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On the Expected Number of k-Sets*

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Abstract. Given a set S of n points in R^d, a subset X of size d is called a *k-simplex* if the hyperplane aff(X) has exactly k points on one side. We study $E_d(k, n)$, the expected number of k-simplices when S is a random sample of n points from a probability distribution P on R^d. When P is spherically symmetric we prove that $E_d(k, n) \leq cn^{d-1}$. When P is uniform on a convex body $K \subset R^2$ we prove that $E_2(k, n)$ is asymptotically linear in the range $cn \leq k \leq n/2$ and when k is constant it is asymptotically the expected number of vertices on the convex hull of S. Finally, we construct a distribution P on R^2 for which $E_2((n-2)/2, n)$ is $cn \log n$.

1. Introduction and Summary

Let X be a set of n points in R^d in general position. The simplex conv(S) (when $S \subset X$ and $|S| = d$) is called a *k-simplex* if X has exactly k points on one side of the hyperplane aff(S). A k-simplex is an (n-d-k)-simplex as well. Although this should not cause any confusion we always try to have $k \leq n - d - k$. In two dimensions a k-simplex is called a *k-segment.*

Write $e_d(k, n)$ for the maximal number of k-simplices over all configurations X of n points in R^d. Most of the previous work has focused on $e_d(k, n)$ because of its connection with *k-sets.* A subset $Y \subset X$ of size k is called a k-set if Y and $X \setminus Y$ are separated by a hyperplane. The question is: how many k-sets may a set X

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possess? It is easy to translate an upper bound for $e_d(k, n)$ into an upper bound on k-sets.

Clearly, $O(n^d)$ provides a trivial upper bound for $e_d(k, n)$. When $d = 2$, nontrivial bounds were obtained by Lovász [15] for halving sets (n even, $k = n/2$), and later, for general $k \leq n/2$, by Erdős *et al.* [12]. A simple construction gives a set S with $n \log k$ k-sets, while a counting argument shows that $e_2(k, n) = O(n\sqrt{k})$. These bounds were rediscovered several times, for example by Edelsbrunner and Welzl [11], but had not been improved until Pach *et al.* [17] reduced the bound to $n\sqrt{k}/\log^* k$. Papers [1], [13], and [22] contain results related to the study of $e_2(k, n)$.

Raimund Seidel (see [10]) extended the Lovász lower bound construction to $d = 3$ and showed that $e_3(k, n) = \Omega(nk \log(k + 1))$. The argument may be applied inductively giving $e_d(k, n) = \Omega(nk^{d-2} \log(k))$.

A nontrivial upper bound for $d = 3$ was recently obtained by Bárány *et al.* [5]. They showed that $e_3(n/2, n) = n^{3-\varepsilon}$, where $\varepsilon > 0$ is some small constant. This, in turn, was improved by Aronov *et al.* [2] to $O(n^{8/3} \log^{5/3} n)$. Dey and Edelsbrunner [9] have been able to remove the logarithmic factors from this bound. Recently, a nontrivial upper bound for $d > 3$ was established via a result of Živaljević and Vrećica [23]. They proved a colored version of Tverberg's theorem which now implies that $O(n^{d-\varepsilon_d})$ is an upper bound for halving sets in R^d, $\varepsilon_d > 0$ being a small constant depending on d.

It appears likely that the truth is near the lower bound. Support comes from the fact that in "typical" cases there are relatively few k-sets. In this paper we study $E_d(k, n)$, the expected number of k-simplices when X is a sample of n random points from a probability measure P on R^d. When there is no confusion we write $E(k, n)$. The following derivation gives an expression for $E_d(k, n)$ that we use throughout. Pick d points $x_1, \ldots, x_d$ independently, according to P. Write l for the hyperplane $\text{aff}(x_1, \ldots, x_d)$. We assume throughout that P vanishes on every hyperplane so l is well defined with probability one. (In particular, P is nonatomic.) Write l^+ and l^- for the open half-spaces on the right and left of l, respectively, and set $F(l) = \min(P(l^+), P(l^-))$, the probability content cut off by l. The random variable $F(l)$ has a distribution function

$$G(t) = P(F(l) \leq t) \tag{1}$$

which determines $E_d(k, n)$ in the following way. Given a sample $X = \{x_1, \ldots, x_n\}$ from P, the expected number of k-simplices is

$$\begin{aligned} E_d(k, n) &= \sum_{1 \leq i_1 < \cdots < i_d \leq n} \text{Prob[there are } k \text{ points on one side of } \text{aff}(x_{i_1}, \ldots, x_{i_d})] \\ &= \binom{n}{d}\binom{n-d}{k} \int_0^{1/2} [t^k(1-t)^{n-d-k} + (1-t)^k t^{n-d-k}]\, dG(t). \end{aligned} \tag{2}$$

Our first result is a simple one about spherically symmetric distributions (the definition is given in Section 2).

Theorem 1. *For a spherically symmetric distribution we have $E_d(k, n) \le c_1 n^{d-1}$, c_1 being a constant depending only on d.*

Next we deal with the case where P is the uniform distribution on a compact, convex body $K \subset R^2$. We assume that $\text{Area}(K) = 1$ so that P coincides with the restriction of Lebesgue measure to K. Define $v: K \mapsto R$ by

$$v(x) = \inf\{\text{Area}(K \cap H): x \in H, H \text{ is a half-plane}\} \tag{3}$$

and $A(t) = A_K(t) = \text{Area}\{x \in K: v(x) \le t\}$. The properties of v and $A(t)$ have been studied in [3], [6], and [21]. Here we prove that G is differentiable and that $G'(t)$ is essentially equal to $A(t)$. This helps establish the following theorem.

Theorem 2. *There are absolute constants c_2 and c_3 such that, for the uniform distribution over any convex set in the plane,*

$$c_2 n A\left(\frac{k+1}{n}\right) \le E_2(k, n) \le c_3 n A\left(\frac{k+1}{n}\right) \tag{4}$$

for every sufficiently large n and every $k = 0, 1, \ldots, \lfloor (n-2)/2 \rfloor$.

Sometimes we express the relation in (4) as

$$E_2(k, n) \sim n A\left(\frac{k+1}{n}\right).$$

Remark 1. Since $t \le A(t)$, we have $c_4 \le A(t) \le 1$ when $t \ge c_4 > 0$. Theorem 2 then shows that when $\frac{1}{2} \ge k/n \ge c_4$,

$$E_2(k, n) \sim n.$$

The behavior of $A(t)$ (for small t) is given by Theorem 7 of [6]

$$c_5 t \log \frac{1}{t} \le A(t) \le c_6 t^{2/3}. \tag{5}$$

Schütt and Werner [21] show that for a function $f(t)$ with $c_5 t \log(1/t) \le f(t) \le c_6 t^{2/3}$ (and some additional properties) there is a convex set $K = K_f \subset R^2$ of area 1 such that $A_K(t) \sim f(t)$. This shows that not only does

$$c_5(k+1) \log \frac{n}{k+1} \le E_2(k, n) \le c_6 n \left(\frac{k+1}{n}\right)^{2/3}$$

hold for P uniform on a convex body, but also, for (almost) any function between these bounds, there is a convex body K with $E_2(k, n)$ behaving like that function.

The special case $k = 0$ is interesting. Then $E_2(k, n)$ equals the expected number of edges of conv(X), which was known to behave like $A(1/n)$ (see [6]). So Theorem 2 says that $E_2(k, n)$ behaves like the expected number of edges of conv(X) when k is a constant, and like n when $k/n \geq t_0$.

Finally, we give an example of a distribution for which $E_2(k, n)$ is large. We consider the case $k = (n-2)/2$ (n even), that is, the expected number of halving segments. We give a distribution P_n such that

$$E_2\left(\frac{m-2}{2}, m\right) \geq c_7 m \log m$$

whenever the sample size m is within a constant factor of n. Then, using P_n, we describe a distribution P for which

$$E_2\left(\frac{n-2}{2}, n\right) \geq c_8 n \log n.$$

Finally, we point out the abstract of [7], where one of the present results was announced, but with an erroneous proof. This is one of the reasons we take some care in establishing the simple statements about E_d. The methods are familiar in geometric probability and integral geometry (see [4], [16], and [19]). Nevertheless, the results seem to be the first ones concerning E_d and in view of the fact that k-sets have applications in computational geometry and machine learning [14], [18], we feel that these theorems are useful and interesting.

2. Spherically Symmetric Continuous Distributions

Suppose that P has a density function $g: R^d \to R$ that only depends on $|x|$, the distance from $x \in R^d$ to the origin. We say that such a P is spherically symmetric. This defines another function $f: R_+ \to R$ by $f(r) = g(|x|)$ when $r = |x|$.

Proof of Theorem 1. Set $\kappa_{d-1} = vol_{d-1}(S^{d-1})$. Clearly,

$$1 = \int_{R^d} g(x)\, dx = \int_{u \in S^{d-1}} \int_{r=0}^{\infty} f(r) r^{d-1}\, dr\, du = \kappa_{d-1} \int_0^{\infty} f(r) r^{d-1}\, dr. \tag{6}$$

Now let $H(t)$ be an open half-space with probability content t, $0 < t \le \frac{1}{2}$, and write $p = p(t)$ for the distance (from the origin) to Π, the bounding hyperplane of $H(t)$. Then

$$t = \int_{H(t)} g(x)\, dx = \int_{r=p}^{\infty} \int_{y \in R^{d-1}} f(\sqrt{r^2 + |y|^2})\, dy\, dr, \tag{7}$$

where r is the length of the component of x parallel to u and $y = x - ur$, u denoting the unit normal to Π.

Claim 1. $G(t + \Delta t) - G(t) \le c_9 \Delta t$.

Theorem 1 follows immediately because, from (2),

$$\begin{aligned} E_d(k, n) &\le \binom{n}{d}\binom{n-d}{k} \int_0^{1/2} [t^k(1-t)^{n-d-k} + (1-t)^k t^{n-d-k}] c_9\, dt \\ &\le c_{10} n^{d-1}; \end{aligned} \tag{8}$$

the last inequality is a consequence of the well-known fact that

$$(m+1)\binom{m}{j} \int_0^1 t^j (1-t)^{m-j}\, dt = 1. \tag{9}$$

Proof of Claim 1. We use the Blaschke–Petkantschin formula (see p. 201 of [20]) which says that

$$dx_1 \cdots dx_d = d!\, vol_{d-1}(\mathrm{conv}\{y_1, \ldots, y_d\})\, dy_1 \cdots dy_d\, du\, dr,$$

where the points $x_1, \ldots, x_d$ lie in the hyperplane $ux = r$ with $u \in S^{d-1}$, the unit sphere in R^d and $r > 0$, and $y_i = x_i - ru$. With this fact,

$$\begin{aligned} G(t + \Delta t) - G(t) &= \int \cdots \int_{t < F(l) \le t + \Delta t} g(x_1) \cdots g(x_d)\, dx_1 \cdots dx_d \\ &= \int_{r = p(t+\Delta t)}^{p(t)} \int_{u \in S^{d-1}} \int_{y_1} \cdots \int_{y_d} f(\sqrt{r^2 + |y_1|^2}) \cdots f(\sqrt{r^2 + |y_d|^2}) \\ &\quad \times d!\, vol_{d-1}(\mathrm{conv}\{y_1, \ldots, y_d\})\, dy_1 \cdots dy_d\, du\, dr \\ &= d!\, \kappa_{d-1} \int_{p(t+\Delta t)}^{p(t)} \int_{R^{d-1}} \cdots \int_{R^{d-1}} f(\sqrt{r^2 + |y_1|^2}) \cdots f(\sqrt{r^2 + |y_d|^2}) \\ &\quad \times vol_{d-1}(\mathrm{conv}\{y_1, \ldots, y_d\})\, dy_1 \cdots dy_d\, dr. \end{aligned}$$

Notice that the innermost d integrals here denote the expectation of the volume of $\text{conv}\{y_1, \ldots, y_d\}$ when the points $y_1, \ldots, y_d$ are distributed on the hyperplane $H = \{x: ux = r\}$ according to density $f(\sqrt{r^2 + |y|^2})$. This is, again, a spherically symmetric distribution in the hyperplane H with center ru which we take for the origin of H and denote by $\bar{0}$. The signed volume of $\text{conv}\{y_1, \ldots, y_d\}$ is

$$\frac{1}{(d-1)!}\det\begin{pmatrix} 1 & \cdots & 1 \\ y_1 & \cdots & y_d \end{pmatrix}$$

but

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ y_1 & \cdots & y_d \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & y_2 & \cdots & y_d \end{pmatrix} + \cdots + \det\begin{pmatrix} 1 & \cdots & 1 & 1 \\ y_1 & \cdots & y_{d-1} & 0 \end{pmatrix}.$$

Consequently, with unsigned volumes,

$$vol(\text{conv}\{y_1, \ldots, y_d\}) \le \sum_{i=1}^{d} vol(\text{conv}\{\{y_1, \ldots, y_d, \bar{0}\} \backslash \{y_i\}\}).$$

Since every term on the right-hand side has the same expectation,

$$E[vol(\text{conv}\{y_1, \ldots, y_d\})] \le dE[vol(\text{conv}\{\bar{0}, y_1, \ldots, y_{d-1}\})].$$

Moreover,

$$vol(\text{conv}\{\bar{0}, y_1, \ldots, y_{d-1}\}) = \frac{1}{(d-1)!}|\det(y_1, \ldots, y_{d-1})| \le \frac{1}{(d-1)!}\prod_{i=1}^{d-1}|y_i|$$

by Hadamard's inequality. This way we get

$$G(t + \Delta t) - G(t) \le d^2 \kappa_{d-1} \int_{r=p(t+\Delta t)}^{p(t)} \left[\prod_{i=1}^{d-1} \int_{y_i} f(\sqrt{r^2 + |y_i|^2})|y_i|\, dy_i\right]$$
$$\times \int_{y_d} f(\sqrt{r^2 + |y_d|^2})\, dy_d\, dr.$$

From (6),

$$\begin{aligned}\int_{R^{d-1}} f(\sqrt{r^2+y_i^2})|y_i|\,dy_i &= \int_{u\in S^{d-2}}\int_{s=0}^{\infty} f(\sqrt{r^2+s^2})s\cdot s^{d-2}\,ds\,du \\ &= \kappa_{d-2}\int_{s=0}^{\infty} f(\sqrt{r^2+s^2})s^{d-1}\,ds \\ &= \kappa_{d-2}\int_{q=r}^{\infty} f(q)(q^2-r^2)^{(d-1)/2}\frac{q\,dq}{\sqrt{q^2-r^2}} \\ &\le \kappa_{d-2}\int_{q=r}^{\infty} f(q)q^{d-1}\,dq \le \frac{\kappa_{d-2}}{\kappa_{d-1}}.\end{aligned}$$

By (7)

$$\int_{r=p(t+\Delta t)}^{p(t)}\int_{R^{d-1}} f(\sqrt{r^2+|y_d|^2})\,dy_d\,dr = \Delta t$$

and therefore

$$G(t+\Delta t)-G(t) \le d^2\kappa_{d-2}\left(\frac{\kappa_{d-2}}{\kappa_{d-1}}\right)^{d-1}\Delta t. \qquad \square$$

3. Uniform Distribution on a Convex Set

Let $K \subset R^2$ be a convex set with Area$(K) = 1$. We are interested in $E_2(k, n)$ when P is the Lebesgue measure restricted to K. Since E_2 is invariant under (non-degenerate) affine transformations of K we may assume that K is in "normal position," i.e., that

$$rB^2 \subset K \subset 2rB^2,$$

where B^2 is the unit disk, centered at the origin, and r is a universal constant (in fact $r = 3^{-3/4}$, but we do not need this precision). The existence of the "normal position" follows from that of the Löwner–John ellipsoid [8].

It is more convenient to work with the directed version of (2). So let $l = \overrightarrow{xy}$ denote the line directed from x to y. Write $F(l)$ for the probability content of the half-plane l^+ on the right of l; this is equal to the area of $K \cap l^+$. Set $G(t) = \text{Prob}[F(l) \le t]$. Then (2) becomes

$$E_2(k, n) = \binom{n}{2}\binom{n-2}{k}\int_0^1 t^k(1-t)^{n-2-k}\,dG(t). \tag{10}$$

We need some further notation. Given $\varphi \in [0, 2\pi]$ and $t \in (0, 1)$ there is a unique directed line $l(\varphi, t)$ with direction φ that has $F(l) = t$. $l(\varphi, t)$ is clearly continuous in both variables and can be extended to $t = 0$ and $t = 1$. In that case $l(\varphi, 0)$, say, denotes the directed line that has K on its left and supports K. The line $l(\varphi, t)$ has signed distance $p(\varphi, t)$ from the origin so that $p(\varphi, t) \geq 0$ if the origin is on the left of $l(\varphi, t)$ (or on $l(\varphi, t)$ itself) and $p(\varphi, t) < 0$ otherwise. Also, for any $\varphi \in [0, 2\pi]$, $p(\varphi, t) \in [p(\varphi, 1), p(\varphi, 0)]$.

The directed line having direction φ and at signed distance p from the origin cuts K into two parts of area $t(\varphi, p)$ on its right and $1 - t(\varphi, p)$ on its left. Define $\psi(\varphi, t)$ as the length of the chord $l(\varphi, t) \cap K$. Clearly, $p = p(\varphi, t(\varphi, p))$ identically. It is evident that $p \mapsto \psi(\varphi, t(\varphi, p))$ is a concave function on $[p(\varphi, 1), p(\varphi, 0)]$.

Recall the definition of $v(x)$ from (3). We write $K(t) = \{x \in K: v(x) \leq t\}$ and $A(t)$ for its area.

Theorem 3. *$G(t)$ is differentiable when $t \in (0, 1)$ and*

$$G'(t) = \tfrac{1}{6} \int_0^{2\pi} \psi^2(\varphi, t)\, d\varphi.$$

Theorem 4. *As $t \to 0$,*

$$G'(t) = \tfrac{4}{3} A(t)(1 + o(1)).$$

We mention here that $G(t) \sim tA(t)$ is proved in [3]. Theorems 3 and 4 establish a different and apparently more subtle property of the function G. We need:

Lemma 1. *For each $t \in (0, 1)$ there is a constant C_t such that*

$$|\psi(\varphi, t) - \psi(\varphi, u)| \leq C_t |t - u|$$

for all $\varphi \in [0, 2\pi]$ and $u \in [0, 1]$. In fact, $C_t = 8r/\min(t, 1 - t)$.

The proof is straightforward using the normal position and the following easy facts (refer to Fig. 1):

1. The chord function $p \mapsto \psi(\varphi, t(\varphi, p))$ is concave in p.
2. For all $s \in (0, 1)$, $4r(p(\varphi, 0) - p(\varphi, s)) \geq s$ and $4r(p(\varphi, s) - p(\varphi, 1)) \geq 1 - s$.
3. $\psi(\varphi, s)(p(\varphi, t) - p(\varphi, u)) \leq 2(u - t)$ if $s = u$ or $s = t$.

We omit the details.

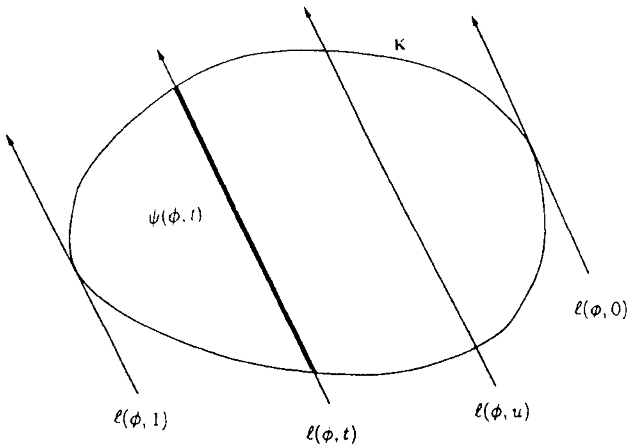


Fig. 1. The chord function $\psi(\varphi, t) = |l(\varphi, t) \cap K|$.

Proof of Theorem 3. Write $l = \overrightarrow{xy}$. By the Blaschke–Petkantschin formula,

$$\begin{aligned} G(t + \Delta t) - G(t) &= \operatorname{Prob}[F(l) \in [t, t + \Delta t)] \\ &= \int_{t \le F(\overrightarrow{xy}) < t + \Delta t} \int 1 \, dx \, dy \\ &= \int_0^{2\pi} \int_{p(\varphi, t+\Delta t)}^{p(\varphi, t)} \iint_{\bar{x} < \bar{y} \in K \cap l} |\bar{x} - \bar{y}| \, d\bar{x} \, d\bar{y} \, dp \, d\varphi. \end{aligned}$$

An elementary computation reveals that

$$\iint_{\bar{x} < \bar{y} \in K \cap l} |\bar{x} - \bar{y}| \, d\bar{x} \, d\bar{y} = \tfrac{1}{6} \chi^3(l),$$

where $\chi(l) = \psi(\varphi, t(\varphi, p))$ is the length of the chord $K \cap l$. So

$$\begin{aligned} G(t + \Delta t) - G(t) &= \tfrac{1}{6} \int_0^{2\pi} \int_{p(\varphi, t+\Delta t)}^{p(\varphi, t)} \psi^3(\varphi, t(\varphi, p)) \, dp \, d\varphi \\ &= \tfrac{1}{6} \int_0^{2\pi} \psi^2(\varphi, t) \int_{p(\varphi, t+\Delta t)}^{p(\varphi, t)} \psi(\varphi, t(\varphi, p)) \, dp \, d\varphi \\ &\quad + \tfrac{1}{6} \int_0^{2\pi} \int_{p(\varphi, t+\Delta t)}^{p(\varphi, t)} [\psi^2(\varphi, t(\varphi, p)) - \psi^2(\varphi, t)] \psi(\varphi, t(\varphi, p)) \, dp \, d\varphi. \end{aligned}$$

The first term here equals $\frac{1}{6}\int_0^{2\pi} \psi^2(\varphi, t)\, d\varphi\, \Delta t$ since trivially (again, see Fig. 1)

$$\Delta t = \int_{p(\psi, t+\Delta t)}^{p(\psi, t)} \psi(\varphi, t(\varphi, p))\, dp. \tag{11}$$

Therefore

$$\begin{aligned}
&\left| \frac{G(t+\Delta t) - G(t)}{\Delta t} - \frac{1}{6}\int_0^{2\pi} \psi^2(\varphi, t)\, d\varphi \right| \\
&\quad \le \frac{1}{6\Delta t}\int_0^{2\pi}\int_{p(\varphi, t+\Delta t)}^{p(\varphi, t)} |\psi^2(\varphi, t(\varphi, p)) - \psi^2(\varphi, t)|\psi(\varphi, t(\varphi, p))\, dp\, d\varphi \\
&\quad \le \frac{1}{6\Delta t}\int_0^{2\pi}\int_{p(\varphi, t+\Delta t)}^{p(\varphi, t)} 8rC_t\Delta t\psi(\varphi, t(\varphi, p))\, dp\, d\varphi \\
&\quad = \frac{8r\pi}{3} C_t \Delta t,
\end{aligned}$$

where we used Lemma 1 in the last inequality and (11) in the last equality. □

Remark 2. We point out that, for $\frac{1}{2} \ge t \ge t_0 > 0$,

$$c_{11} \le G'(t) \le c_{12}. \tag{12}$$

The upper bound is trivial from Theorem 3 because ψ is bounded. For the lower bound it is enough to see that $\psi(\varphi, t) \ge c_{13}t$. This follows easily from the normal position of K.

Before the proof of Theorem 4 we need some preparation. The body

$$K(v \ge t) = \{x \in K\colon v(x) \ge t\}$$

is clearly convex. We assume $t \le t_0 \le 0.01$, say, and then $K(v \ge t)$ is nonempty as well. Thus the boundary of $K(v \ge t)$ is a convex curve $V(t)$ with left and right tangents at every $z \in V(t)$. These tangents coincide at all but countably many $z \in V(t)$.

Fix $t \in (0, t_0]$. Given $\varphi \in [0, 2\pi)$ let $\lambda(\varphi, t)$ be the unique directed line (with direction φ) that is a supporting line to $K(v \ge t)$ and has $K(v \ge t)$ on its left. $\lambda(\varphi, t)$ has exactly one point (to be denoted by $z(\varphi, t)$) in common with $K(v \ge t)$ since, as is proved in [3], $V(t)$ contains no line segment. Call the angle φ *regular* if $\lambda(\varphi, t)$ is tangent (left, right, or both) to the curve $V(t)$ at $z(\varphi, t)$. Write R for the set of regular angles in $[0, 2\pi]$ and NR for its complement. It is not difficult to see that R is a closed set. Therefore NR is a countable union of open intervals; the point in the proof of Theorem 4 is that the total length of these intervals is $O(t)$.

Recall that $l(\varphi, t)$ is a directed line that cuts off area t from K. It follows from the proof of Lemma G in [3] that if φ is regular, then $\lambda(\varphi, t)$ and $l(\varphi, t)$ coincide and $z(\varphi, t)$ is the midpoint of the chord $K \cap l(\varphi, t)$. Finally, let $L(\varphi, t)$ be the length of the segment connecting $z(\varphi, t)$ to the last point on $\lambda(\varphi, t)$ in K. Observe that, for a regular angle, $L(\varphi, t) = \frac{1}{2}\psi(\varphi, t)$.

We omit the simple proof of the following.

Claim 2. *Area*$(K(v \geq t)) = \frac{1}{2}\int_0^{2\pi} L^2(\varphi, t)\, d\varphi$.

Lemma 2. *The total length of the intervals in NR is $O(t)$.*

Proof. Assume that φ is nonregular and let φ^+ and φ^- be the direction of the left and right tangents l^+ and l^- to $V(t)$ at $z(\varphi, t)$. Since $\varphi^+(\varphi^-)$ are regular, $z(\varphi, t)$ is the midpoint of the corresponding chords which we denote by u^+v^+ and u^-v^-, as is shown in Fig. 2. Then u^-u^+ and v^-v^+ span parallel lines. Let S be the strip between them. Clearly,

$$\text{Area}(rB^2 \backslash S) \leq \text{Area}(K \backslash S) \leq 2t.$$

An elementary computation reveals that the width of S is at least $2r - (t^2/(2r))^{1/3} > 0.8$, so

$$\min(|u^- - v^-|, |u^+ - v^+|) \geq 0.8. \tag{13}$$

Moreover,

$$1 = \text{Area}(K) \leq 2t + \tfrac{1}{2}\,\text{diam}^2(K)(\pi - (\varphi^+ - \varphi^-)),$$

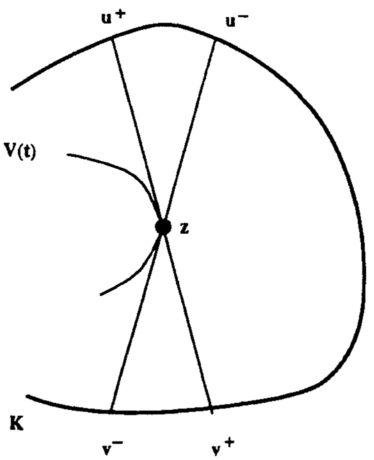


Fig. 2. Tangents at a nonregular point $z = z(\varphi, t)$ on $V(t)$.

because both lines l^+ and l^- cut off a cap from K of area t, and the remainder is contained in a circular sector with center $z(\varphi, t)$, radius equal to diam$(K) \leq 4r$, and angle $\pi - (\varphi^+ - \varphi^-)$. Consequently (see Fig. 2),

$$\varphi^+ - \varphi^- \leq \pi - \frac{1-2t}{8r^2},$$

so $\varphi^+ - \varphi^-$ is separated from π. On the other hand,

$$\begin{aligned} t \geq \text{Area}(\text{conv}\{u^-, u^+, v^-\}) &= \tfrac{1}{2}|u^+ - v^+|\tfrac{1}{2}|u^- - v^-| \sin(\varphi^+ - \varphi^-) \\ &\geq \tfrac{1}{4}(0.8)^2 \sin(\varphi^+ - \varphi^-) > 0.16 \sin(\varphi^+ - \varphi^-) \end{aligned}$$

proving that

$$\varphi^+ - \varphi^- \leq 8t. \tag{14}$$

This shows that NR contains only "short" intervals.

Because $t < 0.01$ a smaller disk, $0.8rB^2$, is contained in $K(v \geq t)$. Call the points $z(\varphi, t)$ nonregular if $\varphi \in NR$, and the other points of $V(t)$ regular. Observe that there are only countably many nonregular points, each one corresponding to an interval from NR. Choose a regular point $z(\varphi_1, t)$ and take φ_1 to be 0. We are going to construct, by induction, a sequence of regular points $z_1 = z(\varphi_1, t), \ldots, z_m = z(\varphi_m, t)$ with $\varphi_1 < \varphi_2 < \cdots < \varphi_m < 2\pi$. Assume $\varphi_1, \ldots, \varphi_i$ have already been constructed. Pick a regular point $z = z(\varphi, t)$ with $\varphi > \varphi_i$ so that $|z - z_i| \in [0.19, 0.20]$. Such a point clearly exists. Further, it can be chosen so that $\varphi - \varphi_i \leq \frac{3}{4}\pi$, as can easily be seen from $0.8rB^2 \subset K(v \geq t)$. Now if $\varphi - \varphi_i \leq \pi/2$, then we define $\varphi_{i+1} = \varphi$ and $z_{i+1} = z$. However, if not, then define φ_{i+1} to be a regular angle very close to $(\varphi + \varphi_i)/2$ and set $\varphi_{i+2} = \varphi$. Since the intervals in NR are shorter than $8t \leq 0.08$ we have $\varphi_{i+1} - \varphi_i \leq \pi/2$ and $\varphi_{i+2} - \varphi_{i+1} \leq \pi/2$. We stop when the next φ, φ_{m+1} is larger than 2π.

It is easy to see now that $m \leq 35$. Indeed, $\varphi - \varphi_i > \pi/2$ can happen at most three times, and in the other cases $|z_{i+1} - z_i| \geq 0.19$. As the perimeter of $V(t)$ is at most $4\pi r$ we get $m \leq 3 + (4\pi r)/0.19 \leq 35$.

Consider now the counterclockwise arc A_i connecting z_i to z_{i+1} on $V(t)$. Let w_1 and w_2 be two nonregular points on A_i, w_1 having the left tangent direction ψ_1 and w_2 having the right tangent direction ψ_2, with $0 < \psi_2 - \psi_1$. The inequality $\psi_2 - \psi_1 < \pi/2$ is automatically satisfied since, by the construction, $\varphi_{i+1} - \varphi_i \leq \pi/2$.

Claim 3. $\psi_2 - \psi_1 < 16t$.

Proof (see Fig. 3). If $w_1 = w_2$, then this follows from (14). Otherwise, let w be the intersection of the tangents at w_1 and w_2. Observe that the angle $w_1 w w_2$ is at least $\pi/2$ (since $\psi_2 - \psi_1 \leq \pi/2$), so

$$|w - w_2| \leq |w_1 - w_2| \leq |z_i - z_{i+1}| \leq 0.20.$$

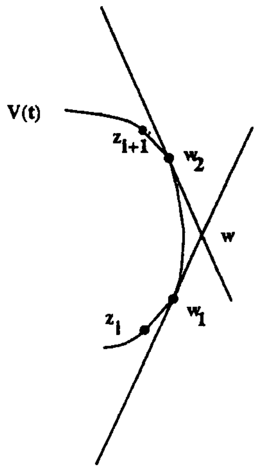


Fig. 3. Left and right tangents at nonregular points w_1 and w_2.

Again, we used $\varphi_{i+1} - \varphi_i \leq \pi/2$. Moreover, writing $u_2 v_2$ for the tangent chord in direction ψ_2 and $u_1 v_1$ for the other tangent chord,

$$\begin{aligned} t \geq \text{Area}(\text{conv}\{u_1, u_2, v_1\}) &= \tfrac{1}{2}|w - u_2|\,|u_1 - v_1| \sin(\psi_2 - \psi_1) \\ &\geq \tfrac{1}{2}(\tfrac{1}{2}|v_2 - u_2| - 0.20)|u_1 - v_1| \sin(\psi_2 - \psi_1) \\ &\geq \tfrac{1}{2}(0.20)(0.8) \sin(\psi_2 - \psi_1) = 0.08 \sin(\psi_2 - \psi_1), \end{aligned}$$

where we used (13) as well. □

Proof of Theorem 4. For a regular direction, $L(\varphi, t) = \frac{1}{2}\psi(\varphi, t)$. Then

$$\begin{aligned} G'(t) &= \tfrac{1}{6} \int_0^{2\pi} \psi^2(\varphi, t)\, d\varphi \\ &= \tfrac{1}{6}\left[\int_R \psi^2(\varphi, t)\, d\varphi + \int_{NR} \psi^2(\varphi, t)\, d\varphi\right] \\ &= \tfrac{1}{6}\left[\int_0^{2\pi} 4L^2(\varphi, t)\, d\varphi + \int_{NR} (\psi^2(\varphi, t) - 4L^2(\varphi, t))\, d\varphi\right] \\ &= \tfrac{4}{3}A(t) + \tfrac{1}{6}\int_{NR} (\psi^2(\varphi, t) - 4L^2(\varphi, t))\, d\varphi. \end{aligned}$$

If all directions are regular, we are finished. Otherwise

$$\begin{aligned} |G'(t) - \tfrac{4}{3}A(t)| &\leq \tfrac{1}{6}\int_{NR} |\psi^2(\varphi, t) - 4L^2(\varphi, t)|\, d\varphi \\ &\leq \frac{16r^2}{6}\, \text{meas}(NR) \leq c_{14} t. \end{aligned}$$

According to [6], $A(t) \geq ct \log(1/t)$ for some absolute constant c, so we get

$$G'(t) = \tfrac{4}{3}A(t)\left(1 + O\left(\frac{1}{\log(1/t)}\right)\right). \qquad \square$$

Theorem 2 follows easily from Theorems 3 and 4 using some properties of $A(t)$, namely:

1. $1 \geq A(t) \geq 0$ and $A(t)$ is monotone increasing.
2. $A(\alpha t) \leq c_{15}\alpha^2 A(t)$, if $\alpha \geq 1$ and $t > 0$ (see [6]).

Proof of Theorem 2. When $k = 0$, $E_2(k, n)$ is the expected number of edges (or vertices) of the convex hull and this case is covered in [6]. So assume $k \geq 1$. It follows from properties 1 and 2 above that $A((k+1)/n) \sim A(k/(n-2))$. We write $m = n - 2$ to simplify the notation. By Theorem 3

$$E_2(k, m+2) = \binom{m+2}{2}\binom{m}{k}\int_0^1 t^k(1-t)^{m-k}G'(t)\,dt.$$

It follows easily from Theorem 4 and the properties of $G'(t)$ and $A(t)$ that

$$\int_0^1 t^k(1-t)^{m-k}G'(t)\,dt \sim \int_0^1 t^k(1-t)^{m-k}A(t)\,dt.$$

Therefore it is enough to show that, for all $k = 1, \ldots, \lfloor m/2 \rfloor$,

$$A\left(\frac{k}{m}\right) \sim (m+1)\binom{m}{k}\int_0^1 t^k(1-t)^{m-k}A(t)\,dt.$$

Write $I(m, k)$ for the expression on the right. $I(m, k)$ would decrease if we only integrated on the interval $[k/m, 1]$, and, since $A(t)$ is increasing, it would decrease further if we replaced $A(t)$ by $A(k/m)$. This shows that

$$I(m,k) \geq A\left(\frac{k}{m}\right)(m+1)\binom{m}{k}\int_{k/m}^1 t^k(1-t)^{m-k}\,dt \geq c_2 A\left(\frac{k}{m}\right).$$

For the last inequality it should be proved that

$$(m+1)\binom{m}{k}\int_{k/m}^1 t^k(1-t)^{m-k}\,dt \geq c_2 > 0$$

for all $k = 1, \ldots, \lfloor m/2 \rfloor$ and for all large enough m. This can be done as follows.

The integrand is maximal at $t = k/m$ and decreases on $[k/m, 1]$. So, for any $T \in [k/m, 1]$,

$$\int_{k/m}^{1} t^k(1-t)^{m-k}\,dt \ge \left(T - \frac{k}{m}\right)T^k(1-T)^{m-k}.$$

Choosing $T = (k + \sqrt{k})/m$ gives a good lower bound for the integral. We omit the technical computations.

For the other inequality we observe that $A(t) \le A(k/m)$ when $t \le k/m$. From property 2 above,

$$A(t) \le c_{15}\left(\frac{tm}{k}\right)^2 A\left(\frac{k}{m}\right),$$

when $t \ge k/m$. This gives

$$\begin{aligned} I(m, k) &\le (m+1)\binom{m}{k}A\left(\frac{k}{m}\right)\left[\int_0^{k/m} t^k(1-t)^{m-k}\,dt + \int_{k/m}^1 t^k(1-t)^{m-k}c_{15}\left(\frac{tm}{k}\right)^2 dt\right] \\ &\le A\left(\frac{k}{m}\right)(m+1)\binom{m}{k}\left[\int_0^1 t^k(1-t)^{m-k}\,dt + c_{15}\frac{m^2}{k^2}\int_0^1 t^{k+2}(1-t)^{m-k}\,dt\right] \end{aligned}$$

and this is less than $c_3 A(k/m)$ by (9). □

3.1. *Higher Dimensions*

We mention a possible generalization to the case $d > 2$. In this case define

$$G(t) = P[F(x_1, \ldots, x_d) \le t],$$

where $F(x_1, \ldots, x_d)$ is the probability content of the half-space on the right-hand side of $\mathrm{aff}(x_1, \ldots, x_d)$. Here $x_1, \ldots, x_d$ are independent random points from P (on R^d). Formula (2) is replaced by its directed version:

$$2\binom{n}{d}\binom{n-d}{k}\int_0^1 t^k(1-t)^{n-d-k}\,dG(t). \tag{15}$$

Let P be the uniform distribution on a convex body $K \subset R^d$. Define v and $A(t)$ as in (3). It is proved in [3] that $G(t) \sim t^{d-1}A(t)$ for any convex body $K \in R^d$ but what we need here is the behavior of the derivative of G. This does not seem to be easy to establish and we could only settle the case when K is smooth (say $\mathscr{C}^3$)

with the Gauss–Kronecker curvature bounded away from zero and infinity. In this case we can prove

$$G'(t) \sim t^{d-2}A(t)$$

and so

$$E_d(k, n) \sim \binom{n}{d}\binom{n-d}{k}\int_0^1 t^{k+d-2}(1-t)^{n-d-k}A(t)\,dt.$$

It is known that, for a $\mathscr{C}^3$ convex body K, $A(t) \sim t^{2/(d+1)}$ which gives

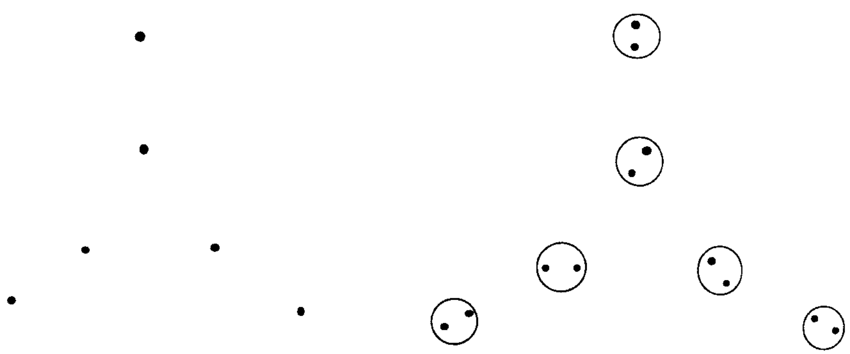
$$E_d(k, n) \sim k^{d-2+2/(d+1)}n^{1-2/(d+1)}$$

in view of (9). This shows, again, that $E_d(k, n)$ behaves like the expected number of facets (or vertices, edges, etc.) of the random polytope inscribed in K when k is constant and like n^{d-1} when $k > cn$. This is probably true for all convex bodies $K \subset R^d$, not only for the $\mathscr{C}^3$ ones.

4. A Distribution with Many Halving Lines

Erdős *et al.* [12] exhibited a set T_i of $n_i = 3 \cdot 2^i$ points which has at least $cn_i \log n_i$ halving segments. We use this example to construct distributions P for which $E_2((n-2)/2, n) \geq c_8 n \log n$. First we review the example of [12] and point out some new features that are needed for the analysis.

The example is sequential. At step $i = 1$ there are $n_1 = 6$ points; three are vertices of an equilateral triangle and three are on rays from the center through these vertices, as in Fig. 4(a). Clearly, there are $h_1 = 6$ halving segments. To form T_2,



(a) (b)

Fig. 4. Sets T_1 and T_2.

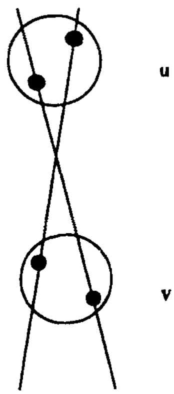


Fig. 5. Halving pair u, $v \in T_i$ begets two pairs in T_{i+1}.

each point $u \in T_1$ splits into two close points u_1, u_2 which are positioned so they define a halving line, as in Fig. 4(b). In addition each pair u, v that defined a halving line in T_1 now defines two halving lines, as shown in Fig. 4(b) (see also Fig. 5). This gives $n_2 = 3 \cdot 2^2 = 12$ points with $h_2 = 18$.

In general, T_i has $n_i = 3 \cdot 2^i$ points. It is shown in [12] that each point $u \in T_i$ may be replaced by two close points u_1, u_2 which can be positioned so that:

1. $u_1 u_2$ is a halving segment in T_{i+1}.
2. If uv was a halving segment in T_i, two *new* halving lines are formed from u_1, u_2, v_1, v_2 (see Fig. 5).

If h_i denotes the number of halving segments in T_i, properties 1 and 2, respectively, show that

$$h_{i+1} = n_i + 2h_i, \qquad h_1 = 6,$$

a recurrence with solution $h_i = 3 \cdot 2^{i-1}(i + 1)$.

To describe our construction we need to know $f_i(j)$, the number of j-segments in T_i, $j = 0, 1, \ldots, n_i/2 - 1$. We have used h_i for $f_i(n_i/2 - 1)$ and we write $h_i^- = f_i(n_i/2 - 2)$ for the number of segments that are one-less-than-halving. From Fig. 5, if uv was a j-segment in T_i, then the four segments u_1v_1, u_1v_2, u_2v_1, u_2v_2 form two $(2j + 1)$-segments and a $2j$-segment and a $(2j + 2)$-segment in T_{i+1}. However, when $j = n_i/2 - 1$, the $2j$-segment and the $(2j + 2)$-segment are both one-less-than-halving. Therefore $f_{i+1}(0) = f_i(0)$ and

$$f_{i+1}(2j) = f_i(j) + f_i(j - 1), \qquad j = 1, \ldots, \frac{n_i}{2} - 2, \tag{16}$$

$$f_{i+1}(2j + 1) = 2f_i(j), \qquad j = 0, 1, \ldots, \frac{n_i}{2} - 1, \tag{17}$$

$$f_{i+1}(2j) = 2f_i(j) + f_i(j-1), \qquad j = \frac{n_i}{2} - 1, \tag{18}$$

$$f_{i+1}(2j+1) = 2f_i(j) + n_i, \qquad j = \frac{n_i}{2} - 1. \tag{19}$$

Equation (19) is the recurrence for h_i, while (18) gives

$$h_{i+1}^- = 2h_i + h_i^-, \qquad h_1^- = 6,$$

a recursion solved by $h_i^- = 3 \cdot 2^i(i-1) + 6$.

A convenient way to represent f_i is via the continuous function g_i on $[0, \frac{1}{2}]$ with values $g_i(0) = 0$ and

$$g_i\left(\frac{j+1}{n_i}\right) = \frac{f_i(j)}{n_i}, \qquad j = 0, 1, \ldots, \frac{n}{2} - 1, \tag{20}$$

and linear between the points j/n_i. Evaluating (19) for $j = n_i/2 - 1$ shows that $g_i(\frac{1}{2}) = h_i/n_i = (i+1)/2$ and for $j = n_i/2 - 2$, that $g_i(\frac{1}{2} - 1/n_i) = h_i^-/n_i \geq i - 1$. From (17) and (18), for $j \leq n_i/2 - 2$, $g_{i+1}((j+1)/n_i) = g_i((j+1)/n_i)$ and this implies $g_{i+k}((j+1)/n_i) = g_i((j+1)/n_i)$. Therefore, for $t \leq t_i = \frac{1}{2} - 1/n_i$, $g_{i+k}(t) = g_i(t)$, by the linearity of g_i. These relations allow the computation of all values of $f_i(j)$.

We now make T_i into a set S_i of positive area by replacing each point $x \in T_i$ by the disk centered at x with radius ε_i, which may be chosen small enough so that the disks are in general position (no three stabbed by a line). It is not surprising that:

Lemma 3. *If P_i is the uniform distribution on S_i, then $E_2((n-2)/2, n) = \Omega(n \log n)$ as long as $an < n_i < bn$, for fixed $0 < a < b < \infty$.*

Proof. We have, according to (2),

$$\begin{aligned}
E_2\left(\frac{n-2}{2}, n\right) &= \binom{n}{2}\binom{n-2}{n/2-1} \int_0^{1/2} 2[t(1-t)]^{n/2-1}\, dG(t) \\
&\geq \binom{n}{2} c_{16} \frac{2^n}{\sqrt{n}} \int_{1/2-1/\sqrt{n_i}}^{1/2} 2[t(1-t)]^{n/2-1}\, dG(t) \\
&\geq \binom{n}{2} c_{16} \frac{2^n}{\sqrt{n}} \int_{1/2-1/\sqrt{n_i}}^{1/2} \frac{c_{17}}{2^n}\left[1 - \frac{4}{n_i}\right]^{n/2-1} dG(t) \\
&\leq c_{18} n^{3/2} e^{-2(n/n_i)} \int_{1/2-1/\sqrt{n_i}}^{1/2} dG(t) \\
&= c_{18} n^{3/2} e^{-2(n/n_i)} \left[G(\tfrac{1}{2}) - G\left(\frac{1}{2} - \frac{1}{\sqrt{n_i}}\right)\right].
\end{aligned} \tag{21}$$

Now let x and y be two points chosen independently and randomly according to P_i. Write D for the event that x, y are not in the same disk; clearly, $\text{Prob}[D] = 1 - 1/n_i$. We have

$$\begin{aligned}
G(\tfrac{1}{2}) - G\left(\frac{1}{2} - \frac{1}{\sqrt{n_i}}\right) &= \text{Prob}\left[F(xy) \in \left[\frac{1}{2} - \frac{1}{\sqrt{n_i}}, \frac{1}{2}\right]\right] \\
&\geq \text{Prob}\left[F(xy) \in \left[\frac{1}{2} - \frac{1}{\sqrt{n_i}}, \frac{1}{2}\right] \middle| D\right] \text{Prob}[D] \\
&\geq \left(1 - \frac{1}{n_i}\right) \sum_{j = n_i/2 - \sqrt{n_i}}^{n_i/2 - 1} \frac{f_i(j)}{\binom{n_i}{2}} \\
&\geq (n_i - 1)\binom{n_i}{2}^{-1} \sum_{j = n_i(1/2 - 1/\sqrt{n_i})}^{n_i/2 - 1} g_i\left(\frac{j+1}{n_i}\right).
\end{aligned}$$

Note that $t_{i/2} = \frac{1}{2} - 1/m_{i/2} > \frac{1}{2} - 1/\sqrt{n_i}$ and that there are 2^{i-k-1} values of j such that $t_k \leq (j+1)/n_i < t_{k+1}$. Therefore,

$$\begin{aligned}
G(\tfrac{1}{2}) - G(\tfrac{1}{2} - 1\sqrt{n_i}) &\geq \frac{2}{n_i} \sum_{k=i/2}^{i-1} \sum_{j=t_k}^{k_{k+1}-1} g_i\left(\frac{j+1}{n_i}\right) \geq \frac{2}{n_i} \sum_{k=i/2}^{i-1} g_i(t_k) 2^{i-k-1} \\
&\geq \frac{2}{n_i} \sum_{k=i/2}^{i-1} (k-1) 2^{i-k-1} \geq \left(\frac{2}{n_i}\right) \frac{i}{4} 2^{i/2}.
\end{aligned}$$

Combined with (21),

$$E_2\left(\frac{n-2}{2}, n\right) \geq c_{18} n^{3/2} e^{-2(n/n_i)} c_{19} \frac{\log n_i}{\sqrt{n_i}}$$

and so $E_2 \geq c_7 n \log n$. □

On the other hand, it is straightforward to show that, as $n \to \infty$, the expected number of halving segments for n points chosen from P_i is $O(n)$. The argument is a simple calculation like the one in (8) using the fact that dG is bounded as n increases.

Next we show that there is a single distribution for which E_2 grows at a superlinear rate. Assume that a sequence $w_m \to 0$ is given. We construct an absolutely continuous distribution P for which $E_2((n-2)/2, n) \geq c_7 w_n n \log n$ for *any* n. We use the same sequence of sets T_i and system of disks S_i as before with the nesting condition that $S_i \supset S_{i+1}$. This can be achieved if, in each step, the radii of the disks are small enough.

Define P by requiring that $P(S_i) = m_i$ with every disk in S_i having probability content m_i/n_i, $i = 1, 2, \ldots$; m_i is specified later. Clearly, $m_1 = 1$ must hold and as $S_i \supset S_{i+1}$ we have $m_i \geq m_{i+1}$. If $m_i > m_{i+1}$ we define P, restricted to $S_i \backslash S_{i+1}$, to be uniform on $S_i \backslash S_{i+1}$. P is a probability measure for every sequence $1 = m_1 \geq m_2 \geq \cdots$ of positive numbers.

Arguing as in (21) we see that

$$E\left(\frac{n-2}{2}, n\right) \geq c_{20} n^{3/2}\left[G(\tfrac{1}{2}) - G\left(\frac{1}{2} - \frac{1}{\sqrt{n}}\right)\right].$$

As in the proof of Lemma 3 we let x, y denote a random pair of points distributed according to P. Define i by requiring $n_i \leq n < n_{i+1}$. Let D_i denote the event that both x and y are in S_i but belong to different disks of S_i. Clearly,

$$\text{Prob}[D_i] = m_i^2\left(1 - \frac{1}{n_i}\right).$$

The previous computation applies now in the following way:

$$\begin{aligned} G(\tfrac{1}{2}) - G\left(\frac{1}{2} - \frac{1}{\sqrt{n}}\right) &\geq \text{Prob}\left[F(xy) \in \left[\frac{1}{2} - \frac{1}{\sqrt{n_i}}, \frac{1}{2}\right] \middle| D_i\right] \text{Prob}[D_i] \\ &\geq \frac{2}{n_i} \frac{i}{4} 2^{i/2} m_i^2 \geq c_7 m_i^2 n \log n. \end{aligned}$$

If we choose $m_i = 1$ for all i, then P is a probability distribution, with support $\bigcap S_i$ and having $E_2 \sim n \log n$. This distribution is concentrated in a small set. If we choose a decreasing sequence m_i slowly tending to zero, then P is an absolutely continuous measure and $E_2 \geq m_n^2 n \log n$.

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