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A NOTE ON SYLVESTER'S FOUR-POINT PROBLEM

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Abstract

Let $p(n,K)$ be the probability that a random sample of n points, chosen uniformly from a convex body $K\subset R^d$ is in convex position. In this paper the order of magnitude of $p(n,K)$ is determined as n goes to infinity.

1. Background

In the 1864 April issue of the Educational Times J. J. Sylvester [9] posed an innocent looking question that read: *"Show that the chance of four points forming the apices of a reentrant quadrilateral is* $1/4$ *if they be taken at random in an indefinite plane."* It was understood within a year that the question is ill-posed. In Sylvester's words: *"This problem does not admit of a determinate solution"*. The culprit is, as we all know by now, the "indefinite plane" since there is no natural probability measure on it.

What can be the next move? Modify the question. That is exactly what Sylvester did. Let K be a convex set in the plane, and choose four points from K randomly, independently, and uniformly. What's the chance, $\mathbf{P}(K)$, that the four points form the apices of a reentrant quadrilateral, or, in more recent terminology, that their convex hull is a triangle? Further, for what K is the probability $\mathbf{P}(K)$ the largest and the smallest? This question became known later as "Sylvester's four-point problem". It took fifty years to find the answer: Blaschke [3] showed that for all convex compact bodies $K\subset R^2$

$$\mathbf{P}(\text{disk})\leqq\mathbf{P}(K)\leqq\mathbf{P}(\text{triangle}).$$

The solution uses the technique of symmetrization and "shaking down" that have become standard tools since.

Sylvester's question and its solution determined the direction of research for a long while. Many papers have been written starting with the setting: let $X_n=\{x_1,\dots,x_n\}$ random, independent, uniform sample of n points from

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some fixed d-dimensional convex body K, and let K_n be their convex hull. K_n is the so called random polytope, or polygon when $d=2$. Sylvester's four-point problem is then to determine the probability that K_4 is a triangle.

More generally, define, in the style of Sylvester's original question, $p(n,K)$ as the probability that X_n is in "convex position", that is, no x_i is in the convex hull of the others. Sylvester's four-point question is just the complementary probability when $n=4$: $\mathbf{P}(K)=1-p(4,K)$. Hostinsky [6], Miles [7], Buchta [5], and others considered the problem of determining $p(n,B^d)$, where B^d is the euclidean unit ball in dimension d. An asymptotic result in this direction (from [2]) is that $p(n,B^d)$ tends to one when $n<d^{-1}2^{d/2}$ and tends to zero when $n>d2^{d/2}$ (as d goes to infinity).

In a different development it was observed that, when K is fixed and n goes to infinity, the random polytope K_n gets closer and closer to K. So the natural questions are: How well K_n approximates K in various measures of approximation? How many vertices, edges, facets does K_n have? This type of questions have been studied since Blaschke [3], and Rényi and Sulanke [8] in infinitely many papers. For a comprehesive survey see [10].

2. Back to Sylvester

Let us return to the planar case. It is clear that $p(n,K)$ is extremely small when n is large. But how small is this extremely small? I have recently proved (cf. [1]) that for all convex $K\subset R^2$ with $\operatorname{Vol}K=1$

$$\lim_{n\to\infty} n^2\sqrt[n]{p(n,K)}=\frac{e^2}{4}A^3(K), \tag{1}$$

where $A(K)$ is the affine perimeter of a unique convex subset K_0 of K. The distinguishing property of K_0 is that its affine perimeter is the largest among all convex subsets of K. (The affine perimeter is the integral of the cubic root of the curvature along the boundary.)

The aim of this note is to extend (1) to higher dimensions in the following weak sense.

THEOREM. *Assume $K\subset R^d$ $(d\geqq 2)$ is a convex body with* $\operatorname{Vol}K=1$. *Then for all $n\geqq n_0$ we have*

$$c_1<n^{\frac{2}{d-1}}\sqrt[n]{p(n,K)}<c_2, \tag{2}$$

where n_0, c_1 and c_2 are positive constants depending only on d.

This is the first step towards the stronger conjecture from [1], namely, that

$$\lim_{n\to\infty} n^{\frac{2}{d-1}}\sqrt[n]{p(n,K)}=c(d)A^{\frac{d+1}{d-1}}(K)$$

which looks hopeless at the moment. (Here $A(K)$ is the same constant as in (1): the supremum of the affine surface areas of all convex subsets of K but we will not need this.)

3. The proof

In what follows c and b_i $(i=0,1,2,3)$ denote constants that depend only on d. We start with a simple fact.

LEMMA. *Assume C,D are convex bodies in R^d with $C\subset D$. Then*
$$p(n,C)\leqq\left(\frac{\operatorname{Vol}D}{\operatorname{Vol}C}\right)^n p(n,D).$$

PROOF. Let X_n be random n-sample from D. Then
$$\begin{aligned}p(n,D)&=\mathbf{P}[X_n\text{ convex}]\geqq\mathbf{P}[X_n\text{ convex and }X_n\subset C]\\&=\mathbf{P}[X_n\text{ convex }|X_n\subset C]\,\mathbf{P}[X_n\subset C]\\&=\left(\frac{\operatorname{Vol}C}{\operatorname{Vol}D}\right)^n p(n,C).\end{aligned}$$
□

Using this lemma we show next that it suffices to prove the theorem in the special case when K coincides with $B=B^d$, the euclidean unit ball of R^d. Note first that $p(n,K)$ is invariant under nondegenerate linear transformations. By the well-known theorem of Fritz John, every convex body K can be sandwiched between concentric ellipsoids: $E_1\subset K\subset E_2$ with blow-up factor at most d. So we may assume that K is sandwiched between B and dB. By the lemma
$$d^{-dn}p(n,B)\leqq p(n,K)\leqq d^{dn}p(n,B),$$
as $d^{-d}\leqq\operatorname{Vol}B/\operatorname{Vol}K\leqq 1$.

Next we prove the upper bound in (2) assuming $K=B$. Write $\mathcal{C}$ for the metric space of all convex subsets of B equipped with the Hausdorff metric. According to an important result of Bronshtein (see [4]), for every $\varepsilon>0$, there is an ε-net $\{C_1,\dots,C_N\}\subset\mathcal{C}$ with $N\leqq\exp\{c\,\varepsilon^{-\frac{d-1}{2}}\}$. That is, for every $D\in\mathcal{C}$ there is a C_j with Hausdorff distance at most ε from D. In particular, the boundary of D is contained in $\partial C_j+\varepsilon B$ for some $j\in\{1,\dots,N\}$.

Now consider n-tuples $(x_1,\dots,x_n)\in K^n=K\times\cdots\times K$ that are in convex position. As $D=\operatorname{conv}\{x_1,\dots,x_n\}\in\mathcal{C}$, the previous observation applies to D and gives
$$\{(x_1,\dots,x_n)\in K^n:\text{ in convex position}\}\subset\bigcup_1^N(\partial C_j+\varepsilon B)^n.$$
As $p(n,B)$ is just the product measure of the left-hand side above (with $\operatorname{Vol}B$ scaled to 1) we get
$$\begin{aligned}p(n,B)&\leqq\operatorname{meas}\{\bigcup_1^N(\partial C_j+\varepsilon B)^n\}\\&\leqq\sum_1^N\operatorname{meas}\{(\partial C_j+\varepsilon B)^n\}\leqq N\max_j\operatorname{meas}\{(\partial C_j+\varepsilon B)^n\}.\end{aligned}$$

Observe now that $\mathrm{meas}\{(\partial C_j + \varepsilon B)^n\} = (\mathrm{Vol}(\partial C_j + \varepsilon B))^n$ and $\mathrm{Vol}(\partial C_j + \varepsilon B) \leqq \mathrm{Vol}(\partial B + \varepsilon B) \leqq b_0\varepsilon$ for some constant b_0 depending only on d. Choosing $\varepsilon = n^{-\frac{2}{d-1}}$ gives $N \leqq e^{cn}$ and so $\sqrt[n]{p(n,B)} \leqq e^c b_0 n^{-\frac{2}{d-1}}$ giving the upper bound in (2).

For the lower bound define $A = A(\varepsilon)$ as the annulus $B \setminus (1-\varepsilon)B$, where $\varepsilon = \triangle n^{-\frac{2}{d-1}}$ with $\triangle$, a small positive constant to be chosen later. For $x \in A$ define

$$G(x) = \{y \in A : \angle x0y \leqq \arccos(1-\varepsilon)\}.$$

Note that, for $x, y \in A$, $x \in G(y)$ if and only if $y \in G(x)$. For $x \in A$ let $C(x)$ be the cap cut off from B by the hyperplane orthogonal to, and passing through, x. Clearly, $C(x) \subset G(x)$.

Now pick the points $x_1, \dots, x_n$ by induction with the following rule. Assuming $x_1, \dots, x_k$ have been chosen, take x_{k+1} uniformly from $A \setminus \bigcup_1^k G(x_i)$ (and independently of the previous choices).

We claim that the points $x_1, \dots, x_n$ are in convex position. The proof is simple: as $C(x_i) \subset G(x_i)$ for all i, $C(x_i)$ contains none of the other x_j. Thus x_i is separated from the other points by the bounding hyperplane of $C(x_i)$.

We estimate next the volume of $A \setminus \bigcup_1^k G(x_i)$. It is evident that $\mathrm{Vol}\, G(x) \leqq b_1 \varepsilon^{\frac{d+1}{2}}$.

$$\begin{aligned}\mathrm{Vol}(A \setminus \bigcup_1^k G(x_i)) &\geqq \mathrm{Vol}\, A - \sum_1^k \mathrm{Vol}\, G(x_i) \geqq b_2\varepsilon - k b_1 \varepsilon^{\frac{d+1}{2}} \\ &\geqq b_2\varepsilon - n b_1 \varepsilon^{\frac{d+1}{2}} \geqq b_2 \triangle n^{-\frac{2}{d-1}} - b_1 \triangle^{\frac{d+1}{2}} n^{1-\frac{d+1}{d-1}} \\ &\geqq n^{-\frac{2}{d-1}} \left(b_2\triangle - b_1 \triangle^{\frac{d+1}{2}} \right) \geqq b_3 n^{-\frac{2}{d-1}},\end{aligned}$$

if the constant $\triangle$ is chosen small enough.

Assuming again that Vol is scaled so that $\mathrm{Vol}\, B = 1$ we have

$$p(n,B) \geqq \prod_0^{n-1} \mathrm{Vol}(A \setminus \bigcup_1^k G(x_i)) \geqq \left(b_3 n^{-\frac{2}{d-1}} \right)^n,$$

finishing the proof. □

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