J. reine angew. Math. **484** (1997), 71—84

Journal für die reine und angewandte Mathematik

Affine perimeter and limit shape

By *Imre Bárány*[1]) at Budapest

Abstract. It is proved here that, as $n \to \infty$, almost all convex $\frac{1}{n}\mathbb{Z}^2$-lattice polygons lying in a given convex compact set $K \subset \mathbb{R}^2$ are very close to a fixed convex set K_0. The distinguishing property of K_0 is that its affine perimeter is the largest among all convex sets contained in K.

1. Main results

For a compact convex set $K \subset \mathbb{R}^2$ let $\mathscr{P}_n(K)$ denote the set of convex polygons $P \subset K$ whose vertices belong to the lattice $\frac{1}{n}\mathbb{Z}^2$. Inspired by a result of Arnold [Ar], Vershik asked about 15 years ago if there is a limit shape to some collections of convex lattice polygons. This was answered in the affirmative in [Bá] and [Ve] for the case $\mathscr{P}_n(K)$ with K the unit square (see also [Si]). Here we extend this result for every convex compact set K with nonempty interior by showing that, as $n \to \infty$, the overwhelming majority of the members of $\mathscr{P}_n(K)$ are very close to a fixed convex set K_0.

This fixed convex set K_0, the limit shape of the elements in $\mathscr{P}_n(K)$, is characterized as the convex subset of K with the largest affine perimeter. Write $\mathscr{C}$ for the family of all convex compact sets in $\mathbb{R}^2$ with nonempty interior, and set $\mathscr{C}(K) = \{S \in \mathscr{C} : S \subset K\}$. The affine perimeter (the definition is in the next section) of $S \in \mathscr{C}$ is denoted by $\mathrm{AP}(S)$. The existence and unicity of K_0 is the content of

Theorem 1. *For every* $K \in \mathscr{C}$ *there is a* $K_0 \in \mathscr{C}(K)$ *such that* $\mathrm{AP}(K_0) > \mathrm{AP}(S)$ *for every* $S \in \mathscr{C}(K)$ *different from* K_0.

Let $\delta(A, B)$ stand for the Haussdorf distance of $A, B \subset R^2$. The limit shape of the members of $\mathscr{P}_n(K)$ is K_0.

[1]) Partially supported by Hungarian Science Foundation Grant T016391.

Theorem 2. *For every $K \in \mathscr{C}$ and every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{|\{P \in \mathscr{P}_n(K) : \delta(P, K_0) < \varepsilon\}|}{|\mathscr{P}_n(K)|} = 1 .$$

Almost all the paper is devoted to the proof of this limit shape theorem. We will need to know, asymptotically at least, the size of $\mathscr{P}_n(K)$. We state this as

Theorem 3. *For each $K \in \mathscr{C}$,*

$$\lim_{n\to\infty} n^{-2/3} \log |\mathscr{P}_n(K)| = 3\zeta \mathrm{AP}(K_0) .$$

Here and in what follows

(1.1) $$\zeta = \sqrt[3]{\frac{\zeta(3)}{4\zeta(2)}}$$

with $\zeta(x)$ being Riemann's zeta function. The theorem shows that $|\mathscr{P}_n(K)|$ remains essentially the same after any area-preserving affine transformation, which is a priori not obvious (at least for the author). Clearly $|\mathscr{P}_n(K)|$ is invariant under lattice preserving affine transformations and of course, $(LK)_0 = L(K_0)$ for every nonsingular affine transformation L.

In the next two sections we define the affine perimeter and prove Theorem 1. We prove and use the fact (Lemma 1 and (3.3)) that the affine perimeter is a concave functional on $\mathscr{C}$ with respect to Minkowski addition. This has been known for Blaschke addition in stronger form (see Lutwak [Lu]) and in any dimension. In section 4 we give the proof that $\liminf n^{-2/3} \log |\mathscr{P}_n(K)| \geqq 3\zeta \mathrm{AP}(K_0)$ and a sketch of the proof of the other "half" of Theorem 3, namely, that the lim sup is at most $3\zeta \mathrm{AP}(K_0)$. The details are presented in the next section. We conclude with the proof of the limit shape theorem.

2. Affine perimeter

The affine perimeter (and more generally, the affine arclength) can be defined in many ways (cf. [Bl], [Le], [Lu], [Sc]) the most pleasant for us being the following (see [Bl] for the facts cited below). Given $S \in \mathscr{C}$ choose a subdivison $x_1, \dots, x_m, x_{m+1} = x_1$ of the boundary ∂S and lines ℓ_i supporting S at x_i for all $i \in [m]$ where $[m] = \{1, \dots, m\}$. Write y_i for the intersection of ℓ_i and ℓ_{i+1} (if $\ell_i = \ell_{i+1}$, then y_i can be any point between x_i and x_{i+1}). Let T_i denote the triangle with vertices x_i, y_i, x_{i+1} and also its area. The definition is:

(2.1) $$\mathrm{AP}(S) = 2 \lim \sum_1^m \sqrt[3]{T_i}$$

where the limit is taken over a sequence of subdivisions with $\max_{1,\dots,m} |x_i - x_{i+1}| \to 0$. The existence of the limit, and its independence of the sequence chosen, follow from the fact that $\sum_1^m \sqrt[3]{T_i}$ decreases as the subdivision is refined. Consequently

(2.2) $$\mathrm{AP}(S) = 2\inf \sum_1^m \sqrt[3]{T_i}.$$

We record further properties of the map $\mathrm{AP}: \mathscr{C} \to R$, (see [Bl], [Le]):

(2.3) $$\mathrm{AP}(\lambda S) = \lambda^{2/3}\mathrm{AP}(S), \quad \text{when } \lambda > 0,$$

(2.4) $$\mathrm{AP}(LS) = (\det L)^{1/3}\mathrm{AP}(S), \quad \text{when } L: R^2 \to R^2 \text{ is linear},$$

(2.5) $$\mathrm{AP}(S) = \int_{\partial S} \kappa^{1/3} ds = \int_0^{2\pi} r^{2/3} d\phi,$$

where κ is the curvature and $r = r(\phi) = \kappa^{-1}$ is the radius of curvature at the boundary point with outer normal $\omega(\phi) = (\cos\phi, \sin\phi)$. In (2.5), of course, ∂S has to be sufficiently smooth.

The affine length of a convex curve is defined analogously. We will need the following fact. Given a triangle $T = \mathrm{conv}\{a, b, c\}$, let M be the unique parabola which is tangent to ac at a and to bc at b.

(2.6) Among all convex curves connecting a and b within T the arc of the parabola M has the largest affine length.

3. Proof of Theorem 1

Define $\mathrm{AP}^*(K) = \sup\{\mathrm{AP}(S): S \in \mathscr{C}(K)\}$. There is a sequence $S_k \in \mathscr{C}(K)$ with $\mathrm{AP}(S_k) \to \mathrm{AP}^*(K)$. Choose a convergent (in Haussdorf metric) subsequence $S_{k_j} \to K_0$. Obviously $K_0 \in \mathscr{C}(K)$. Since $\mathrm{AP}: \mathscr{C} \to R$ is upper semicontinuous (see [Lu] or find a simple proof for the planar case), $\mathrm{AP}(K_0) = \mathrm{AP}^*(K)$, so existence is easy. For unicity we need two properties of K_0. The first is:

(3.1) ∂K_0 contains no line segment.

Proof. Assume the contrary and let x be the midpoint of the line segment $\ell \subset \partial K_0$. Consider a parabola M touching ℓ at x (and on the same side of ℓ as K_0). Let $y \in M \cap \partial K_0$ be the point so that the arc M_{xy} of M between x and y lies entirely in $\mathrm{int}\, K_0$. Consider all parabola-arcs connecting x to y and touching ℓ at x. This family can be parametrized by the tangent direction at y so it has a "last" element Q_{xy} which is contained in K_0. Let $z \in Q_{xy}$ be the first element on ∂K_0 different from x ($z = y$ is possible). Then Q_{xy} and ∂K_0 have common tangent at z. According to (2.6), the affine length of Q_{xz} is larger than that of the corresponding arc of ∂K_0. So replacing this arc by Q_{xz} results in a convex set contained in K and having larger affine perimeter than $\mathrm{AP}(K_0) = \mathrm{AP}^*(K)$. □

Evidently $\partial K_0 \cap \partial K \neq \emptyset$ as otherwise a slightly enlarged copy of K_0 would be contained in K and would have larger affine perimeter than $\mathrm{AP}^*(K)$. Thus $\partial K_0 \backslash \partial K$ consists of (countably many) convex arcs $A_1, A_2, \ldots$ to be called free arcs. The second property we need is:

(3.2) Each free arc A is an arc of a parabola whose tangents at the endpoints are tangent to K_0 as well.

Proof. Let $x, y \in A$ be so close that the intersection point of the tangents to A at x and y lies in K. Replacing the arc of A between x and y by the suitable parabola-arc would increase the affine perimeter. Thus each small enough subarc of A is an arc of a parabola and so A itself is an arc of a parabola as well. The second half of (3.2) is easy. □

Lemma 1. *Assume $H_i = \operatorname{conv}\{a_i, b_i, c_i\}$ for $i = 1, 2$ are triangles with $a_1 c_1$ and $a_2 c_2$, (and $b_1 c_1$ and $b_2 c_2$, resp.) parallel and of the same direction. Let $H_0 = \operatorname{conv}\{a_0, b_0, c_0\}$ where $a_0 = \frac{1}{2}(a_1 + a_2)$, $b_0 = \frac{1}{2}(b_1 + b_2)$, $c_0 = \frac{1}{2}(c_1 + c_2)$. Then*

$$H_0^{1/3} \geqq \frac{1}{2}(H_1^{1/3} + H_2^{1/3}).$$

Proof. The statements is "affinely invariant" so we may assume that

$$c_0 = c_1 = c_2 = (0,0) \quad \text{and} \quad a_0 = (a,0),\ b_0 = (0,b) \quad \text{with} \quad a, b > 0.$$

Then $a_1 = (a+h, 0)$, $a_2 = (a-h, 0)$ and $b_1 = (0, b+k)$, $b_2 = (0, b-k)$ with

$$|h| < a, \quad |k| < b.$$

The areas are $H_0 = \frac{1}{2}ab$, $H_1 = \frac{1}{2}(a+h)(b+k)$, and $H_2 = \frac{1}{2}(a-h)(b-k)$ and the lemma follows from the concavity of the map $(x,y) \mapsto \sqrt[3]{xy}$. □

The lemma implies that $\mathrm{AP} : \mathscr{C} \to R$ is a concave functional when $\mathscr{C}$ is equipped with Minkowski addition; i.e., for S_1, $S_2 \in \mathscr{C}$ and with $S_0 = \frac{1}{2}(S_1 + S_2)$

$$\mathrm{AP}(S_0) \geqq \frac{1}{2}(\mathrm{AP}(S_1) + \mathrm{AP}(S_2)). \tag{3.3}$$

For the proof take $\eta > 0$ and a subdivision $x_1, \ldots, x_m$ of ∂S_0 with corresponding triangles $T_1, \ldots, T_m$ so that $\mathrm{AP}(S_0) + \eta \geqq 2\sum_1^m T_i^{1/3}$. There are corresponding subdivisions $x_1(j), \ldots, x_m(j)$ of ∂S_j with the same tangent directions at $x_i(j)$ as at x_i and triangles $T_1(j), \ldots, T_m(j)$, for $j = 1, 2$. Further, $x_i = \frac{1}{2}(x_i(1) + x_i(2))$ and Lemma 1 can be applied giving

$$\mathrm{AP}(S_0) + \eta \geqq 2\sum_1^m T_i^{1/3} \geqq \sum_1^m T_i(1)^{1/3} + \sum_1^m T_i(2)^{1/3} \geqq \frac{1}{2}(\mathrm{AP}(S_1) + \mathrm{AP}(S_2)).$$

For later reference we repeat the inequality

$$2\sum_1^m T_i^{1/3} \geqq \sum_1^m T_i(1)^{1/3} + \sum_1^m T_i(2)^{1/3}. \tag{3.4}$$

We will need a slight strengthening of (3.3) when S_1 and S_2 have smooth enough boundaries and well-defined bounded radius of curvature $r(\phi)$. In this case (3.3) holds with equality if and only if S_1 is a translate of S_2. For the proof one notices that $r_0(\phi) = \frac{1}{2}(r_1(\phi) + r_2(\phi))$ and then, to have equality in (3.3), one has to have

$$r_0^{2/3} = \frac{1}{2}(r_1^{2/3} + r_2^{2/3})$$

for every ϕ, according to (2.5). This implies $r_1 = r_2$ for every ϕ proving the claim.

Assume now $S_1, S_2 \in \mathscr{C}(K)$ with $\mathrm{AP}^*(K) = \mathrm{AP}(S_1) = \mathrm{AP}(S_2)$. Define

$$K_1 = \frac{3}{4}S_1 + \frac{1}{4}S_2, \quad K_0 = \frac{1}{2}S_1 + \frac{1}{2}S_2, \quad K_2 = \frac{1}{4}S_1 + \frac{3}{4}S_2.$$

Clearly, all of them belong to $\mathscr{C}(K)$ and have affine perimeter equal to $\mathrm{AP}^*(K)$. (3.1) shows that the point $z_i(\phi) \in \partial K_i$ (and $s_j(\phi) \in \partial S_j$, resp.) where the outward normal to K_i (S_j) is $\omega(\phi)$ is uniquely determined for all $\phi \in [0, 2\pi)$ and $i = 0, 1, 2, j = 1, 2$. Of course, $\frac{1}{2}(z_1(\phi) + z_2(\phi)) = z_0(\phi)$. The advantage of using K_i (instead of S_j) is that the set $\{\phi \in [0, 2\pi) : z_i(\phi) \in \operatorname{int} K\}$ is the same for $i = 0, 1, 2$; we denote it by G. G is open in $[0, 2\pi)$, its complement $F = [0, 2\pi) \backslash G$ is the disjoint union of

$$F_e = \{\phi \in F : z_1(\phi) = z_2(\phi)\} \quad \text{and} \quad F_d = \{\phi \in F : z_1(\phi) \neq z_2(\phi)\}.$$

Every $\phi \in F_d$ is an isolated point of F. This is so since $s_1(\phi), z_1(\phi), z_0(\phi), z_2(\phi), s_2(\phi)$ are all on the boundary of K, are distinct, and are contained in the line segment between $s_1(\phi)$ and $s_2(\phi)$. Consequently, $z_i(\psi) \in \operatorname{int} K$ for every $\psi \neq \phi$ but close enough to ϕ.

Evidently F is nonvoid. Even $|F| \geqq 3$ follows as the outer normal to a parabola-arc changes less than π.

We claim that $F_d = \emptyset$ which implies $K_1 = K_2$ and so $S_1 = S_2$, and so the theorem. Assume, on the contrary, that some $\phi_0 \in F_d$. Then, for $i = 0, 1, 2$, $z_i(\phi_0)$ is the endpoint of a free arc of ∂K_i, whose other endpoint is $z_i(\phi_1)$, say. If $\phi_1 \in F_d$, then $z_i(\phi_1)$ is the endpoint of another free arc of ∂K_i, whose other endpoint is $z_i(\phi_2)$, etc. This sequence is either finite and ends with ϕ_0 (which happens if and only if $F_e = \emptyset$), or finite and ends with a point $\psi_+ \in F_e$, or infinite and its limit is a point $\psi_+ \in F_e$, again. We can start the other direction from ϕ_0 and define ψ_- similarly. The integrals (where $\psi_- = 0$ and $\psi_+ = 2\pi$ in case $F_e = \emptyset$)

$$\int_{\psi_-}^{\psi_+} r^{2/3} d\phi$$

represent the affine perimeter of ∂K_i between $z_i(\psi_-)$ and $z_i(\psi_+)$. So they have to be equal for $i = 0, 1, 2$. (The $r_i(\phi)$ exist because ∂K_i here are just parabola-arcs.) But then, as we have seen, ∂K_1 between $z_i(\psi_-)$ and $z_i(\psi_+)$ is a translated copy of ∂K_2 between the same two points, so they coincide. Consequently, $\phi_0 \in F_e$. □

Remark 1. It is not clear whether the unicity part of Theorem 1 remains true in higher dimensions. The problem is to find an addition of convex bodies such that the affine surface area is concave and convex combination of two bodies in $\mathscr{C}(K)$ is again in $\mathscr{C}(K)$. Simple examples (with tedious computations) show that the Minkowski sum does not satisfy the concavity requirement. Lutwak [Lu] proved a stronger inequality than (3.3) for the Blaschke sum. But the Blaschke convex combination of two convex bodies from $\mathscr{C}(K)$ need not be a subset of K. Luckily, the Blaschke and Minkowski sums coincide in the plane and that is what works for Theorem 1.

The same proof gives the following "pointed" extension of Theorem 1 that we will need later. Write $\mathscr{C}(K,x) = \{C \in \mathscr{C}(K) : x \in C\}$.

Theorem 4. *For every $K \in \mathscr{C}$ and every $x \in K$ there is a $K_0(x) \in \mathscr{C}(K,x)$ such that* $\mathrm{AP}(K_0(x)) > \mathrm{AP}(S)$ *for every $S \in \mathscr{C}(K,x)$ different from $K_0(x)$.*

4. Sketch of the proof of Theorem 3

We need a theorem of Vershik (Theorem 2.3 in [Ve]) which, when applied to the convex body $C \in \mathscr{C}$, says

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} n^{-2/3} \log |\{P \in \mathscr{P}_n : \delta(P,C) < \varepsilon\}| = 3\zeta \mathrm{AP}(C) \tag{4.1}$$

with the same ζ as in (1.1). In fact in [Ve] the right hand side is $3\zeta \int \kappa^{1/3} ds$ but a similar proof gives (4.1) (see also Remark 2 at the end of section 5).

Now shrink K_0 by a factor of $\lambda < 1$ with center in $\operatorname{int} K_0$ and apply (4.1). Using (2.3), for small enough ε, all $P \in \mathscr{P}_n$ with $\delta(P, \lambda K_0) < \varepsilon$ are contained in K, showing that

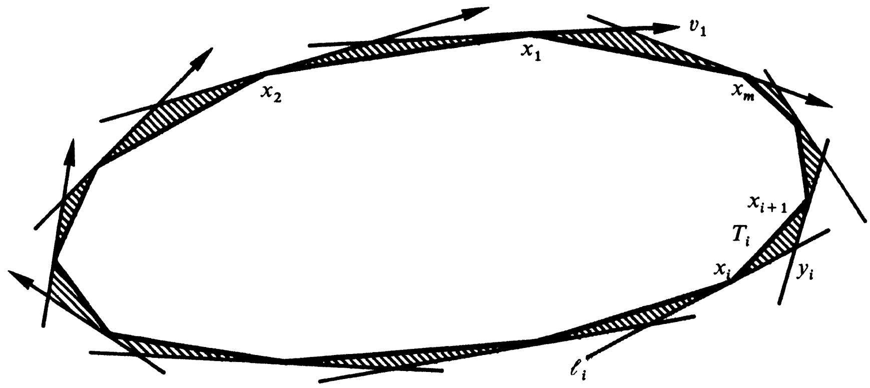
$$\liminf n^{-2/3} \log |\mathscr{P}_n(K)| \geqq \lambda^{2/3} 3\zeta \mathrm{AP}(K_0)\,.$$

Since $\lambda < 1$ is arbitrary,

$$\liminf n^{-2/3} \log |\mathscr{P}_n(K)| \geqq 3\zeta \mathrm{AP}(K_0)\,, \tag{4.2}$$

which is "half" of what we have to prove.

The idea of the other half is simple. Fix a set of primitive vectors $v_1, \dots, v_m \in \mathbb{Z}^2$ whose lengths are about the same and whose directions are distributed fairly evenly in $[0, 2\pi)$, i.e., the angle between v_i and $(1,0)$ is essentially $i\frac{2\pi}{m}$. ($m = m(n)$ is to be chosen later.) Pick a line ℓ_i with direction v_i that passes through a point $x_i \in \frac{1}{n}\mathbb{Z}^2 \cap K$ so that $X = \{x_1, \dots, x_m\}$ is in *convex position* (see the Figure). This gives rise to a convex polygon $P(X)$ whose ith edge is contained in ℓ_i and contains x_i.



Figure

Write $N(X) = |\{C \in \mathscr{P}_n(P(X)) : X \subset \partial C\}|$. Evidently

$$(4.3) \qquad |\mathscr{P}_n(K)| \leqq \sum N(X)$$

where the summation is taken over all $X = \{x_1, \ldots, x_m\} \subset \frac{1}{n}\mathbb{Z}^2 \cap K$ in convex position with respect to $v_1, \ldots, v_m$. (The overcounting is due to the possibility that $P(X)$ is not contained in K but the error is small if m is large.) Now

$$(4.4) \qquad N(X) = \prod_1^m p(T_i)$$

where $T_i = \operatorname{conv}\{x_i, y_i, x_{i+1}\}$ is a triangle, y_i being the intersection of ℓ_i and ℓ_{i+1} and $p(T_i)$ denotes the number of convex $\frac{1}{n}\mathbb{Z}^2$-lattice curves connecting x_i to x_{i+1} within T_i. We are going to use good estimates for $p(T_i)$ from [Bá] which we now describe.

Let T be the triangle $\operatorname{conv}\{z, z+\alpha a, z+\alpha a+\beta b\}$ where $a, b \in \mathbb{Z}^2$ are primitive vectors $(a \neq \pm b)$, $\alpha, \beta > 0$, and $z, z+\alpha a+\beta b \in \frac{1}{n}\mathbb{Z}^2$. Define

$$(4.5) \qquad \det(T) = |\det(a,b)|, \quad \eta(T) = \frac{1}{n}\det(T)\max(\alpha\beta^{-2}, \beta\alpha^{-2}).$$

We have from Theorems B and D of [Bá] (cf. [Ve] and [Si] as well)

$$(4.6) \qquad p(T) \leqq n^2 T \exp\{6\zeta n^{2/3} T^{1/3}(1 + 100\det(T)\eta^{1/4}(T))\},$$

$$(4.7) \qquad p(T) \leqq \exp\{12\sqrt[3]{\zeta(3)}\, n^{2/3} T^{1/3}\}.$$

The explicit (and non-important) constant $12\sqrt[3]{\zeta(3)}$ in (4.7) comes from [BP] and [BV].

In the "generic" case (when all the $\eta(T_i)$ are small) we have

$$N(X) \leqq \exp\left\{6\zeta n^{2/3}\left(\sum_1^m T_i^{1/3} + o(1)\right)\right\}. \tag{4.8}$$

As (3.4) shows, the map $X \mapsto \sum_1^m T_i^{1/3}$ is concave and for large enough m, its maximum is reached when $P(X)$ is very close to K_0. Then $\sum_1^m T_i^{1/3} \leqq \mathrm{AP}(K_0) + \eta$, for any $\eta > 0$, implying

$$N(X) \leqq \exp\{3\zeta n^{2/3} \mathrm{AP}(K_0)(1 + o(1))\}. \tag{4.9}$$

This would prove the other "half" of the theorem since the number of terms in (4.3) is less than $\binom{n^2 K}{m} \leqq \exp\{m \log(n^2 K)\}$ and this is small compared to $N(X)$ if $m = m(n)$ is much less than $n^{2/3}$.

Details of this argument, with a separate treatment of the nongeneric cases, is presented in the next section.

5. Proof of Theorem 3

For each $i \in [m]$ choose a primitive vector $v_i \in \mathbb{Z}^2$ whose direction is between $\left(i - \frac{1}{2}\right)\frac{2\pi}{m}$ and $i\frac{2\pi}{m}$ and whose length, $|v_i|$, is between $.7m$ and $1.3m$. The proof of the existence of such vectors is elementary and is therefore omitted.

Without loss of generality we can assume Area $K = 1$. E, the largest area ellipse inscribed in K satisfies $\mathrm{AP}(E) \geqq \frac{2\pi}{\sqrt{3}}$, and $\mathrm{AP}^*(K) \geqq \mathrm{AP}(E)$.

Next choose a set $X = \{x_1, \ldots, x_m\} \subset \frac{1}{n}\mathbb{Z}^2 \cap K$ in convex position with respect to $v_1, \ldots, v_m$ (see the Figure). This gives rise to the convex polygon $P(X)$ and triangles $T_i = \mathrm{conv}\{x_i, y_i, x_{i+1}\}$. Set $e_i = |x_i - x_{i+1}|$ and $p(X) = \sum_1^m e_i$; $p(X)$ is the usual perimeter of conv X and is always smaller than $p(K)$, the perimeter of K.

We first handle the case when $p(X)$ is small.

Lemma 2. *If* $p(X) < \frac{1}{25}(\mathrm{AP}^*(K))^{3/2}$, *then* $N(X) \leqq \exp\{3\zeta n^{2/3}\mathrm{AP}^*(K)\}$.

Proof. Let ϕ_i denote the angle between v_i and v_{i+1}; clearly

$$\frac{\pi}{m} \leqq \phi_i \leqq \frac{3\pi}{m}. \tag{5.1}$$

Then e_i is the longest side of T_i and one easily proves $T_i \leqq \frac{5}{4m} e_i^2$. The inequality between the 2/3- and the arithmetic mean implies

$$\sum_1^m e_i^{2/3} \leqq \sqrt[3]{m}\left(\sum_1^m e_i\right)^{2/3}. \tag{5.2}$$

Consequently

$$\sum_1^m T_i^{1/3} < \left(\frac{5}{4m}\right)^{1/3} \sum_1^m e_i^{2/3} \leqq \left(\frac{5}{4}\right)^{1/3} (p(X))^{2/3} < \left(\frac{1}{500}\right)^{1/3} \mathrm{AP}^*(K).$$

Using this in (4.7) and (4.3) we obtain

$$\begin{aligned} N(X) &\leqq \prod_1^m p(T_i) \leqq \exp\left\{12\sqrt[3]{\zeta(3)}\, n^{2/3} \sum_1^m T_i^{1/3}\right\} \\ &\leqq \exp\left\{12\sqrt[3]{\zeta(3)}\left(\frac{1}{500}\right)^{1/3} n^{2/3} \mathrm{AP}^*(K)\right\} \\ &< \exp\{3\zeta n^{2/3} \mathrm{AP}^*(K)\}. \quad \square \end{aligned} \tag{5.3}$$

From now on we consider $X = \{x_1, \ldots, x_m\}$ with $p(X)$ large, namely

$$p(X) \geqq \frac{1}{25}(\mathrm{AP}^*(K))^{3/2} \geqq \frac{1}{25}\left(\frac{2\pi}{\sqrt{3}}\right)^{3/2} > \frac{1}{4}, \tag{5.4}$$

where we made use if the inequality $\mathrm{AP}^*(K) \geqq \frac{2\pi}{\sqrt{3}}$.

We now define three subsets I_1, I_2, I_3 of $[m]$; the numbers m_1 (large) and μ (small) will be specified later. In the triangle T_i we have $y_i - x_i = \alpha_i v_i$ and $x_{i+1} - y_i = \beta_i v_{i+1}$ with $\alpha_i, \beta_i \geqq 0$. Set

$$I_1 = \{i \in [m] : \alpha_i > m_1 \beta_i\}, \quad I_2 = \{i \in [m] : \beta_i > m_1 \alpha_i\},$$

$$I_3 = \left\{i \in [m] \setminus (I_1 \cup I_2) : e_i < \mu \frac{p(X)}{m}\right\}.$$

Lemma 3. $\sum_{i \in I_1} T_i^{1/3} \leqq \left(\frac{3\pi}{m_1}\right)^{1/3} (p(X))^{2/3}$.

Proof. As $m_1 < \frac{\alpha_i}{\beta_i} = \frac{|\alpha_i v_i|}{|\beta_i v_{i+1}|} \frac{|v_{i+1}|}{|v_i|}$ and $\frac{|v_{i+1}|}{|v_i|} \leqq \frac{1.3}{.7}$, we get

$$T_i = \frac{1}{2}\sin\phi_i |\alpha_i v_i| |\beta_i v_{i+1}| \leqq \frac{3\pi}{2m} |\alpha_i v_i|^2 \frac{|\beta_i v_{i+1}|}{|\alpha_i v_i|} < \frac{3.9\pi}{1.4 m m_1} e_i^2 < \frac{3\pi}{m m_1} e_i^2.$$

Inequality (5.2) implies $\sum_{I_1} e_i^{2/3} \leqq |I_1|^{1/3} (\sum_{I_1} e_i)^{2/3}$, thus showing

$$\sum_{I_1} T_i^{1/3} \leqq \left(\frac{3\pi}{m_1}\right)^{1/3} \left(\frac{|I_1|}{m}\right)^{1/3} (p(X))^{2/3}$$

as required. □

Lemma 3 and (4.7) and $p(X) \leqq p(K)$ imply with an absolut constant c_1

$$(5.5) \quad \prod_{I_1} p(T_i) \leqq \exp\left\{12\sqrt[3]{\zeta(3)}\, n^{2/3} \sum_1^m T_i^{1/3}\right\} \leqq \exp\{c_1 m_1^{-1/3} (p(K))^{2/3} n^{2/3}\}.$$

Lemma 4. $\sum_{i \in I_3} T_i^{1/3} \leqq \left(\frac{5}{4}\right)^{1/3} \mu^{2/3} (p(X))^{2/3}.$

Proof. Since $T_i \leqq \frac{5}{4m} e_i^2$ we infer

$$\sum_{i \in I_3} T_i^{1/3} \leqq \left(\frac{5}{4m}\right)^{1/3} \sum_{I_3} e_i^{2/3} \leqq \left(\frac{5}{4m}\right)^{1/3} |I_3| \left(\mu \frac{p(X)}{m}\right)^{2/3}. \quad \square$$

Similarly to (5.5), we get with a constant c_3

$$(5.6) \quad \prod_{I_3} p(T_i) \leqq \exp\left\{12\sqrt[3]{\zeta(3)}\, n^{2/3} \sum_1^m T_i^{1/3}\right\} \leqq \exp\{c_3 \mu^{-2/3} (p(K))^{2/3} n^{2/3}\}.$$

Finally we compute the error term in (4.8) when $i \in I_0 = [m] \setminus (I_1 \cup I_2 \cup I_3)$. First $\det(v_i v_{i+1}) < 10m$ is straightforward, so

$$\eta(T_i) = \frac{1}{n} \det(v_i v_{i+1}) \max(\alpha_i \beta_i^{-2}, \beta_i \alpha_i^{-2}) \leqq \frac{10 m m_1}{n} \max(\alpha_i^{-1}, \beta_i^{-1}).$$

Next,

$$\frac{1}{\alpha_i} = \frac{e_i}{|\alpha_i v_i|} \frac{|v_i|}{e_i} \leqq \frac{|\alpha v_i + \beta_i v_{i+1}|}{|\alpha_i v_i|} \frac{|v_i|}{e_i} \leqq \left(1 + \frac{1.3}{.7} m_1\right) \frac{1.3m}{e_i} \leqq 2.6 m m_1 \frac{m}{\mu p(X)},$$

where the last inequality follows from $i \notin I_3$. Consequently

$$\eta(T_i) \leqq \frac{26}{n} \frac{m^3 m_1^2}{\mu p(X)} < \frac{104}{n} \frac{m^3 m_1^2}{\mu}$$

since $p(X) > 1/4$, and sum of the error terms in (4.8) for $i \in I_0$ is

$$\sum_{I_0} 100 \det(T_i)(\eta(T_i))^{1/4} \leqq |I_0| 1000 m \left(\frac{104}{n}\right)^{1/4} \left(\frac{m^3 m_1^2}{\mu}\right)^{1/4}.$$

Now fix $m_1 = m$ and $\mu = 1/m_1$. By (4.6) we have

$$\prod_{I_0} p(T_i) \leqq \left(\prod_{I_0} n^2 T_i\right) \exp\{3\zeta n^{2/3} (\sum_{I_0} T_i^{1/3} + c_0 n^{-1/4} m^{7/2})\}.$$

Using this, (5.5) and (5.6) in (4.4) we get

$$(5.7)\qquad N(X)=\prod_1^m p(T_i)\leqq\prod_{j=0}^{3}\prod_{i\in I_j}p(T_i)$$
$$\leqq\exp\left\{3\zeta n^{2/3}\sum_1^m T_i^{1/3}+\text{error terms}\right\}.$$

According to Lemma 1 or rather (3.4), the mapping $X\mapsto\sum_1^m T_i^{1/3}$ is concave on m-tuples X in convex position. Therefore for each m, it has a maximum at some X_m. The limit of a suitable subsequence of $P(X_m)$ is a convex set $K^*\in\mathscr{C}(K)$ with

$$\mathrm{AP}(K^*)\geqq\limsup\mathrm{AP}(P(X_m))$$

since AP is upper semicontinuous. But $\mathrm{AP}^*(K)\geqq\mathrm{AP}(K^*)$ so $K^*=K_0$. Then for every $\eta>0$ there is m_0 such that for $m>m_0$, and for every m-tuple $X\subset K$ in convex position, $2\sum_1^m T_i^{1/3}\leqq\mathrm{AP}^*(K)+\eta$.

Now set $m=n^{1/16}$. The error term in (5.7) is

$$n^{2/3}\{c_0n^{-1/4}m^{7/2}+(c_1+c_2)(p(K))^{2/3}m^{-1/3}+c_3(p(K))^{2/3}m^{-2/3}\}+\sum_{I_0}\log(n^2),$$

which is less than $cn^{2/3-1/48}$ with the constant c depending only on the perimeter of K. Thus

$$N(X)\leqq\exp\{3\zeta n^{2/3}(\mathrm{AP}^*(K)+\eta)\}\exp\{cn^{2/3-1/34}\}$$
$$\leqq\{\exp 3\zeta\, n^{2/3}(\mathrm{AP}^*(K)+2\eta)\},$$

if n is large enough. Combining this with Lemma 2 and (4.3), and using Area $K=1$, we have for large enough n,

$$|\mathscr{P}_n(K)|\leqq\sum N(X)\leqq\sum\exp\{3\zeta n^{2/3}(\mathrm{AP}^*(K)+2\eta)\}$$
$$\leqq\binom{n^2}{m}\exp\{3\zeta n^{2/3}(\mathrm{AP}^*(K)+2\eta)\}\leqq\exp\{3\zeta n^{2/3}(\mathrm{AP}^*(K)+3\eta)\}.\quad\square$$

We will need a "pointed" version of Theorem 3 (cf. Theorem 4). For $x\in K\in\mathscr{C}$ write $\mathscr{P}_n(K,x)=\{P\in\mathscr{P}_n(K):x\in P\}$. The above argument can be used to prove

Theorem 5. *For each $K\in\mathscr{C}$ and each $x\in K$*

$$\lim_{n\to\infty}n^{-2/3}\log|\mathscr{P}_n(K,x)|=3\zeta\,\mathrm{AP}(K_0(x)).$$

Remark 2. A slight extension of Vershik's theorem can be proved along similar lines. Specifically, let γ be a (bounded) convex curve in the plane, and let $\mathscr{P}_n(\gamma,\varepsilon)$ denote the collection of convex $\frac{1}{n}\mathbb{Z}^2$-lattice curves C with $\delta(C,\gamma)<\varepsilon$. Then

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} n^{-2/3} \log |\mathscr{P}_n(\gamma, \varepsilon)| = 3\zeta \,\mathrm{AP}(\gamma).$$

In the proof one has to use the fact that $\log p(T) = 6\zeta n^{2/3} T^{1/3}(1 + o(1))$ under suitable conditions on the triangle T (see Theorem A of [Bá], Theorem 3.1 of [Ve], or Theorem 1 of [Si]).

6. Proof of the limit shape theorem

We have to show that

$$\mathscr{F} = \mathscr{F}_{n,\varepsilon} = \{P \in \mathscr{P}_n(K) : \delta(K_0, P) \geqq \varepsilon\}$$

is a "small" subset of $\mathscr{P}_n(K)$. We will do so by partitioning $\mathscr{F}$ into finitely many sets that are small as a consequence of Theorems 3 and 5.

Assuming Area $K = 1$, let E be the maximum area ellipse contained in K. It is well known that Area $E \geqq \frac{\pi}{3\sqrt{3}}$. Clearly $\mathrm{AP}(K_0) = \mathrm{AP}^*(K) \geqq \mathrm{AP}(E)$. The affine isoperimetric inequality implies Area $K_0 \geqq$ Area $E \geqq \frac{\pi}{3\sqrt{3}}$ and so the maximum area ellipse E_0 in K_0 has area at least $\frac{\pi^2}{27}$.

We prove the theorem first in the special case when E_0 coincides with a circle of radius r centered at the origin (and explain the general case later). Clearly, $r \geqq \sqrt{\pi/27} > 1/3$ which implies, in turn, that diam $K < 3$.

It is clear that there is a small $\eta = \eta(\varepsilon) > 0$ such that $C \in \mathscr{C}(K)$ and $\delta(K_0, C) \geqq \varepsilon$ imply

(6.1) $$\text{either} \quad (1-\eta) K_0 \not\subset C \quad \text{or} \quad C \not\subset (1+\eta) K_0 .$$

For the partitioning of $\mathscr{F}$ we need

Lemma 5. *There are halfplanes $H_1, \ldots, H_p$, each with $(1-\eta/2)K_0 \not\subset H_i$, and points $x_1, \ldots, x_q \in K \cap \partial(1+\eta/2)K_0$, where p and q depend only on η, such that the following holds. For every $C \in \mathscr{C}(K)$ satisfying* (6.1) *either there is an H_i with $C \subset H_i$ or there is an x_j with $x_j \in C$.*

The lemma implies that $\mathscr{F} = \bigcup_1^p \mathscr{H}_i \cup \bigcup_1^q \mathscr{F}_j$ where

$$\mathscr{H}_i = \{C \in \mathscr{F} : C \subset H_i\} \quad \text{and} \quad \mathscr{F}_j = \{C \in \mathscr{F} : x_j \in C\}.$$

Here $\mathscr{H}_i \subset \mathscr{C}(H_i \cap K)$ and $\mathrm{AP}^*(H_i \cap K) < \mathrm{AP}^*(K)$, as K_0 is unique and is not contained in $H_i \cap K$. Then for all i and for a suitably small δ_1 depending only on η,

$$\mathrm{AP}^*(H_i \cap K) \leqq \mathrm{AP}^*(K) - 2\delta_1 .$$

So by Theorem 3, for large enough n,

$$(6.3)\qquad |\mathscr{H}_i| \leqq \exp\{3\zeta n^{2/3}(\mathrm{AP}^*(H_i \cap K) + \delta_1)\} \leqq \exp\{3\zeta n^{2/3}(\mathrm{AP}^*(K) - \delta_1)\}.$$

To estimate the size of $\mathscr{F}_j$ we use Theorems 4 and 5. Obviously $\mathscr{F}_j \subset \{S \in \mathscr{C}(K) : x_j \in S\}$ and $\mathrm{AP}(K_0(x_j)) < \mathrm{AP}(K_0)$, since K_0 is unique and $x_j \notin K_0$. So for all j and for a suitably small δ_2 depending only on η, $\mathrm{AP}(K_0(x_j)) \leqq \mathrm{AP}(K_0) - 2\delta_2$. Then, by Theorem 5, for large enough n,

$$(6.4)\qquad |\mathscr{F}_j| \leqq \exp\{3\zeta n^{2/3}(\mathrm{AP}(K_0(x_j)) + \delta_2)\} \leqq \exp\{3\zeta n^{2/3}(\mathrm{AP}^*(K) - \delta_2)\}.$$

Together with (6.3) this finishes the proof of the limit shape theorem in the special case when E_0 is a circle. To prove the general case we apply the affine map L carrying E_0 to the circle, to K, K_0, E_0. We then find the halfspaces H_i and points x_j via Lemma 5, and use $L^{-1}H_i$ and $L^{-1}x_j$ to obtain the analogons of (6.3) and (6.4).

Proof of Lemma 5. We let $\omega(\alpha)$ denote the vector $(\cos\alpha, \sin\alpha) \in \mathbb{R}^2$. Write $H(\alpha)$ for the halfplane with $0 \in H(\alpha)$ and outer normal $\omega(\alpha)$ whose bounding line is tangent to the circle of radius $1/10$ centered at the origin. We claim that every $C \in \mathscr{C}(K)$ with $0 \notin C$ is contained in some of the halfplanes $H\left(i\frac{2\pi}{100}\right)$, $i = 1, \dots, 100$.

By separation there is a halfplane H with $0 \in \partial H$ and $C \subset H$. Denote its outer normal by $\omega(\phi)$. Elementary computations (using $\operatorname{diam} K < 3$) show that $C \subset H(\alpha)$ holds for all α with $|\alpha - \phi| < 1/30$. But every interval $[\phi - 1/30, \phi + 1/30]$ (mod 2π) contains an angle of the form $i\frac{2\pi}{100}$.

Next write $H'(\alpha)$ for the halfplane with $0 \in H'(\alpha)$ and outer normal $\omega(\alpha)$ whose bounding line is tangent to $(1 - \eta/2)K_0$. We claim that if $0 \in C \in \mathscr{C}(K)$ but $(1 - \eta)K_0 \not\subset C$, then there is a $\beta \in [0, 2\pi)$ such that $C \subset H'(\alpha)$ for every $\alpha \in [\beta - \eta/6, \beta + \eta/6]$ which is, again, understood mod 2π.

Indeed, there is a halfplane H' containing C whose bounding line is tangent to $(1 - \eta)K_0$ since otherwise $(1 - \eta)K_0 \subset C$. Then $C \subset H'(\alpha)$ whenever $K \cap H' \subset H'(\alpha)$. An elementary computation (using $\operatorname{diam} K < 3$ and the "nice" position of K_0) shows that $K \cap H' \subset H'(\alpha)$ holds for all $\alpha \in [\beta - \eta/6, \beta + \eta/6]$ where $\omega(\beta)$ is the outer normal of H'.

Now define the halfplanes H_i by $H_i = H\left(i\frac{2\pi}{100}\right)$, if $i = 1, \dots, 100$ and

$$H_i = H'((i - 101)\eta/3),$$

if $i = 101, \dots, 101 + \left[\frac{6\pi}{\eta}\right]$. So with $p = 101 + \left[\frac{6\pi}{\eta}\right]$, for any $C \in \mathscr{C}(K)$ with

$$(1-\eta) K_0 \not\subset C,$$

there is an $i \in \{1, \dots, p\}$ so that $C \subset H_i$.

Finally, consider a subdivision $x_1, \dots, x_q, x_{q+1} = x_1$ of the boundary of $(1+\eta/2) K_0$ with the property that $|x_j - x_{j+1}| < \eta/6$. We claim that every $C \in \mathscr{C}(K)$ with $(1-\eta) K_0 \subset C$ and $C \not\subset (1+\eta) K_0$ contains at least one of the points $x_1, \dots, x_q$. Every such C contains some point y from the boundary of $(1+\eta) K_0$, so $y = (1+\eta) x$ with $x \in \partial K_0$. The length of the arc of $C \cap \partial(1+\eta/2) K_0$ containing $(1+\eta/2) x$ is at least $\eta/6$, as an elementary computation quickly reveals. □

Acknowledgement. Part of this work was done on a very pleasant visit at Université de Marne-la-Vallée. The author thanks Matthieu Meyer and Alain Pajor for their hospitality, and for useful and illuminating discussions.

References

[Ar] *V. I. Arnold*, Statistics of integral lattice polytopes, Func. Anal. Appl. **14** (1980), 1–3 (in Russian).
[Bá] *I. Bárány*, The limit shape of convex lattice polygons, Discrete Comp. Geom. **13** (1995), 270–295.
[BP] *I. Bárány*, *J. Pach*, On the number of convex lattice polygons, Comb. Prob. Comp. **1** (1992), 295–302.
[BV] *I. Bárány*, *A. M. Vershik*, On the number of convex lattice polytopes, GAFA J. **2** (1992), 381–393.
[Bl] *W. Blaschke*, Vorlesungen über Differentialgeometrie II. Affine Differentialgeometrie, Springer, Berlin 1923.
[Le] *K. Leichtweiss*, Zur Affinoberfläche konvexer Körper, Manuscr. Math. **56** (1986), 429–464.
[Lu] *E. Lutwak*, Extended affine surface area, Adv. Math. **85** (1991), 39–68.
[Sc] *C. Schütt*, On the affine surface area, Proc. Amer. Math. Soc. **118** (1993), 1213–1218.
[Si] *Ya. G. Sinai*, Probabilistic approach to analyse the statistics of convex polygonal curves, Func. Anal. Appl. **28** (1994), 41–48 (in Russian).
[Ve] *A. M. Vershik*, The limit shape for convex lattice polygons and related topics, Func. Anal. Appl. **28** (1994), 16–25 (in Russian).

Mathematical Institute of the Hungarian Academy of Sciences, P.O.B. 127, H-1364 Budapest
e-mail: barany@math-inst.hu

Eingegangen 7. März 1996