

Universal points of convex bodies and bisectors in Minkowski spaces

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Abstract

We deal with different properties of a smooth and strictly convex body that depend on the behavior of the planar sections of the body parallel to and close to a given tangent plane. The first topic is boundary points where any given convex domain in the tangent plane can be approximated by a sequence of suitably rescaled planar sections (so-called p -universal points). In the second topic, the given convex body is the unit ball of a Minkowski space, and oscillation properties of bisectors and trisectors in that space are considered. In either case it turns out that geometrically irregular behavior is typical, when considered from the viewpoint of Baire category.

1 Introduction

As the title indicates, this paper treats two seemingly unrelated topics. However, the proof schemes in both cases are so similar that a joint treatment is appropriate. Both topics are manifestations of the repeatedly observed phenomenon that certain ‘pathological’ objects, constructed with some effort in special cases, turn out to be typical, and often in a stronger form, when considered under the aspect of Baire categories.

The space \mathcal{K}^d of convex bodies (nonempty, compact, convex sets) in Euclidean space \mathbb{R}^d , equipped with the Hausdorff metric δ , is a complete metric space and thus a Baire space. This means that in \mathcal{K}^d the intersection of countably many dense open sets is dense. A subset of \mathcal{K}^d is called *meager* or *of first category* if it is a countable union of nowhere dense subsets. The complement of a meager set contains the intersection of countably many dense open sets and hence is dense. Therefore, meager subsets of a Baire space can be considered as ‘small’. If P is a property that elements of a Baire space can have, one says that *most* elements have property P , or that a *typical* element has this property, if the subset of elements not having property P is meager. In that case, property P is also called *generic*. As an example, we mention that most convex bodies in \mathcal{K}^d are smooth and strictly convex (‘smooth’ means that at each boundary point of K there is a unique support plane). In fact, the set \mathcal{K}_*^d of smooth and strictly convex bodies is a dense G_δ set in \mathcal{K}^d (a G_δ set is, by definition, the intersection of countably many open sets), and thus itself a Baire space (a fact which will be used below). For surveys of Baire category results in convexity, we refer to Zamfirescu [16, 17, 18] and Gruber [4, 5].

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Our first topic is universal points of convex bodies. Let K be a smooth convex body in \mathbb{R}^3 . Let z be a boundary point of K , let T be the tangent plane (unique support plane) of K at z , and let $T(t)$ be the plane parallel to T , at distance $t > 0$ from it, and intersecting K (supposing that t is sufficiently small). A convex body $M \subset T$ is called a *limit section* of K at z if there is a null sequence $(t_i)_{i \in \mathbb{N}}$ such that suitable homothets of the intersections $T(t_i) \cap K$ converge to M . For example, if the boundary surface of K is of class C^2 with positive curvatures, then every limit section is an ellipse, homothetic to the Dupin indicatrix (see, e.g., [13]). The point z is called a *p-universal* point of K if every convex body $M \subset T$ is a limit section of K at z . (The ‘p’ refers to parallel sections.) This concept was introduced by Melzak [10]. He constructed a convex body in \mathbb{R}^3 for which the set of p-universal points is dense in the boundary. We show that this behavior, in a stronger form, is typical.

Theorem 1. *For most convex bodies in \mathcal{K}_*^3 , most boundary points are p-universal.*

Here the second ‘most’ refers, of course, to the boundary of a convex body, which is itself a Baire space.

The theorem and its proof extend to higher dimensions, but since no essentially new ideas are required, we refrain from this. The proof is given in Section 2.

Our second topic concerns bisectors in Minkowski spaces, that is, in finite-dimensional real normed spaces. A comprehensive survey of bisectors and related notions in Minkowski spaces is given in Section 4 of the article by Martini and Swanepoel [9]. Here we first restrict ourselves to the two-dimensional case. Let $\|\cdot\|$ be a norm on \mathbb{R}^2 , with unit ball K , which is assumed to be smooth and strictly convex. For different points $p, q \in \mathbb{R}^2$ the (p, q) -bisector is the set

$$B_{p,q} := \{x \in \mathbb{R}^2 : \|x - p\| = \|x - q\|\}.$$

If the norm is Euclidean, then all bisectors are straight lines, and consequently the Voronoi diagram of a locally finite point set has convex cells. Motivated by applications to algorithms, Klein [7, 8], (see also [1]), investigated Voronoi diagrams for more general metrics and suggested (among other conditions) that ‘nice’ metrics should have the property that the intersection of any two bisectors has only finitely many connected components. Corbalan, Mazon and Recio [3] constructed a norm on \mathbb{R}^2 which is not nice in this sense. To explain this, let $K \in \mathcal{K}^2$ be the unit ball of a norm, and for given different points $p, q \in \mathbb{R}^2$ let G_p, G_q be the lines through p and q , respectively, whose directions are normal to $p - q$. The latter means (see [14]) that the direction is parallel to z , where z is a point of K where the tangent line is parallel to $p - q$; see Figure 1. The bisector $B_{p,q}$ is contained in the open strip $S_{p,q}$ bounded by G_p and G_q .

We say that z is a *point of strong oscillation* if any line in the open strip $S_{p,q}$ intersects the bisector $B_{p,q}$ at arbitrarily large distances from p, q . (If one pair (p, q) has this property, then all homothetic pairs have the same property.) If this holds, then the intersection of the bisectors of p, q and of $p + z, q + z$ has infinitely many components. Corbalan, Mazon and Recio constructed a unit ball K with a point z with a slightly weaker property, from which they concluded that the corresponding norm is not ‘nice’. We prove here a stronger result. We denote the set of unit balls of smooth and strictly convex norms on \mathbb{R}^d by \mathcal{K}_{*o}^d .

Theorem 2. *For most convex bodies in \mathcal{K}_{*o}^2 , most boundary points are points of strong oscillation.*

We prove this result in Section 3. The reader will notice that here the symmetry of the

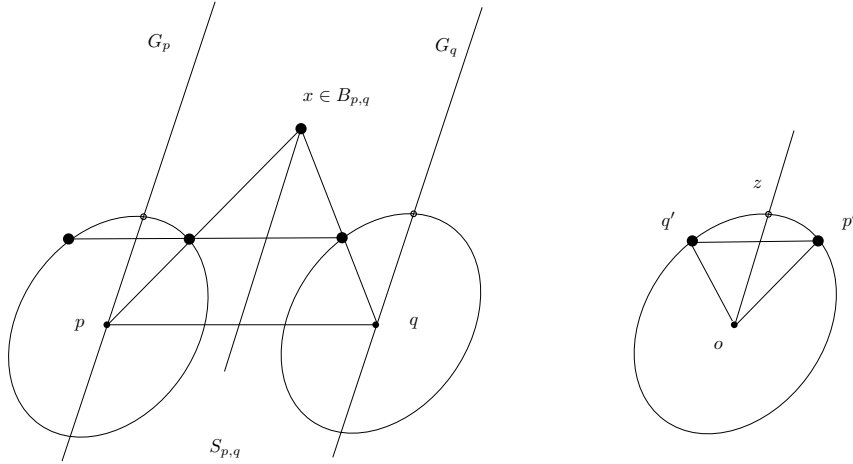


Figure 1: The triangle $op'q'$ on the right is similar to the triangle xpq on the left, and p' (resp. q') is the unit vector in direction $x - p$ (and $x - q$).

norm is not essential, if bisectors $B_{p,q}$ are only considered on one side of the line through p and q . For simplicity, we present only the symmetric case.

For bisectors in Minkowski spaces of dimension $d > 2$, some typical irregularity properties can be deduced by considering two-dimensional subspaces and using the methods applied for Theorem 2. Which more general irregularity properties are typical, has not been investigated.

In three-dimensional space, we shall also consider ‘trisectors’ in \mathbb{R}^3 , that is, the sets of points having the same distance from three given, affinely independent points, with respect to a given norm. In the typical case, they have even more bizarre properties than bisectors, as will be made precise and proved in Section 5. In this case, an extension to higher dimensions meets with some difficulties, as is briefly explained at the end of Section 5.

2 Proof of Theorem 1

On \mathbb{R}^d , we use the standard scalar product $\langle \cdot, \cdot \rangle$ and its induced norm, denoted by $|\cdot|$. By $B_0(x, r)$ we denote the open ball in \mathbb{R}^d with center x and radius $r > 0$.

The proof scheme for our theorems is modelled after that for Theorem 2.6.4 in [12], which in turn used ideas from Schneider [11] and Zamfirescu [16]. For the sake of easier reading, we first describe this proof scheme in general terms and in a simplified version, neglecting the refinements that will later be necessary. Let P denote a property that a boundary point x of a convex body K can have. For a boundary point x of $K \in \mathcal{K}_*^d$, we write $P[K, x]$ if x has the property P , and $\neg P[K, x]$ otherwise. In each of the considered cases, we aim to show that $P[K, x]$ holds for most boundary points x of most convex bodies K . For this, we define, for each $K \in \mathcal{K}_*^d$, closed sets $A_k(K) \subset \text{bd } K$, $k \in \mathbb{N}$, with

$$K(\neg P) := \{x \in \text{bd } K : \neg P[K, x]\} = \bigcup_{k \in \mathbb{N}} A_k(K).$$

Then we define the sets

$$\mathcal{C}_{k,m} := \{K \in \mathcal{K}_*^d : \exists x \in \text{bd } K \text{ such that } B_0(x, 1/m) \cap \text{bd } K \subset A_k(K)\}$$

for $m \in \mathbb{N}$ and show that $\mathcal{C}_{k,m}$ is closed and nowhere dense in \mathcal{K}_*^d , and that

$$\mathcal{K}_*^d(\neg P) := \{K \in \mathcal{K}_*^d : K(\neg P) \text{ is not meager in } \text{bd} K\} \subset \bigcup_{k,m \in \mathbb{N}} \mathcal{C}_{k,m}.$$

Since each $\mathcal{C}_{k,m}$ is closed and nowhere dense, the set $\mathcal{K}_*^d(\neg P)$ is meager in \mathcal{K}_*^d . Hence, most bodies $K \in \mathcal{K}_*^d$ belong to $\mathcal{K}_*^d \setminus \mathcal{K}_*^d(\neg P)$, that is, they have the property that $K(\neg P)$ is meager in $\text{bd} K$, in other words, that P holds for most $x \in \text{bd} K$.

In the subsequent applications of this scheme, the essential differences are in the proof that $\mathcal{C}_{k,m}$ is nowhere dense, which requires explicit geometric constructions.

To begin with the proof of Theorem 1, we fix a unit vector $u \in \mathbb{R}^3$ and let U be the two-dimensional subspace of \mathbb{R}^3 orthogonal to u . By $\mathcal{N}(U)$ we denote the set of normalized convex bodies in U , that is, those with mean width one and Steiner point (see [12], p. 42, for these notions) at the origin. Each convex body which lies in a plane parallel to U and is not a singleton, has a unique homothetic copy in $\mathcal{N}(U)$.

Recall that δ denotes the Hausdorff metric, defined by

$$\delta(L, M) := \max\{\max_{x \in L} \min_{y \in M} |x - y|, \max_{x \in M} \min_{y \in L} |x - y|\}$$

for convex bodies L, M . If L, M are not singletons, we define their *homothetic distance* by

$$h(L, M) := \delta\left(\frac{L - s(L)}{w(L)}, \frac{M - s(M)}{w(M)}\right), \quad (1)$$

where s denotes the Steiner point and w is the mean width.

By $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ we denote the unit sphere of \mathbb{R}^3 . On $S^2 \setminus \{\pm u\}$ we can choose two continuous vector functions A, B such that for each $v \in S^2 \setminus \{\pm u\}$, the triple $(v, A(v), B(v))$ is a positively oriented orthonormal frame (with respect to some fixed orientation of \mathbb{R}^3). If we fix a positively oriented orthonormal frame (u, a, b) (so that $a, b \in U$), there is a unique continuous mapping $\rho : S^2 \setminus \{\pm u\} \rightarrow \text{SO}(3)$ such that $\rho(v)(u, a, b) = (v, A(v), B(v))$.

Now we fix a convex body $L \in \mathcal{N}(U)$, and for $v \in S^2 \setminus \{\pm u\}$, we define

$$L(v) := \rho(v)L + v.$$

Then $L(v)$ is a congruent copy of L lying in the tangent plane of S^2 at v and having its Steiner point at v . Moreover, $L(v)$ depends continuously on v .

Let $K \in \mathcal{K}_*^3$. At each point $z \in \text{bd} K$, there is a unique outer unit normal vector to K , which we denote by $\nu(z)$ (in this notation, we suppress the dependence on K , which will be clear from the context). This defines a continuous mapping $\nu : \text{bd} K \rightarrow S^2$, also known as the Gauss map of K . For $n \in \mathbb{N}$, we define

$$\text{bd}_n K := \{z \in \text{bd} K : |\langle \nu(z), u \rangle| \leq 1 - (1/n)\}.$$

This is a closed set. For $z \in \text{bd} K$ and $t > 0$, let $T_{K,z,t}$ be the plane that is parallel to the (unique) support plane of K at z , at distance t from it and on the same side as K . Its intersection with K will play an essential role in the following, we denote it by

$$K(z, t) := T_{K,z,t} \cap K.$$

For $n, k \in \mathbb{N}$ we define

$$A_{n,k}(K) := \{z \in \text{bd}_n K : h(K(z, t), L(\nu(z))) \geq 1/k \forall t \in (0, 1/k]\}.$$

(Formally, we define $h(\emptyset, \cdot) := -1$.)

Claim 1. $A_{n,k}(K)$ is closed.

Proof. Let $n, k \in \mathbb{N}$. Let $(z_j)_{j \in \mathbb{N}}$ be a sequence in $A_{n,k}(K)$ converging to z ; then $z \in \text{bd}_n K$. Let $t \in (0, 1/k]$; then $h(K(z_j, t), L(\nu(z_j))) \geq 1/k$ for all j (in particular, $K(z_j, t) \neq \emptyset$). For $j \rightarrow \infty$, we have $T_{K, z_j, t} \rightarrow T_{K, z, t}$ (in the usual topology for the space of planes) and hence $K(z_j, t) \rightarrow K(z, t)$ (by [12], Theorem 1.8.8). Since the function $z \mapsto L(\nu(z))$ and the homothetic distance h are continuous, it follows that $h(K(z, t), L(\nu(z))) \geq 1/k$ and hence that $z \in A_{n,k}(K)$. This proves the claim. \square

For $n, k, m \in \mathbb{N}$ we define

$$\mathcal{C}_{n,k,m} := \{K \in \mathcal{K}_*^3 : \exists x \in \text{bd}_n K \text{ with } B_0(x, 1/m) \cap \text{bd}_n K \subset A_{n,k}(K)\}.$$

Claim 2. $\mathcal{C}_{n,k,m}$ is closed in \mathcal{K}_*^3 .

Proof. Let $(K_j)_{j \in \mathbb{N}}$ be a sequence in $\mathcal{C}_{n,k,m}$ converging to some $K \in \mathcal{K}_*^3$. For $j \in \mathbb{N}$ we can choose $x_j \in \text{bd}_n K_j$ such that $B_0(x_j, 1/m) \cap \text{bd}_n K_j \subset A_{n,k}(K_j)$. The sequence $(x_j)_{j \in \mathbb{N}}$ has a convergent subsequence, and we may assume that the sequence itself converges to some point x . Then $x \in \text{bd}_n K$. Let $y \in B_0(x, 1/m) \cap \text{bd} K$. For each $j \in \mathbb{N}$, we can choose a point $y_j \in \text{bd} K_j$ such that $y_j \rightarrow y$ for $j \rightarrow \infty$ (cf. [12], Theorem 1.8.7). Let $0 < t \leq 1/k$. For sufficiently large j we have $|x_j - y_j| < 1/m$ and thus $y_j \in A_{n,k}(K_j)$. Therefore, $h(K_j(y_j, t), L(\nu(y_j))) \geq 1/k$. Since $K_j(y_j, t) \rightarrow K(y, t)$ for $j \rightarrow \infty$, it follows that $h(K(y, t), L(\nu(y))) \geq 1/k$ and hence that $y \in A_{n,k}(K)$. Since $y \in B_0(x, 1/m) \cap \text{bd}_n K$ was arbitrary, we obtain that $B_0(x, 1/m) \cap \text{bd} K \subset A_{n,k}(K)$. This means that $K \in \mathcal{C}_{n,k,m}$. We have proved that $\mathcal{C}_{n,k,m}$ is closed in \mathcal{K}_*^3 . \square

Claim 3. $\mathcal{C}_{n,k,m}$ is nowhere dense in \mathcal{K}_*^3 .

Proof. Since $\mathcal{C}_{n,k,m}$ is closed, the assertion says that $\mathcal{C}_{n,k,m}$ has no interior points relative to \mathcal{K}_*^3 . Let $n, k, m \in \mathbb{N}$. For given $K \in \mathcal{K}_*^3$ and given $\epsilon > 0$, we have to show that the ϵ -neighborhood of K contains some element of $\mathcal{K}_*^3 \setminus \mathcal{C}_{n,k,m}$. Without loss of generality, we assume that $\epsilon < 1/m$.

By a *cap* of K we understand here the nonempty intersection of K with an open halfspace. We can obviously choose finitely many caps C_1, \dots, C_p of K such that the following holds:

- (a) $C_i \cap C_j = \emptyset$ for $i \neq j$,
- (b) the convex body $K' := K \setminus \bigcup_{i=1}^p C_i$ satisfies $\delta(K, K') < \epsilon/2$,
- (c) for each $x \in \text{bd} K$, the ball $B_0(x, 1/(2m))$ contains at least one of the caps C_1, \dots, C_p ,
- (d) the distance of any point of any C_i from $\text{bd} K$ is at most $1/(4m)$.

Let $i \in \{1, \dots, p\}$. Let H_i be the plane determining C_i , let u_i be its unit normal vector pointing from H_i to the cap. We choose a homothetic copy L_i of $L(u_i)$ with $L_i \subset \text{relint}(H_i \cap K)$. Then we choose a point $c_i \in \text{relint} L_i$ and numbers $0 < \lambda_i < \mu_i$ such that $L_i + \lambda_i u_i \subset \text{int} C_i$, that $c_i + \mu_i u_i \in \text{int} C_i$ and that the cone with apex $c_i + \mu_i u_i$ spanned by $L_i + \lambda_i u_i$ contains $K \setminus C_i$ in its interior. Then the convex body

$$C'_i := \text{conv} \{(H_i \cap K) \cup (L_i + \lambda_i u_i) \cup \{c_i + \mu_i u_i\}\}$$

is contained in the closure of the cap C_i . Further, the body

$$K'' := K' \cup C'_1 \cup \dots \cup C'_p$$

is convex and satisfies $\delta(K'', K) < \epsilon/2$ (by (b) above). The body K'' has the following property. Let $i \in \{1, \dots, p\}$, and let T_i be the support plane of K'' parallel to H_i and touching K'' at the point $c_i + \mu_i u_i$. All planes parallel to T_i , on the same side of T_i as H_i and at a sufficiently small distance from T_i , intersect K'' in a convex set that is homothetic to $L(u_i)$.

By a smoothing procedure, for example the one described in [12], pp. 158–160, we can, for every $\alpha > 0$, obtain a convex body $K^\alpha \in \mathcal{K}_*^3$ such that $\delta(K'', K^\alpha) < \alpha$. By choosing $\alpha < \epsilon/2$, we achieve that $\delta(K^\alpha, K) < \epsilon$.

For $i \in \{1, \dots, p\}$, let T_i^α be the support plane of K^α parallel to H_i and such that T_i lies between T_i^α and H_i . Let z_i^α be the (unique) point where it touches K^α . We can choose $0 < \alpha < 1/(4m)$ so small that the following holds. For each i , there is a number $t_i < 1/k$ such that

$$h(K^\alpha(z_i^\alpha, t_i), L(u_i)) < 1/k.$$

This means that $z_i^\alpha \notin A_{n,k}(K^\alpha)$.

Now let $x \in \text{bd } K^\alpha$. There exists a point $x'' \in \text{bd } K''$ with $|x - x''| \leq \alpha < 1/(4m)$. If x'' is contained in one of the caps C_1, \dots, C_p , then by (d) above there is a point $y \in \text{bd } K$ with $|x'' - y| < 1/(4m)$. If x'' is not in one of the caps, then $x'' \in \text{bd } K$, and we can take $y = x''$. In either case, $|x - y| < 1/(2m)$. By (c) above, the ball $B_0(y, 1/(2m))$ contains one of the caps, say C_i , hence it contains the point z_i^α . We have $|x - z_i^\alpha| \leq |x - y| + |y - z_i^\alpha| < 1/m$. Thus, each ball $B_0(x, 1/m)$ with $x \in \text{bd } K^\alpha$ contains some boundary point of K^α which is not in $A_{n,k}(K^\alpha)$. Therefore, $K^\alpha \notin \mathcal{C}_{n,k,m}$. Since $\delta(K, K^\alpha) < \epsilon$, this proves that $\mathcal{C}_{n,k,m}$ is nowhere dense in \mathcal{K}_*^3 . This finishes the proof of Claim 3. \square

From now on, our notation exhibits the dependence on L , for later purposes. For $K \in \mathcal{K}_*^3$, we denote by $\text{bd}' K$ the set of all boundary points of K except those with outer normal vectors $\pm u$. For $K \in \mathcal{K}_*^3$, we define

$$B_L(K) := \{z \in \text{bd}' K : \liminf_{t \rightarrow 0} h(K(z, t), L(\nu(z))) > 0\}.$$

We state that

$$B_L(K) = \bigcup_{n,k \in \mathbb{N}} A_{n,k}(K). \quad (2)$$

Indeed,

$$\begin{aligned} z \in B_L(K) &\Leftrightarrow \exists n \in \mathbb{N} : z \in \text{bd}_n K \wedge \liminf_{t \rightarrow 0} h(K(z, t), L(\nu(z))) > 0 \\ &\Leftrightarrow \exists n \in \mathbb{N} : z \in \text{bd}_n K \exists \epsilon > 0 \exists t_0 > 0 \forall t < t_0 : h(K(z, t), L(\nu(z))) \geq \epsilon \\ &\Leftrightarrow \exists n \in \mathbb{N} : z \in \text{bd}_n K \exists k \in \mathbb{N} \forall t < 1/k : h(K(z, t), L(\nu(z))) \geq 1/k \\ &\Leftrightarrow \exists n \in \mathbb{N} \exists k \in \mathbb{N} : z \in A_{n,k}(K). \end{aligned}$$

This proves (2).

For the set

$$\tilde{\mathcal{K}}_L := \{K \in \mathcal{K}_*^3 : B_L(K) \text{ is not meager in } \text{bd } K\},$$

we state that

$$\tilde{\mathcal{K}}_L \subset \bigcup_{n,k,m \in \mathbb{N}} \mathcal{C}_{n,k,m}. \quad (3)$$

In fact, let $K \in \tilde{\mathcal{K}}_L$. Then, by definition and (2), $\bigcup_{n,k} A_{n,k}(K)$ is not meager in $\text{bd } K$. Therefore, not all sets $A_{n,k}(K)$ can be nowhere dense. Let n, k be numbers such that $A_{n,k}(K)$ is not nowhere dense. Since $A_{n,k}(K)$ is closed, it has nonempty interior relative to $\text{bd}_n K$. Thus, there are a point $x \in \text{bd}_n K$ and a number $m \in \mathbb{N}$ such that $B_0(x, 1/m) \cap \text{bd}_n K \subset A_{n,k}(K)$, which implies that $K \in \mathcal{C}_{n,k,m}$ and thus proves (3). Since we have proved that the sets $\mathcal{C}_{n,k,m}$ are closed and nowhere dense in \mathcal{K}_*^3 , the set $\tilde{\mathcal{K}}_L$ is meager in \mathcal{K}_*^3 . Hence, for most $K \in \mathcal{K}_*^3$, the set $B_L(K)$ is meager in $\text{bd } K$, which means that most points $z \in \text{bd } K$ have the property that

$$\liminf_{t \rightarrow 0} h(K(z, t), L(\nu(z))) = 0.$$

Now let \mathcal{L} be a countable set of two-dimensional convex bodies which is dense in the space $\mathcal{N}(U)$. The set

$$\tilde{\mathcal{K}} := \bigcup_{L \in \mathcal{L}} \tilde{\mathcal{K}}_L$$

is a countable union of meager sets and hence is itself meager in \mathcal{K}_*^3 . Therefore, most $K \in \mathcal{K}_*^3$ belong to the set

$$\bigcap_{L \in \mathcal{L}} (\mathcal{K}_*^3 \setminus \tilde{\mathcal{K}}_L)$$

and hence have the property that $B_L(K)$ is meager in $\text{bd } K$, for all $L \in \mathcal{L}$, and thus the property that the set

$$B(K) := \bigcup_{L \in \mathcal{L}} B_L(K)$$

is meager in $\text{bd } K$. This means that most $K \in \mathcal{K}_*^3$ have the property that, for most points $z \in \text{bd } K$,

$$\liminf_{t \rightarrow 0} h(K(z, t), L(\nu(z))) = 0 \quad \text{for all } L \in \mathcal{L}. \quad (4)$$

Let $K \in \mathcal{K}_*^3$ and $z \in \text{bd } K$ be such that (4) holds. Let M be any convex body in $\mathcal{N}(U)$. To each $n \in \mathbb{N}$, there exists a convex body $L_n \in \mathcal{L}$ such that $\delta(M, L_n) < 1/n$. With $\nu(z)$ being the outer unit normal vector of K at z , this implies that

$$\delta(\rho(\nu(z))M, \rho(\nu(z))L_n) < 1/n.$$

By (4),

$$\liminf_{t \rightarrow 0} h(K(z, t), \rho(\nu(z))L_n) = 0,$$

hence there exists $t_n \in (0, 1/n)$ with $h(K(z, t_n), \rho(\nu(z))L_n) < 1/n$. By the definition of the homothetic distance, this yields the existence of a homothety ϕ_n such that

$$\delta(\phi_n K(z, t_n), \rho(\nu(z))L_n) < 1/n,$$

which gives

$$\delta(\phi_n K(z, t_n), \rho(\nu(z))M) < 2/n.$$

Thus, we have proved the existence of a null sequence $(t_n)_{n \in \mathbb{N}}$ and a sequence $(\phi_n)_{n \in \mathbb{N}}$ of homotheties such that

$$\lim_{n \rightarrow \infty} \phi_n K(z, t_n) = \rho(\nu(z))M.$$

Since here $\rho(\nu(z))M$ can be a homothet of any convex body of positive dimension in the tangent plane to K at z (and since singletons are trivially limit sections), this completes the proof of Theorem 1. \square

Remark. In this proof we needed $S^2 \setminus \{\pm u\}$ to make the map $L(\cdot)$ continuous. This cannot be done if L is defined on the whole of S^2 . But we could have used $S^2 \setminus \{u\}$ just as well.

3 Proof of Theorem 2

The setting for Theorem 2 is \mathbb{R}^2 , and we recall that \mathcal{K}_{*o}^2 denotes the set of unit balls of smooth, strictly convex norms on \mathbb{R}^2 . Thus, \mathcal{K}_{*o}^2 is the set of smooth and strictly convex bodies in \mathbb{R}^2 with the origin o as center of symmetry and interior point.

First we remark that, for given K , the qualitative behavior of the bisector $B_{p,q}$ depends only on the direction of the line through p and q : if $p - q = \lambda(p' - q')$ with $\lambda > 0$, then the homothety carrying (p, q) to (p', q') maps $B_{p,q}$ to $B_{p',q'}$.

In the following proof of Theorem 2, we use the standard Euclidean structure of \mathbb{R}^2 for auxiliary purposes (except that bisectors refer to the norm $\|\cdot\|$ induced by a given element of \mathcal{K}_{*o}^2). Let $K \in \mathcal{K}_{*o}^2$ be given. We fix an orientation of \mathbb{R}^2 and equip $\text{bd } K$ with the induced counterclockwise cyclic order. Let $z \in \text{bd } K$, and let T be a support line of K at z . For sufficiently small $t > 0$, the line $T(t)$ parallel to T , at distance t from it, and on the same side of T as K , intersects the segment $[o, z]$ in a point $c(z, t)$. Let $a(z, t), b(z, t)$ be the intersection points of $T(t)$ with $\text{bd } K$ such that $a(z, t), z, b(z, t)$ follow each other in this order. Define

$$R(K, z, t) := |a(z, t) - c(z, t)|, \quad L(K, z, t) := |b(z, t) - c(z, t)|.$$

It is not difficult to see (or compare [3]) that z is a point of strong oscillation for K if and only if

$$\liminf_{t \rightarrow 0} \frac{R(K, z, t)}{L(K, z, t)} = 0 \quad \text{and} \quad \limsup_{t \rightarrow 0} \frac{R(K, z, t)}{L(K, z, t)} = \infty. \quad (5)$$

For $k \in \mathbb{N}$ we define the set

$$A_k(K) := \left\{ z \in \text{bd } K : \frac{R(K, z, t)}{L(K, z, t)} \geq \frac{1}{k} \text{ for } 0 < t \leq 1/k \right\}.$$

(If the quotient $R(K, z, t)/L(K, z, t)$ is not defined since t is too large, we set it formally equal to -1 .)

Claim 4. $A_k(K)$ is closed.

Proof. Since K is smooth and strictly convex, it is easy to see that the points $c(z, t)$, $a(z, t)$, $b(z, t)$ defined above are continuous functions of z and t . Therefore, also $R(K, z, t)$ and $L(K, z, t)$ are continuous. Now let $(z_j)_{j \in \mathbb{N}}$ be a sequence in $A_k(K)$ converging to $z \in \text{bd } K$. For fixed $t \in (0, 1/k]$, we have $R(K, z_j, t)/L(K, z_j, t) \geq 1/k$ for all $j \in \mathbb{N}$, and from the mentioned continuity it follows that $R(K, z, t)/L(K, z, t) \geq 1/k$. Since this holds for all $t \in (0, 1/k]$, we have $z \in A_k(K)$. Thus, $A_k(K)$ is closed. \square

For $k, m \in \mathbb{N}$ we define

$$C_{k,m} := \left\{ K \in \mathcal{K}_{*o}^2 : \exists x \in \text{bd } K \text{ such that } B_0(x, 1/m) \cap \text{bd } K \subset A_k(K) \right\}.$$

Claim 5. $C_{k,m}$ is closed in \mathcal{K}_*^2 .

Proof. Let $(K_j)_{j \in \mathbb{N}}$ be a sequence in $C_{k,m}$ converging to some $K \in \mathcal{K}_{*o}^2$. For $j \in \mathbb{N}$, we can choose $x_j \in \text{bd } K_j$ such that $B_0(x_j, 1/m) \cap \text{bd } K_j \subset A_k(K_j)$. The sequence $(x_j)_{j \in \mathbb{N}}$ has a convergent subsequence, and we may assume that the sequence itself converges to some point x . Then $x \in \text{bd } K$. Let $y \in B_0(x, 1/m) \cap \text{bd } K$. For each $j \in \mathbb{N}$, we can choose a point $y_j \in \text{bd } K_j$ such that $y_j \rightarrow y$ for $j \rightarrow \infty$. For sufficiently large j we have $|x_j - y_j| < 1/m$ and thus $y_j \in A_k(K_j)$. Therefore, $R(K_j, y_j, t)/L(K_j, y_j, t) \geq 1/k$ for $0 < t \leq 1/k$. Let $0 < t \leq 1/k$. It follows that $R(K, y, t)/L(K, y, t) \geq 1/k$ and hence that $y \in A_k(K)$. Thus, $B_0(x, 1/m) \cap \text{bd } K \subset A_k(K)$. Therefore, $K \in C_{k,m}$. We have proved that $C_{k,m}$ is closed in \mathcal{K}_*^2 . \square

Claim 6. $\mathcal{C}_{k,m}$ is nowhere dense in \mathcal{K}_*^2 .

Proof. Since $\mathcal{C}_{k,m}$ is closed, we have to verify that $\mathcal{C}_{k,m}$ does not have interior points relative to \mathcal{K}_{*o}^2 . Let $k, m \in \mathbb{N}$ be given. Let $K \in \mathcal{K}_*^2$ be a given convex body, and let $\epsilon > 0$. There exists a o -symmetric convex polygon P with $\delta(K, P) < \epsilon/2$ and such that for each boundary point y of P , the ball $B_0(y, 1/(3m))$ contains a vertex of P . In the present situation, we can choose a rather elementary smoothing process. We replace each edge of P by a circular arc of sufficiently large radius R , connecting its endpoints, so that the resulting figure remains convex. Then we take the outer parallel body of the obtained convex body at distance $\alpha > 0$ and denote the resulting convex body by $K(R, \alpha)$. It belongs to \mathcal{K}_{*o}^2 . Choosing R sufficiently large and α sufficiently small, we can achieve that

$$\delta(K(R, \alpha), P) < \min \left\{ \frac{\epsilon}{2}, \frac{1}{3m} \right\}.$$

Then $\delta(K(R, \alpha), K) < \epsilon$. This estimate remains true if we further increase R and decrease α .

Let v be a vertex of P . Since P is a polygon, there is a line L (parallel to a support line of P at v and cutting P close to the vertex v) with the following properties. The line L intersects the segment $[o, v]$ at a point c and the boundary of P at two points a and b , such that a, v, b follow each other in this cyclic order on $\text{bd } P$ and that

$$\frac{|a - c|}{|b - c|} < \frac{1}{2k}.$$

The support line S to $K(R, \alpha)$ parallel to L , and leaving v between L and S , touches $K(R, \alpha)$ at a unique point $z(R, \alpha)$. We translate S by the vector from v to its nearest point in L , to obtain a line $L(R, \alpha)$. This line intersects the segment $[o, z(R, \alpha)]$ at a point $c(R, \alpha)$ and the boundary of $K(R, \alpha)$ at two points $a(R, \alpha)$ and $b(R, \alpha)$, such that $a(R, \alpha), z(R, \alpha), b(R, \alpha)$ follow each other in this cyclic order on $\text{bd } K(R, \alpha)$. For $R \rightarrow \infty$ and $\alpha \rightarrow 0$ we have

$$z(R, \alpha) \rightarrow v, \quad c(R, \alpha) \rightarrow c, \quad a(R, \alpha) \rightarrow a, \quad b(R, \alpha) \rightarrow b.$$

Hence, for sufficiently large R and sufficiently small α we get

$$|z(R, \alpha) - v| < \frac{1}{3m}$$

and

$$\frac{|a(R, \alpha) - c(R, \alpha)|}{|b(R, \alpha) - c(R, \alpha)|} < \frac{1}{k}. \tag{6}$$

Since P has only finitely many vertices, we can assume that these estimates hold independently of the choice of the vertex v .

Now let $x \in \text{bd } K(R, \alpha)$. Let x' be the point in P nearest to x . By the choice of P , there exists a vertex v of P with $v \in B_0(x', 1/(3m))$. For this vertex, the above estimates are valid, and we use the notation that was used above. By (6), the point $z(R, \alpha)$ does not belong to the set $A_k(K(R, \alpha))$. We have

$$|x - z(R, \alpha)| \leq |x - x'| + |x' - v| + |v - z(R, \alpha)| \leq \delta(K(R, \alpha), P) + \frac{1}{3m} + \frac{1}{3m} < \frac{1}{m},$$

thus $z(R, \alpha) \in B_0(x, 1/m)$. This proves that

$$B_0(x, 1/m) \cap \text{bd } K(R, \alpha) \not\subset A_k(K(R, \alpha)).$$

Since $x \in \text{bd } K(R, \alpha)$ was arbitrary, we have shown that $K(R, \alpha) \notin \mathcal{C}_{k,m}$. On the other hand, $K(R, \alpha)$ is in the ϵ -neighborhood of K . Since $\epsilon > 0$ was arbitrary, this completes the proof of Claim 6. \square

For $K \in \mathcal{K}_{*o}^2$ we define the boundary set

$$B_K := \left\{ z \in \text{bd } K : \liminf_{t \rightarrow 0} \frac{R(K, z, t)}{L(K, z, t)} > 0 \right\}.$$

With this, we define

$$\tilde{\mathcal{K}} := \{K \in \mathcal{K}_{*o}^2 : B_K \text{ is not meager in } \text{bd } K\}.$$

For $z \in \text{bd } K$ we have

$$\begin{aligned} \liminf_{t \rightarrow 0} \frac{R(K, z, t)}{L(K, z, t)} > 0 &\Leftrightarrow \exists k \in \mathbb{N} : \frac{R(K, z, t)}{L(K, z, t)} \geq \frac{1}{k} \forall t \in (0, 1/k] \\ &\Leftrightarrow \exists k \in \mathbb{N} : z \in A_k(K). \end{aligned}$$

Therefore,

$$\tilde{\mathcal{K}} = \left\{ K \in \mathcal{K}_{*o}^2 : \bigcup_{k \in \mathbb{N}} A_k(K) \text{ is not meager in } \text{bd } K \right\}.$$

Let $K \in \tilde{\mathcal{K}}$. Then not all the sets $A_k(K)$ are nowhere dense. Hence, there is a number $k \in \mathbb{N}$ such that $A_k(K)$ is not nowhere dense. Since the set $A_k(K)$ is closed, it has nonempty interior relative to $\text{bd } K$. Thus, there are a point $x \in \text{bd } K$ and a number $m \in \mathbb{N}$ such that $B_0(x, 1/m) \cap \text{bd } K \subset A_k(K)$, which means that $K \in \mathcal{C}_{k,m}$. We conclude that

$$\tilde{\mathcal{K}} \subset \bigcup_{k,m \in \mathbb{N}} \mathcal{C}_{k,m}. \quad (7)$$

Since the sets $\mathcal{C}_{k,m}$ are nowhere dense, the set $\tilde{\mathcal{K}}$ is meager. Thus, most convex bodies $K \in \mathcal{K}_{*o}^2$ have the property that B_K is meager. In other words, most convex bodies $K \in \mathcal{K}_{*o}^2$ have the property that most of its boundary points z satisfy

$$\liminf_{t \rightarrow 0} \frac{R(K, z, t)}{L(K, z, t)} = 0. \quad (8)$$

In the same way (interchanging R and L in the definition of $A_k(K)$), we obtain that most convex bodies $K \in \mathcal{K}_{*o}^2$ have the property that most of its boundary points z satisfy

$$\liminf_{t \rightarrow 0} \frac{R(K, z, t)}{L(K, z, t)} = \infty. \quad (9)$$

Hence, for most convex bodies $K \in \mathcal{K}_{*o}^2$, most boundary points z have both properties, (8) and (9). This proves Theorem 2.

4 A common extension

The proof of Theorem 2 is based on the fact (expressed in (5)) that the point $c(z, t)$ can divide the segment $K(z, t) = [a(z, t), b(z, t)]$ in arbitrary ratios. Theorem 1 can be strengthened to give this statement as well in the following way. First one has to deal not just with convex bodies L but pairs (L, ℓ) where L is a convex body (which is not a singleton), and $\ell \in \text{relint}L$. The homothetic distance of two such pairs (L, ℓ) and (M, m) is defined, in analogy with (1), as

$$H((L, \ell), (M, m)) := \delta \left(\frac{L - \ell}{w(L)}, \frac{M - m}{w(M)} \right),$$

where w is the mean width, again. Next, define \mathcal{K}_o^d as the set of all $K \in \mathcal{K}_*^d$ such that $o \in \text{int}K$. Note that \mathcal{K}_o^d is a Baire space as it is G_δ in \mathcal{K}_*^d . For $z \in \text{bd}K$ (and for small enough $t > 0$) the section $K(z, t)$ gives rise to the pair $(K(z, t), s(z, t))$ where $s(z, t)$ is the intersection of $K(z, t)$ and the segment $[o, z]$. The point $z \in \text{bd}K$ is called *P-universal* if for every pair (L, z) where L lies in the (unique) tangent plane to K at z , there is a null sequence $(t_i)_{i \in \mathbb{N}}$ such that $H((L, z), (K(z, t_i), s(z, t_i)))$ tends to zero as $i \rightarrow \infty$.

With these definitions, here comes the strengthening of Theorem 1. We state it for arbitrary dimension $d \geq 2$.

Theorem 3. *For most convex bodies $K \in \mathcal{K}_o^d$ ($d \geq 2$), most boundary points are P-universal.*

The proof is essentially the same as that of Theorem 1 and is therefore omitted. Note that the case $d = 2$ contains the basic fact proved for Theorem 2.

5 Trisectors

The notion of bisector calls for a generalization. Let $\|\cdot\|$ be a norm on \mathbb{R}^d . For $k \in \{2, \dots, d\}$ and k affinely independent points $p_1, \dots, p_k \in \mathbb{R}^d$, the (p_1, \dots, p_k) -*equidistant set* is defined as

$$B_{p_1, \dots, p_k} := \{x \in \mathbb{R}^d : \|x - p_1\| = \|x - p_2\| = \dots = \|x - p_k\|\}.$$

We consider here only the case $d = k = 3$.

The setting now is \mathbb{R}^3 . A (p, q, r) -equidistant set $B_{p, q, r}$ is called a *trisector*. Since we will deal with positions of points in a plane relative to a given triangle, it is convenient to use barycentric coordinates. Let (p, q, r) be an ordered triple of affinely independent points in \mathbb{R}^3 . A point $x \in \text{aff}\{p, q, r\}$ has a unique representation

$$x = \alpha p + \beta q + \gamma r \quad \text{with } \alpha + \beta + \gamma = 1.$$

The vector

$$V_{(p, q, r)}(x) := (\alpha, \beta, \gamma) \in \mathbb{R}^3$$

is called the *barycentric coordinate vector* of x with respect to (p, q, r) . We denote by

$$B := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha + \beta + \gamma = 1\}$$

the set of all possible barycentric coordinate vectors. By B^+ we denote the subset of B where at most one coordinate is non-positive.

Let $\|\cdot\|$ be a norm on \mathbb{R}^3 , with unit ball $K \in \mathcal{K}_{o*}^3$. With (p, q, r) being an ordered triple of affinely independent points, let $E := \text{aff}\{p, q, r\}$, and let $z \in \text{bd} K$ be one of the two points where a tangent plane parallel to E touches K . The direction of z is called *normal* to the plane E . If x is a point of the trisector $B_{p,q,r}$, then the intersection point of the line through x normal to E with the plane E is called the *normal projection* of x .

We now call the point z a *point of strong oscillation* if the following holds: in each plane E for which z is normal, there is a dense set of affinely independent triples such that for each triple (p, q, r) in this set, the normal projection of the trisector $B_{p,q,r}$ is dense in the region

$$\{s \in E : V_{(p,q,r)}(s) \in B^+\}.$$

Theorem 4. *For most convex bodies $K \in \mathcal{K}_*^3$, most boundary points are points of strong oscillation.*

For the proof, we need a lemma on two-dimensional convex sets. If $L \in \mathcal{K}^2$ is a convex body with interior points and τ is triple of affinely independent points in \mathbb{R}^2 , there may exist a unique triple homothetic to τ in $\text{bd} L$. If this is the case, we denote this triple (also called *inscribed to L*) by $\Delta(L, \tau)$.

Lemma 1. *If $L \in \mathcal{K}_*^2$, then $\Delta(L, \tau)$ exists for each affinely independent triple τ . The map $\Delta(\cdot, \tau)$ from \mathcal{K}_*^2 into the space of triples homothetic to τ is continuous.*

Proof. Let an affinely independent triple $\tau = (p, q, r)$ in \mathbb{R}^2 be given. Existence and uniqueness of $\Delta(K, \tau)$ were proved in [2], pp. 25–26, for strictly convex K of class C^2 . It is easy to see that the C^2 assumption in the proof can be replaced by smoothness. Now let a sequence $(L_j)_{j \in \mathbb{N}}$ in \mathcal{K}_*^2 converge to a convex body $L \in \mathcal{K}_*^2$. Let $\Delta(L_j, \tau) = (p_j, q_j, r_j)$. Successively choosing subsequences, we find a sequence $(j_i)_{i \in \mathbb{N}}$ such that $p_{j_i} \rightarrow p_L$, $q_{j_i} \rightarrow q_L$, and $r_{j_i} \rightarrow r_L$ for $i \rightarrow \infty$, with suitable points p_L, q_L, r_L . Note that p_L, q_L, r_L are distinct points as otherwise all three would coincide with some $z \in \text{bd} K$ and then the tangent to K at z would be parallel with the segments $[p, q]$ and $[p, r]$ which is impossible. Then (p_L, q_L, r_L) is inscribed to L and is homothetic to (p, q, r) . Suppose that $(p_j)_{j \in \mathbb{N}}$ has a subsequence converging to some other point p'_L . Then, again successively choosing convergent subsequences, we find points q'_L, r'_L such that (p'_L, q'_L, r'_L) is a different triple homothetic to τ and inscribed to L , a contradiction. Since all convergent subsequences of $(p_j)_{j \in \mathbb{N}}$ converge to the same point, this sequence converges, and the same holds for the sequences $(q_j)_{j \in \mathbb{N}}$ and $(r_j)_{j \in \mathbb{N}}$. The assertion of Lemma 1 follows. \square

The definition of $\Delta(L, \tau)$ and the lemma extend, obviously, to two-dimensional convex bodies L and triples τ lying in parallel planes of \mathbb{R}^3 .

The proof of Theorem 4 uses a scheme similar to that for Theorem 1, and we borrow some of the constructions employed there. We fix a unit vector $u \in \mathbb{R}^3$ and denote by U the linear subspace of \mathbb{R}^3 orthogonal to it. The continuous map $\rho : S^2 \setminus \{\pm u\} \rightarrow \text{SO}(3)$ is the same as in the proof of Theorem 1.

We fix an affinely independent triple τ in the subspace U , and for $v \in S^2 \setminus \{\pm u\}$ we define

$$\tau(v) := \rho(v)\tau + v.$$

We also fix a point b in the subset B^+ of the space of barycentric coordinates.

For a convex body $K \in \mathcal{K}_{*o}^3$, the Gauss mapping $\nu : \text{bd} K \rightarrow S^2$, the boundary set $\text{bd}_n K$ for $n \in \mathbb{N}$, the plane $T_{K,z,t}$ parallel to the tangent plane at $z \in \text{bd} K$ and at distance t from

it, and the section $K(z, t) = T_{K, z, t} \cap K$ are all defined as in the proof of Theorem 1. By $s(z, t)$ (the dependence on K is suppressed in this notation) we denote the intersection point of $T_{K, z, t}$ and the segment $[o, z]$.

For $K \in \mathcal{K}_{*o}^3$ and for $n, k \in \mathbb{N}$ we define

$$A_{n, k}(K) := \{z \in \text{bd}_n K : |V_{\Delta(K(z, t), \tau(\nu(z)))}(s(z, t)) - b| \geq 1/k \forall t \in (0, 1/k]\}.$$

Note that $\Delta(K(z, t), \tau(\nu(z)))$ is the unique triple homothetic to the triple $\tau(\nu(z))$ (lying in the tangent plane to S^2 at $\nu(z)$) that is inscribed to the section $K(z, t)$ (lying in a parallel plane). Then $V_{\Delta(K(z, t), \tau(\nu(z)))}(s(z, t))$ is the barycentric coordinate vector of the intersection point $s(z, t)$ with respect to this inscribed triple. The defining condition requires that this coordinate triple stays away, by a certain amount, from the given coordinate triple b .

Claim 7. $A_{n, k}(K)$ is closed.

Proof. The proof is so similar to the corresponding one for Claim 1 that we need not carry it out. The essential point now to observe is that

$$\Delta(K(z, t), \tau(\nu(z))) = \Delta(\rho(\nu(z))^{-1}K(z, t), \tau)$$

is a continuous function of z , due to Lemma 1, and the fact that barycentric coordinates of a point depend continuously on the point and on the reference triple. \square

For $n, k, m \in \mathbb{N}$ we define

$$\mathcal{C}_{n, k, m} := \{K \in \mathcal{K}_{*o}^3 : \exists x \in \text{bd}_n K \text{ with } B_0(x, 1/m) \cap \text{bd}_n K \subset A_{n, k}(K)\}.$$

Claim 8. $\mathcal{C}_{n, k, m}$ is closed in \mathcal{K}_{*o}^3 .

Proof. The proof can again be essentially copied from that of Claim 2. One additionally has to observe that the point $s(z, t)$, which for a convex body K we now denote by $s(K, z, t)$, depends continuously on K and $z \in \text{bd} K$. \square

We need another simple lemma.

Lemma 2. *Let (p, q, r) be an affinely independent triple in \mathbb{R}^2 , and let $s \in \mathbb{R}^2$ be a point with $V_{(p, q, r)}(s) \in B^+$. Then there exists a convex body $L \in \mathcal{K}_*^2$ such that (p, q, r) is inscribed to L and that $s \in \text{int} L$.*

Proof. Since $V_{(p, q, r)}(s) \in B^+$, we can choose a triangle D such that $s \in \text{int} D$, $D \subset \text{int} B^+$, and that p, q, r are vertices of the polygon $P := \text{conv}(\{p, q, r\} \cup D)$. By a result of Weil [15], there exists a convex body $L \subset \mathbb{R}^2$ of class C^∞ with positive curvature such that all vertices of P are on the boundary of L . This proves Lemma 2. \square

Claim 9. $\mathcal{C}_{n, k, m}$ is nowhere dense in \mathcal{K}_{*o}^3 .

Proof. Let $n, k, m \in \mathbb{N}$, a body $K \in \mathcal{K}_{*o}^3$, and $0 < \epsilon < 1/m$ be given. Precisely as in the proof of Claim 3, we define the caps C_i , with corresponding hyperplanes H_i and unit normal vectors u_i , $i = 1, \dots, p$. Let $i \in \{1, \dots, p\}$. Recall that the triple $\tau(u_i) = \rho(u_i)\tau$ lies in a plane parallel to H_i . By Lemma 2, there is a smooth and strictly convex body L_i in that plane such that $\tau(u_i)$ is inscribed to L_i and that the point s_i with $V_{\tau(u_i)}(s_i) = b$ lies in the relative interior of L_i . We choose a homothety ϕ_i with $\phi_i L_i \subset \text{int} C_i$. Let G_i be the line trough o and

the point $\phi_i s_i$. On this line, we choose a point $c_i \in \text{int } C_i$ such that $\phi_i s_i$ is between c_i and o and that the cone with apex c_i spanned by $\phi_i L_i$ contains $K \setminus C_i$ in its interior.

The convex body

$$C'_i := \text{conv} \{(H_i \cap K) \cup \phi_i L_i \cup \{c_i\}\}$$

is contained in the closure of the cap C_i . Further, the body

$$K'' := K' \cup C'_1 \cup \dots \cup C'_p$$

is convex and satisfies $\delta(K'', K) < \epsilon/2$. The body K'' has the following property. Let $i \in \{1, \dots, p\}$, and let T_i be the support plane of K'' parallel to H_i and touching K'' at the point c_i . Let H be a plane parallel to T_i , on the same side of T_i as H_i and at a sufficiently small distance from T_i . Let $H \cap G_i = \{s_H\}$. Then

$$V_{\Delta(H \cap K'', \tau(u_i))}(s_H) = b.$$

From here on, we can proceed similarly as in the proof of Theorem 1, and so finish the proof that $\mathcal{C}_{n,k,m}$ is nowhere dense in \mathcal{K}_{*o}^3 . \square

Also the rest of the proof follows a similar scheme. Defining

$$B_{\tau,b}(K) := \left\{ z \in \text{bd}' K : \liminf_{t \rightarrow 0} |V_{\Delta(K(z,t), \tau(\nu(z)))}(s(z,t)) - b| > 0 \right\},$$

we have

$$B_{\tau,b}(K) = \bigcup_{n,k \in \mathbb{N}} A_{n,k}(K).$$

For the set

$$\tilde{K}_{\tau,b} := \{K \in \mathcal{K}_{*o}^3 : B_{\tau,b}(K) \text{ is not meager in } \text{bd } K\}$$

we obtain

$$\tilde{K}_{\tau,b} \subset \bigcup_{n,k,m \in \mathbb{N}} \mathcal{C}_{n,k,m}$$

and, therefore, that $\tilde{K}_{\tau,b}$ is meager in \mathcal{K}_{*o}^3 . Hence, for most bodies $K \in \mathcal{K}_{*o}^3$, most points $z \in \text{bd } K$ have the property that

$$\liminf_{t \rightarrow 0} |V_{\Delta(K(z,t), \tau(\nu(z)))}(s(z,t)) - b| = 0.$$

Now let \mathcal{T} be a countable, dense set of affinely independent triples in U , and let \mathcal{B} be a countable, dense set of points in B^+ . The set

$$\tilde{K} := \bigcup_{\tau \in \mathcal{T}, b \in \mathcal{B}} \tilde{K}_{\tau,b}$$

is meager in \mathcal{K}_{*o}^3 . Therefore, most $K \in \mathcal{K}_{*o}^3$ have the property that $B_{\tau,b}(K)$ is meager in $\text{bd } K$, for all $\tau \in \mathcal{T}$ and all $b \in \mathcal{B}$, and hence the property that the set

$$B(K) := \bigcup_{\tau \in \mathcal{T}, b \in \mathcal{B}} B_{\tau,b}(K)$$

is meager. Thus, most $K \in \mathcal{K}_{*o}^3$ have the property that, for most points $z \in \text{bd } K$,

$$\liminf_{t \rightarrow 0} |V_{\Delta(K(z,t), \tau(\nu(z)))}(s(z,t)) - b| = 0 \quad \forall \tau \in \mathcal{T} \quad \forall b \in \mathcal{B}. \quad (10)$$

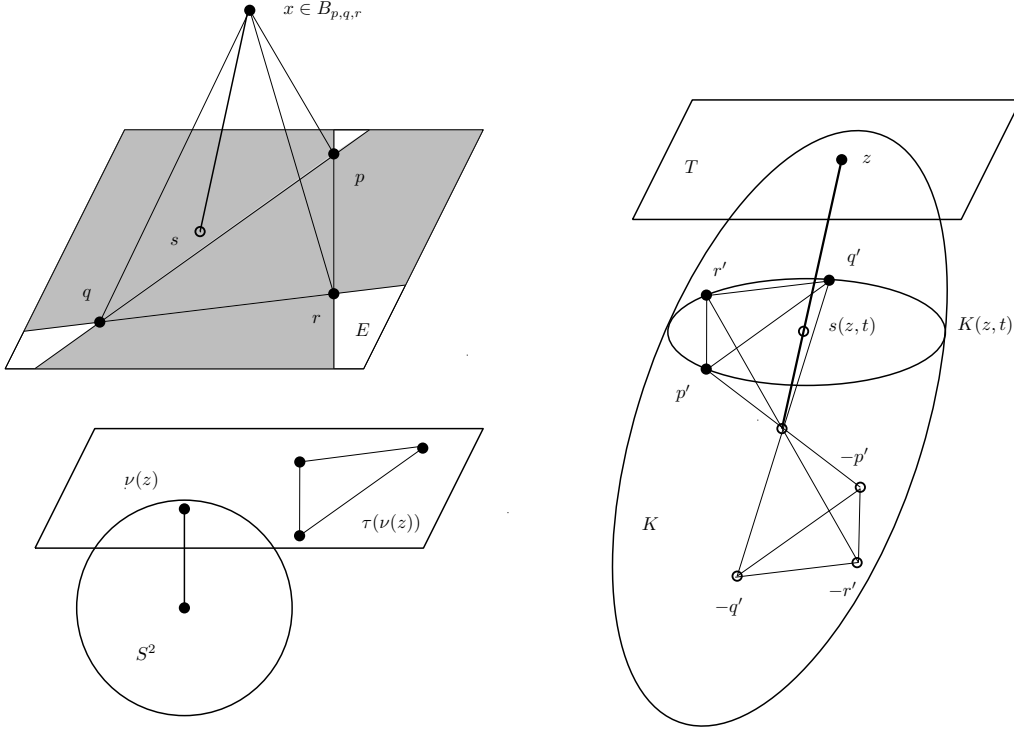


Figure 2: Trisector

Finally, let $K \in \mathcal{K}_{*o}^3$ and $z \in \text{bd } K$ be such that (10) is satisfied. Let E be a plane that is parallel to the tangent plane T of K at z (so that z is normal to E), see Figure 2. Let (p, q, r) be a triple in E that is homothetic to $-\tau(\nu(z))$ for some $\tau \in \mathcal{T}$. Let $y \in E$ be a point such that $V_{(p,q,r)}(y) \in B^+$, and let $\epsilon > 0$. We can choose a point $b \in \mathcal{B}$ with $|V_{(p,q,r)}(y) - b| < \epsilon$. According to (10), there exists a number $t > 0$ such that

$$|V_{\Delta(K(z,t), \tau(\nu(z)))}(s(z, t)) - b| < \epsilon.$$

The triple $\Delta(K(z, t), \tau(\nu(z))) =: (p', q', r')$ is homothetic to $\tau(\nu(z))$, hence there is a homothety ϕ carrying $-(p', q', r')$ to (p, q, r) . The point $x := \phi o$ has equal distance (with respect to the norm with unit ball K) from p, q and r and hence belongs to the trisector $B_{p,q,r}$. The normal projection of x in E is the point $\phi(-s(z, t)) =: s$, and we have $V_{(p,q,r)}(s) = V_{\Delta(K(z,t), \tau(\nu(z)))}(s(z, t))$, hence $|V_{(p,q,r)}(s) - b| < \epsilon$. It follows that $|V_{(p,q,r)}(y) - V_{(p,q,r)}(s)| < 2\epsilon$.

We have shown that, for every point of the set $\{y \in E : V_{(p,q,r)}(y) \in B^+\}$, every neighborhood contains a point in the normal projection of the trisector $B_{p,q,r}$. This is true for a dense set of triples (p, q, r) in E . This completes the proof of Theorem 4.

Concerning the possibility of extending the result to higher dimensions, we point out the following. In [2], p. 25, an example is given of a convex body $K \in \mathcal{K}_*^3$ to which two different translates of the same tetrahedron can be inscribed. Thus, Lemma 2, which is an essential ingredient of the preceding proof, has no immediate extension to higher dimensions. That the higher-dimensional situation is more complicated, is also indicated by the following observation. The mentioned example can be used to construct a convex body $K \in \mathcal{K}_{o*}^4$ with the following property. There are a point $z \in \text{bd } K$ and a quadruple (p_1, \dots, p_4) of affinely independent points in the tangent hyperplane to K at z such that the following holds. For t in some interval $[t_0, t_1)$, the section $K(z, t)$ has a unique inscribed quadruple homothetic

to (p_1, \dots, p_4) , but not for $t \in (0, t_0)$. Therefore, the equidistant set B_{p_1, p_2, p_3, p_4} cannot be a one-dimensional manifold. This is in contrast to the case of bisectors, where Horváth [6] has shown that for strictly convex norms, they are homeomorphic to hyperplanes.

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