On the lattice diameter of a convex polygon

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Abstract

The lattice diameter, \( \ell(P) \), of a convex polygon \( P \) in \( \mathbb{R}^2 \) measures the longest string of integer points on a line contained in \( P \). We relate the lattice diameter to the area and to the lattice width of \( P \), \( w(P) \). We show, e.g., that \( w \leq \frac{\ell}{2} + 1 \), thus giving a discrete analogue of Blaschke’s theorem. © 2001 Elsevier Science B.V. All rights reserved.

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1. The area of lattice polygons

Let \( P \) be a convex, closed, non-empty lattice polygon, i.e., \( P = \text{conv}(P \cap \mathbb{Z}^2) \). The lattice diameter, \( \ell(P) \), measures the longest string of integer points on a line contained in \( P \)

\[
\ell(P) = \max \{|P \cap \mathbb{Z}^2 \cap L| - 1 : L \text{ is a line}\}.
\]

Thus \( \ell(P) = 0 \), if and only if \( P \) consists of a single lattice point, and for the square \( Q^1 = [0, \ell] \times [0, \ell] \) and for the special pentagon \( Q^2 = \text{conv}\{(x, y) \in \mathbb{Z}^2 : 0 \leq x, y \leq \ell\} \cup \{(\ell + 1, 0), (0, 0)\} \) \( \ell(Q^1) = \ell(Q^2) = \ell \). (See Fig. 1.) This definition is due to Stolarsky and Corzatt [5] who proved several properties of \( \ell(P) \).

The lattice diameter is invariant under the group of unimodular affine transformations

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Fig. 1.

SL(2, \mathbb{Z}); these are lattice preserving mappings $R^2 \to R^2$ also preserving parallel lines and area.

A simple consequence of the definition is the following fact on lattice points contained in $P$ which first appeared in the literature in Rabinowitz [10].

$$ (P \cap \mathbb{Z}^2) \cap ((\ell(P) + 1)z + (P \cap \mathbb{Z}^2)) = \emptyset \quad \text{for every } z \in \mathbb{Z}^2, \ z \neq (0, 0). \quad (1) $$

To see this we note that the common point to $P$, $\mathbb{Z}^2$, and $(\ell(P) + 1)z + P$ would be of the form $(\ell(P) + 1)z + x$ with $x \in (P \cap \mathbb{Z}^2)$ implying that the string of $\ell(P) + 2$ integer points $x, x + z, \ldots, x + (\ell(P) + 1)z$ all belong to $P$ contradicting the definition of the lattice diameter. Eq. (1) implies that \{$(\ell(P) + 1)z + (P \cap \mathbb{Z}^2)$\}$z \in \mathbb{Z}^2$ form a “packing” in $\mathbb{Z}^2$ which shows, in turn, that $P$ contains at most $(\ell(P) + 1)^2$ lattice points,

$$ |P \cap \mathbb{Z}^2| \leq (\ell(P) + 1)^2. \quad (2) $$

An elementary argument and (1) imply that $(\ell(P) + 1)\mathbb{Z}^2 + P$ is a packing in $R^2$ by translates of $P$ so that

$$ \text{area}(P) \leq (\ell(P) + 1)^2. \quad (3) $$

For higher dimension the volume of $P$ is not bounded by a function of $\ell(P)$; there are empty simplices $S \subset R^d$ (i.e., $S \cap \mathbb{Z}^d = \text{vert}(S)$) having arbitrarily large volume (see [11,4,12]), e.g., one can take (in $R^3$) $S = \text{conv}((0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)).$

Let $a(k)$ denote the maximal area a convex lattice polygon $P$ with $\ell(P) \leq k$ can have. The square, i.e., Example $Q^1$, implies $a(k) \geq k^2$. Alarcon [1] observed that this is far from being optimal, area($Q^2$) = $k^2 + k - 1/2$. He also showed $a(1) = 1.5$, $a(2) = 5.5$, $a(3) = 11.5$ and $a(4) = 21$, and improved (3) to $a(k) \leq k^2 + 2k - 2$ for $k \geq 5$. Our first result is that $a(k)$ is very close to the upper bound (3).

**Theorem 1.** For $k \geq 5$ there exists a convex lattice polygon $Q^3$ with $\ell(Q^3) = k$ and area($Q^3$) = $k^2 + 2k - 4$.

The construction $Q^3 = Q^3(k)$ is an octagon with vertices $(-1,0), (0,k - 1), (2,k), (k - 1,k + 1), (k + 1,k), (k,k - 2,0)$, and $(1,-1)$, see Fig. 2. In fact, for $k > 5$ the polygon $Q^3$ is indeed an octagon with only these eight vertices on its boundary and with $(k + 1)^2 - 8$ interior points. For $k = 5$ two of its boundary points, $(2,k)$ and...
Fig. 2.

$(k - 2, 0)$, are not vertices, it becomes a hexagon. Thus Pick’s theorem [9] on the area of lattice polygons, i.e.,

$$\text{area}(P) = |\text{int}(P) \cap \mathbb{Z}^2| - 1 + \frac{|\partial(P) \cap \mathbb{Z}^2|}{2}$$

implies $\text{area}(Q^3) = (k^2 + 2k - 7) - 1 + \frac{k}{2}$, as claimed. (This can be shown directly as well.) Alarcon’s improvement of (3) also utilizes Pick’s theorem, he shows that a maximal $P$ has at least 4 vertices. We conjecture that $Q^3$ is extremal, $a(k) = k^2 + 2k - 4$.

2. Slopes of diameters

Bang [2] solved Tarski’s plank problem by showing that if a compact convex set in $\mathbb{R}^2$ can be covered by $n$ strips of widths $w_1, w_2, \ldots, w_n$ then it can be covered with one strip of width $\sum_{1 \leq i \leq n} w_i$. Corzatt [5] conjectured the following discrete analogue. If the set of lattice points contained in the lattice polygon $P$ can be covered by $n$ lines, $(P \cap \mathbb{Z}^2) \subset (L_1 \cup L_2 \cup \cdots \cup L_n)$, then there exists a set of covering lines $\mathcal{L} = \{L'_1, \ldots, L'_n\}$, $(P \cap \mathbb{Z}^2) \subset (L'_1 \cup L'_2 \cup \cdots \cup L'_n)$ such that the lines in $\mathcal{L}'$ have at most four different slopes. This problem motivated Alarcon [1] to ask the maximum number of diameter directions of a lattice polygon.

A non-zero vector $u \in \mathbb{Z}^2$ is a diameter direction for the convex lattice polygon $P$ if there is an integer $z$ such that $z, z + u, \ldots, z + \ell(P)u$ all belong to $P$. Such a $u$ is necessarily a primitive vector, i.e., its coordinates are coprime. Write $N(P)$ for the number of diameter directions of $P$. The triangle with vertices $(-1, -1), (1, 0), (0, 1)$ and baricenter $(0,0)$ has 6 different diameter directions. Here we prove that

$$N(P) \leq 4,$$
for all convex lattice polygons with $\ell(P) > 1$. This is done by a good description (Theorem 2 below) of convex lattice polygons $P$ that are maximal to containment with respect to $\ell(P) = \ell$.

Write $\mathcal{M}_\ell$ for the collection of maximal convex lattice polygons, i.e., $P \in \mathcal{M}_\ell$ if $\ell(P) = \ell$, and for any convex lattice polygon $P'$ properly containing $P$, $\ell(P') > \ell$.

One more definition: given primitive vectors $u, b \in \mathbb{Z}^2$ (non-parallel) and $z \in \mathbb{Z}^2$, the half-open slab $S(u, b, z)$ is defined as

$$S(u, b, z) = \{z + xu + \beta b : 0 \leq x < \ell + 1, -\infty < \beta < +\infty\}.$$

**Theorem 2.** If $P \in \mathcal{M}_\ell$ then one of the following 3 cases holds.

(i) $P$ has exactly two diameter directions, $u_1$ and $u_2$, say. They form a basis of $\mathbb{Z}^2$. Further, there are points $z_1, z_2 \in \mathbb{Z}^2$ and primitive vectors $b_1$ and $b_2$ such that $z_i, z_i + u_i, \ldots, z_i + \ell u_i \in P$ and

$$P = \text{conv}(\mathbb{Z}^2 \cap S(u_1, b_1, z_1) \cap S(u_2, b_2, z_2)).$$

(ii) $P$ has exactly three diameter directions, $u_1$, $u_2$, $u_3$. Any two of them form a basis of $\mathbb{Z}^2$ thus $u_3 = \pm u_1 \pm u_2$. Further, there are points $z_i \in \mathbb{Z}^2$ and primitive vectors $b_i (i = 1, 2, 3)$ such that $z_i, z_i + u_i, \ldots, z_i + \ell u_i \in P$ and

$$P = \text{conv}\left(\mathbb{Z}^2 \cap \bigcap_{1 \leq i \leq 3} S(u_i, b_i, z_i)\right).$$

(iii) $P$ has exactly four diameter directions. Then (mod $\text{SL}(2, \mathbb{Z})$, i.e., up to a lattice preserving affine transformation) $P$ is either the square $Q^1$ or the special pentagon $Q^2$. (See again Fig. 1.)

The proof is postponed to Section 4.

### 3. Width and covering radius

The lattice diameter is the natural counterpart of the lattice width, $w_l(P)$, which is defined as

$$w_l(P) = \min_{u \in \mathbb{Z}^2 u \neq (0,0)} \left(\max_{x, y \in P} u(x - y)\right).$$

The lattice width is also invariant under the group of unimodular affine transformations $\text{SL}(2, \mathbb{Z})$. Thus $w_l(P) = 0$ if and only if $P$ can be covered by a single line. For the square we have $w_l(Q^1) = \ell$ and for the special pentagon $Q^2$ in Example 1, we have $w_l(Q^2) = \ell + 1 > \ell(Q^2) = \ell$. In general, in Section 5, we prove the following consequence of Theorem 2.

**Theorem 3.** $w_l(P) \leq \left\lceil \frac{4}{3} \ell(P) \right\rceil + 1$ and for given $\ell$ this upper bound is best possible.
The following example, $Q^4$, shows that here equality can hold if $\ell$ is of the form $3t + 1$. The polygon $Q^4 = Q^4(t)$ is a triangle with vertices $(0, 0), (4t + 2, 2t + 1)$, and $(2t + 1, 4t + 2)$; it has lattice diameter $\ell = 3t + 1$ and lattice width $w_1(Q^4) = 4t + 2$. For other values of $\ell$ we obtain equality by considering the triangle $(0, 0), (t, 2t + 1), (2t + 1, t + 1)$. Its width is $2t + 1$ and its diameter is $\lfloor (3t + 1)/2 \rfloor$.

The following example, $Q^5$, shows that there are other completely different polygons with almost equality in Theorem 3. Let $Q^5 = Q^5(t)$ be a hexagon with vertices $(0, 0), (13t, -3t), (9t, 13t), (-3t, 9t)$, where $t \in \mathbb{Z}^+$. It contains the inscribed square $(0, 0), (10t, 0), (10t, 10t), (0, 10t)$ and its covering radius is $\mu_2 = 1/(10t)$. On the other hand, it is easy to see that $\mu_1 = 16t$ is at least 1.2 times larger than $\ell(Q^5) = |(40/3)t|$. We conjecture that in general Schnell’s bound is at most $(1 + \sqrt{2})/2 = 1.207\ldots$ times larger than $\ell(C)$.

Another upper bound for the lattice width is due to Fejes-Tóth and Makai [6]

$$w_1(C) \leq \sqrt{\frac{2}{3}} \text{area}(C).$$

This is also sharp for some cases, like for the triangle $(0, t), (t, 0), (-t, -t)$, but again $Q^6$ shows that it could exceed the bound of Theorem 3 by more than 50%.

4. The maximal polygons, the Proof of Theorem 2

We start with a statement that applies to every convex lattice polygon.

**Lemma 1.** Assume $P$ is a convex lattice polygon and $u \in \mathbb{Z}^2$, $u \neq (0, 0)$. Then there is a longest segment $[z, v]$ contained in $P$ and parallel with $u$ such that $z$ is a vertex of $P$. Further, for every such longest segment $[z, v]$, $v$ lies on an edge $[v_1v_2]$ of $P$ so that the line through $z$ and parallel with $[v_1v_2]$ is tangent to $P$.

The proof is simple and can be found in [3].
Consider now \( P \in \mathcal{M}_\ell \) (with \( \ell \geq 1 \)) and let \( u \) be a diameter direction for \( P \). Apply Lemma 1 to get a longest segment \([z,v]\) with \( z \) a vertex. As \([z,v]\) is a longest segment in direction \( u \), \( z, z+u, \ldots, z+\ell u \in P \cap \mathbb{Z}^2 \). Thus \([z,v]\) contains a lattice diameter.

Applying a suitable lattice preserving affine transformation we may assume \( u = (0,1) \), \( z = (0,0) \) and \( v_2 - v_1 = b = (b_x, b_y) \) with \( 0 \leq 2b_y \leq b_y \), here \([v_1,v_2]\) is the edge of \( P \) specified by Lemma 1. We conclude that \( P \) lies in the half-open slab \( S(u,b,z) \), see Fig. 3.

As the area of the \( z,v_1,v_2 \) triangle is at most \( \text{area}(P) \leq (\ell + 1)^2 \) by (3) and the area of the \( z + (\ell + 1)u,v_1,v_2 \) triangle is at least \( \frac{1}{2} \), we obtain that \( P \) is contained in the slightly narrower half-open slab

\[
S'(u,b,z) := \left\{ z + xu + \beta b : 0 \leq x < \ell + 1 - \frac{1}{2\ell + 2}, -\infty < \beta < +\infty \right\}.
\] (8)

It follows from (1) that \((\pm(\ell + 1),k) \notin P \) for all \( k \in \mathbb{Z} \). Assume now that some \( q = (q_x,q_y) \in \mathbb{Z}^2 \) with \( q_x > \ell + 1 \) belongs to \( P \). The triangle \( T := \text{conv}\{(0,0),(0,\ell),q\} \) meets the line \( x = \ell + 1 \) in a segment of length \( \ell(q_x - \ell - 1)/q_x \). This segment must be lattice point free, so its length is less than 1, implying \( q_x < \ell + 3 \) for \( \ell > 2 \). The case \( \ell < 2 \) is obvious, so from now on we always suppose \( \ell > 2 \). A simple computation reveals that \( T \) contains a lattice point from the line \( x = \ell + 1 \) unless \( q = (\ell + 2,\ell + 1) \).

We treat first this case \( q = (\ell + 2,\ell + 1) \in P \) (which leads to case (iii) as we shall see soon). First \( \text{conv}\{(0,0),v,q\} \subset P \) shows \((0,\ell),(1,\ell),\ldots,(\ell,\ell) \in P \) and \((0,0),(1,1),\ldots,(\ell,\ell) \in P \). So \((0,1),(1,0)\) and \((1,1)\) are diameter directions. As the line \( x = \ell + 1 \) contains no lattice point of \( P \) we have \((\ell + 1,\ell + 1) \in P \) and \((\ell + 1,\ell) \notin P \cap \mathbb{Z}^2 \). As \((\ell + 2,\ell + 1) \in P \) this implies that \((k,\ell + 1) \notin P \) and \((k,k - 1) \notin P \) for all \( k \leq \ell + 1 \). We obtain that for all \((x,y) \in P \cap \mathbb{Z}^2 \) other than \((\ell + 2,\ell + 1) \) we have \( y \leq \ell \) and \( x \leq y \). Further, \((k,-1) \notin P \) and \((k,\ell + 1 + k) \notin P \) for all \( k \in \{-1,-2,\ldots,-(\ell + 1)\} \). Also, \((-\ell,0) \notin P \) since otherwise \((-\ell,0),(-\ell + 2,1),\ldots,(\ell + 2,\ell + 1) \) all belong to \( P \) implying \( \ell(P) > \ell \). Fig. 4 shows the room left for \( P \) after these restrictions.
The maximality of $P$ implies now that $P$ equals $\text{conv}\{(\ell + 2, \ell + 1), (0, 0), (-\ell + 1, 0), (-\ell + 1, 1), (0, \ell)\}$. This is one of the special cases of (iii), the lattice preserving affine transformation $(x, y) \rightarrow (x - y + \ell, y)$ carries $P$ to the “almost-square” special pentagon $Q_2$ of Fig. 1.

From now on we assume that $|x| \leq \ell$ for all $(x, y) \in P$. Thus $P$ is confined to the parallelogram of Fig. 3 bounded by the lines $x = \pm \ell$ and two other lines parallel to $b$. There are only six lattice directions in this parallelogram which can have a chord containing $\ell + 1$ integer points. They are $(0, 1), (1, 0), (1, 1), (1, -1), (2, 1)$ and $(2, -1)$.

To simplify matters we state

**Claim 2.** If $u_1$ and $u_2$ are diameter directions with $\det(u_1, u_2) = 2$ of $P \in M_\ell$ then the diameter segments $[z_1, z_1 + \ell u_1]$ and $[z_2, z_2 + \ell u_2]$ meet either at their midpoints or one segment is off by $u_i$. In these cases $(\mod \text{SL}(2, \mathbb{Z}^2)) P$ is either the square $Q_1$ or the almost square, $Q_2$, cf. Fig. 1.

**Proof.** As we have seen above, we may suppose that $u_1 = (0, 1), P \subset Q$ as in Fig. 3 and $u_2 = (2, 1)$ or $u_2 = (2, -1)$. The latter case leads to the square with vertices $(-\ell, \ell), (0, 0), (\ell, 0)$, and $(0, \ell)$. When $u_2 = (2, 1)$ the diameters are $\{(0, 0), (0, 1), \ldots, (0, \ell)\}$ and $\{(-\ell, i), (-\ell + 2, i + 1), \ldots, (\ell, i + \ell)\} \subset P$. Considering the string of $\ell + 2$ lattice points from $(-1, i - 1)$ to $(\ell, i + \ell)$ it follows that $(-1, i - 1) \notin P$. Since $(-1, 1) \in \text{conv}((-\ell, i), (0, 0), (0, \ell)) \subset P$, it follows $i \leq 1$. Using a symmetric argument we obtain that $i \in \{-1, 0, 1\}$ and can finish the proof as in the case $q = (\ell + 2, \ell + 1) \in P$ above. □

Assume now that $P$ has exactly $k$ diameter directions, $u_1, \ldots, u_k$. Assume that $P$ is not affinely equivalent to $Q_1$ neither $Q_2$. Then by the above Claim $\det(u_i, u_j) = \pm 1$ for any two diameter directions. This implies that $k \leq 3$. The diameters are $z_i, z_i + u_i, \ldots, z_i + \ell u_i$ $(i = 1, \ldots, k)$ with suitable directions $b_i$ of the edge opposite to $z_i$ of $P$ (see Lemma 1).
Define
\[ Q = \bigcap_{1 \leq i \leq k} S(u_i, b_i, z_i). \]

Clearly \( P \subset Q \). We claim \( \ell(Q) = \ell \), so again by the maximality of \( P \), \( P = \text{conv}(Q \cap \mathbb{Z}^2) \), finishing the proof.

Assume, on the contrary, that there exists a lattice point \( q \in (Q \setminus P) \), and suppose that among these points \( q \) is one of the closest to \( P \). Add this point to \( P \), consider \( P' := \text{conv}(P \cup \{q\}) \). So \( q \) is the only new lattice point in \( P' \), \( P' \cap \mathbb{Z}^2 = P \cap \mathbb{Z}^2 \cup \{q\} \). The maximality of \( P \) implies that \( \ell(P') > \ell(P) \), thus \( q \) creates a new longer diameter segment \( q, q + u, \ldots , q + (\ell + 1)u \in P' \cap \mathbb{Z}^2 \) with \( u \neq (0, 0) \). As \( \ell + 1 \) of these points belong to \( P \), we obtain that \( u \) is a diameter direction of \( P \), too. However \( S(u_i, b_i, z_i) \) contains no segments of direction \( u_i \) longer than \( \ell \). Thus \( u \) has to be different from \( u_1, \ldots , u_k \), contradicting that \( P \) has exactly \( k \) diameter directions. Evidently, since \( P \) is not infinite, there are at least two diameter directions, \( k = 2 \) or 3. \( \square \)

5. Bounding the width, the Proof of Theorem 3

As \( w_l \) is an integer for a lattice polygon we have to prove only \( w_l < (\frac{4}{3})(\ell + 1) \).

We give a sketch for the convex set
\[ Q = \bigcap_{1 \leq i \leq k} S_i, \]
where \( S_i = S'(u_i, b_i, z_i) \) are the half-open slabs in (4) and (5) of Theorem 2 modified in (8). Denote the width of the slabs by \( L \). By (8) we have \( L = \ell + 1 - 1/(2\ell + 2) < \ell + 1 \).

Applying a suitable \( \text{SL}(2, \mathbb{Z}^2) \) mapping we may assume that \( u_1 = (1, 0), u_2 = (0, 1) \) and \( u_1 \), if exists, is \( (1, 1) \) or \( (1, -1) \). We will use the fact (which is easy to establish) that the lattice width of \( Q \) is realized in one of the directions \( (0, 1), (1, 0), (1, 1), \) and \( (1, -1) \). The lattice width of \( Q \) in direction \( q \in \mathbb{Z}^2 \) is \( w_l(q, Q) := \max_{x, y \in Q} q(x - y) \).

In case (i) of Theorem 2 (see Fig. 5) \( x = u \) follows from computing the area of \( Q \) in two ways. Similarity of triangles implies \( z : x = (L - x) : y \). We get
\[ w_l((1, 0), Q) = L + y - x, \quad w_l((0, 1), Q) = L + z - x, \]
\[ w_l((-1, 1), Q) = 2L + y - 2x - z, \quad w_l((1, 1), Q) = 2L + z - 2x - y. \] \( \tag{9} \)

Then
\[ w_l(Q) = \min(L + y - x, L + z - x) = L - x + \min\left(y, \frac{(L - x)x}{y}\right) \]
and a simple analysis shows
\[ w_l(Q) \leq \frac{1 + \sqrt{2}}{2} L \approx 1.207 \ldots L. \]
In case (ii) see Fig. 6.

For the left-hand-side hexagon note that the position of $S_3$ does not influence the width of $Q$ as long as $S_3$ cuts off two opposite vertices of the parallelogram $S_1 \cap S_2$. So we may place $S_3$ so as to contain the isosceles and right angle triangle of Fig. 6. Reflecting inwards the three small triangles and comparing areas gives

$$\frac{1}{2}m_1 L + \frac{1}{2}m_2 L + \frac{1}{2}m_3 \sqrt{2} L \leq \frac{1}{2} L^2$$

implying

$$\min(m_1, m_2, \sqrt{2}m_3) \leq \frac{1}{4} L.$$

Further, $w_l((1,0), Q) = L + m_2$, $w_l((0,1), Q) = L + m_1$, and $w_l((-1,1), Q) = L + \sqrt{2}m_3$. So $w_l(Q) \leq \frac{1}{4} L$.

For the other hexagon of Fig. 6 the computations in (9) can easily be applied.
References