BARYCENTRIC SUBDIVISION OF TRIANGLES AND SEMIGROUPS OF MÖBIUS MAPS

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§1. *Introduction.* The following question of V. Stakhovskii was passed to us by N. Dolbilin [4]. Take the barycentric subdivision of a triangle to obtain six triangles, then take the barycentric subdivision of each of these six triangles and so on; is it true that the resulting collection of triangles is dense (up to similarities) in the space of all triangles? We shall show that it is, but that, nevertheless, the process leads almost surely to a flat triangle (that is, a triangle whose vertices are collinear).

THEOREM 1. *Successive barycentric subdivisions of a non-flat triangle contain triangles which, to within a similarity, approximate arbitrarily closely any given triangle.*

We can produce a random sequence of triangles by successively choosing one of the six triangles from each barycentric subdivision. So, if T_n is a triangle, we choose T_{n+1} at random from the six triangles into which T_n is subdivided. This gives a Markov chain (T_n). In contrast to Theorem 1 we shall show that the shapes of these triangles converge, almost surely, towards the flat triangles as $n \to \infty$.

THEOREM 2. *Given a triangle, we repeatedly subdivide it barycentrically and randomly choose one of the resulting triangles. Then, with probability one, the shapes of the triangles will converge to flat triangles.*

Any triangle T is similar to a triangle with vertices 0, 1 and σ in the complex plane. The complex number σ represents the shape of the triangle. Let T_j $(j=1, \ldots, 6)$ be the six triangles of the barycentric subdivision of T. Then the shape σ_j of T_j is given by $g_j(\sigma)$ for some Möbius map g_j of the upper half-plane H^+ onto itself. In this way the proof of Theorem 1 reduces to showing that the orbit of a point in H^+ under the semigroup generated by $g_1, \ldots, g_6$ is dense in H^+. Although much has been written on discrete subgroups of the group Aut (H^+) of Möbius transformations of H^+ onto itself (see, for example, [1]), almost nothing seems to have been written on semigroups. With this in mind, the following result may be of independent interest.

THEOREM 3. *Suppose that g and h are non-commuting elements of* Aut (H^+), *and that h is an elliptic element of infinite order. Then the semigroup generated by g and h is dense in* Aut (H^+).

The proofs of Theorems 1, 2 and 3 are given in Sections 3, 4 and 2, respectively.

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§2. *The proof of Theorem* 3. Our proof is based on hyperbolic geometry, but it is more convenient to work with the unit disc Δ as the hyperbolic plane rather than H^+. Let g and h be non-commuting elements of Aut (Δ) with h elliptic of infinite order. Then h is a hyperbolic rotation and, by conjugation, we may assume that h fixes the origin. Since $g \in \mathrm{Aut}\,(\Delta)$ we have

$$g(z) = \frac{az + \bar{c}}{cz + \bar{a}}$$

for some a and c satisfying $|a|^2 - |c|^2 = 1$.

Now let S be the closure (in the topological group Aut (Δ)) of the semigroup generated by g and h. If S contains an elliptic element f of infinite order, then S contains all iterates f^n and so, being closed, S contains all hyperbolic rotations about the fixed point of f. We shall denote the group of hyperbolic rotations about w by R_w so, in particular, $R_0 \subset S$.

We now know that S contains the maps

$$h_t(z) = \frac{e^{it}z + 0}{0z + e^{-it}}, \qquad \text{and} \qquad h_t g(z) = \frac{e^{it}az + e^{it}\bar{c}}{e^{-it}cz + e^{-it}\bar{a}}$$

for all real t, and it is clear that for a suitable choice of t,

$$\text{trace}\,(h_t g) = 2\,\mathrm{Re}\,(e^{it}a) = 2\cos\tfrac{1}{2}\theta,$$

where θ is some irrational multiple of π. For such t, the transformation $h_t g$ is a rotation of infinite order. Now $h_t g$ has a fixed point w, say, in Δ and we conclude, as above, that $R_w \subset S$. It is important to note that $w \neq 0$ (else $c = 0$ and then g and h would commute); thus S contains both R_0 and R_w, where $w \neq 0$.

Now take any point z in Δ, not on the geodesic through 0 and w, and form the hyperbolic triangle with vertices z, 0 and w. Let α_z, α_0 and α_w be the hyperbolic reflections across the side of the triangle opposite z, 0 and w, respectively. As

$$\alpha_0\alpha_w = \alpha_0(\alpha_z\alpha_z)\alpha_w = (\alpha_0\alpha_z)(\alpha_z\alpha_w) \in R_w R_0,$$

we see that $\alpha_0\alpha_w$ is in S. Also $\alpha_0\alpha_w$ is a rotation through some angle $\varphi(z)$ about z. Since φ is a continuous function but is not constant, it follows that the angle $\varphi(z)$ is an irrational multiple of π for a dense set of z. Then $\alpha_0\alpha_w$ is a rotation of infinite order. Thus S contains the group R_z for a dense set of points z in Δ and, as S is closed, we now see that S contains all hyperbolic rotations.

Finally, it is known that any element f in Aut (Δ) can be expressed in the form $\alpha\beta$, where α and β, are reflections in some geodesics A and B. Draw a geodesic C which crosses both A and B and let γ denote reflection in C. Then $f=\alpha\beta=(\alpha\gamma)(\gamma\beta)\in S$. This proves that $S=\mathrm{Aut}\,(\Delta)$.

§3. *The proof of Theorem* 1. The idea of describing the shape of a triangle by a single complex parameter σ was developed by D. G. Kendall [**6**]. For this it is convenient to regard a triangle T as an *ordered* triple of distinct points (a, b, c) which are its vertices. For example, we distinguish between the triangles (a, b, c) and (b, c, a). Of course, in this sense, six different triangles now correspond to the same triangular set of points. The *shape* σ of the triangle (a, b, c) is $(c-a)/(b-a)$. This is invariant under all of the similarity maps $z\mapsto \alpha z+\beta$ (with $\alpha\neq 0$). In particular, the map $g(z)=(z-a)/(b-a)$ maps the triangle (a, b, c) to the triangle $(0, 1, \sigma)$. Furthermore, the triangle T has shape σ if, and only if, it is similar under some orientation preserving map to $(0, 1, \sigma)$.

Suppose that T is the triangle (a, b, c). If the triangle T reduces to a point then it has no shape. Otherwise its shape σ is a point of the extended complex plane. The shapes 0, 1 and ∞ correspond to triangles with $a=c$, $c=b$ and $b=a$ respectively. The triangle T is *flat* when its shape σ lies in the extended real line. The vertices a, b and c occur anticlockwise around the boundary of T precisely when σ lies in the upper half-plane $H^+=\{\sigma\in\mathbb{C}: \mathrm{Im}\,(\sigma)>0\}$. Finally note that permuting the vertices of the triangle T gives the six shapes:

$$\sigma,\qquad \frac{1}{1-\sigma},\qquad \frac{\sigma-1}{\sigma},\qquad 1-\sigma,\qquad \frac{1}{\sigma},\qquad \frac{\sigma}{\sigma-1}.$$

These arise from the Möbius group which permutes 0, 1 and ∞.

Let $T=(a, b, c)$ have shape σ. Among the triangles of the barycentric subdivision of T are the two triangles

$$\left(a, \frac{a+b}{2}, \frac{a+b+c}{3}\right),\qquad \left(\frac{a+c}{2}, a, \frac{a+b+c}{3}\right)$$

with shapes σ_1 and σ_2 respectively. An easy computation shows that $\sigma_1=g(\sigma)$ and $\sigma_2=h(\sigma)$ for the two elements g and h of Aut (H^+):

$$g(z)=\frac{2z+2}{3},\qquad h(z)=\frac{z-2}{3z}.$$

Repeated subdivision of T will give all of the triangles $f(\sigma)$ for f in the semigroup generated by g and h, so Theorem 1 will follow once we have checked that g and h satisfy the hypotheses of Theorem 3. It is easy to check that g and h do not commute (this also follows from the fact that they have no common fixed point). A computation shows that

$$h^2(z)=\frac{5z+2}{-3z+6}=\frac{\frac{5}{6}z+\frac{2}{6}}{-\frac{3}{6}z+\frac{6}{6}}$$

(where the final form has determinant 1). Hence trace $(h^2)=\frac{11}{6}\in(-2,2)$ and so h^2 is a rotation of angle ψ, say. Consequently, $2\cos\frac{1}{2}\psi=\text{trace }(h^2)=\frac{11}{6}$. The only rational values of x and y with $\cos\pi x=y$ are those with $y=0,\ \pm\frac{1}{2}$, ± 1 (see, for example, [3]), so we find that h^2, and hence h itself, is a rotation of infinite order. (Alternatively, if h were of finite order, its trace would be $\omega+\omega^{-1}$ for a root of unity ω. Hence $\tau=\text{trace }(g)=1/\sqrt{6}$ would be an algebraic integer. The minimal polynomial for τ in $\mathbb{Z}[X]$ is $6X^2-1$ so τ is not an algebraic integer.) This completes the proof of Theorem 1.

(There are other ways to represent the shapes of triangles. For example, if the side lengths of the triangle are r, s and t, then the point $(r^2:s^2:t^2)$ in the real projective plane determines the unoriented shape of the triangle. Moreover barycentric subdivision of the triangle acts as a linear map on the vector (r^2,s^2,t^2). We may use this to give a different proof of Theorem 1 in a similar way to that described above.)

Remark. Let Q be a fixed, non-flat triangle. Theorem 1 shows that, if we begin with a non-flat triangle T, there are triangles T_n in successive barycentric subdivisions of T which have shapes arbitrarily close to that of Q. This result can be strengthened by insisting that the sides of T_n are almost parallel to those of Q. So T_n is arbitrarily close to a homothetic copy of Q.

To prove this, note that an affine linear map A transforms the barycentric subdivision of a triangle T into the barycentric subdivision of $A(T)$. So it suffices to demonstrate the result for one fixed triangle T. Let T be the triangle with vertices (a,b,c) and shape $\sigma=\frac{1}{6}(1+i\sqrt{23})$. This shape is chosen because it is fixed by the map $h\colon z\mapsto(z-2)/3z$ considered above. One of the triangles in the barycentric subdivision of T is $T'=(\frac{1}{2}(a+c),\,b,\,\frac{1}{3}(a+b+c))$ which also has the same shape σ as T. However, the sides of T' need not be parallel to those of T. Indeed T' is obtained from T by rotating by the complex number

$$\omega=\left(\frac{b-\frac{1}{2}(a+c)}{|b-\frac{1}{2}(a+c)|}\right)\left(\frac{|b-a|}{b-a}\right)=-\frac{\sigma}{|\sigma|}=\frac{1+i\sqrt{23}}{\sqrt{24}}.$$

Now ω is not a root of unity, since $\omega+\omega^{-1}=1/\sqrt{6}$ is not an algebraic integer. So powers of ω are dense in the unit circle. Repeating this process we obtain a sequence of triangles $T^{(k)}$ each with shape σ but rotated by ω^k from T. For any rotation R we can thus find a k with $T^{(k)}$ arbitrarily close to a homothetic copy of $R(T)$.

Theorem 1 shows that we can find a triangle T_n, by successively subdividing T, which has shape arbitrarily close to Q. Thus there is a rotation R with $R(T_n)$ arbitrarily close to a homothetic copy of Q. Hence, by subdividing $T^{(k)}$ instead of T, we obtain a triangle S arbitrarily close to a homothetic copy of Q. The triangle S is one from the barycentric subdivisions of T.

§4. *The proof of Theorem* 2. When we barycentrically subdivide a triangle T with shape $\sigma\in H^+$, we obtain six smaller triangles. We can order the vertices of each of these anticlockwise in three different ways to get 18 different shapes all lying in H^+. As above, these are given by $g_j(\sigma)$ for 18 Möbius transformations g_j ($j=1, 2, \ldots, 18$). It is important to note that the g_j are independent of the original triangle.

The process of repeatedly dividing a triangle barycentrically defines a random process on the upper half-plane H^+ representing the shape of the resulting triangles. For suppose that we begin with the triangle T having shape $\sigma \in H^+$. Barycentric subdivision gives rise to the 18 new triangles with shapes $(g_j(\sigma): j = 1, 2, \ldots, 18)$. Choose one of these, $g_{j(1)}(\sigma)$ say, at random and repeat the process. This gives rise to a random sequence of triangles with shapes

$$g_{j(n)} g_{j(n-1)} \ldots g_{j(1)}(\sigma).$$

More formally, let μ be the probability measure on Aut (H^+) which assigns probability $1/18$ to each of the transformations g_j. Then let $(X_i : i \in \mathbb{N})$ be independent random variables which take values in Aut (H^+) and have the distribution μ. The sequence

$$\sigma_n = X_n X_{n-1} \ldots X_1(\sigma)$$

then defines a Markov process in H^+. We have shown that the union of the paths of this Markov process are dense in H^+. However, Theorem 2 claims that almost every path converges to the boundary $\mathbb{R} \cup \{\infty\}$ as $n \to \infty$.

Remark. We will, in fact, prove more. Let ρ be the hyperbolic metric on H^+. Then we will show that there is a $\lambda > 0$ with

$$\frac{\rho(\sigma_n, i)}{n} \to \lambda \qquad \text{almost surely as } n \to \infty.$$

The number λ is the (first) *Lyapounov exponent* for the process. This result will follow from the law of large numbers for non-commuting random products proved by H. Furstenberg [**5**]. (See also the accounts of this result given by P. Bougerol [**2**] and F. Ledrappier [**7**].)

In order to apply Furstenberg's results we wish to consider random products of unimodular matrices. So, rather than working in Aut (H^+), we will work in the double cover SL $(2, \mathbb{R})$. We will alter the notation slightly and think of μ as a probability measure on SL $(2, \mathbb{R})$ and X_n as independent random variables in SL $(2, \mathbb{R})$ with distribution μ. Each element $g \in$ SL $(2, \mathbb{R})$ acts as a linear map on $\mathbb{R}^2$ and we will denote its norm by $\|g\|$. Since μ is supported on only finitely many points, the integral $\int \log \|g\| d\mu(g)$ is certainly finite. Theorem 1 shows that the smallest closed semigroup containing the support of μ is all of SL $(2, \mathbb{R})$. Hence the smallest closed group containing the support of μ is SL $(2, \mathbb{R})$. This clearly acts irreducibly on $\mathbb{R}^2$ and all finite index subgroups do so also. Hence we can apply Theorems 8.5 and 8.6 of [**5**]. These show that there is a strictly positive constant α so that, almost surely,

$$\frac{1}{n}\log \|X_n X_{n-1} \ldots X_1 u\| \to \alpha$$

as $n \to \infty$ for all non-zero $u \in \mathbb{R}^2$. By separating $u \in \mathbb{C}^2$ into real and imaginary parts we easily see that this conclusion still holds for non-zero $u \in \mathbb{C}^2$.

Consider the matrix

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}\,(2, \mathbb{R})$$

and the corresponding Möbius transformation

$$g : \sigma \mapsto \frac{a\sigma + b}{c\sigma + d}.$$

It is easy to check that

$$\left\| G\begin{pmatrix} \sigma \\ 1 \end{pmatrix} \right\|^2 = |a\sigma + b|^2 + |c\sigma + d|^2 = \frac{(|g(\sigma)|^2 + 1)}{\mathrm{Im}\,(g(\sigma))}\,\mathrm{Im}\,(\sigma)$$

$$= 2\cosh \rho(g(\sigma), i)\,\mathrm{Im}\,(\sigma).$$

Therefore

$$\frac{1}{2n}\log\,(2\cosh \rho(\sigma_n, i)\,\mathrm{Im}\,(\sigma)) \to \alpha$$

almost surely as $n \to \infty$. Consequently,

$$\frac{\rho(\sigma_n, i)}{n} \to 2\alpha > 0$$

almost surely as $n \to \infty$. Of course, the triangle inequality shows that the same conclusion holds with i replaced by any point of H^+.

A related application of Furstenberg's work to shapes of triangles was given by D. Mannion [**8**] and [**9**].

Furstenberg's results also show that the sequence σ_n converges almost surely to a shape $\sigma_\infty \in \partial H^+$. It would be interesting to know the distribution of the limiting shape σ_∞ and the value of the Lyapounov exponent 2α.

The argument used to prove Theorem 2 requires little information about the transition distribution μ on Aut (H^+). So there are similar results for certain other stationary Markov chains of triangles. However, there are related chains which do not converge to flat triangles. As a simple example, consider the sequence of triangles T_n with shape σ_n defined by choosing T_{n+1} as one of the triangles in the barycentric subdivision of T_n which has shape closest in the hyperbolic metric to $\omega = \frac{1}{2}(1 + i\sqrt{3})$. The number ω is the shape of an equilateral triangle. It is easy to check that

$$K = \{\sigma \in H^+ : \rho(\sigma, g_j^{-1}(\omega)) > \rho(\sigma, \omega) \text{ for } j = 1, 2, \ldots, 18\}$$

is bounded for the hyperbolic metric in H^+. If $\sigma_n \notin K$, then there is a j with

$$\rho(g_j(\sigma_n), \omega) = \rho(\sigma_n, g_j^{-1}(\omega)) \leqslant \rho(\sigma_n, \omega)$$

so $\rho(\sigma_{n+1}, \omega) \leqslant \rho(\sigma_n, \omega)$. It follows that the sequence σ_n remains bounded and so cannot converge to the flat triangle shapes in ∂H^+.

References

1. A. F. Beardon. *The geometry of discrete groups*, *Graduate Texts*, 91 (Springer-Verlag, 1983).
2. P. Bougerol and J. Lacroix. Products of random matrices with applications to Schrödinger operators. *Progress in Probability and Statistics*, 8 (Birkhäuser, 1985).
3. J. H. Conway and A. J. Jones. Trigonometric Diophantine equations (on vanishing sums of roots of unity). *Acta Arithmetica*, 30 (1976), 229–240.
4. N. Dolbilin. Private correspondence.
5. H. Furstenberg. Noncommuting random products. *Trans. Amer. Math. Soc.*, 108 (1963), 377–428.
6. D. G. Kendall. Shape manifolds, Procrustean metrics, and complex projective space. *Bull. London Math. Soc.*, 16 (1984), 81–121.
7. F. Ledrappier. *Quelques propriétés des exposants caractéristiques*, (from *École d'été de Saint-Flour, XII*, edited by P. L. Hennequin) *Lecture Notes in Mathematics* 1097 (Springer-Verlag, 1984).
8. D. Mannion. Convergence to collinearity of a sequence of random triangle shapes. *Adv. Appl. Prob.*, 22 (1990), 831–844.
9. D. Mannion. The invariant distribution of a sequence of random collinear triangle shapes. *Adv. Appl. Prob.*, 22 (1990), 845–865.

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