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Random Convex Hulls: Floating Bodies and Expectations

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Recently Bárány and Larman have shown that the volume of the convex hull of a point sample taken uniformly within a convex body can be approximated by the volume of a related "floating body." Here we show that, in the sense of sets, the floating body and the expectation of the convex hull are close. As an immediate consequence of the floating body as intermediary, we observe that the expectation of the volume of the convex hull is approximately the same as the volume of its expectation, an issue related to the Brunn–Minkowski inequality.

1. INTRODUCTION

Beginning with the classic work of Rényi and Sulanke [6–8], the nature of the convex hull of a collection of random points has been intensively studied. A survey appears in Schneider [10]. Recently, Bárány and Larman [3] have shown that, in the case of points selected uniformly within a fixed parent body, the expected volume of the convex hull is

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closely approximated by the volume of a (non-random) "floating body." The purpose here is to link up this result with the topic of random sets, in particular, the idea of a *set-valued* expectation, for which background and applications can be found in Artstein and Vitale [1], Aumann [2], Mecke [4], Vitale [12–15], and Wieacker [16]. Our result is that the floating body essentially coincides with the expectation of the convex hull. This is of interest in view of the fact that the expectation is defined in completely different terms, and also for statistical applications [9, 5].

2. Preliminaries

In R^d, we fix a compact, convex K with non-empty interior. Points $X_1, X_2, \ldots$ are generated independently and uniformly in K, yielding the successive convex hulls $\mathbf{X}_n = \text{conv}\{X_1, \ldots, X_n\}$, $n = 1, 2, \ldots$. The expectation $E\mathbf{X}_n$ of $\mathbf{X}_n$ is a compact, convex subset of K and is conveniently described in support function terms:

$$\begin{aligned} h_{E\mathbf{X}_n}(u) &= \max\{u \cdot y \mid y \in EX_n\} \\ &= Eh_{\mathbf{X}_n}(u) = E\max\{u \cdot X_1, \ldots, u \cdot X_n\} \qquad \|u\| = 1. \end{aligned} \tag{2.1}$$

For $x \in K$, we define $v(x) = \min\{\text{vol}(K \cap H) \mid x \in H,\ H \text{ a half space}\}$ and for $\varepsilon > 0$ $K(\varepsilon) \equiv \{x \in K \mid v(x) \leqslant \varepsilon\}$. The associated floating body is $K \backslash K(\varepsilon)$.

3. Floating Bodies and Expectations

Our result comparing floating bodies and expectations is stated in terms of a natural scaling for the approach of $\mathbf{X}_n$ to K.

THEOREM. *There are constants $0 < a < b < \infty$ such that for all n*

$$K(a/n) \subseteq K \backslash E\mathbf{X}_n \subseteq K(b/n). \tag{3.1}$$

Before proceeding to the proof, we mention an immediate consequence of this result (and Theorems 1 and 7 of Bárány and Larman [3]).

COROLLARY. *There are constants $0 < c_1 < c_2 < \infty$ such that for all n*

$$c_1 \leqslant \frac{\text{vol}\, K - E\,\text{vol}\,\mathbf{X}_n}{\text{vol}(K \backslash E\mathbf{X}_n)} \leqslant c_2. \tag{3.2}$$

We return to this in Remark (3) in Section 4.

Proof of Theorem. First we arrange a convenient set-up. Given $\|u\|=1$, translate K so that $0 \in K$ and $u \cdot x \leqslant 0\ \forall x \in K$. Set $F(t)=\operatorname{vol}\{x \in K \mid -u \cdot x \leqslant t\}$. Without loss of generality, assume that $\operatorname{vol} K=1$, so that F is the distribution function of $-u \cdot X$, where X is uniformly distributed in K (at $t=T \equiv \max\{-u \cdot x \mid x \in K\}$, $F(T)=1$). Let $A(t)$ stand for the $(d-1)$-dimensional volume of the cross-section $K \cap \{x \in R^d \mid -u \cdot x=t\}$ so that $F(t)=\int_0^t A(s)\, ds$. If $0 \leqslant t^* \leqslant T$ and $A^*=\max\{A(t) \mid 0 \leqslant t \leqslant t^*\}$, we record the following estimates for $t^* \leqslant t \leqslant T$:

$$F(t^*) \leqslant A^* t \tag{3.3}$$

$$\frac{1}{d} A^*(t) \leqslant F(t) \tag{3.4}$$

and

$$F(t) \leqslant \left(\frac{t}{t^*}\right)^d F(t^*). \tag{3.5}$$

By definition,

$$h_{E\mathbf{X}_n}(u)=E \max\{u \cdot X_1, \ldots, u \cdot X_n\}=-E \min\{-u \cdot X_1, \ldots, -u \cdot X_n\}$$

so that

$$\begin{aligned} h_K(u)-h_{E\mathbf{X}_n}(u) &= E \min\{-u \cdot X_1, \ldots, -u \cdot X_n\} \\ &= \int_0^T [1-F(t)]^n\, dt. \end{aligned} \tag{3.6}$$

Now we establish (3.1) by estimating (3.6). Let t_n satisfy $F(t_n)=1/nd$ and observe that

$$\begin{aligned} \int_0^T [1-F(t)]^n\, dt &\geqslant \int_0^{t_n} [1-F(t)]^n\, dt \\ &\geqslant \int_0^{t_n} [1-F(t_n)]^n\, dt = t_n \left(1-\frac{1}{nd}\right)^n. \end{aligned}$$

Applying $F(\cdot)$ and using (3.5) gives

$$\begin{aligned} F\left(\int_0^T [1-F(t)]^n\, dt\right) &\geqslant F\left(t_n\left(1-\frac{1}{nd}\right)^n\right) \geqslant \left(1-\frac{1}{nd}\right)^{nd} F(t_n) \\ &= \left(1-\frac{1}{nd}\right)^{nd} \frac{1}{nd} \geqslant \frac{1}{4nd}. \end{aligned}$$

(the extreme inequality holding in the exceptional case $nd=1$ by direct calculation: $F(\int_0^T [1-F(t)]\,dt)=(1/T)\int_0^T [1-(t/T)]\,dt=1/2\geq 1/4$). Since this is independent of u, it follows that $K(a/n)\subseteq K\backslash E\mathbf{X}_n$ with $a=1/4d$.

For the other inclusion in (3.1), we estimate (3.6) from above:

$$\int_0^T [1-F(t)]^n\,dt=\int_0^{t_n} [1-F(t)]^n\,dt+\int_{t_n}^T [1-F(t)]^n\,dt$$
$$\leq t_n+\int_{t_n}^T \left[1-\frac{A^*}{d}t\right]^n dt,$$

where we have identified $t_n=t^*$ and used (3.4). Making the change of variable $y=t/t_n$, and using (3.3), (3.5), and other estimates, we bound the integral,

$$t_n\int_1^{T/t_n}\left[1-\frac{A^*}{d}t_n y\right]^n dy\leq t_n\int_1^{T/t_n}\left[1-\frac{F(t_n)}{d}y\right]^n dy$$
$$\leq t_n\int_1^{T/t_n}\left[1-\frac{1}{nd^2}y\right]^n dy$$
$$\leq t_n\int_1^{T/t_n} e^{-y/d^2}\,dy\leq d^2 t_n$$

so that

$$F\left(\int_0^T [1-F(t)]^n\,dt\right)\leq F((1+d^2)\,t_n)$$
$$\leq (1+d^2)^d\,F(t_n)=(1+d^2)^d\frac{1}{nd}.$$

It follows that $K\backslash E\mathbf{X}_n\subseteq K(b/n)$ with $b=(1+d^2)^d\,(1/d)$. This completes the proof.

4. Remarks

(1) With a more detailed analysis, one can show that, for sufficiently large n, (3.1) holds with $b=10\log d$. The interesting question of (asymptotically) optimal constants however remains open.

(2) A related result for the rate of convergence of the mean width of $E\mathbf{X}_n$ to that of (sufficiently smooth) K can be seen in Schneider and Wieacker [11]. It is enough to observe that by Fubini's theorem the

expected mean width of $\mathbf{X}_n$, which they treat, is the same as the mean width of $E\mathbf{X}_n$.

(3) The fact that $\operatorname{vol}(E\mathbf{X}_n)$ and $E\operatorname{vol}(\mathbf{X}_n)$ show similar rates of increase to vol K ((3.2)) is of interest from a general point of view. For an arbitrary random compact, convex set $\mathbf{X}$ with expectation $E\mathbf{X}$, there is the Brunn–Minkowski inequality [14]

$$\operatorname{vol}^{1/d}(E\mathbf{X}) \geqslant E\operatorname{vol}^{1/d}(\mathbf{X})$$

as well as the usual moment inequality

$$[E\operatorname{vol}\mathbf{X}]^{1/d} \geqslant E\operatorname{vol}^{1/d}(\mathbf{X}).$$

The question of which is sharper is the obviously same as which of $\operatorname{vol}(E\mathbf{X})$ and $E\operatorname{vol}(\mathbf{X})$ is larger. By example either can happen, and while we have not resolved the issue here, we have at least shown that (for the convex hull mechanism) they are of comparable size.

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