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The topological structure of maximal lattice free convex bodies: The general case

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Abstract

Given a generic $m \times n$ matrix A, the simplicial complex $\mathcal{K}(A)$ is defined to be the collection of simplices representing maximal lattice point free convex bodies of the form $\{x : Ax \leqslant b\}$. The main result of this paper is that the topological space associated with $\mathcal{K}(A)$ is homeomorphic with R^{m-1}. © 1998 The Mathematical Programming Society, Inc. Published by Elsevier Science B.V.

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1. Introduction

Let A be a real $m \times n$ matrix with rows $a_1, \dots, a_m$ and consider the family of integer programming problems

$$\begin{aligned} &\min\ a_1 \cdot z \ \text{ subject to} \\ &\qquad a_i \cdot z \leqslant b_i \ \text{for } i = 2, \dots, m \text{ and } z \in \mathbb{Z}^n, \end{aligned} \tag{1.1}$$

obtained by selecting arbitrary right hand sides b_i.

A generic matrix A has associated with it a unique, finite set of lattice points, $N(A)$, called the neighbors of the origin. The neighbors form a test set for this family of integer programs, in the sense that an integer vector z, satisfying the constraints of any one of these problems will be optimal for that problem if for every $h \in N(A)$, the vector $z + h$

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is either infeasible or yields an inferior value of the objective function than does z. Moreover, if a lattice point is eliminated from $N(A)$, there will be an integer program obtained by specifying a particular value of the right hand side, and some feasible but not optimal lattice point z, whose lack of optimality cannot be detected by using this smaller test set.

The neighbors of the origin are simple to describe for certain matrices. A generic matrix of size 3 by 2 will have six neighbors, which are easy to calculate. A general matrix with two columns may have quite a few neighbors – if the entries of A are integers, the number of neighbors may be exponential in the bitsize of the matrix, but they can always be organized into a "small" number of linear segments. Another simple case arises when the matrix has $n+1$ rows, n columns, and the following sign pattern

$$\begin{bmatrix} + & \cdots & - \\ \vdots & \ddots & \vdots \\ - & \cdots & + \\ - & \cdots & - \end{bmatrix}$$

with $\sum_j a_{ij} > 0$ for $i = 1, \ldots, n$. In this case, the neighbors of the origin consist of all of the vertices of the unit cube in n space, and their negatives.

In the general case, the set of neighbors can be calculated using Groebner basis algorithms borrowed from algebraic geometry. They can also be calculated by what might be thought of as homotopy methods. To calculate $N(A)$, for a particular matrix A, one can find a matrix of the same size B whose neighbors are known, and examine the linear family of matrices $A(t) = tA + (1-t)B$. It is quite simple to find those particular values of the parameter t where the set of neighbors makes a discontinuous change, and it is possible to describe the precise changes that take place in the set of neighbors as t passes through this discontinuity.

Neither of these procedures is particularly fast; for integer matrices they are not polynomial in the size of A. But there is considerable theoretical evidence to suggest that the set of neighbors can be examined rapidly for a fixed number of variables. Unfortunately we do not know, at this moment, how to find a polynomial structure for the set of neighbors; the best that we can do is to look for any structural properties of this set of lattice points that can augment the current state of our knowledge.

In this paper, we construct a canonical simplicial complex associated with a generic matrix A. The simplices in the complex have lattice points as vertices, and lattice translates of a simplex will also be in the complex. In the $n+1$ by n matrix given above, with a particular sign pattern and with positivity of the sum of the entries in the first n rows, the simplicial complex consists of the most conventional simplicial subdivision of the cube, translated by $\mathbb{Z}^n$,

The importance of the complex is that the set of neighbors is precisely the collection of edges emanating from the origin in those simplices containing the origin as one of its vertices. The main theorem of the paper characterizes the global structure of the general complex, and shows, in fact, that from a topological point of view, the complex is as simple as it can possibly be. It is our hope that this result will lead to concrete structural features of the set of neighbors, and ultimately to improvements in our ability to solve difficult integer programming problems. Perhaps the geometric representation of the complex, via the exponential map, will turn out to be useful: this is a highly symmetric polyhedron where the additive structure of $\mathcal{K}(A)$ is reflected multiplicatively.

We make the following assumptions on A.

A1. There is a strictly positive row vector $\lambda \in R^m$ with $\lambda A = 0$.

A2. If, for some $i \in \{1, \dots, m\}$ and $z \in \mathbb{Z}^n$ $a_i z = 0$, then $z = 0$.

A3. The $n \times n$ minors of A are all nonsingular.

The first and the third condition imply that for any $b \in R^m$ the convex set

$$K_b = \{x \in R^m : Ax \leqslant b\} \tag{1.2}$$

is bounded. Condition A2 asserts that the hyperplane $a_i x = \beta_i$ contains at most one lattice point. This condition is much more stringent than necessary for our analysis and can be relaxed to allow an open set of matrices containing A in its interior. Condition A3 can be replaced by the assumption that $\lambda A = 0$, and $\lambda \geqslant 0$ implies that at least $n+1$ components of λ are strictly positive.

Definition. K_b is a maximal lattice free convex body (or MLFC body, for short), if

(1) K_b has no lattice points in its interior,

(2) any closed convex body which properly contains K_b does have a lattice point in its interior.

By A1 and A3, K_b is a convex polytope. Notice that if K_b is a MLFC body, then so is $z + K_b$ for every $z \in \mathbb{Z}^n$.

Condition (A2) implies that every facet of a MLFC body K_b contains a unique lattice point in its relative interior. Let z^i be this lattice point when the facet is defined by the ith inequality $a_i x \leqslant \beta_i$. Some inequalities $a_i x \leqslant \beta_i$ may not define a facet of K_b in which case the inequality $a_i x \leqslant \beta_i$ can be replaced by $a_i x \leqslant \overline{\beta}_i$ with any $\overline{\beta}_i > \beta_i$ without changing K_b. Thus different right-hand sides (i.e., different b's) may give rise to the same MLFC body.

To avoid this ambiguity we set $\overline{\beta}_i = +\infty$ for an inequality that does not define a facet. A convenient way to do this is to introduce "ideal points" $w^1,\ w^2, \dots, w^m$ by defining

$$a_i w^j = \begin{cases} +\infty & \text{if } i = j, \\ -\infty & \text{otherwise.} \end{cases}$$

Let $W = \{w^1, \dots, w^m\}$.

Assume now that K_b is a MLFC body. We shall represent it by an m-element set $\sigma \subset \mathbb{Z}^n \cup W$ in the following way. For $i = 1,\ 2, \dots, m$ define

$$s^i = \begin{cases} z^i & \text{if } a_i x \leqslant \beta_i \text{ defines a facet, and } z^i \in \mathbb{Z}^n \text{ is on this facet,} \\ w^i & \text{otherwise.} \end{cases}$$

Let $\sigma = \{s^1,\ s^2, \dots, s^m\}$.

On the other hand, an m-element set $\sigma \subset \mathbb{Z}^n \cup W$ determines a convex set K_b via

$$\beta_i = \max\{a_i s : s \in \sigma\}, \quad \text{and} \quad b = (\beta_1, \ldots, \beta_m)^{\mathrm{T}}.$$

The set K_b is a MLFC body if the elements of σ can be indexed as $\sigma = \{s^1, s^2, \ldots, s^m\}$ so that the following holds: $\beta_i = a_i s^i$ $(i = 1, \ldots, m)$, $a_i s^j < \beta_i$, if $j \neq i$, and there is no $z \in \mathbb{Z}^n$ with $a_i z < \beta_i$ for all $i = 1, \ldots, m$.

Define now the complex $\mathcal{K}(A)$ associated with this collection of MLFC bodies as the simplicial complex whose simplices are the finite sets σ representing MLFC bodies together with their subsimplices. The vertex set of $\mathcal{K}(A)$ is $\mathbb{Z}^n \cup W$ so it is infinite. Given a simplex $\sigma = \{z^1, \ldots, z^p, w^{j_1}, \ldots, w^{j_q}\} \in \mathcal{K}(A)$ with $p \geqslant 1$, its cell, $|\sigma|$, is the set of all abstract mixed combinations from σ that are defined as

$$x = \sum_{k=1}^{p} \gamma(k) z^k + \sum_{l=1}^{q} \beta(j_l) w^{j_l} \tag{1.3}$$

where $\gamma(k), \beta(j_l) \geqslant 0$ and $\sum_1^p \gamma(k) = 1$. Notice that $|\sigma|$ is not a subset of R^n since the points z^i and w^j are thought of as abstract points.

The body of $\mathcal{K}(A)$, $|\mathcal{K}(A)|$, is the union of cells of simplices σ containing at least one non-ideal point. This is not the usual definition of the body of a simplicial complex but it suits our purposes well.

We will show later (Lemma 2 in Section 5) that every point of $|\mathcal{K}(A)|$ is contained in finitely many cells of $\mathcal{K}(A)$, i.e., $\mathcal{K}(A)$ is locally finite except possibly at the ideal points. This implies that the topology of $|\mathcal{K}(A)|$ is well defined.

Now we can state our main result.

Theorem 1. *$|\mathcal{K}(A)|$ is homeomorphic to R^{m-1}.*

This theorem is a generalization of a result from [1] where the case $m = n + 1$ is considered. The constructions and the proofs of this paper take their origin from [1], but a different and novel approach is needed here at several places: Assumption A3 is necessary here to ensure local finiteness of $\mathcal{K}(A)$; there are no ideal points when $m = n + 1$; and the geometric realization of $\mathcal{K}(A)$ (see Section 7) is simpler in [1].

2. Examples

Before presenting further theorems and the proofs it is instructive to consider a few examples.

When $m = n + 1$, ideal points are not needed since every MLFC body is a simplex. When $n = 2$ and $m = 3$, $\mathcal{K}(A)$ has a particularly simple structure (cf. [7]). Namely, there is a basis, e^1, e^2, of the lattice $\mathbb{Z}^2$ such that the simplices of $\mathcal{K}(A)$ are lattice translates of $\{0, e^1, e^1 + e^2\}$ and $\{0, e^2, e^1 + e^2\}$. The corresponding triangles and their lattice translates form a tiling of the whole plane and constitute a simple geometric realization of $\mathcal{K}(A)$ as R^2 (see Fig. 1).

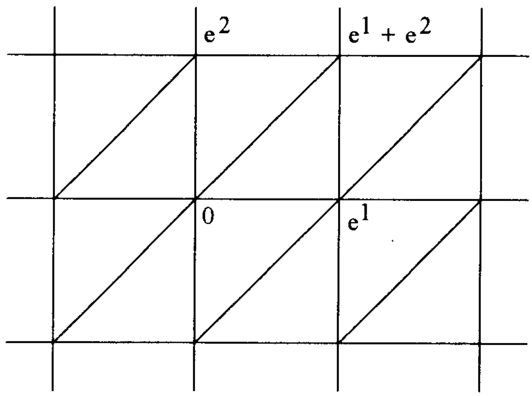

Fig. 1. The 3 × 3 case.

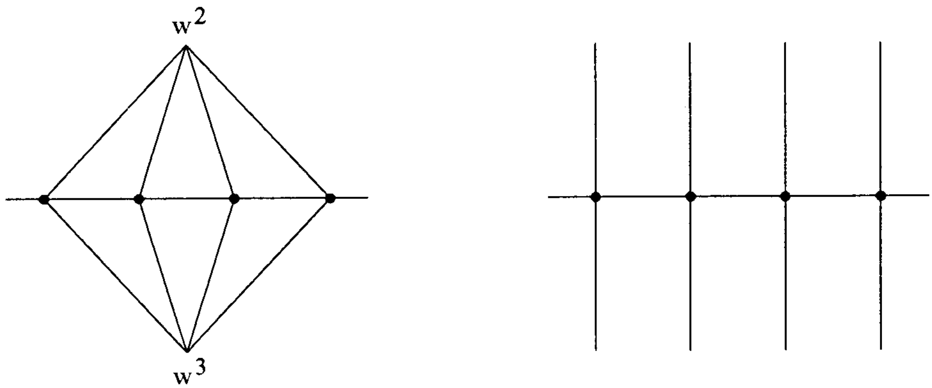

Fig. 2. The 3 × 1 case.

When $n = 1$ and $m = 3$ the inequalities in the system (1.1) can be put in the form $-x \leqslant \beta_1$, $x \leqslant \beta_2$, $x \leqslant \beta_3$. The MLFC bodies are the intervals $[k, k+1]$ $(k \in \mathbb{Z})$. They are represented by simplices of $\mathcal{K}(A)$ of the form

$$\{k,\ w^2,\ k+1\} \quad \text{and} \quad \{k,\ k+1,\ w^3\}.$$

The ideal point w^1 does not appear in any simplex of $\mathcal{K}(A)$. $|\mathcal{K}(A)|$ is given in two ways in Fig. 2: first the ideal points are in the plane, and, second, they are placed at infinity.

The case $n = 2$, $m = 4$ can be treated using results of [7]. In this case some three of the inequalities in (1.1), $a_1x \leqslant \beta_1$, $a_2x \leqslant \beta_2$, $a_3x \leqslant \beta_3$, say, determine a bounded region and the 3 by 2 case applies. Each of the two types of simplices obtained from these three inequalities alone is augmented by w^4 in order to get a maximal simplex in $\mathcal{K}(A)$. Some other three inequalities, $a_2x \leqslant \beta_2$, $a_3x \leqslant \beta_3$, $a_4x \leqslant \beta_3$ say, also determine a bounded region, and the 3 by 2 case applies again. Of the ideal points only w^1 and w^4 are needed and they only appear in this way. The remaining maximal lattice free bodies do not involve the ideal points; the four lines corresponding to the four inequalities are

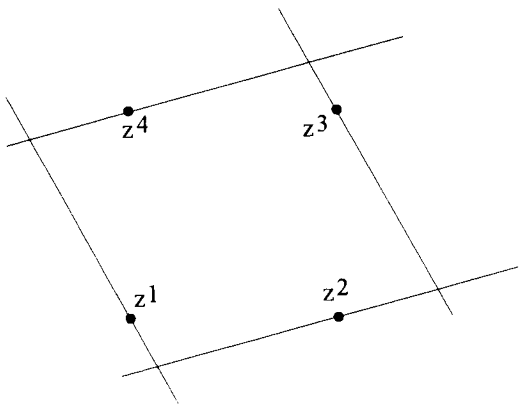

Fig. 3. The 4×2 case.

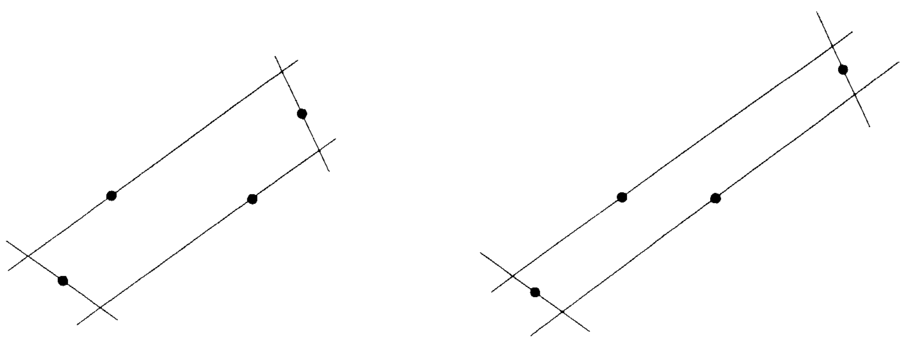

Fig. 4. $\mathcal{K}(A)$ is not locally finite.

placed at four lattice points z^1, z^2, z^3, z^4 whose convex hull is a parallelogram of unit area. One can visualize the abstract simplicial complex $\mathcal{K}(A)$ as the collection of "3-dimensional" parallelograms, with vertices coming from $\mathbb{Z}^2$. The boundary of their union consists of two pieces: each piece is homeomorphic to R^2 and corresponds to the tiling (of R^2) by triangles from the 3 by 2 submatrices. (Above each tiling there is a suspension to infinity by w^1 and w^4.) This is what we like to call the quilted paplan.

As these simple examples, show not all ideal points belong to simplices of $\mathcal{K}(A)$. On the other hand, a result of Doignon [3], Scarf [6], and Bell [2] states that a MLFC body can have at most 2^n facets. Thus for a maximal dimensional simplex $\sigma = \{z^1, z^2, \ldots, z^k, w^{j_1}, \ldots, w^{j_{m-k}}\} \in \mathcal{K}(A)$ one has $n + 1 \leqslant k \leqslant 2^n$.

As we mentioned, the well-conditioning assumption A3 ensures the local finiteness of $\mathcal{K}(A)$. An example due to Lovász [5] shows that if A3 does not hold, then $\mathcal{K}(A)$ may not be locally finite. The example (Fig. 3) is for the 4 by 2 case: two of the vectors, say a_1 and a_2 are opposite ($a_1 + a_2 = 0$) and have irrational slope. Fig. 4 depicts two parallelograms $\{z^1, z^2, z^3, z^4\} \in \mathcal{K}(A)$, from an infinite sequence of parallelograms that contain the point $z^4 = 0$ and correspond to MLFC bodies. A3 is violated here by the 2 by 3 minor $[a_1, a_2]^T$ of A.

We mention further that the same well-conditioning assumption A3 was needed in [5] in order to show that the "shapes" of the MLFC bodies of the type K_b (with A fixed, again) can be approximated by the shapes of a finite subset of this type. Details can be found in [5].

3. The exponential map

The proof of Theorem 1 will be based on a geometric realization of $\mathcal{K}(A)$. The key construction is the exponential map $E : R^n \times (0, \infty) \to R^m$ defined by

$$E(x, t) = (\exp\{ta_1x\},\ \exp\{ta_2x\}, \dots, \exp\{ta_mx\})^{\mathrm{T}}.$$

Quite often the parameter $t \in (0, \infty)$ is not important and we simply write $E_t(x)$ or $E(x)$.

Consider now $\lambda \in R^m_+$ from condition A1 and set

$$M = \left\{y \in R^m_+ : \prod_{i=1}^m y_i^{\lambda_i} = 1\right\}. \tag{3.1}$$

Notice that M is the boundary of the strictly convex set $\{y \in R^m_+ : \Pi y_i^{\lambda_i} \geqslant 1\}$. Further, $E_t(x) \in M$ for every $x \in R^m$.

We remark that, more generally, for a row vector $\mu \in R^m_+$ with $\mu A = 0$, one could define

$$M(\mu) = \left\{y \in R^m_+ : \prod_1^m y_i^{\mu_i} = 1\right\}.$$

It follows then that $E_t(x) \in M(\mu)$ for every such μ so that E_t maps R^n to $\bigcap M(\mu)$. In what follows, however, we will only make use of this fact with $\mu = \lambda$.

Define now $V_t = E_t(\mathbb{Z}^n)$. Obviously $V_t \subset M$. Moreover, no point of V_t is contained in the convex hull of other points of V_t. Define

$$C_t = R^m_+ + \operatorname{conv} V_t,$$

a convex set that has extreme points $y \in V_t$. Denote the standard basis of R^m by $\{e(1), \dots, e(m)\}$.

Let $v^1, \dots, v^p \in V_t$ $(p \geqslant 1)$ and $j_1, \dots, j_q \in \{1, \dots, m\}$ $(q \geqslant 0)$ and define

$$F = \operatorname{conv}\{v^1, \dots, v^p\} + \operatorname{pos}\{e(j_1), \dots, e(j_q)\} \tag{3.2}$$

where $\operatorname{conv} X$ and $\operatorname{pos} X$ denote the set of convex combinations and non-negative combinations, respectively, of the elements of X. Clearly, F lies in C_t and is a convex polyhedron. F will be called a face of C_t if it is the intersection of C_t with a supporting hyperplane. In this way we can define vertices, edges, ..., facets of C_t as well. It is easy to see that the vertices of C_t are the points in V_t.

The connection between $\mathcal{K}(A)$ and the facets of C_t is established in the following theorem.

Theorem 2. *There is a $t_0 > 0$ such that for $t > t_0$ the following statements are equivalent.*

(1) *$\sigma = \{z^1, \dots, z^p, w^{j_1}, \dots, w^{j_q}\}$ is a maximal simplex of $\mathcal{K}(A)$ (i.e., $p + q = m$).*
(2) *$F = \operatorname{conv}\{E_t(z^1), \dots, E_t(z^p)\} + \operatorname{pos}\{e(j_1), \dots, e(j_q)\}$ is a facet of C_t.*

It follows from Theorem 2 that for $t \geqslant t_0$, $p + q = m$ holds for the facet F in (3.2).

The boundary of C_t is going to be a geometric realization of the complex $\mathcal{K}(A)$. In order to show this we have to prove that the boundary of C_t consists of faces of the type (3.2).

Theorem 3. *C_t is a closed set. Its boundary consists of faces of the form* (3.2) *with $v^i = E_t(z^i)$ for some $z^i \in \mathbb{Z}^n$ $(i = 1, \dots, p)$.*

Notice that every point of C_t is of the form $\sum \alpha_i E_t(z^i) + \sum \beta_j e(j)$ where the first sum is a convex combination and the second is a nonnegative combination. Thus the first part of Theorem 3 implies the second. We mention further that Theorems 2 and 3 show that the combinatorial structure of the face lattice of C_t stabilizes after $t > t_0$.

4. $\mathbb{Z}^n$ acts on $\mathcal{K}(A)$ and C

We mentioned earlier that $\mathcal{K}(A)$ is invariant under translations by integers. Precisely, given $z \in \mathbb{Z}^n$ define

$$T_z(x) = \begin{cases} z + x & \text{when } x \in R^n, \\ x & \text{when } x \in W. \end{cases}$$

The group of translations $T^n = \{T_z : z \in \mathbb{Z}^n\}$ is isomorphic to $\mathbb{Z}^n$ and leaves $\mathcal{K}(A)$ invariant, i.e., if $\sigma \in \mathcal{K}(A)$, then $T_z(\sigma) = \{s + z : s \in \sigma\} \in \mathcal{K}(A)$ as well. The orbit of $\sigma \in \mathcal{K}(A)$ under T^n is the set of all simplices of the form $T\sigma$ with $T \in T^n$. Moreover, T^n acts transitively on the vertices of $\mathcal{K}(A)$ (belonging to $\mathbb{Z}^n$), i.e., for every pair $z, v \in \mathbb{Z}^n$ there is a $T \in T^n$ with $z = Tv$. So we have the following simple

Lemma 1. *The orbit of every $\sigma \in \mathcal{K}(A)$ with $\sigma \cap \mathbb{Z} \neq \emptyset$ contains a simplex with a vertex at the origin.*

$\mathbb{Z}^n$ acts on the convex set C_t as well in the following way. Given $z \in \mathbb{Z}^n$ define the $m \times m$ diagonal matrix D_z as

$$D_z = \operatorname{diag}(\exp\{t a_1 z\}, \dots, \exp\{t a_m z\}).$$

$D_z : R^m \to R^m$ is a nonsingular linear map and $D^n = \{D_z : z \in \mathbb{Z}^n\}$ is a group isomorphic to $\mathbb{Z}^n$. Notice that D_z leaves V_t and R^m_+ invariant since

$$D_z E_t(z_0) = E_t(z + z_0) \quad \text{and} \quad D_z R^m_+ = R^m_+.$$

It follows that $D_z C_t = C_t$ so that C_t is invariant under the group $\mathbb{D}^n$ of linear transformation. This implies that if F is a face of C_t then so is $D_z F$. It is clear, moreover, that $\mathbb{D}^n$ acts transitively on the vertices of C_t and therefore C_t looks the same at every one of its vertices. Thus C_t is a highly symmetric convex set which is, as we shall see later, locally a polytope.

As the group T^n acts on $|\mathcal{K}(A)|$ one can factor it out to obtain the topological space $|\mathcal{K}(A)|/T^n$. We shall prove

Theorem 4. *$|\mathcal{K}(A)|/T^n$ is homeomorphic to the direct product of the n-torus and R^{m-n-1}.*

This result is the natural extension of Theorem 2 from [1]. Its proof uses equivariance as well but this time the exponential map is not simplicial and we have to use an unusual extension of E, cf. (8.1).

5. Auxiliary results and proof of Theorem 3

We will need a few properties of the complex $\mathcal{K}(A)$. The first is local finiteness which we state in the form of

Lemma 2. *Each lattice point $z \in \mathbb{Z}^n$ is contained in a finite number of simplices of $\mathcal{K}(A)$.*

Proof. It is enough to prove this for $z = 0$. Assume, to the contrary, that an infinite number of maximal dimensional simplices, $\sigma(1), \sigma(2), \ldots \in \mathcal{K}(A)$ contain 0. We can further assume (after possibly reordering the rows of A and deleting some of the $\sigma(k)$) that each $\sigma(k)$ is of the form

$$\sigma(k) = \{z^1(k), \ldots, z^p(k), w^{p+1}, \ldots, w^m\}$$

where $z^1(k) = 0$ $(\forall k)$ and

$$\max_{j=1,\ldots,p} a_i z^j(k) = a_i z^i(k) =: \beta_i(k) \ (i = 1, \ldots, p).$$

As the sequence $\sigma(k)$ is infinite, some of the $\beta_i(k)$ cannot be bounded. Assume (again by deleting some of the $\sigma(k)$) that

$$\beta_i(k) \to \beta_i \text{ for } i = 1, 2, \ldots, p', \quad \text{and}$$

$$\beta_i(k) \nearrow \infty \text{ for } i = p' + 1, \ldots, p$$

where $0 < p' < p$ and $\beta_i < \infty$ for $i = 1, \ldots, p'$. Notice that $\beta_i(k) = a_i z^i(k) > a_i z^0(k)$ so that

$$\beta_i \geqslant 0 \text{ for } i = 1, \ldots, p'.$$

Moreover, the sets

$$Q(k) = \{x \in R^n : a_i x \leqslant \beta_i(k),\ i = 1, \ldots, p'\}$$

cannot be bounded (they contain the infinite sequence $z^p(k)$). Consequently the cone

$$Q(0) = \{x \in R^n : a_i x \leqslant 0,\ i = 1, \ldots, p'\} \subset Q(k)$$

is not bounded. Now condition A3 readily implies that int $Q(0) \neq \phi$. Then $Q(0)$ contains infinitely many lattice points. But the sets

$$Q(0) \cap \{x \in R^n : a_i x \leqslant \beta_i(k),\ i = p' + 1, \ldots, p\}$$

form an increasing sequence as $k \to \infty$ (since $\beta_i(k) \nearrow \infty$) and cannot be lattice point free. This contradiction demonstrates Lemma 2. □

Remark. The Lemma is equivalent to the fact that the number of one-dimensional simplices of the form $\{0,\ z\} \in \mathcal{K}(A)$ is finite. Such a $z \in \mathbb{Z}^n$ is a neighbor of the origin (cf. [7]). Therefore Lemma 2 says that there are finitely many neighbors of the origin if A is well conditioned, i.e., it satisfies A3; similar statements were proved in [9,7].

We mention further that Theorems 2, 3, and Lemma 2 show that C_t is locally a polytope (when $t > t_0$). Indeed, every point of ∂C_t belongs to some facet by Theorem 3; and every facet comes from a maximal simplex of $\mathcal{K}(A)$ by Theorem 2. Then, by Lemma 2, any vertex v of C_t is contained in finitely many facets; C_t has the structure of a polytope at any one of its vertices.

We need two more properties of the sets K_b. Both of them are stated in [1] for the $n+1$ by n case. The proof given there extends without difficulty and is, therefore, omitted.

Lemma 3. *There is a $\delta_1 > 0$ (depending only on A) with the following property. Let S be a finite set of lattice points and define*

$$K = \{x \in R^n : Ax \leqslant b\} \quad \textit{where } \beta_i = \max\{a_i z : z \in S\}.$$

If K contains a lattice point in its interior, then it contains a lattice point z such that $a_i z < \beta_i - \delta_1$ for all $i = 1, \ldots, m$.

Lemma 4. *There is $\delta_2 > 0$ (depending only on A) such that if $\sigma = \{z^1, \ldots, z^p, w^{j_1}, \ldots, w^{j_q}\} \in \mathcal{K}(A)$ with $p + q = m$ and z is a lattice point different from $z^1, \ldots, z^p$, then for some $i \in \{1, \ldots, m\} \setminus \{j_1, \ldots, j_q\}$*

$$a_i z \geqslant \max_{j=1,\ldots,p} a_i z^j + \delta_2. \quad \square$$

Proof of Theorem 3. We prove that C_t is closed. We may assume $t = 1$.

Notice that V is discrete, i.e., every compact set contains only finitely many elements of V. By the definition of C, every element $c \in C$ can be written as a mixed combination $\sum \alpha_i v^i + \sum \beta_j e(j)$, i.e., the first sum is a convex combination of some $v^i \in V$ and the second is a nonnegative combination. As $V \subset R^m_+$, $\sum \beta_j e(j)$ and every $\alpha_i v^i$ is less (componentwise) than c.

Assume now that c is from the boundary of C. Then $c = \lim_{k\to\infty} c(k)$ with $c(k) = v(k) + f(k)$ where $v(k) \in \operatorname{conv} V$ and $f(k) \in R^m_+$ for all $k = 1, 2, \ldots$. The sequence $f(k)$ must be bounded so we may assume (by considering a subsequence if necessary) that $\lim f(k)$ exists and equals $f \in R^m_+$, say. Then $\lim v(k)$ exists and equals $v = c - f$. As $v(k) \in \operatorname{conv} V \subset R^m$, every $v(k)$ can be written as a convex combination of $m+1$ elements of V:

$$v(k) = \sum_{i=0}^{m} \alpha_i(k) v^i(k).$$

Considering a subsequence if necessary we assume that $\lim \alpha_i(k) = \alpha_i$ for $i = 0, 1, \ldots, m$. Clearly $\alpha_i \geqslant 0$ and $\sum_0^m \alpha_i = 1$. To have convenient notation assume $\alpha_i > 0$ for $i = 0, 1, \ldots, j$ and $\alpha_i = 0$ for $i = j+1, \ldots, m$. Then, for $i = 0, 1, \ldots, j$, the sequence $v^i(k)$ must be bounded and we may assume that $\lim v^i(k) = v^i$. Since V is discrete, $v^i \in V$. Thus $\lim \sum_0^j \alpha_i(k) v^i(k) = \sum_0^j \alpha_i v^i = u$, say. Consequently $v - u = \lim \sum_{j+1}^m \alpha_i(k) v^i(k)$ and the limit is in R^m_+ since every summand is there. Thus $c = u + (v - u) + f$ and here u is of the form $\sum_0^j \alpha_i v^i$, a convex combination, and $(v - u) + f \in R^m_+$. □

6. Proof of Theorem 2

We essentially repeat the argument for the $(n+1) \times n$ case from [1] with the necessary modifications.

We show first that (2) implies (1). Let h be the normal to C at F, i.e.,

$$hy \geqslant 1 \text{ for all } y \in C, \text{ with equality for } y \in F. \tag{6.1}$$

Clearly $h = (h_1, \ldots, h_m)^{\mathrm{T}}$ is nonnegative and $h_i = 0$ if and only if F is parallel with $e(i)$. To simplify notation assume $j_1 = m$, $j_2 = m-1$, ..., $j_q = m - q + 1$. Thus $h_i = 0$ if $i \geqslant m - q + 1$ and we rewrite (6.1) as

$$\sum_{i=1}^{m-q} h_i \exp\{t a_i z\} \geqslant 1 \text{ for all } z \in \mathbb{Z}^n, \text{ with equality for } z = z^1, \ldots, z^p. \tag{6.2}$$

It follows from the equality case that $h_i \exp\{t a_i z^j\} \leqslant 1$ $(i = 1, \ldots, m-q,\ j = 1, \ldots, p)$, implying $a_i z^j \leqslant -\frac{1}{t} \log\ h_i$

$$\max_{j=1,\ldots,p} a_i z^j \leqslant -\frac{1}{t} \log\ h_i \ (i = 1, \ldots, m-q). \tag{6.3}$$

We wish to show that $\sigma = \{z^1, \ldots, z^p, w^{m-q+1}, \ldots, w^m\} \in \mathcal{K}(A)$ (in particular $p + q = m$), i.e., there are no lattice points other than $z^1, \ldots, z^p$ in

$$K = \{x \in R^n : a_i x \leqslant \beta_i,\ i = 1, \ldots, m - q\}$$

where $\beta_i = \max\{a_i z^j : j = 1, \ldots, p\}$ and, further, that $z^1, \ldots, z^p$ are on distinct facets of K. Let z be a lattice point satisfying $a_i z < \beta_i$ for $i = 1, \ldots, m - q$ ($z = z^j$ is possible). Then, by Lemma 3, for $i = 1, \ldots, m - q$

$$a_i z \leqslant \max\{a_i z^j : j = 1, \ldots, p\} - \delta_1. \tag{6.4}$$

On the other hand, (6.2) shows that there is an $i \in \{1, \ldots, m - q\}$ with

$$h_i \exp\{t a_i z\} \geqslant \frac{1}{m - q}, \text{ or, } a_i z \geqslant -\frac{1}{t}(\log h_i + \log (m - q)).$$

Thus by (6.3)

$$\begin{aligned} a_i z &\geqslant -\frac{1}{t} \log h_i - \frac{1}{t} \log(m - q) \\ &\geqslant \max\{a_i z^j : j = 1, \ldots, p\} - \frac{1}{t} \log(m - q), \end{aligned}$$

contradicting (6.4) if $t > t_1 = \frac{1}{\delta_1} \log(m - q)$.

It follows that K is a MLFC body and there is at most one z_i on every one of its facets implying $p \leqslant m - q$. Finally, $p + q \geqslant m$ follows from the fact that F is a facet.

We now turn to the second part of the argument and show that (1) implies (2). Assume

$$\sigma = \{z^1, \ldots, z^p, w^{p+1}, \ldots, w^m\} \in \mathcal{K}(A) \tag{6.5}$$

(using convenient notation, again). Let $h \in R^m_+$ satisfy $h_i = 0$ for $i = p + 1, \ldots, m$ and

$$h E_t(z^j) = 1 \text{ for } j = 1, \ldots, p. \tag{6.6}$$

We will show the existence of a t_2 such that $hE_t(z) \geqslant 2$ for every $t > t_2$ and $z \in \mathbb{Z}^n$, different from $z^1, \ldots, z^p$. Assume the vertices have been permuted so that $a_i z^i = \max\{a_i z^j : j = 1, \ldots, p\}$.

We compute $h_1, \ldots, h_p$ from the system of linear equations (6.6). By Cramer's rule we have

$$h_1 = \frac{\det N}{\det(\exp\{t a_i z^j\})}$$

where N is the matrix obtained by replacing the first row by (1, ..., 1) in the matrix appearing in the denominator. The determinant in the denominator can be written as the sum of $p!$ terms, each one based on a permutation of $\{1, \ldots, p\}$. But for each permutation π, other than the identity, the corresponding term is $(\Pi \exp\{a_i z^{\pi(i)}\})^t$ which is strictly less than $(\Pi \exp\{a_i z^i\})^t$ so that for large t this single term will be the asymptotic value of the denominator. Similarly, the numerator is asymptotically equal to the same product with index ranging from 2 to p. Thus we get that

$$h_1 = (1 + \varepsilon_1(t)) \exp\{-ta_1 z^1\}$$

with $\varepsilon_1(t) \to 0$ as $t \to \infty$. An identical argument gives that for $i = 1, \dots, p$

$$h_i = (\ 1 + \varepsilon_i(t)) \exp\{-ta_i z^i\}$$

with $\varepsilon_i(t) \to \infty$ as $t \to \infty$. In particular, there is a t_2 so that for all $t \geqslant t_2$ we have

$$h_i \geqslant 2\ \exp\{-ta_i z^i - t\delta_2\} \text{ for } i = 1, \dots, p \tag{6.7}$$

with δ_2 the constant in Lemma 4 since $1 + \varepsilon_i(t) \geqslant 2\exp\{-t\delta_2\}$ for large enough t.

Assume now that $v = E_t(z)$ and $z \in \mathbb{Z}^n$ is distinct from $z^1, \dots, z^p$. We have to show that $hv \geqslant 2$ for $t \geqslant t_2$. But using Lemma 4 we get that

$$hv = \Sigma h_i v_i \geqslant \Sigma\ 2\exp\{-t(a_i z^i + \delta_2)\} \exp\{ta_i z\} \geqslant 2.$$

In this argument the value of t_2 depends on the particular simplex $\sigma \in \mathcal{K}(A)$. In order to complete the proof of Theorem 3 we must show that a single value suffices for all simplices. To see this recall that if σ is the simplex in (6.5), then $\sigma_0 = T_z\sigma$ is a simplex of $\mathcal{K}(A)$ again. It is an easy matter to check now that if t_2 is the value given by the above argument for σ, then the same value will do for σ_0 as well. This means that a single value of t_2 suffices for the orbit (under the group T^n) of a simplex. By Lemma 1 every such orbit contains a simplex with one vertex at the origin. Lemma 2 implies that there are finitely many simplices in $\mathcal{K}(A)$ containing 0 and consequently, finitely many such orbits. □

7. Proof of Theorem 1

Assuming $t > t_0$ we suppress t from the notation. Theorem 1 gives a geometric realization of $|\mathcal{K}(A)|$ as the boundary of the convex set C in the following way. We define a map $f : |\mathcal{K}(A)| \to C$. Let

$$\sigma = \{z^1, \dots, z^p,\ w^{j_1}, \dots, w^{j_q}\} \in \mathcal{K}(A)$$

be a simplex with $p \geqslant 1$. The abstract mixed combination from (1.3)

$$x = \sum_{k=1}^{p} \gamma(k) z^k + \sum_{l=1}^{q} \beta(j_l) w^{j_l} \tag{7.1}$$

(which is a point of the cell $|\sigma|$ in $|\mathcal{K}(A)|$) is mapped to

$$f(x) = \sum_{k=1}^{p} \gamma(k) E(z^k) + \sum_{l=1}^{q} \beta(j_l) e(j_l). \tag{7.2}$$

One can see easily that f is well defined, i.e., if x belongs to two simplices of $\mathcal{K}(A)$ then the corresponding definitions coincide. Now $f : |\mathcal{K}(A)| \to \partial C$ is one-to-one by Theorem 3. Moreover f is continuous in both directions as one can readily check. Thus f is a geometric realization of $|\mathcal{K}(A)|$, and so $|\mathcal{K}(A)|$ and ∂C are homeomorphic. But ∂C is homeomorphic to R^{m-1} so Theorem 1 follows. □

8. Proof of Theorem 4

Assume again $t > t_0$. We need to define an equivariant extension

$$E^* : |\mathcal{K}(A)| \to \partial C$$

of the exponential map $E : \mathcal{K}(A) \to \partial C$. Equivariance here simply means that $E^*(T_z x) = D_z E^*(x)$ for all $x \in \mathcal{K}(A)$ and all $z \in \mathbb{Z}^n$.

It is easy to see that f in (7.2) is not equivariant since $D_z e(j) = \exp\{a_j z\} e(j)$. As E is simplicial on the simplices σ without ideal points, for these simplices the extension of E is the usual simplicial one: for x in (7.1) with $q = 0$ we have $E^*(x) = \sum_{k=1}^{p} \gamma(k) E(z^k)$. For a generic point $x \in |\mathcal{K}(A)|$ which is of the form (7.1) define

$$E^*(x) = \sum_{k=1}^{p} \gamma(k) E(z^k) + \sum_{l=1}^{q} \beta(j_l) \sum_{k=1}^{p} \gamma(k) \exp\{a_{j_l} z^k\} e(j_l). \tag{8.1}$$

It is not difficult to check that E^* is equivariant, one-to-one, and continuous in both directions.

Next, we define a map $g : \partial C \to M$ which is equivariant with respect to D_z, i.e., $D_z g(y) = g(D_z y)$ for every $y \in \partial C$ and every $z \in \mathbb{Z}^n$. Let $R(y)$ be the ray starting at the origin and passing through y and define simply

$$g(y) = M \cap R(y)$$

which is clearly a point in M. g is equivariant since $R(D_z y) = D_z R(y)$ and M is invariant under D_z. We see now that the following diagram

$$\begin{array}{ccccc} |\mathcal{K}(A)| & \xrightarrow{E^*} & \partial C & \xrightarrow{g} & M \\ T_z \downarrow & & D_z \downarrow & & D_z \downarrow \\ |\mathcal{K}(A)| & \xrightarrow{E^*} & \partial C & \xrightarrow{g} & M \end{array}$$

commutes for every $z \in \mathbb{Z}^n$ implying that the quotient space $|\mathcal{K}(A)|/T^n$ is homeomorphic to M/D^n.

M is homeomorphic to R^{m-1} and a natural homeomorphism $M \to R^{m-1}$ is the componentwise logarithm of $y \in M$. Write D^* for the set of all m by m diagonal matrices whose diagonal entries, $d_1, \ldots, d_m$, are positive and satisfy $\Pi_1^m d_k^{\lambda_k} = 1$ (cf. (3.1)). D^* acts on M as the group T^* of all translations acts on R^{m-1}. D_n is a discrete subgroup of D^* and the natural isomorphism $D^* \to T^*$ (taking componentwise logarithm of the diagonal entries) maps D_n onto an n-dimensional lattice of T^*, isomorphic to $\mathbb{Z}^n$. Thus the quotient space M/D_n is homeomorphic to $R^{m-1}/\mathbb{Z}^n$ proving the theorem. $\square$

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