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On the Exact Constant in the Quantitative Steinitz Theorem in the Plane*

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Abstract. We determine the maximal value of r with the following property. If the convex hull of a set in R^2 contains a unit circle B, then a subset of at most four points can be selected so that the convex hull of this subset contains the circle of radius r concentric with B. That the result is sharp is shown by the example when the original set is the set of vertices of a regular pentagon circumscribed around B.

1. Introduction

Steinitz proved [S] a long time ago that if the interior of the convex hull of a set $X \subset R^d$ contains the point p, then there is a subset $Y \subset X$, of cardinality at most $2d$, such that $p \in \text{int conv } Y$. This result is made "quantitative" in [BKP] in the sense that, for every $d \geq 2$, there is a constant $c(d)$ with following property. If $\text{conv } X$ contains a closed ball of radius r with center p, then $Y \subset X$, $|Y| \leq 2d$, exists such that $\text{conv } Y$ contains the ball of radius $c(d)r$ with center p. It is shown in [BKP] that, in fact, $c(d) > d^{-2d}$. The aim of this paper is to find the best constant $c(d)$ when $d = 2$.

It turns out that, as expected, the worst case is when $\text{conv } X$ is a regular pentagon, p is its center, and the ball is the inscribed circle of $\text{conv } X$.

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To simplify the presentation we assume that $X \subset R^2$ is finite, then $P = \operatorname{conv} X$ is a convex polygon. Let B denote the unit circle centered at the origin and assume $B \subset P$. rB is the circle of radius r concentric with B. Define

$$r(X) = \max_{Y \subset X, |Y| \le 4} \max\{\rho: \rho B \subset \operatorname{conv} Y\},$$

or, as in [KMY], define, more generally,

$$r_k(X) = \max_{Y \subset X, |Y| \le k} \max\{\rho: \rho B \subset \operatorname{conv} Y\}.$$

When X is the set of the vertices of a regular $(k+1)$-gon with inscribed ball B we get

$$r_k(X) = r_k^* = \frac{\cos(2\pi/(k+1))}{\cos(\pi/(k+1))} = \left(1 - \frac{3\pi^2}{2(k+1)^2}\right)(1 + o(1))$$

as $k \to \infty$.

Theorem 1. *Under the above hypothesis*

$$r(X) = r_4(X) \ge r_4^* = \frac{\cos 2\pi/5}{\cos \pi/5} = \frac{3 - \sqrt{5}}{2} = 0.381966\ldots.$$

Equality holds if and only if conv X *is a regular pentagon with inscribed circle* B.

At the end of this paper we indicate how to prove the same result without assuming that X is finite.

It is proved in [KMY] that, for all $X \subset R^2$ with $B \subset \operatorname{conv} X$,

$$r_k(X) \ge 1 - \frac{2\pi^2}{k^2}.$$

It seems likely that, for every $k \ge 4$, $r_k(X) \ge r_k^*$ with equality only when P is the regular $(k+1)$-gon. Theorem 1 proves this for $k = 4$, i.e., $r_4(X) \ge r_4^*$ and, using the methods of this paper, we can extend this to $k = 5$:

Theorem 2. *If* $X \subset R^2$ *is finite and* $B \subset \operatorname{conv} X$, *then*

$$r_5(X) \ge r_5^* = \frac{\cos 2\pi/6}{\cos \pi/6} = \frac{1}{\sqrt{3}}.$$

Equality holds if and only if conv X *is a regular hexagon with inscribed circle* B.

We can further show the validity of the inequality $r_k(X) \geq r_k^*$ for the case when $|X| = k + 1$:

Theorem 3. *If $X \subset R^2$ is a set of $k + 1$ points and $B \subset \text{conv } X$, then*

$$r_k(X) \geq r_k^*.$$

2. Proof of Theorem 3

Let $X = \{a_1, \ldots, a_{k+1}\}$ (where $a_{k+i+1} = a_i$) be the set of vertices of a convex $(k + 1)$-gon P with $B \subset P$, the vertices indexed in cyclic order. Write $r = r_k^*$. We show that there is an i such that

$$rB \subset \text{int conv}(X \backslash \{a_i\})$$

unless P is a regular $(k + 1)$-gon circumscribed around B.

If some diagonal of P is disjoint from rB, the statement is trivially true, therefore we assume that, for all i,

$$a_i a_{i+2} \cap rB \neq \varnothing.$$

Write b_i and c_i for the projection of the origin o onto the line passing through $a_i a_{i+1}$ and $a_i a_{i+2}$, respectively. Set

$$\alpha_i = \angle a_i o b_i, \qquad \beta_i = \angle b_i o a_{i+1},$$

$$\varphi_i = \angle a_i o c_i, \qquad \psi_i = \angle c_i o a_{i+2}.$$

These angles are taken to be signed. First we assume that

$$o \notin \text{int conv}\{a_i a_{i+1} a_{i+2}\}, \qquad i = 1, \ldots, k + 1, \tag{2.1}$$

consequently, $\varphi_i, \psi_i \geq 0$. The conditions imply that

$$|b_i| = |a_i| \cos \alpha_i = |a_{i+1}| \cos \beta_i \geq 1 \quad \text{and} \quad |c_i| = |a_i| \cos \varphi_i = |a_{i+2}| \cos \psi_i \leq r.$$

So we have

$$\cos \varphi_i \leq \frac{r}{|a_i|} \leq r \cos \alpha_i \quad \text{or} \quad \varphi_i \geq \text{arc cos}(r \cos \alpha_i),$$

$$\cos \psi_i \leq \frac{r}{|a_{i+2}|} \leq r \cos \beta_{i+1} \quad \text{or} \quad \psi_i \geq \text{arc cos}(r \cos \beta_{i+1}).$$

The second derivative of the function $f(t) = \arccos(r \cos t)$ is positive when $t \in (-\pi/2, \pi/2)$. So $f(t)$ is convex and the Jensen inequality implies that

$$4\pi = \sum_{1}^{k+1} (\varphi_i + \psi_i) \geq \sum_{1}^{k+1} (\arccos(r \cos \alpha_i) + \arccos(r \cos \beta_{i+1}))$$

$$\geq 2(k+1) \arccos\left(r \cos \frac{2\pi}{2(k+1)}\right)$$

$$= 2(k+1) \arccos\left(\cos \frac{2\pi}{k+1}\right) = 4\pi.$$

Thus equality holds in all the inequalities above. This proves the claim under assumption (2.1).

Suppose now that assumption (2.1) does not hold for some "exceptional" values of i. We show the existence of another $(k+1)$-gon P' containing B that has fewer exceptional i values. Further, P' has the property that if the convex hull of a k-subset of X contains ρB, then there are k vertices of P' such that their convex hull contains ρB.

Let $i = 1$ be an exceptional value, i.e., $o \in \operatorname{int} \operatorname{conv}\{a_1 a_2, a_3\}$. Let L be the line, parallel to $a_1 a_3$ and passing through the origin. It meets the lines $a_k a_1$ and $a_5 a_3$ in points a_1' and a_3', respectively. Now replace a_1 by a_1' and a_3 by a_3' to get P'. Clearly, $P \subset P'$ and, except for $a_1 a_3$, all diagonal lines of type $a_i a_{i+2}$ remain the same. From this all the above claims follow, concluding the proof of Theorem 3. □

3. Proof of Theorem 1

In the proof r stands for r_4^*. We assume, without loss of generality, that X coincides with the set of vertices of an n-gon P. We call the elements of X vertices (of P) and denote them by $a, b, c, \ldots$. The diagonal ab is called an *inner* (*outer*) *diagonal* if $ab \cap rB \neq \emptyset$ is true (false).

We assume that

$$rB \text{ is } \textit{not} \text{ contained in int conv } Y \qquad \text{for any} \quad Y \subset X, \quad |Y| \leq 4. \qquad (*)$$

We derive several properties of X and finally conclude that P has to be a regular pentagon circumscribed around B.

Claim 1. *To every vertex a two distinct inner diagonals adjacent to a exist.*

Proof. The $(n-3)$ diagonals starting at a dissect P into $(n-2)$ triangles. If the minimal number of those (consecutive) triangles which cover rB is less than three, then we are in contradiction with $(*)$. Else we have at least two inner diagonals. □

Definition. A sequence $a_1, a_2, \ldots, a_k$ of consecutive vertices is a *critical sequence* if $a_i a_j$ is an inner diagonal if and only if $\{i, j\} = \{1, k\}$.

For a critical sequence, $k \geq 3$ holds, since no edge of P can intersect rB. Further, for any $i \in \{2, \ldots, k-1\}$, the diagonals $a_1 a_i$ and $a_i a_k$ avoid rB. Now let a_k, $a_{k+1}, \ldots, a_n, a_1$ be the rest of the vertices listed consecutively from a_k to a_1. This sequence cannot be critical since if it was, then by the previous remark, for any $i \in \{2, \ldots, k-1\}$ and $j \in \{k, \ldots, n\}$,

$$rB \subset \operatorname{int} \operatorname{conv}\{a_1, a_i, a_k, a_j\},$$

contradicting (*). So we may and do speak of a *critical diagonal* and of its *critical side* since this sequence (side) is uniquely determined.

Remark 1. It is evident that on both sides of an inner diagonal ab there is (at least) one critical diagonal (one of them, actually, may coincide with it). Claim 1 implies that at least two critical diagonals exist.

We need some further notation. If $x, y \in R^2$ $(x \neq y)$, then we write $\overline{xy}$ for the line they span and $\overrightarrow{xy}$ for the half-line starting at x and containing y. Let $\operatorname{cone}(\overrightarrow{xy}, \overrightarrow{uv})$ denote the convex cone whose boundary is contained in $\overrightarrow{xy} \cup \overrightarrow{uv}$. (This makes sense only if the half-lines $\overrightarrow{xy}$ and $\overrightarrow{uv}$ intersect.) Define the angle γ by $\sin \gamma = r$, $0 < \gamma < \pi/2$. It is clear that, for any inner diagonal ab,

$$\angle aob \geq \pi - 2\gamma.$$

Here and in what follows $\angle aob$ denotes the (unsigned) angle of the half-lines $\overrightarrow{oa}$ and $\overrightarrow{ob}$. As luck would have it,

$$\gamma < \frac{\pi}{8}.$$

In fact $\gamma = 0.391922\ldots$ and $\pi/8 = 0.392699\ldots$, which can be verified by simple calculation. We refer quite often to this fact. We also need the following:

Lemma 1. *Assume ab is an outer diagonal or an edge of P. Let $\overrightarrow{ax}$ and $\overrightarrow{by}$ be tangent half-lines to rB such that the triangle determined by $\overline{ax}$, $\overline{by}$, and $\overline{ab}$ contains rB. If the intersection of the lines $\overline{ax}$, $\overline{by}$ is in* int B, *then there are vertices $u, v \in X$ such that*

$$rB \subset \operatorname{int} \operatorname{conv}\{a, b, u, v\}.$$

Proof. The apex of the cone $C = \operatorname{cone}(\overrightarrow{ax}, \overrightarrow{by})$ is in int B. Then C intersects the boundary of P, and it intersects the interior of some edge uv of P, and Lemma 1 follows immediately. □

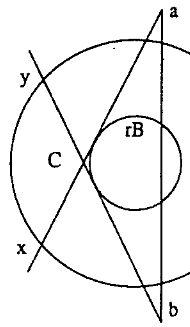

Fig. 1

Now let ab be a critical diagonal. Draw a half-line starting from a (resp. from b) tangent to rB and toward the noncritical side of ab. Denote the last point of this half-line in B by x (resp. y) (see Fig. 1).

Claim 2. *No vertex of P lies in the interior of $C = \text{cone}(\overrightarrow{ax}, \overrightarrow{by})$.*

Proof. Assume $u \in X \cap \text{int } C$ and let c be any vertex on the critical side of ab. Then $rB \subset \text{conv}\{a, b, c, u\}$. □

Claim 3. *If ab is a critical diagonal, then the origin lies on its noncritical side.*

Proof. Assume the contrary and consider the previous picture. It is easily seen (as a consequence of the inequality $\gamma < \pi/8$) that the apex of the cone $C = \text{cone}(\overrightarrow{ax}, \overrightarrow{by})$ is in int B. As C intersects the boundary of P, and, by Claim 2, there is no vertex in int C, an edge uv crossing C (and not intersecting B) exists. We choose the notation so that au is a diagonal meeting rB. Let $\overrightarrow{uu'}$ and $\overrightarrow{vv'}$ be half-lines, tangent to rB such that the triangle determined by $\overline{uv}$, $\overline{uu'}$, and $\overline{vv'}$ contains rB. We choose u' and v' as the last point in B, respectively (see Fig. 2).

We show now that the lines $\overline{uu'}$ and $\overline{vv'}$ meet in int B. In view of Lemma 1 this will prove Claim 3. We compute angles: $\angle aou \geq \pi - 2\gamma$, $\angle uou' \geq \pi - 2\gamma$, therefore $\angle aou' \leq 4\gamma$. Similarly, $\angle bov' \leq 4\gamma$. Since o is on the same side of ab as u' and v', we get

$$\angle aou' + \angle bov' + \angle aob \leq \pi + 8\gamma < 2\pi,$$

consequently the lines $\overline{uu'}$ and $\overline{vv'}$ intersect in int B. □

Claim 4. *There is at most one critical diagonal on each side of any line L passing through the origin.*

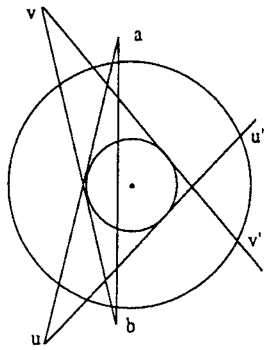

Fig. 2

Proof. Let p and q be the points where L crosses the boundary of P and assume there are two critical diagonals $a_1 b_1$ and $a_2 b_2$ on one side of pq (Fig. 3). The order of the points in question along the boundary of P is p, a_1, a_2, b_1, b_2, q ($p = a_1$ and $q = b_2$ are allowed). By Claim 3, each of $a_1 b_1$ and $a_2 b_2$ is critical with respect to the side not containing the origin. Then $b_1 a_2 \cap rB = \emptyset$ and we can apply Lemma 1 if the corresponding tangents $\overline{a_2 x}$ and $\overline{b_1 y}$ meet in int B. (x and y are chosen to be again on the boundary of B so that both cone$(\overrightarrow{b_1 y}, \overrightarrow{b_1 a_2})$ and cone$(\overrightarrow{a_2 x}, \overrightarrow{a_2 b_1})$ contain rB.) We compute angles:

$$\angle qox = 2\pi - (\angle qob_2 + \angle b_2 oa_2 + \angle a_2 ox) \leq 4\gamma$$

and

$$\angle poy = 2\pi - (\angle poa_1 + \angle a_1 ob_1 + \angle b_1 oy) \leq 4\gamma,$$

and so

$$\angle qox + \angle poy \leq 8\gamma < \pi,$$

consequently, the lines $\overline{a_2 x}$ and $\overline{b_1 y}$ meet in int B. □

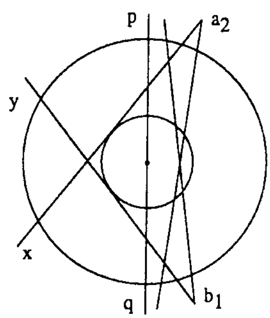

Fig. 3

Claim 5. *For any inner diagonal there is exactly one critical diagonal on the side not containing the origin.* (*The two diagonals may coincide.*)

Proof. Immediate from Claims 3 and 4. □

We say that two diagonals are *disjoint* if they have no *inner* point in common.

Claim 6. *To every critical diagonal ab there are at most two critical diagonals disjoint from it.*

Proof. If there were three such diagonals, then the three lines they span would intersect the half-lines $\overrightarrow{ab}$ and $\overrightarrow{ba}$ in two or three points (one can be parallel to ab). Then one of them, say $\overrightarrow{ab}$ is met by them in at most one point. We show that this is impossible.

Assume that the lines of two critical diagonals ce and df do not meet $\overrightarrow{ab}$ (as in Fig. 4). Then $de \cap rB = \varnothing$. Consider, again, the tangent half-lines $\overrightarrow{dx}$ and $\overrightarrow{ey}$ of rB, with x and y being the last points of the half-lines in B, and rB lying in the triangle determined by the lines $\overline{dx}$, $\overline{ey}$, $\overline{de}$. We show that these two half-lines meet inside B. Then Lemma 1 can be applied and Claim 6 follows. Let $a'b'$ and $a''b''$ be the two chords of B tangent to rB and parallel to ab on the critical and noncritical side, respectively, a' and a'' being closer to a than to b; further, let c', d', and e' be the last points of the half-lines $\overrightarrow{ec}$, $\overrightarrow{fd}$, and $\overrightarrow{ce}$ in B, respectively. Clearly, $c' \in \operatorname{arc}(a', a'')$ and $d' \in \operatorname{arc}(a', a'')$. This implies, on the one hand, $x \in \operatorname{arc}(s, b')$, and, on the other hand, $e' \in \operatorname{arc}(t, b') \Rightarrow y \in \operatorname{arc}(t', a')$, where $s\ (\neq b'')$, $t\ (\neq b')$, and $t'\ (\neq a')$ denote the last points in B of the half-lines $\overrightarrow{a''s}$, $\overrightarrow{a't}$, and $\overrightarrow{tt'}$, all tangent to rB. From the relations

$$\angle a'os = \pi - 4\gamma > \frac{\pi}{2} \quad \text{and} \quad \angle t'oa' = 4\gamma < \frac{\pi}{2}$$

our claim follows directly. □

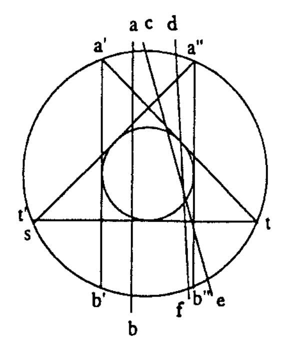

Fig. 4

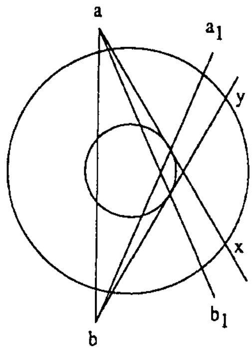

Fig. 5

Claim 7. *To every critical diagonal ab there are exactly two critical diagonals disjoint from it.*

Proof. By Claim 1 from each end of ab a further inner diagonal starts (necessarily into the noncritical side.) Let x and y be the last point within B of the half-lines starting from a and b, tangent to rB on the noncritical side of ab, respectively, further, let b_1 (a_1) be the "last" vertex for which ab_1 (ba_1) is an inner diagonal, i.e., $\operatorname{cone}(\overrightarrow{ba_1}, \overrightarrow{by})$ and $\operatorname{cone}(\overrightarrow{ab_1}, \overrightarrow{ax})$ contain no vertex besides a_1 and b_1, respectively (Fig. 5). By Remark 1 to both ab_1 and ba_1 there is a critical diagonal on the non-ab side. We show that they do not coincide. Since by Claim 2 int $\operatorname{cone}(\overrightarrow{ax}, \overrightarrow{by})$ is free from vertices, there is no vertex in int $\operatorname{cone}(\overrightarrow{ab_1}, \overrightarrow{ba_1})$. Thus a_1b_1 is an edge and by this the last possibility for a common critical diagonal is excluded. □

Remark 2. The proof shows that if there are two critical diagonals ce and df disjoint from ab, then none of them is parallel to ab, $\overline{ce}$ intersects one of the half-lines $\overrightarrow{ab}$ and $\overrightarrow{ba}$, and $\overline{df}$ intersects the other one.

We say that the two critical diagonals *intersect* if they are not disjoint.

Claim 8. *There are no three critical diagonals that pairwise intersect.*

Proof. Assume, to the contrary, that ab, cd, ef are pairwise intersecting critical diagonals, and let a, c, e, b, d, f be the order of these vertices along the perimeter of P. We assume, without loss of generality, that e is on the critical side of ab and b is on the critical side of ef. There are two cases to consider now:

Case 1: a and f are on the critical side of cd. Then, by Claim 3, the origin lies in the triangle determined by the lines $\overline{ab}$, $\overline{cd}$, $\overline{ef}$ and this triangle is contained in $\operatorname{conv}\{a, e, d\}$ (Fig. 6). However, ae, ed, da avoid rB so

$$rB \subset \operatorname{int} \operatorname{conv}\{a, e, d\},$$

contradicting (∗).

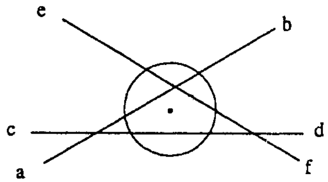

Fig. 6

Case 2: e and b are on the critical side of cd. Then $o \in \text{conv}\{a, e, b, f\}$ since otherwise we separate $\text{conv}\{a, e, b, f\}$ from o by a line L passing through the origin and observe that two (in fact, three) critical diagonals lie on one side of L, contradicting Claim 4. If $af \cap rB = \emptyset$, then

$$rB \subset \text{int conv}\{a, e, b, f\}$$

and we are finished. However, if $af \cap rB \neq \emptyset$, then there is a critical diagonal on the nonzero side of af (by Claim 5), and there are three critical diagonals on the zero side of that diagonal, contradicting Claim 7. □

Define now a graph Γ whose vertex set consists of the critical diagonals, and two of them are joined by an edge if the two critical diagonals are disjoint.

Claim 9. *Γ is a five-cycle.*

Proof. By Claim 7 every degree in Γ is two, thus Γ is the union of finitely many cycles.

A critical diagonal is seen from o at an angle at least $\pi - 2\gamma$. Then three pairwise disjoint diagonals would give a total of at least $3(\pi - 2\gamma) \geq 2\pi$, showing that there is no triangle C_3 among the cycles of Γ. A four-cycle C_4 in Γ corresponds to critical diagonals D_1, D_2, D_3, D_4 where D_1 intersects D_2 and D_3 intersects D_4 and the rest of the pairs are disjoint. Set $D_i = a_i b_i$. Then $\text{conv}\{a_1, b_1, a_2, b_2\}$ and $\text{conv}\{a_3, b_3, a_4, b_4\}$ can be separated by a line L, and the nonzero side of L would contain two critical diagonals which is not allowed by Claim 4. So there is no C_4 in Γ. A k-cycle with $k \geq 6$ would contain an independent set of three which is excluded by Claim 8. Thus Γ consists of five-cycles only. However, two distinct five-cycles would again contain an independent triple. Consequently, Γ is a single five-cycle. □

We are homing in on the target now. Let D_i, $i = 1, \ldots, 5$ ($D_{i+5} = D_i$), be the five critical diagonals, the consecutive pairs disjoint in cyclic order (Fig. 7). Set $D_i = a_i b_i$. D_2 is disjoint from D_1 and D_3 and both of them are on the noncritical side of D_2. In view of Remark 2 (following Claim 7) we can choose the notation so that $\overrightarrow{a_1 b_1}$ intersects $\overrightarrow{b_2 a_2}$, and $\overrightarrow{a_3 b_3}$ intersects $\overrightarrow{a_2 b_2}$. D_4 is disjoint from D_3 and intersects D_1 and D_2. Let b_4 (resp. a_4) be the critical side of $a_2 b_2$ ($a_1 b_1$). D_5 is

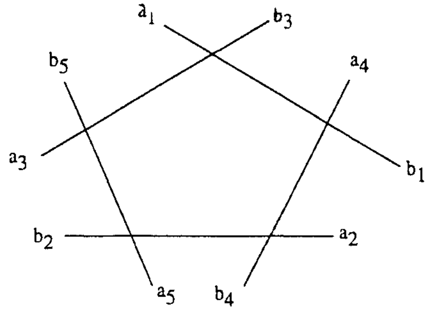

Fig. 7

disjoint from D_4 and D_1 and it intersects a_3b_3. Thus the order of these points on the boundary of P is $a_1, b_3, a_4, b_1, a_2, b_4, a_5, b_2, a_3, b_5$.

It follows from Remark 2 that $\overrightarrow{a_ib_i}$ intersects $\overrightarrow{b_{i+1}a_{i+1}}$. Let c_i denote the point of intersection. Notice that $c_i = b_i = a_{i+1}$ is possible. We show next that there is no vertex in the interior of the cone $(\overrightarrow{b_ia_i}, \overrightarrow{a_{i+2}b_{i+2}})$ $(i = 1, \ldots, 5)$. Assume, to the contrary, that v is a vertex in int cone $(\overrightarrow{b_1a_1}, \overrightarrow{a_3b_3})$, say. Then all four edges of $\text{conv}\{v, b_2, a_5, a_2\}$ are disjoint from rB and since a_2b_2 is an inner diagonal, $rB \subset \text{int conv}\{v, b_2, a_5, a_2\}$ contradicting $(*)$. This means that all vertices are contained in the star-pentagon $c_1c_2c_3c_4c_5$. Set $Q = \text{conv}\{c_1, \ldots, c_5\}$. Observe that

$$Q \supset P \supset B$$

and the first inclusion is strict unless $c_i = b_i = a_{i+1}$ for every $i = 1, \ldots, 5$. Then, by Theorem 3, either Q is a regular pentagon circumscribed around B or it has a diagonal not meeting rB. Since every diagonal of Q intersects rB, the first alternative holds. As B is the incircle of Q the inclusion $P \subset Q$ cannot be strict, consequently $P = Q$. □

4. Extensions

Sketch of the Proof of Theorem 1 when X is not Finite. Write $P = \text{conv } X$, $\bar{X} = \text{cl } X$, $\bar{P} = \text{cl conv } X = \text{cl } P$. There are two cases to consider.

Case 1: X is bounded. Then P is bounded, too, and $\bar{X}$, $\bar{P}$ are compact. It can be readily seen that the proof of Theorem 1 goes through (almost without change) for the compact set $\bar{X}$. Then either $\bar{P}$ is a regular pentagon circumscribed around B or there are points $y_1, y_2, y_3, y_4 \in X$ such that $rB \subset \text{int conv}\{y_1, \ldots, y_4\}$. As $y_i \in \bar{X}$, there are points in X arbitrarily close to y_i. So for suitable $x_1, x_2, x_3, x_4 \in X$ we have $rB \subset \text{int conv}\{x_1, \ldots, x_4\}$.

When $\bar{P}$ is a regular pentagon, then so is P as well but with some points on its boundary missing from P. A simple analysis using the fact that P contains the closed disk B finishes the proof.

Case 2: X is not bounded. We may assume that B is not contained in the convex hull of any bounded subset of X, i.e., for all $R > 0$,

$$B \not\subset \text{conv}(RB \cap X).$$

It can be shown, then, that as $R \to \infty$ the sets

$$\text{cl}(B \backslash \text{conv}(RB \cap X))$$

shrink to one or two points on the boundary of B. Let b be this point (or one of these points). Clearly, $b \in X$ must hold. Further, one of the half-lines, L, starting from b and tangent to B, has the property that X contains points arbitrarily close to L and arbitrarily far from b. Let $\overrightarrow{bx}$ be the half-line tangent to rB (so that $rB \subset \text{cone}(L, \overrightarrow{bx})$), where x is a point on the boundary of B. We show easily that $x \in \text{conv}\{a, b, c\}$ for some $a, c \in X$. Then, for a suitable $y \in X$ far from b but close to L, we get

$$rB \subset \text{int conv}\{a, b, c, y\}.$$

Sketch of the Proof of Theorem 2. Let X be finite and set $r = r_5^*$. The definition of a critical diagonal is the same as in Theorem 1. Let ab be a critical diagonal and let ac (resp. bd) be the last inner diagonal adjacent to a (b) and distinct from ab (see Claim 1 from the proof of Theorem 1). The lines $\overline{ac}$ and $\overline{bd}$ intersect outside B since otherwise $rB \subset \text{int conv}\{a, b, c, d, x\}$ for any $x \in X$ from the critical side of ab. Then the nonzero side of ac (bd) contains a critical diagonal a_1c_1 (b_1d_1). Thus the critical diagonals $D_1 = a_1b_1$, $D_2 = a_1c_1$, and $D_3 = b_1d_1$ are pairwise disjoint. We can find three pairwise disjoint critical diagonals E_1, E_2, and E_3 so that E_i is disjoint from D_j if and only if $i = j$. (We omit the details.) These six diagonals form a star-hexagon. The rest of the proof is analogous to that of Theorem 1.

Some related problems are discussed in [RZ].

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