On Some Combinatorial Questions in Finite-Dimensional Spaces

I. Bárány
Mathematical Institute of the Hungarian Academy of Sciences
1053 Budapest, Reáltanoda u. 13-15, Hungary

and

V. S. Grinberg
Arm'anskij per. 1/3 kv. 9
310003 Kharkov, U.S.S.R.

Submitted by Richard A. Brualdi

ABSTRACT

Bounds are given for $\sup\inf\|\sum_{i=1}^{p}v_i\|$, where sup is taken over all set systems $V_1,\dots,V_p$ of R^n with $0\in\bigcap_{i=1}^{p}\operatorname{conv}V_i$ and $\sup_{v\in V_i}\|v\|\leqslant 1$ for $i=1,\dots,p$, and inf is taken over all possible choices $v_i\in V_i$ for $i=1,\dots,p$. Another similar problem is considered. The bounds are sharp.

1. INTRODUCTION

In this paper we are concerned with the following question. Let a norm of R^n be given with unit ball B^n. Suppose $C_i\subseteq B^n$ and $0\in\operatorname{conv}C_i$ for $i\in[p]$ (here and throughout the paper we write $[p]$ for the set $\{1,\dots,p\}$). Questions: Are there elements $c_i\in C_i$ $(i\in[p])$ such that $\|\sum_1^p c_i\|$ is less than a constant depending only on n and the norm (and not depending on p and the sequence C_i)? Are there elements $c_i\in C_i$ $(i\in[p])$ such that $\max_{q\in[p]}\|\sum_{i=1}^{q}c_i\|$ is less than a constant depending only on n and the norm? How large are these constants?

These questions came from other questions and facts. In 1963 Dvoretzky [3] asked what $\max\min\|\pm x_1\pm x_2\pm\cdots\pm x_p\|$ equals, where max is taken over all p-tuples $\{x_1,\dots,x_p\}$ of the unit sphere of R^n and min is taken over all possible signs. Taking max over all p-tuples of B^n and putting $C_i=\{x_i,-x_i\}$ $(i\in[p])$, we get a special case of our question. In 1976 Spencer [5] considered several "balancing games." In one of his games the first player picks a sequence $x_1,x_2,\dots$ from B^n, and then the second player picks $\varepsilon_i=1$ or -1

52 Vanderbilt Ave., New York, NY 10017 0024-3795/81/070001+9$02.50

($i=1,2,\dots$) so as to minimize $\sup_{q=1,\dots}\|\sum_{i=1}^{q}\varepsilon_i x_i\|$. Spencer showed that this supremum is less than a constant depending only on n and the norm. Our second theorem will generalize and sharpen Spencer's result. We remark that our theorems have interesting applications to some problems in the theory of planning.

A few results of this paper cover some theorems of [1]. However the results here are stronger and the proof method of this paper is different. The idea of using the theory of linear inequalities comes from S. V. Sevast'yanov [4], who proved the Steinitz lemma with the help of this theory.

2. MAIN RESULTS

As we shall see, our theorems hold not only for norms but for "symmetric and/or non-symmetric seminorms" as well. A nonsymmetric seminorm is a map $h: R^n \to R^1$ such that

(i) $h(x+y) \leqslant h(x)+h(y)$ ($x, y \in R^n$),
(ii) $h(\alpha x)=\alpha h(x)$ ($x \in R^n$; $\alpha \geqslant 0$).

A symmetric seminorm satisfies (i) and, further,

(iii) $h(\alpha x)=|\alpha| h(x)$ ($x \in R^n$, $\alpha \in R^1$).

We write $B^n=\{x \in R^n : h(x) \leqslant 1\}$.

THEOREM 1. *Let h be a nonsymmetric seminorm. Suppose that $C_i \subseteq B^n$ and $0 \in \operatorname{conv} C_i$ ($i \in [p]$). Then there exists $c_i \in C_i$ ($i \in [p]$) such that $h(\sum_{i=1}^{p} c_i) \leqslant n$.*

This theorem can be expressed in the following way. Put

$$E(n,h)=\sup\inf h\left(\sum_{i=1}^{p} c_i\right),$$

where sup is taken over all sequences of sets $C_i \subseteq B^n$ with $0 \in \operatorname{conv} C_i$ ($i \in [p]$) and inf is taken over all choices $c_i \in C_i$ ($i \in [p]$). Then the theorem states that $E(n,h) \leqslant n$ for every nonsymmetric seminorm h. Theorem 1 is sharp in the sense that with the l_1 norm, $E(n, \| \ \|_1)=n$. To see this consider $C_i=\{e_i, -e_i\}$ ($i \in [n]$), where e_i is the ith basic vector of R^n. Then, clearly, for any choice $c_i \in C_i$ one has $\|\sum_{i=1}^{n} c_i\|_1=n$.

To prove the theorem we need a lemma. This lemma is a generalization of the well-known theorem of Carathéodory [2]. We write $\operatorname{pos} C$ ($\operatorname{lin} C$) for the cone (linear) hull of $C \subset R^n$.

LEMMA 1. *Suppose* $U, V_i \subset R^n$ $(i \in [k])$, *U consists of linearly independent vectors, and* $v \in \operatorname{lin} U + \operatorname{pos} V_i$ *for* $i \in [k]$. *Assume further, that* $|U| + k = n$. *Then there exist vectors* $v_i \in V_i$ *for* $i \in [k]$ *such that* $v \in \operatorname{lin} U + \operatorname{pos}\{v_1, \ldots, v_k\}$.

Proof. For any choice $v_1, \ldots, v_k$ with $v_i \in V_i$ $(i \in [k])$ put $d(v_1, \ldots, v_k) = \min\{\|v - x\| : x \in \operatorname{lin} U + \operatorname{pos}\{v_1, \ldots, v_k\}\}$, where $\| \;\|$ stands for the Euclidean norm. By Carathéodory's theorem we may suppose that each V_i is finite. Then there exists a choice $v_1, \ldots, v_k$ for which $d = d(v_1, \ldots, v_k)$ is minimal. We claim that $d = 0$. This will prove the lemma.

Suppose, on the contrary, that $d > 0$. Then there exists a (unique) $z \in \operatorname{lin} U + \operatorname{pos}\{v_1, \ldots, v_k\} = D$ with $d = \|v - z\|$; this z is the projection of the point v onto the convex cone D. Thus putting $w = v - z$, the hyperplane $\langle w, x \rangle = 0$ separates v and D, i.e., $\langle w, v \rangle > 0$, $\langle w, z \rangle = 0$, $\langle w, u \rangle = 0$ for $u \in U$, and $\langle w, v_i \rangle \leqslant 0$ for $i \in [k]$.

Clearly z can be expressed as $z = u + \sum_{i=1}^{k} \gamma_i v_i$, where $u \in \operatorname{lin} U$ and $\gamma_i \geqslant 0$; moreover, this representation can be chosen so that $\gamma_j = 0$ for some $j \in [k]$. This is true either because D is n-dimensional, and consequently the minimum of $\|v - x\|$ for $x \in D$ is attained on the boundary of D, or because D lies in a hyperplane, and then any point of D can be expressed with some $\gamma_j = 0$. Suppose, without loss of generality, that $\gamma_1 = 0$. Now $v \in \operatorname{lin} U + \operatorname{pos} V_1$ implies that there is a $v_1' \in V_1$ such that $\langle w, v_1' \rangle > 0$.

Now we prove that $d(v_1', v_2, \ldots, v_k) < d$. Indeed, for $t \in [0,1]$, $z + t(v_1' - z) \in \operatorname{lin} U + \operatorname{pos}\{v_1', v_2, \ldots, v_k\}$, and

$$\begin{aligned} d^2(v_1', v_2, \ldots, v_k) &\leqslant \|v - [z + t(v_1' - z)]\|^2 \\ &= \|(v - z) - t(v_1' - z)\|^2 \\ &= d^2 - 2t\langle w, v_1' - z \rangle + t^2 \|v_1' - z\|^2, \end{aligned}$$

and this is less than d^2 if $t > 0$ is sufficiently small, because $\langle w, v_1' - z \rangle = \langle w, v_1' \rangle > 0$. ■

Proof of Theorem 1. We suppose again that every C_i is finite. Consider the following optimization problem:

$$\begin{array}{lll} \text{maximize} & \sum_{i=1}^{p} \gamma_i & \\ \text{subject to} & 0 \leqslant \gamma_i \leqslant 1, & i \in [p], \\ & c_i \in C_i, & i \in [p], \\ & \sum_{i=1}^{p} \gamma_i c_i = 0. & \end{array}$$

The maximum is clearly attained for some system $\bar{\gamma}_i$ $(i \in [p])$ with $\bar{c}_i \in C_i$. We choose this system so that $A_{01} = \{i: 0 < \bar{\gamma}_i < 1\}$ will be minimal. Put further $A_1 = \{i: \bar{\gamma}_i = 1\}$ and $A_0 = \{i: \bar{\gamma}_i = 0\}$.

First we claim that the vectors $\bar{c}_i$ $(i \in A_{01})$ are linearly independent. Suppose they are not; then $\sum_{i \in A_{01}} \alpha_i \bar{c}_i = 0$ and $\sum_{i \in A_{01}} \alpha_i \geqslant 0$ (say) for some (not all zero) numbers α_i. Put $\alpha_i = 0$ for $i \notin A_{01}$ and $\gamma_i = \bar{\gamma}_i + t\alpha_i$. Choosing here $t > 0$ appropriately (i.e. so that $\gamma_i \in [0,1]$ for each $i \in [p]$, and $\gamma_i = 0$ or 1 for some $i \in A_{01}$), we have a system with $\sum_1^p \gamma_i \bar{c}_i = 0$, $0 \leqslant \gamma_i \leqslant 1$, and $\sum_1^p \gamma_i \geqslant \sum_1^p \bar{\gamma}_i$, but with a smaller set $A'_{01} = \{i: 0 < \gamma_i < 1\}$.

If $c \in R^n$ we write $(c;1)$ for the vector in R^{n+1} whose jth component equals that of c for $j \in [n]$ and 1 for $j = n+1$. If $C \subset R^n$ we use $(C;1)$ as a shorthand for the set $\{(c;1): c \in C\}$.

Secondly we claim that $|A_{01} \cup A_0| \leqslant n$. Suppose, on the contrary, that $|A_{01} \cup A_0| \geqslant n+1$; then because of $|A_{01}| \leqslant n$, there is a set $A \subseteq A_0$ with $|A_{01} \cup A| = n+1$. Put $U = \{(\bar{c}_j;1): j \in A_{01}\}$ and $V_i = (C_i;1)$ for $i \in A$. Now we can apply the lemma (even if $A_{01} = \varnothing$) with $v = (0;1)$, because

$$(0;1) \in \operatorname{pos} V_i \subseteq \operatorname{lin} U + \operatorname{pos} V_i \qquad \text{for each} \quad i \in A$$

and $|A_{01} \cup A| = n+1$. This yields coefficients β_i for $i \in A_{01} \cup A$ with $\beta_i \geqslant 0$ if $i \in A$, and vectors $c_i \in C_i$ $(i \in A)$ such that

$$\sum_{i \in A_{01}} \beta_i(\bar{c}_i;1) + \sum_{i \in A} \beta_i(c_i;1) = (0;1).$$

Put now $\beta_i = 0$ if $i \notin A \cup A_{01}$, and $c_i = \bar{c}_i$ if $i \notin A$. Then for some small $t > 0$, $\gamma_i = \bar{\gamma}_i + t\beta_i$ satisfies

$$0 \leqslant \gamma_i \leqslant 1 \quad \text{and} \quad \sum_{i=1}^{p} \gamma_i c_i = 0,$$

but $\sum \gamma_i = \sum \bar{\gamma}_i + t > \sum \bar{\gamma}_i$, a contradiction.

So $|A_{01} \cup A_0| \leqslant n$, and consequently $|A_1| \geqslant p - n$. This implies that $\sum_1^p \bar{\gamma}_i \geqslant p - n$.

Now we prove that $\bar{c}_i \in C_i$ is the choice whose existence is claimed in the theorem. Indeed,

$$\sum_{i=1}^{p} \bar{c}_i = \sum_{i=1}^{p} \bar{c}_i - \sum_{i=1}^{p} \bar{\gamma}_i \bar{c}_i = \sum_{i=1}^{p} (1 - \bar{\gamma}_i)\bar{c}_i$$

and

$$h\left(\sum_{i=1}^{p} \bar{c}_i\right) \leqslant \sum_{i=1}^{p} (1 - \bar{\gamma}_i) \leqslant p - (p-n) = n. \qquad \blacksquare$$

For symmetric seminorms we can weaken the assumptions of Theorem 1. For $A, B \subseteq R^n$ put $A+B=\{a+b : a\in A,\ b\in B\}$.

THEOREM 2. *Let h be a symmetric seminorm, and suppose that $C_i \subset B^n$ for $i\in[p]$ and $0\in\sum_{i=1}^{p}\operatorname{conv} C_i$. Then there exist vectors $c_i\in C_i$ $(i\in[p])$ such that*

$$h\left(\sum_{i=1}^{p} c_i\right)\leqslant n.$$

Proof. The condition means that the system with unknowns $\alpha_i(x)$ given by

$$\sum_{i=1}^{p}\sum_{x\in C_i}\alpha_i(x)x=0,$$

$$\sum_{x\in C_i}\alpha_i(x)=1 \qquad \text{for} \quad i\in[p],$$

$$\alpha_i(x)\geqslant 0 \qquad \text{for} \quad i\in[p], \quad x\in C_i$$

has at least one solution, so that the solution set of this system is nonempty and, of course, convex compact. Take any extreme point $\bar{\alpha}_i(x)$ and put $a_i=|\{x\in C_i : \bar{\alpha}_i(x)>0\}|$. (Again we suppose that each C_i is finite.) It is evident that the number of slack inequalities in this point, $\sum_{i=1}^{p}a_i$, is at most $n+p$, and $a_i\geqslant 1$ $(i\in[p])$. Now let c_i be any element of C_i for which $\bar{\alpha}_i(c_i)=\max\{\bar{\alpha}_i(x) : x\in C_i\}$ $(i\in[p])$. Clearly $\bar{\alpha}_i(c_i)\geqslant 1/a_i$. Now

$$\sum_{i=1}^{p} c_i=\sum_{i=1}^{p}\left(c_i-\sum_{x\in C_i}\bar{\alpha}_i(x)x\right)$$

$$=\sum_{i=1}^{p}\left([1-\bar{\alpha}_i(c_i)]c_i-\sum_{\substack{x\in C_i\\ x\neq c_i}}\bar{\alpha}_i(x)x\right),$$

so using the inequality $1-1/a_i\leqslant(a_i-1)/2$, which is true for $a_i=1,2,\ldots,$ we get

$$h\left(\sum_{i=1}^{p} c_i\right)\leqslant\sum_{i=1}^{p}\left([1-\bar{\alpha}_i(c_i)]+\sum_{\substack{x\in C_i\\ x\neq c_i}}\bar{\alpha}_i(x)\right)=2\sum_{i=1}^{p}[1-\bar{\alpha}_i(c_i)]$$

$$\leqslant 2\sum_{i=1}^{p}\left(1-\frac{1}{a_i}\right)\leqslant\sum_{i=1}^{p}(a_i-1)\leqslant n. \qquad ■$$

We have seen that for the l_1 norm of R^n, $E(n, \|\ \|_1) = n$. This, of course, does not mean that one cannot do better for other norms. For instance, for the Euclidean norm we have $E(n, \|\ \|_2) \geq \sqrt{n}$, this lower bound being reached here when $p = n$ and $C_i = \{e_i, -e_i\}$ $(i \in [n])$. V. V. Grinberg has informed us that he has proved $E(n, \|\ \|_2) \leq \sqrt{n}$ (unpublished). From the point of view of applications it would be interesting to know more about $E(n, \|\ \|_\infty)$.

We finish this section with open questions. Suppose we are given another seminorm h' of R^n, and further, that the unit balls of both seminorms h and h' are compact. Put

$$E(n, h, h') = \sup\inf h'\left(\sum_{i=1}^{p} c_i\right),$$

where sup and inf are taken over the same sets as in the definition of $E(n, h)$. Clearly

$$E(n, h, h') \leq E(n, h) \sup\{h'(x) : x \in B^n\},$$

but we think that in general one can do better. In connection with this there is a striking question due to J. Komlós (private communication). He asks whether there is a universal constant c, such that for any n and p and any set of vectors $\{x_1, \ldots, x_p\} \subset R^n$ with $\|x_i\|_2 \leq 1$ $(i \in [p])$, one can find signs $\varepsilon_1, \ldots, \varepsilon_p = 1$ such that $\|\sum_{i=1}^{p} \varepsilon_i x_i\|_\infty \leq c$.

3. A VARIANT PROBLEM

In this section we shall prove the following theorem.

THEOREM 3. *Let h be a symmetric seminorm of R^n with unit ball B^n. Suppose $C_i \subset B^n$ and $0 \in \operatorname{conv} C_i$ for $i = 1, 2, \ldots$. Then there exist elements $c_i \in C_i$ $(i = 1, 2, \ldots)$ such that*

$$h\left(\sum_{i=1}^{p} c_i\right) \leq 2n \qquad \text{for} \quad p = 1, 2, \ldots .$$

We need the following simple lemma.

LEMMA 2. *Suppose $v \in \sum_{i=1}^{n+1} \operatorname{conv} V_i$, where $V_i \subset R^n$ for $i \in [n+1]$. Then there exist an index $j \in [n+1]$ and element $v_j \in V_j$ such that $v \in \operatorname{conv} V_1 + \cdots + \operatorname{conv} V_{j-1} + v_j + \operatorname{conv} V_{j+1} + \cdots + \operatorname{conv} V_{n+1}$.*

Proof of the lemma. The solution set of the system with unknowns $\alpha_i(x)$ $(i\in[n+1], x\in V_i)$

$$\sum_{i=1}^{n+1}\sum_{x\in C_i}\alpha_i(x)x=0,$$

$$\sum_{x\in C_i}\alpha_i(x)=1 \qquad \text{for} \quad i\in[n+1],$$

$$\alpha_i(x)\geqslant 0 \qquad \text{for} \quad i\in[n+1], \quad x\in V_i$$

is convex, compact, and, in view of the assumption, nonempty. Then at the extreme point $\bar{\alpha}_i(x)$ at most $2n+1$ inequalities are strict. In each V_i there is at least one element $v\in V_i$ with $\bar{\alpha}_i(v)>0$. This implies that for some $j\in[n+1]$ and $v_j\in V_j$, $\bar{\alpha}_j(v_j)=1$, and then, of course, $\bar{\alpha}_j(v)=0$ if $v\in V_j\setminus\{v_j\}$. This proves the lemma. ■

Proof of Theorem 3. First we shall construct a sequence $A_0\subset A_1\subset A_2\subset\cdots$ with $A_j\subset[j+n]$ and $|A_j|=j$, and choose an element $c_i\in C_i$ for each $i\in\bigcup_{j=d}^{\infty}A_j$ such that putting $B_j=[j+n]\setminus A_j$,

$$0\in\sum_{i\in A_j}c_i+\sum_{i\in B_j}\operatorname{conv}C_i. \tag{*}$$

This is done by induction on j.

$j=0$. Put $A_j\neq\varnothing$; then $B_j=[n]$ and (*) is fulfilled.

$j\to j+1$. Write $D=B_j\cup\{j+n+1\}$. In view of (*) and the assumption of the theorem we have

$$-\sum_{i\in A_j}c_i\in\sum_{i\in D}\operatorname{conv}C_i,$$

and $|D|=n+1$. Thus by the lemma there is an index $i_0\in D$ and an element $c_{i_0}\in C_{i_0}$ such that

$$-\sum_{i\in A_j}c_i\in c_{i_0}+\sum_{i\in D\setminus\{i_0\}}\operatorname{conv}C_i.$$

Putting $A_{j+1}=A_j\cup\{i_0\}$, we are through: c_{i_0} is just the element needed.

Now the sequence whose existence is claimed in the theorem is "almost" defined, for there are at most n natural numbers not belonging to $\bigcup_{j=0}^{\infty} A_j$. For these indices i let c_i be an arbitrary element of C_i. Let us put $p=j+n$; then

$$\sum_{i=1}^{p} c_i = \sum_{i\in A_j} c_i + \sum_{i\in B_j} c_i,$$

and by (*) $\sum_{i\in A_j} c_i \in -\sum_{i\in B_j}\operatorname{conv} C_i$. But $c_i\in B^n$ and $-\operatorname{conv} C_i\subset B^n$ for $i\in B_j$ and $|B_j|=n$, whence

$$\sum_{i=1}^{p} c_i \in \sum_{i\in B_j} c_i + \sum_{i\in B_j} -\operatorname{conv} C_i \subset 2nB^n.$$

If $p<n$, then $h(\sum_{i=1}^{p} c_i)\leqslant p<n$. ■

Again, Theorem 3 can be expressed as $F(n,h)\leqslant 2n$ where $F(n,h)= \sup\inf\sup_{p=1,2,\ldots} h(\sum_{i=1}^{p} c_i)$; here the first sup is taken over all sequences $C_i\subset B^n$, $0\in\operatorname{conv} C_i$ $(i=1,2,\ldots)$, and inf is taken over all choices $c_i\in C_i$ $(i=1,2,\ldots)$. With some additional effort we can prove here $F(n,h)\leqslant 2n-1$. On the other hand the best lower bound known to the authors is $n\leqslant F(n,\|\ \|_1)$. This lower bound is reached by the same construction as in Theorem 1.

Finally we present an example showing that $F(2,h)$ [and so $F(n,h)$] is not bounded in general when h is nonsymmetric. To this end let $h(x,y)= \max\{0,-x,-y\}$ for $(x,y)\in R^2$ be the nonsymmetric seminorm, and put $C_i=\{a_i,b_i\}$, where $a_i=(-1,2^i)$ and $b_i=(2^{-i},-1)$ for $i=1,2,\ldots$. Clearly $h(a_i)=h(b_i)=1$ and $0\in\operatorname{conv} C_i$. In this case $h(\sum_{i=1}^{p} c_i)$ tends to infinity as $p\to\infty$ for any choice $c_i\in C_i$. Indeed, if $c_i=a_i$ for infinitely many indices i, then the first component of $\sum_{i=1}^{p} c_i$ tends to $-\infty$, and if $c_i=a_i$ for finitely many times only, then the second component of the sum tends to $-\infty$.

From this example it is not difficult to show that for any $N>0$ there exist a nonsymmetric seminorm h of R^2 with compact unit ball B and sets $C_i\subset B$ for $i\in[p]$ with $0\in\operatorname{conv} C_i$ such that for any choice $c_i\in C_i$ $(i\in[p])$,

$$\max_{1\leqslant k\leqslant p} h\left(\sum_{i=1}^{k} c_i\right)\geqslant N.$$

REFERENCES

1 I. Bárány, System of representatives for sets whose convex hull contains zero, in *Proceedings of the Conference on Algebraic Methods in Graph Theory, Szeged, 1978*, to appear.

2 L. Danzer, B. Grünbaum, and V. Klee, Helly's theorem and its relatives, in *Proceedings of Symposia in Pure Mathematics, Vol. 7. Convexity*, Amer. Math. Soc., Providence, RI, 1963, pp. 101–180.
3 A. Dvoretzky, Problem, in *Proceedings of Symposia in Pure Mathematics, Vol. 7. Convexity*, Amer. Math. Soc., Providence, RI, 1963, p. 496.
4 S. V. Sevast'yanov, On approximate solutions of scheduling problems, *Methody Discretnogo Analiza* (Novosibirsk) 32:66–75 (1978).
5 J. Spencer, Balancing games, *J. Combinatorial Theory Ser. B* 23:68–74 (1977).

Received 4 June 1980; revised 5 December 1980