ge f \ge f_0$.

Integrating on the associated caps is going to be simple. Note first that $V(F) \ge 2^{-(f+1)}$ because of the maximality of f, and so $(1 - V(F))^{n-2d+k} \le (1 - 2^{-f-1})^{n-2d+k}$. Integrating with respect to the variables $x_1, \dots, x_k, y$ on $C_i^f \cap C_j^g$ gives at most $V(C_i^f \cap C_j^g)^{k+1}$, and integrating with respect to the variables $x_{k+1}, \dots, x_d$ on C_i^f gives $\ll (2^{-f})^{d-k+1}$, the extra 1 in the exponent comes from the factor $V(F) \le V(C_i^f)$. Similarly integrating with respect to the variables $x_{d+1}, \dots, x_{2d-k}$ on C_i^f gives $\ll (2^{-g})^{d-k+1}$, the extra 1 in the exponent is due to $V(G) \le V(C_j^g)$. All in all, for a fixed pair C_i^f, C_j^g, the above integral can be bounded as

$$\ll \left(1 - 2^{-f-1}\right)^{n-2d+1} (2^{-f})^{d-k+1} (2^{-g})^{d-k+1} V(C_i^f \cap C_j^g)^{k+1}. \tag{5.1}$$

Thus for fixed f and g the integral with all caps $C_i^f \in \mathcal{M}_f$, $C_j^g \in \mathcal{M}_g$ is bounded by

$$\left(1-2^{-f-1}\right)^{n-2d+1}(2^{-f})^{d-k+1}(2^{-g})^{d-k+1}\sum_{C_i^f\in\mathcal{M}_f,\,C_j^g\in\mathcal{M}_g}V(C_i^f\cap C_j^g)^{k+1}.$$

We bound the sum in the last line using the cap covering theorem again. Let $z\in C_i^f\cap C_j^g$ be the point where the function $v(.)$ takes its maximal value on $C_i^f\cap C_j^g$. Now $C_i^f\cap C_j^g$ is convex and disjoint from $K(v>v(z))$ which is also convex. So they can be separated by a hyperplane. This hyperplane cuts off a small cap off K, whose volume is at most $dv(z)$ by Lemma 3.1. Thus, again by (iv) of the cap covering theorem, there is a maximal integer h such that this cap is contained in some $C_\ell^h\in\mathcal{M}_h$. Of course $h\geq g$, and also, $C_i^f\cap C_j^g\subset C_\ell^h$.

We have to estimate next how many pairs C_i^f, C_j^g go with the same cap $C_\ell^h\in\mathcal{M}_h$. This is easy when K is smooth because then every point $z\in K$ is contained in $\ll 1$ caps from the cap covering $\mathcal{M}_h$.

This is the point where we need to use the fact that the mother body is a polytope P. We use Lemma 3.3 saying that the point z is contained in $\ll F(P)(\log T/v(z))^{d-1}$ caps from a cap covering with parameter T. This bound gives $\ll F(P)(h-f)^{d-1}$ for the number of caps C_i^f containing z provided $h>f$, and $\ll F(P)(h-g)^{d-1}$ for the number of caps C_j^g containing z provided $h>g$.

A little extra care is to be exercised when $h=g$ (or $h=f$). In that case, by (iv) of the cap covering theorem, each cap of volume $\leq 2^{-g}$ is contained in some cap of $\mathcal{M}_{g-1}$. It is clear that each cap in $\mathcal{M}_{g-1}$ contains $\ll 1$ caps from $\mathcal{M}_g$. By Lemma 3.3 the point z is contained in $\ll F(P)(\log 2^{-(g-1)}/v(z))^{d-1}\ll F(P)(h-(g-1))^{d-1}$ caps from $\mathcal{M}_{g-1}$, and then it is contained in $\ll F(P)(1+h-g)^{d-1}$ caps from $\mathcal{M}_g$. Similarly, if $h=f$, then z is contained in $\ll F(P)(1+h-f)^{d-1}$ caps from $\mathcal{M}_f$.

Thus the number of pairs C_i^f, C_j^g with $z\in C_i^f\cap C_j^g$ is $\ll F(P)^2(1+h-f)^{d-1}(1+h-g)^{d-1}$ even if $h=g$ or $h=f$. This is also an upper bound, for fixed $C_\ell^h\in\mathcal{M}_h$, on the number of pairs C_i^f, C_j^g with $C_i^f\cap C_j^g\subset C_\ell^h$.

We use these estimates when the pair f,g is fixed:

$$\begin{aligned}\sum_{C_i^f\in\mathcal{M}_f,\,C_j^g\in\mathcal{M}_g} & V(C_i^f\cap C_j^g)^{k+1}\\ &\ll \sum_{h\geq g}\sum_{C_\ell^h\in\mathcal{M}_h}F(P)^2(1+h-f)^{d-1}(1+h-g)^{d-1}(2^{-h})^{k+1}\\ &\ll \sum_{h\geq g}F(P)^2(1+h-f)^{d-1}(1+h-g)^{d-1}(2^{-h})^{k+1}|\mathcal{M}_h|\\ &\ll \sum_{h\geq g}F(P)^3(1+h-f)^{d-1}(1+h-g)^{d-1}(2^{-h})^{k+1}h^{d-1}.\end{aligned}$$

Here $|\mathcal{M}_h|\ll F(P)(\log 2^h)^{d-1}\ll F(P)h^{d-1}$ follows from (1.1) and (3.3).

The rest of the proof is a straightforward estimation of the infinite sums that come up. It is not hard to see that the last sum is dominated by its first term for instance by checking that the ratio of the $(h+1)$st and hth terms is smaller than 0.9, say. This gives that

$$\sum_{C_i^f \in \mathcal{M}_f,\, C_j^g \in \mathcal{M}_g} V(C_i^f \cap C_j^g)^{k+1} \ll F(P)^3 (1+g-f)^{d-1} (2^{-g})^{k+1} g^{d-1}.$$

Then comes summation for all $g \geq f$. We see again that the corresponding sum is dominated by its first term, and so

$$\sum_{g \geq f} (2^{-g})^{d-k+1} F(P)^3 (1+g-f)^{d-1} (2^{-g})^{k+1} g^{d-1} \ll F(P)^3 (2^{-f})^{d+2} f^{d-1}.$$

Here the factor $(2^{-g})^{d-k+1}$ comes from (5.1). So we have, finally,

$$\Sigma_k \ll F(P)^3 \sum_{f \geq f_0} (1 - 2^{-f-1})^{n-2d+k} (2^{-f})^{2d-k+3} f^{d-1}.$$

Define here f_1 by $2^{-f_1} = 1/n$. We split the last sum into two parts: the first one with $f \geq f_1$ and second one with $f_1 > f \geq f_0$. In the first sum the factor $(1-2^{-f-1})^{n-2d+k} \leq 1$, and without this factor it is dominated by its first term, again:

$$\sum_{f \geq f_1} (1 - 2^{-f-1})^{n-2d+k} (2^{-f})^{2d-k+3} f^{d-1} \ll n^{-2d+k-3} (\log n)^{d-1}.$$

In the second sum we define $f = \lfloor f_1 \rfloor - s$. Then f_1 is almost precisely $\log_2 n$ and s runs from 0 to $\log_2(c \log n)$. With this notation we have

$$\begin{aligned}
\sum_{f_0}^{f_1} & (1 - 2^{-f-1})^{n-2d+k} (2^{-f})^{2d-k+3} f^{d-1} \\
\ll & \sum_{s=0}^{\log_2(c\log n)} \exp\left\{ -\frac{n-2d+k}{2n} 2^s \right\} \left(\frac{2^s}{n} \right)^{2d-k+3} (\log_2 n - s)^{d-1} \\
\ll & \frac{(\log n)^{d-1}}{n^{2d-k+3}} \sum_{s=0}^{\log_2(c\log n)} \exp\{-2^{s-1}\} 2^{(2d+3)s} \left(\frac{\log n - s}{\log n} \right)^{d-1} \\
\ll & \frac{(\log n)^{d-1}}{n^{2d-k+3}} \sum_{s=0}^{\infty} \exp\{-2^{s-1}\} 2^{(2d+3)s} \ll \frac{(\log n)^{d-1}}{n^{2d-k+3}},
\end{aligned}$$

because the sum in the last line is bounded by a constant depending only on d.

We have shown now that $\Sigma_k \ll F(P)^3 \frac{(\log n)^{d-1}}{n^{2d-k+3}}$. Then

$$\sum_{k=0}^{d} 2 \binom{n}{d} \binom{d}{k} \binom{n-d}{d-k} \Sigma_k \ll F(P)^3 \frac{(\log n)^{d-1}}{n^3}.$$

This proves Theorem 1.1 □

To end this section we offer a geometric conjecture that would imply, up to order of magnitude, the same upper bound for $\mathrm{Var} V(K_n))$ and $\mathrm{Var} f_\ell(K_n)$ as the lower bound in Theorem 1.3 for all convex bodies of volume one.

Conjecture. For every $d \geq 2$ there are numbers $T_0 > 0$ and $q > 1$ such that for all convex bodies $K \subset \mathbb{R}^d$ of volume one, and for all $T \in (0, T_0]$, and for all $t \in (0, qT]$ the following holds. Let $D_1, \ldots, D_{m(T)}$, resp. $C_1, \ldots, C_{m(t)}$ be the covering caps for $K(v \leq T)$ and $K(v \leq t)$ from Theorem 3.2. Then

$$\sum_{i=1}^{m(T)} V(K(v \leq t) \cap D_i) \ll \sum_{i=1}^{m(T)} \sum_{j=1}^{m(t)} V(C_j \cap D_i) \ll \sum_{i=1}^{m(T)} V(K(v \leq t) \cap D_i).$$

The lower bound here follows from $V(K(v \leq t) \cap D_i) \ll \sum_1^{m(t)} V(C_j \cap D_i)$ (valid for all i) which is a simple consequence of the cap covering theorem. So the question is the upper bound. The simpler conjecture $\sum_1^{m(t)} V(C_j \cap D_i) \ll V(K(v \leq t) \cap D_i)$ is true in dimension 2 (details will appear elsewhere), but fails in dimension 3 and higher.

Here is a quick **sketch** how the conjecture would imply the upper bound for the variance of $V(K_n)$ for general convex bodies. The proof is the same as above up to (5.1) with the sole exception that this time $\mathcal{M}_f$ is the cap covering with parameter $t = q^{-f}$, and, of course, f_0 is defined by $q^{-f_0} = \frac{c \log n}{n}$. We sum first for fixed f and fixed g the terms

$$\begin{aligned} & (q^{-g})^{d-k+1} \sum_{C_i^f \in \mathcal{M}_f} \sum_{C_j^g \in \mathcal{M}_g} V(C_i^f \cap C_j^g)^{k+1} \\ \leq \quad & (q^{-g})^{d+1} \sum_{C_i^f \in \mathcal{M}_f} \sum_{C_j^g \in \mathcal{M}_g} V(C_i^f \cap C_j^g) \\ \ll \quad & (q^{-g})^{d+1} \sum_{C_i^f \in \mathcal{M}_f} V(K(v \leq q^{-g}) \cap C_i^f) \end{aligned}$$

where the last inequality is implied by the Conjecture. Summing this for all $g \geq f$ is easy because the first term dominates the sum, and we have

$$\begin{aligned} \Sigma_k \quad & \ll \quad \sum_{f \geq f_0} \left(1 - q^{-f}\right)^{n-2d+k} (q^{-f})^{2d-k+2} \sum_{C_i^f \in \mathcal{M}_f} V(K(v \leq q^{-f}) \cap C_i^f) \\ & \ll \quad \sum_{f \geq f_0} \left(1 - q^{-f}\right)^{n-2d+k} (q^{-f})^{2d-k+2} V(K(v \leq q^{-f})). \end{aligned}$$

Splitting the last sum into two parts at f_1 with $q^{-f_1} = 1/n$ shows, the same way as above, that $\Sigma_k \ll n^{-2d+k-2} V(K(v \leq n^{-1}))$. This implies that $\mathrm{Var} V(K_n) \ll n^{-1} V(K(v \leq n^{-1}))$, as promised.

6 Sketch of proof for $f_\ell(K_n)$.

For the **proof** of second part of Theorem 1.3 we use exactly the same method with one exception: instead of choosing one random point Z in $\triangle_0(y_j)$ we choose two random points Z_1, Z_2. Observe that $[Z_1, Z_2, z_1, \dots, z_d]$ can either be a simplex or can have both points Z_1, Z_2 as vertices and thus $f_\ell([Z_1, Z_2, z_1, \dots, z_d])$ attains at least two distinct values with positive probability. The essential change is that now

$$\mathrm{Var}_{Z_1,Z_2} f_\ell([Z_1, Z_2, z_1, \dots, z_d]) \gg 1$$

for all $\ell = 0, \dots, d-1$. For $j \in \{1, \dots, m\}$ let A_j be the event that exactly two random points, from the random sample $\{x_1, \dots, x_n\}$, are contained in the simplex $\triangle_0(y_j)$ and one in each $\triangle_i(y_j)$, $i = 1, \dots, d$, and no further random point is contained in $C^2(y_j)$. Then, the same way as before, $\mathbb{P}(A_j) \gg 1$, and analogously we obtain

$$\mathrm{Var} f_\ell(P_n) \gg nV\left(P\left(v \leq \frac{1}{n}\right)\right).$$

The **proof** of Theorem 2.1 for $\mathrm{Var} f_\ell(\Pi_n)$ uses the above argument. Here $\mathbb{P}(A_j) \gg 1$ follows the same way as in the proof of Theorem 2.1 for $\mathrm{Var} V(\Pi_n)$.

For the **proof** of the remaining part of Theorem 1.1 we use again the Efron-Stein jackknife inequality in the form

$$\mathrm{Var} f_\ell(K_n) \leq (n+1) \cdot \mathbb{E}\left(f_\ell(K_{n+1}) - f_\ell(K_n)\right)^2.$$

In the same way as previously it suffices to give an upper bound on the expectation $\mathbb{E}\left(f_\ell(K_{n+1}) - f_\ell(K_n)\right)^2 \mathbf{1}\{T_n\}$ where T_n is the same event as before.

We use again a coupling argument, and the same notation $y = x_{n+1}$ and $\mathcal{F}$ for the facets of K_n disappearing with the appearance of y. Nothing changes if $y \in K_n$, but if $y \notin K_n$, then some new ℓ-dimensional faces are created, and some old ℓ-dimensional faces disappear. It is not hard to see, using the fact that K_n is simplicial, that $|f_\ell(K_{n+1}) - f_\ell(K_n)| \ll |\mathcal{F}|$. So we are to estimate

$$\mathbb{E}(|\mathcal{F}|^2 \mathbf{1}(T_n)) = \mathbb{E}(\sum_I \mathbf{1}\{F_I \in \mathcal{F}\})^2 \mathbf{1}\{T_n\},$$

where the summation is taken over all d-element subsets of $[n]$ and F_I is the convex hull of $\{x_i : i \in I\}$. Again, the square in this expectation can be written as

$$\left(\sum_I \mathbf{1}\{F_I \in \mathcal{F}\}\right)\left(\sum_J \mathbf{1}\{F_J \in \mathcal{F}\}\right).$$

We let k run from 0 to d and separate the terms here with $|I \cap J| = k$. By symmetry I can be taken to be $\{1, \dots, d\}$, J to be $\{1, \dots, k, d+1, \dots, 2d-k\}$, and setting $F = F_I$ and $G = F_J$ with this I, J we get

$$\mathbb{E}(|\mathcal{F}|^2 \mathbf{1}(T_n)) = \sum_{k=0}^{d} \binom{n}{d} \binom{d}{k} \binom{n-d}{d-k} \mathbb{E}\mathbf{1}\{F \in \mathcal{F}\}\mathbf{1}\{G \in \mathcal{F}\}\mathbf{1}\{T_n\}.$$

Denote the last expectation by Σ_k^* and write $C(F)$ resp., $C(G)$ for the minimal caps containing F and G, and $V(F)$, $V(G)$ for their volume. Then we have, using the symmetry of F and G the same way as before, that

$$\begin{aligned} \Sigma_k^* &= \mathbb{E}\mathbf{1}\{F \in \mathcal{F}\}\mathbf{1}\{G \in \mathcal{F}\}\mathbf{1}\{T_n\} \\ &\leq 2\mathbb{E}\mathbf{1}\{F \in \mathcal{F}\}\mathbf{1}\{G \in \mathcal{F}\}\mathbf{1}\{V(G) \leq V(F) \leq \frac{c \log n}{n}\} \\ &\leq 2 \int_K \dots \int_K (1 - V(F))^{n-2d+k} \, \mathbf{1}\{y \in C(F) \cap C(G)\} \times \\ &\times \quad \mathbf{1}\{V(G) \leq V(F) \leq \frac{c \log n}{n}\} dx_1 \dots, dx_{2d-k} dy. \end{aligned}$$

This is the same as the formula for Σ_k in the previous proof, only the factor $V(F)V(G)$ is missing here. The remaining arguments are the same as before and we get

$$\Sigma_k^* \ll F(P)^3 \frac{(\log n)^{d-1}}{n^{2d-k+1}}.$$

This finishes the proof for $\mathrm{Var} f_\ell(P_n)$. □

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