International Journal of Game Theory (1992) 21: 267–290

Classification of Two-Person Ordinal Bimatrix Games

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Abstract: The set of possible outcomes of a strongly ordinal bimatrix games is studied by imbedding each pair of possible payoffs as a point on the standard two-dimensional integral lattice. In particular, we count the number of different Pareto-optimal sets of each cardinality; we establish asymptotic bounds for the number of different convex hulls of the point sets, for the average shape of the set of points dominated by the Pareto-optimal set, and for the average shape of the convex hull of the point set. We also indicate the effect of individual rationality considerations on our results. As most of our results are asymptotic, the appendix includes a careful examination of the important case of 2×2 games.

1 Introduction

The two person bimatrix game is one of the most fundamental constructions of game theory. Indeed, even the 2×2 bimatrix game has been used extensively in economics and political science to illustrate problems and paradoxes in competition and cooperation.

A basic class of bimatrix games which is studied here is given by considering payoffs to each player as being purely ordinal. An $r \times m$ bimatrix game, where player I (the row player) has r strategies and player II (the column player) has m strategies, has $n = rm$ cells or outcomes. Restricting ourselves to ordinal payoffs, where we assume that there are no ties in the preferences, we can represent the preference ordering of each player by the integers 1, 2, ..., n (denoted henceforth by $[n]$), where x is preferred to y if $x > y$. Such a game is represented by a *bimatrix* (A, B) for which the entry (a_{ij}, b_{ij}) in row i and column j represents the payoffs to players I and II, respectively, if player I plays strategy i, and player II plays strategy j.

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0020-7276/92/3/267-290 \$ 2.50

There are many different ways in which one can attempt to categorize bimatrix games with strongly ordinal payoffs. Rapoport, Gordon, and Guyer (1976) list all 78 "strategically different" 2×2 strongly ordinal bimatrix games, accounting for reordering strategies and players. Rapoport et al. (1976) suggest that these 78 classes of games can be further aggregated into 24 classes. Although such enumerative taxonomies may shed some light on the 2×2 game, their utility for games having more strategies is not evident. Factors that can be used to classify bimatrix games include properties of the Pareto-optimal set, the nature of optimal responses, and properties of the set of non-cooperative equilibria. O'Neill (1981) has considered some aspects of the Pareto-optimal set. In particular, he determined the distribution of the cardinality of the Pareto-optimal set.

In the present paper, we consider properties of the Pareto-optimal set and of the entire payoff set. In Section 2, we establish a formula for the number of different Pareto-optimal sets of each cardinality. We can imbed the n payoffs as points in $[1, n] \times [1, n]$ and consider the associated convex hulls. Section 3 contains asymptotic bounds on the number of different convex hulls that may arise. In Section 4, we consider the average shape of the convex hull of the imbedded Pareto-optimal payoffs as well as that of the entire payoff set. We discuss the impact of individual rationality considerations in Section 5. In Section 6, we provide a brief description of a combinatorial model of the best-response structure of a game. As most of our results are asymptotic, we discuss games with few strategies in the appendix.

The class of payoffs of the games that we consider is in one-to-one correspondence with the set of perfect matchings of the complete bipartite graph $K_{n,n}$ ($n = rm$) with vertices $[n]$ on each side of the bipartition, where an edge indicates a pair of payoffs corresponding to an outcome of the games. A perfect matching M of $K_{n,n}$ is identified with an n-element *outcome set* $O(M)$ of R^2:

$$O(M) = \{(x, y) \in \mathbf{R}^2 \colon \text{ the edge } (x, y) \text{ is in } M\}.$$

The point (x, y) *dominates* (x', y') if $x > x'$ and $y > y'$. The *Pareto(-optimal) set* $P(M)$ consists of those points of $O(M)$ that are not dominated by any other point of $O(M)$. We write $P_<(M)$ for the set of non-negative points dominated by some point of $P(M)$. Finally, we denote the convex hull of $O(M)$ by $C(M)$. See Figure I for an example.

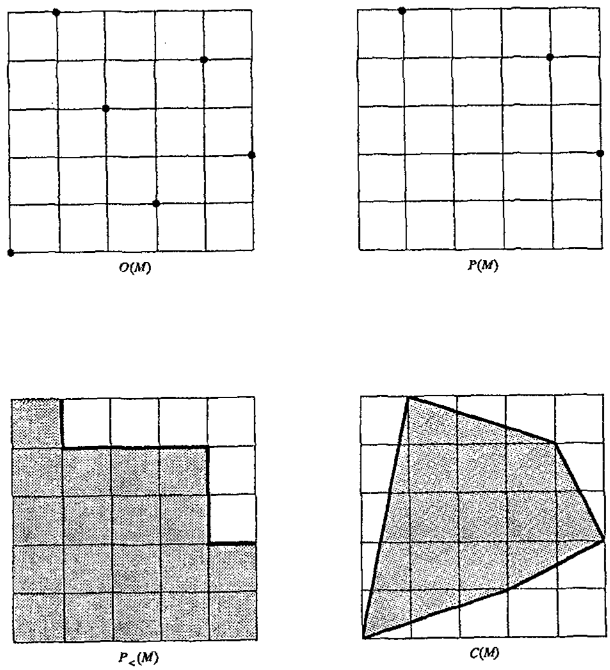


Fig. 1

2 The Number of Pareto Sets

A simple measure of the intrinsic level of competitive structure of a game is the number of points in the Pareto set. The most competitive games have the greatest number of points in the Pareto set. Let $f_n(k)$ be the number of different Pareto sets of cardinality k.

Theorem 1.

$$f_n(k) = \frac{1}{n+1}\binom{n-1}{k-1}\binom{n+1}{k}.$$

Proof. Let $P=\{p_i=(x_i, y_i): i=1, 2, \ldots, k\}$ be a subset of $[n]\times[n]$. We are interested in the number of such k-sets P that can be Pareto subsets $P(M)$ of $O(M)$ for some perfect matching M. We may take the points to be ordered $1\le x_1<x_2<\cdots<x_{k-1}<x_k=n$. It must be, then, that $n=y_1>y_2>\cdots>y_k\ge 1$. Such a P is a Pareto subset of $O(M)$ for some perfect matching M if and only if the bipartite graph $G(P)=([n], [n]; E(P))$ has a perfect matching. Here $(x, y)\in E(P)$ if either $x=x_i$ for some $i=1, \ldots, k$, and then $y=y_i$, or $x_i<x<x_{i+1}$ for some $i=0, \ldots, k-1$ ($x_0=0$ by definition), and then $y=y_{i+1}$.

The necessary and sufficient condition for this to happen is given by the König-Hall theorem: For any subset $X\subseteq[n]$, the set $y(X)$ of $y\in[n]$, such that $(x, y)\in E(P)$

for some $x \in X$, must have cardinality at least $|X|$. Taking into account the special structure of P and $E(P)$, the König-Hall condition reduces to

$$|y(x_i+1, \ldots, x_k)| = x_k - x_i \geq y_{i+1} \quad \text{for } i = 1, 2, \ldots, k,$$

or

$$x_i + y_{i+1} \geq n \text{ for } i = 1, 2, \ldots, k-1.$$

Let $\alpha_i = n - x_{k-i}$ $(i = 1, \ldots, k-1)$ and $\beta_i = y_{k-1+1}$ $(i = 1, 2, \ldots, k-1)$. Then we have that the number of k-sets that are the Pareto subset of some set $O(M)$ is precisely the number of sets $\{(\alpha_i, \beta_i): i = 1, \ldots, k-1\}$ that satisfy

$$\begin{aligned} &1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{k-1}, \\ &\quad \beta_1 < \beta_2 < \cdots < \beta_{k-1} \leq n-1, \\ &\quad \alpha_i \leq \beta_i \textit{ for } i = 1, \ldots, k-1. \end{aligned} \tag{2.1}$$

Considering the multiset $\{\alpha_i : i \in [k-1]\} \cup \{\beta_i : i \in [k-1]\}$, we observe that each value (in $[n-1]$) appears at most twice. Let ℓ be the number of doubletons. There are $\binom{n-1}{2(k-1-\ell)}$ choices for the singletons. The number of ways to bicolor the singletons, aqua and blue, so that for every $j \in [n-1]$, the number of blue elements less than j never exceeds the number of aqua elements less than j is the $(k-\ell-1)^{\text{st}}$ Catalan number: $\binom{2(k-\ell-1)}{k-\ell-1}/(k-\ell)$ (see Feller (1968)). Associated with each such coloring of the $2(k-1)$ element multiset is a unique labeling of the aqua elements as $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ and the blue elements as $\beta_1, \beta_2, \ldots, \beta_{k-1}$ satisfying (2.1). Hence

$$f_n(k) = \sum_{\ell=0}^{k-1} \binom{n-1}{2(k-1-\ell)} \binom{n-1-2(k-1-\ell)}{\ell} \binom{2(k-\ell-1)}{k-\ell-1} \frac{1}{k-\ell}.$$

It is easy to see that this is the same as

$$\begin{aligned} \sum_{\substack{S \subseteq [n-1] \\ |S| = k-1}} \sum_{\substack{T \subseteq [n-1] \\ |T| = k-1}} \frac{1}{|S \backslash T| + 1} &= \binom{n-1}{k-1} \sum_{\substack{T \subseteq [n-1] \\ |T| = k-1}} \frac{1}{|[k-1] \backslash T| + 1} \\ &= \binom{n-1}{k-1} \sum_{\ell=0}^{k-1} \binom{n-k}{k-1-\ell} \binom{k-1}{\ell} \frac{1}{k-\ell} \\ &= \binom{n-1}{k-1} \sum_{\ell=0}^{k-1} \frac{1}{n-k+1} \binom{n-k+1}{k-\ell} \binom{n-1}{\ell} \\ &= \binom{n-1}{k-1} \frac{1}{n-k+1} \binom{n}{k} \\ &= \frac{1}{n+1} \binom{n-1}{k-1} \binom{n+1}{k}. \end{aligned}$$

■

From this it is easy to compute the total number of Pareto sets.

Corollary 1.

$$\sum_{k=1}^{n} f_n(k) = \frac{1}{n+1}\binom{2n}{n} = \frac{4^n}{(n+1)\sqrt{\pi n}}(1+o(1)).$$

Proof. Indeed,

$$\sum_{k=1}^{n} f_n(k) = \frac{1}{n+1}\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{n+1}{k} = \frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+1}{k+1}$$
$$= \frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+1}{n-k} = \frac{1}{n+1}\binom{2n}{n}.$$

The last approximation in the statement of the result follows from Stirling's formula. ■

Let $F_n(k)$ denote the number of outcome sets (perfect matchings) with exactly k points in the Pareto set. Alternatively, consider the set of all perfect matchings as a probability space equipped with uniform distribution and the random variable $\xi = \xi(M) = |P(M)|$. Clearly

$$\frac{1}{n!}F_n(k) = \mathrm{Prob}(\xi = k).$$

O'Neill (1981) determined the exact value of $F_n(k)$: the signless Stirling number of the first kind (see Riordan (1958))

$$F_n(k) = (-1)^{k+n}s(n, k).$$

The expectation and variance of ξ are

$$E\xi = \log n + C$$

and

$$D\xi = \log n + C - \frac{\pi^2}{6},$$

respectively, where C is Euler's constant. Moreover, the distribution of $(\xi - E\xi)/\sqrt{D\xi}$ tends to the standard normal distribution as $n \to \infty$. In particular

$$\mathrm{Prob}(\xi < \log n + \lambda\sqrt{\log n}) = \frac{1}{n!}\sum_{k \le \log n + \lambda\sqrt{\log n}} F_n(k) \approx \frac{1}{\sqrt{2n}}\int_{-\infty}^{\lambda} e^{-x^2/2}dx.$$

This follows from Feller (1968), as noted by O'Neill (1981).

3 The Number of Convex Hulls

Let v_n denote the number of different convex hulls $C(M)$ of $O(M)$ as M ranges over the set of all perfect matchings of $K_{n,n}$. In this section, we establish asymptotic bounds on v_n.

Theorem 2. For sufficiently large n,

$$\frac{2}{5}n^{2/3} \leq \log v_n \leq 11n^{2/3}.$$

Proof. We first prove the upper bound. Let ω_n be the number of piecewise linear monotone increasing convex functions $g:[1,n]\to[1,n]$ having $g(1)=1$, $g(n)=n$, and breakpoints at integer values x such that $g(x)$ is an integer. We have that ω_n^4 is an upper bound for v_n as the graph of such a function g has the shape of the "the south-east corner" of the boundary of $C(M)$.

Let $p_0=(1,1)$, $p_1, \ldots, p_{k-1}$, $p_k=(n,n)$ be the breakpoints of g in consecutive order. Then the set

$$U=\{p_1-p_0, p_2-p_1, \ldots, p_k-p_{k-1}\}$$

is comprised of nonnegative (and non-zero) integer vectors whose sum is equal to $(n-1, n-1)$. Different g functions produce different sets U. So ω_n is not larger than the number of such sets U.

Only p_1-p_0 can have zero as a second component, and there are at most $n-2$ choices for the first component (if the second is zero). Write k_i for the number of vectors $u=(u_1,u_2)\in U$ with $u_2>0$ and $u_1+u_2=i$. There are i vectors with this property. Clearly $\sum_{i=1}^{2(n-1)} ik_i \leq 2(n-1)$. Hence

$$\omega_n \leq \sum_{k_1,k_2,\ldots,k_n}\left\{(n-2)\prod_{i=1}^{2(n-1)}\binom{i}{k_i} : \sum_{i=1}^{2(n-1)} ik_i \leq 2(n-1)\right\}. \tag{3.1}$$

We will evaluate this upper bound on ω_n by first establishing

$$\prod_{i=1}^{2(n-1)}\binom{i}{k_i} \leq \exp\{2.53n^{2/3}\}, \tag{3.2}$$

where $k_i \geq 0$ $(i\in[2(n-1)])$ and $\sum_{i=1}^{2(n-1)} ik_i \leq 2(n-1)$. We note that for integers i and x with $0 \leq x \leq i$,

$$\binom{i}{x} \leq \frac{i^i}{x^x(i-x)^{i-x}},$$

where $x^x = 1$ if $x = 0$, as usual. Hence the left-hand side of (3.2) is bounded from above by

$$\sup \prod_{i=1}^{2(n-1)} \frac{i^i}{x_i^{x_i}(i-x_i)^{i-x_i}}$$
$$\text{subject to } \sum_{i=1}^{2(n-1)} ix_i \le 2(n-1),\ 0 \le x_i \le i,\ i \in [2(n-1)]. \tag{3.3}$$

Observe that this supremum is reached at some point $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2(n-1)})$. Notice, also, that $\bar{x}_i \neq 0$ for all i, since otherwise $\bar{x}_i = 0$ for some i and $\bar{x}_j \neq 0$ for some j. Then we would have larger objective value for the feasible point

$$(\bar{x}_1, \ldots, \bar{x}_i + j\varepsilon, \ldots, \bar{x}_j - i\varepsilon, \ldots, \bar{x}_{2(n-1)})$$

than at $(\bar{x}_1, \ldots, \bar{x}_{2(n-1)})$. This follows from a simple computation. It is also clear that

$$\sum_{i-1}^{2(n-1)} i\bar{x}_i = 2(n-1). \tag{3.4}$$

At the optimal solution

$$\frac{\partial}{\partial x_j}\left\{\sum_{i=1}^{2(n-1)} \frac{i^i}{x_i^{x_i}(i-x_i)^{i-x_i}} + \sum_{i=1}^{2(n-1)} ix_i\right\} = 0$$

must hold. From this it follows that

$$\bar{x}_i = \frac{i}{1+e^{\mu i}}, \quad i \in [2(n-1)],$$

for some suitable $\mu > 0$. By (3.4),

$$2(n-1) = \sum_{i=1}^{2(n-1)} i\bar{x}_i = \sum_{i=1}^{2(n-1)} \frac{i^2}{1+e^{\mu i}}.$$

The function $x^2(1+e^{\mu x})^{-1}$ has its maximum when $2 = \mu x e^{\mu x}(1+e^{\mu x})^{-1}$. The unique solution of this latter equation satisfies $\mu x \in [2.217, 2.218]$. It follows that the maximum value of $x^2(1+e^{\mu x})^{-1}$ is less than μ^{-2}. We have

$$\left|\sum_{i=1}^{2(n-1)} \frac{i^2}{1+e^{\mu i}} - \int_0^{2(n-1)} \frac{x^2}{1+e^{\mu x}}\,dx\right| < \mu^{-2}.$$

Hence,

$$2(n-1)=\sum_{i=1}^{2(n-1)} \frac{i^2}{1+e^{\mu i}} > -\mu^{-2}+\int_0^{2(n-1)} \frac{x^2\,dx}{1+e^{\mu x}}$$
$$=-\mu^{-2}+\mu^{-3}\int_0^{2\mu(n-1)} \frac{y^2\,dy}{1+e^y}.$$

It is easily seen that μ is about $n^{-1/3}$, and numerical integration gives $\mu<0.97n^{-1/3}$. Now

$$\prod_{i=1}^{2(n-1)} \frac{i^i}{\bar{x}_i^{\bar{x}_i}(i-\bar{x}_i)^{i-\bar{x}_i}} = \prod_{i=1}^{2(n-1)} (1+e^{\mu i})^{\frac{i}{1+e^{\mu i}}}(1+e^{-\mu i})^{\frac{i}{1+e^{-\mu i}}}$$
$$=\exp \sum_{i=1}^{2(n-1)} \left(\frac{i\log(1+e^{\mu i})}{1+e^{\mu i}}+\frac{i\log(1+e^{-\mu i})}{1+e^{-\mu i}}\right)$$
$$<\exp \int_0^{\infty} \left(\frac{x\log(1+e^{\mu x})}{1+e^{\mu x}}+\frac{x\log(1+e^{-\mu x})}{1+e^{-\mu x}}\right)dx$$
$$=\exp\left\{\mu^{-2}\int_0^{\infty} \left(\frac{y\log(1+e^{y})}{1+e^{y}}+\frac{y\log(1+e^{-y})}{1+e^{-y}}\right)dy\right\}$$
$$<\exp\{2.71\mu^{-2}\}<\exp\{2.53n^{2/3}\},$$

with numerical integration again.

Hence every summand in (3.1) is less than $(n-2)\exp\{2.53n^{2/3}\}$, for sufficiently large n. The number of different partitions of $2(n-1)$ is asymptotically (see Rademacher (1969))

$$\frac{1}{4\sqrt{3}}\,\frac{\exp\{\pi\sqrt{2/3}\,\sqrt{2(n-1)}\}}{2(n-1)}<\frac{\exp\left\{\frac{2\pi}{\sqrt{3}}\sqrt{n}\right\}}{n-1}.$$

So we have that

$$\omega_n\le\exp\{2.53n^{2/3}\}\exp\left\{\frac{2\pi}{\sqrt{3}}\sqrt{n}\right\},$$

which implies $v_n<\exp\{11n^{2/3}\}$ for n sufficiently large.

For the lower bound, we consider a subset $U=\{u_1, u_2, \ldots, u_k\}$ of the primitive vectors from $[0, t]^2\cap\mathbf{Z}^2$ where the vectors are indexed according to increasing slope. The points 0, u_1, u_1+u_2, $u_1+u_2+u_3$, $\ldots$, $u_1+u_2+\cdots+u_k$ form the vertices of a monotone increasing, convex polygonal path L which we translate by an integral vector a so that $0+a$ is on the line $y=1$ and $u_1+\cdots+u_k+a$ is on the line $x=n$.

Clearly

$$\sum_{i=1}^{k} u_i \leq \sum \{x : x \in [0, t]^2 \cap \mathbf{Z}^2\} = \left(\frac{t(t+1)^2}{2}, \frac{t(t+1)^2}{2}\right)$$

which is no more than $(n-1,\ n-1)$ if $t = n^{1/3}$ (for sufficiently large n). An easy application of the König-Hall theorem shows now that $L + a$ is the "south-east corner" of the boundary of $C(M)$ for some perfect matching M. It is also clear that different sets U give rise to different polygonal paths L.

Now the number of primitive vectors in $[0, t]^2 \cap \mathbf{Z}^2$ is asymptotic to $6\pi^{-2}t^2$, hence the number of such sets U is at least

$$2^{6/\pi^2 n^{2/3}} = \exp\{\log\{2^{6/\pi^2}\} n^{2/3}\} > \exp\{0.42 n^{2/3}\}. \qquad \blacksquare$$

We remark that with a little more care the lower bound can be improved to $\exp\{1.6n^{2/3}\}$.

We note that everything above is true for cardinal payoffs as well since the Pareto set does not change slope when the payoffs are rescaled keeping the same order.

4 Average Shapes

Recall that $P_<(M)$ denotes the set of points $z = (x, y) \in \mathbf{R}^2$, $(x, y) \geq 0$ that are dominated by some point in the Pareto set $P(M)$. The *average Pareto shape* is defined as the average of the characteristic functions of the $P_<(M)$:

$$f_{\mathrm{Par}}(z) = \frac{1}{n!} \sum_M \chi_{P_<(M)}(z).$$

Alternatively, $f_{\mathrm{Par}}(z) = \mathrm{Prob}(z \in P_<(M))$ where Prob is meant with z fixed and M varying over all perfect matchings. One can explicitly compute $f_{\mathrm{Par}}(z)$.

Theorem 3. Let $z = (a, b)$ with $a + b \geq n$, $b \geq a$ and a, b integers. Let $c = n - (a-1)$ and $d = n - (b-1)$. If $c, d \to \infty$ and $c, d = o(n^{2/3})$, as $n \to \infty$, then

$$f_{\mathrm{Par}}(z) = 1 - \exp\left\{-\frac{cd}{n}(1 + o(1))\right\}.$$

Proof.

$$\mathrm{Prob}(z \notin P_<(M)) = \frac{(a-1)(a-2)\cdots(a-(n-(b-r)))}{n(n-1)\cdots(n-(n-(b-1)))}.$$

(The formula for non-integral a, b is obtained by replacing a and b by $\lfloor a \rfloor$ and $\lfloor b \rfloor$, respectively.) We have

$$\operatorname{Prob}(z \notin P_<(M)) = \frac{(n-c)\cdots(n-c-d+1)}{n\cdots(n-d+1)}$$
$$= \prod_{i=1}^{d}\left(1-\frac{c}{n+1-i}\right) = \exp\sum_{i=1}^{d}\log\left(1-\frac{c}{n+1-i}\right).$$

Since $|\log(1-c/y)|$ is monotone decreasing in y for $y \in [n-d, n]$, we have

$$\left|\sum_{i=1}^{d}\log\left(1-\frac{c}{n+1-i}\right) - \int_{n-d}^{n}\log\left(a-\frac{c}{y}\right)dy\right| < \left|\log\left(1-\frac{c}{n-b}\right)\right|$$

$$\int_{n-d}^{d}\log\left(1-\frac{c}{y}\right)dy =$$

$$-c\log\left(1+\frac{d}{n-c-d}\right) - d\log\left(1+\frac{c}{n-c-d}\right) + n\log\left(1+\frac{cd}{n(n-c-d)}\right).$$

Now assume that $d \le c \le n/4$. Using the inequality $|\log(1+h)-h| < h^2$, which is valid for $|h| \le 1/2$, we get

$$\left|\sum_{i=1}^{d}\log\left(1-\frac{c}{n+1-i}\right) + \frac{cd}{n-c-b}\right| < \frac{cd^2+c^2d}{(n-c-d)^2} + \frac{c^2d^2}{n(n-c-d)^2} + \left(\frac{c}{n-d}\right)^2.$$

The result follows. ∎

So the average shape is almost one when cd is much larger than n and almost zero when cd is much less than n.

We note that the average number of vertices of $P_<(M)$ is $\sum_{k=1}^{n} k \operatorname{Prob}(\xi = k)$. This is asymptotic to $\log n$ minus a constant.

Next we consider the average convex shape defined as

$$f_{\text{conv}}(z) = \frac{1}{n!}\sum_{M}\chi_{C(M)}(z).$$

We will give asymptotic estimates of $f_{\text{conv}}(z)$ (and not precise asymptotic bounds). As f_{conv} has the symmetry of the square, we only consider the north-east corner,

Theorem 4. Let $z=(a, b)$ with $(a, b) \ge (n/2, n/2)$ and a, b integers. Let $c = n-(a-1)$ and $d = n-(b-1)$. If $c, d \to \infty$ and $c, d = o(n^{2/3})$, as $n \to \infty$, then

$$1-2\exp\left\{-\frac{cd}{n}(1+o(1))\right\} \le f_{\text{conv}}(z) \le 1-\exp\left\{-\frac{4cd}{n}(1+o(1))\right\}.$$

Proof. Clearly, if $z \notin C(M)$, then $O(M)$ has no point in one of the four rectangles in Figure II.

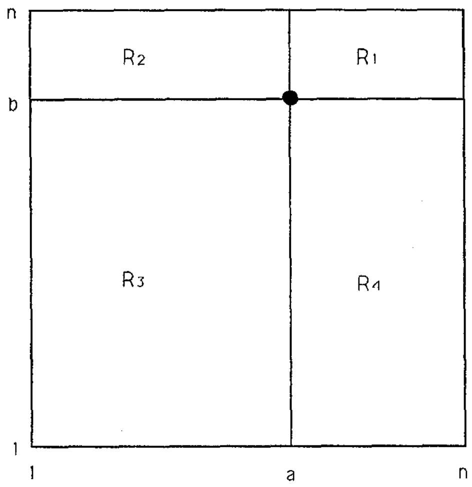


Fig. II

Assume $n/2<a<b$, then the rectangles R_3 and R_4 of Figure II contain points of $O(M)$. Thus

$$
\begin{aligned}
\operatorname{Prob}(z \notin C(M)) &\leq \operatorname{Prob}(O(M) \cap R_1 = \emptyset) + \operatorname{Prob}(O(M) \cap R_2 = \emptyset) \\
&\leq 2 \operatorname{Prob}(O(M) \cap R_1 = \emptyset) \\
&= 2 \frac{(a-1)(a-2)\cdots(a-(n-(b-1)))}{n(b-1)\cdots(n-(n-(n-1)))} \\
&= 2 \exp\left\{-\frac{cd}{n}(1+o(1))\right\}
\end{aligned}
$$

when $c=n-(a-1)$ and $d=n-(b-1)$ provided $c, d \to \infty$ and $c, d = o(n^{2/3})$ as $n \to \infty$.

On the other hand if $O(M)$ has no point in the rectangle $R = \operatorname{conv}\{(2a-n, 2b-n), (2a-n, n), (n, n), (n, 2b-n)\}$, then $z=(a, b) \notin C(M)$. So

$$
\begin{aligned}
\operatorname{Prob}(z \notin C(M)) &\geq \operatorname{Prob}(O(M) \cap R = \emptyset) \\
&= \exp\left\{-\frac{4cd}{n}(1+o(1))\right\},
\end{aligned}
$$

with the same assumptions as above. The result follows. ■

5 Individual Rationality

It is natural to restrict our attention to payoffs that are individually rational for each of the two players. We define the *individually rational levels*

$$\max\min A = \max_i \min_j a_{ij},$$

$$\max\min B = \max_i \min_j b_{ij},$$

and the *individually rational zone*

$$IR = \{(x, y) \in \mathbf{R}^2 : x \geq \max\min A,\ y \geq \max\min B\}.$$

In considering different solution concepts, it is standard to restrict ourselves to outcomes in IR. Hence, we should actually consider $O(M) \cap IR$, $P(M) \cap IR$, $P_<(M) \cap IR$ and $C(M) \cap IR$ rather than $O(M)$, $P(M)$, $P_<(M)$ and $C(M)$. However, it turns out that the individually rational levels are quite small, so that the difference is slight. We demonstrate this with a heuristic argument which can be made precise. The argument will indicate that with high probability

$$\max\min A \in [(1-\varepsilon) r \log r,\ (1+\varepsilon) r \log r],$$

$$\max\min B \in [(1-\varepsilon) m \log m,\ (1+\varepsilon) m \log m],$$

as $r, m \to \infty$, provided $\log r = o(m)$ and $\log m = o(r)$.

Replace the random matrix A having elements from $[rm]$ with a random matrix $\tilde{A}$ having entries uniformly and independently distributed in $[0, rm]$. It is clear that the average behavior of the maxmin for A and $\tilde{A}$ is quite similar (This is where the argument is heuristic). Now for $t \in [0, 1]$

$$\mathrm{Prob}(\min_j \tilde{a}_{i,j} \geq rmt) = (1-t)^m,$$

and so

$$\mathrm{Prob}(\max_i \min_j \tilde{a}_{i,j} < rmt) = (1-(1-t)^m)^r.$$

When $rmt = (1 \pm \varepsilon) r \log r$, then

$$(1-(1-t)^m)^r \approx (1 - e^{(1\pm\varepsilon)\log r})^r = (1 - r^{1\pm\varepsilon})^r \approx \exp\{-r^{\pm\varepsilon}\}.$$

Hence,

$$\mathrm{Prob}(\max\min A \in [(1-\varepsilon) r \log r,\ (1+\varepsilon) r \log r]) \approx \exp\{-r^{-\varepsilon}\} - \exp\{-r^{\varepsilon}\},$$

which is very close to one. Hence under these asymptotic assumptions, we expect the area of IR to be approximately $(rm)^2 - rm \log r \log m$.

6 The Best Response Structure

The sets $P(M)$ and $O(M)$ do not reflect the matrix structure of the game (A, B), only the pairings of the payoffs. Let X and Y denote the disjoint copies of $\{1, \ldots, n\}$ that form the vertices of the bipartite graph $K_{n,n}$. We can capture the matrix structure of the game by partitioning $X \times Y$ into r sets $R_1, \ldots, R_r$, each of size m, corresponding to the rows $1, \ldots, r$, and in sets $C_1, \ldots, C_m$, each of size r, corresponding to the columns, so that $|R_i \cap C_j| = 1$ for all i, j. The number of such partitionings, ignoring the labeling of the rows and columns, is $(n!)^2/(r!m!)$.

We can capture the "best response structure" of the game by considering a directed bipartite graph D, with nodes $\{R_1, \ldots, R_r\}$ and $\{C_1, \ldots, C_m\}$, with one arc leaving every node. (R_i, C_j) is an arc if $b_{ij} > b_{ik}$ for all $k \neq j$. Similarly, (C_j, R_i) is an arc if $a_{ij} > a_{kj}$ for all $k \neq i$. For an example, see Figure III in the appendix. The graph D describes the best response structure of the game, in that if play is sequential with players announcing their strategy choice, a directed path in D corresponds to a sequence of best responses by two players who always believe that the game will end after their move. We note that D has $r+m$ arcs, while M has rm edges. We also mention that each connected component of D has the structure of a directed (simple) cycle with a set of merging directed paths, each terminating at a vertex of the cycle. Each cycle of length 2 corresponds to a noncooperative equilibrium.

7 Conclusion

We have demonstrated that for an $r \times m$ strongly ordinal bimatrix game, with $n = rm$, asymptotically,

$$\#\text{Games} \approx n^{2n} \gg \#\text{Outcome Sets} \approx n^n \gg \#\text{Pareto Sets} \approx 4^n \gg \#\text{Convex Hulls} \approx e^{n^{2/3}}.$$

Moreover, these results are not appreciably altered by the restriction to individually rational outcomes. Hence, depending on the type of game theoretic information that one is interested in, the number of different sets that must be considered varies dramatically.

Appendix

In the text we have considered properties for general $r \times m$ two person strongly ordinal bimatrix games. Many examples and experiments in the behavioral sciences have utilized the 2×2 matrix[4]. Fortunately, the number of strategically different games of this size is sufficiently small that their properties can be examined and enumerated exhaustively without resorting to approximations. This is no longer true for even the 3×3 game. The 2×2 structure gives rise to 78 strategically different games whereas the 3×3 matrix produces over 65 billion distinct games.

As the 2×2 game is considered so frequently in both experimentation and exposition, this appendix is presented as a convenience to those using 2×2 games for these purposes and adds somewhat to the previous work of Rapoport et al. (1976), O'Neill (1988) and others involved in considering the number of ordinal matrix games and their properties.

There are $(4!)^2=576$ ordinal 2×2 games, of which 78 are strategically different. This reduction is obtained by considering the permutation of rows and columns which does not change the game and by considering the interchange of the row and column players. The latter gives a reduction only if the game is not symmetric and then at best by a factor of 2. Thus in the approximations for larger games, it is of little significance.

The structure of the payoff set for any matrix game can be represented by a point set on the lattice of a two dimensional grid. The $r\times r$ game produces $r^2!$ such sets, thus for the 2×2, there are 24 payoff sets associated with the 78 strategically different games. These are illustrated in Figures 1–24. In the figures we note, below each one, the number assigned by Rapoport et al., for ease of reference. Above each figure the different payoff matrices are given and the noncooperative equilibria are indicated by a circle. A letter M above a game indicates that there is no pure strategy equilibrium point.

Pareto Optimality. A simple measure of the intrinsic level of the cooperative or competitive structure of a game is the number of points in the Pareto set. The more points there are in the Pareto set, the more intrinsically competitive is the payoff structure. Figure 1 shows the most cooperative, and Figure 24 the most competitive class of games.

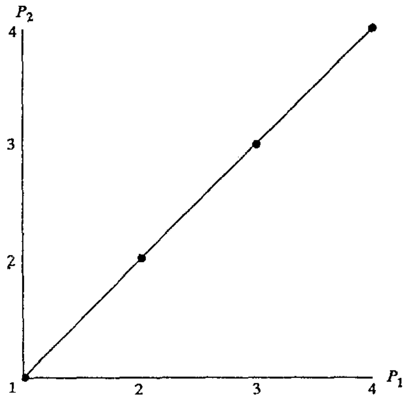
An exhaustive examination of the payoff structures indicate that there are:

6 structures with a 1-point Pareto set,
13 structures with a 2-point Pareto set,
4 structures with a 3-point Pareto set,
1 structure with a 4-point Pareto set.

Convex Hulls. In Figures 1–24, lines have been drawn connecting the points to form the edges of the convex hull for each game. Without considering cardinal payoffs, little significance can be attached to them. But with cardinal payoffs, they do provide a indication of the size of the domain of payoffs arising from the use of mixed strategies (including correlated mixed strategies).

4 From the point of view of experimentation and work in psychology, the 2×2, 2×3, and 3×3 matrix games deserve special attention as human perceptions and abilities to plan appear to be limited as the size of the strategy and outcome sets grow. The bounds suggested sby Miller (1956) are exhausted by the 3×3 matrix with 18 numbers or 9 pairs.

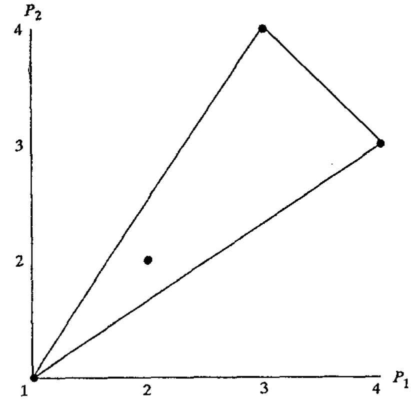
$$\begin{bmatrix} 4,4 & 3,3 \\ 2,2 & 1,1 \end{bmatrix} \begin{bmatrix} 4,4 & 3,3 \\ 1,1 & 2,2 \end{bmatrix} \begin{bmatrix} 4,4 & 2,2 \\ 1,1 & 3,3 \end{bmatrix}$$



1, 23, 59

Fig. 1

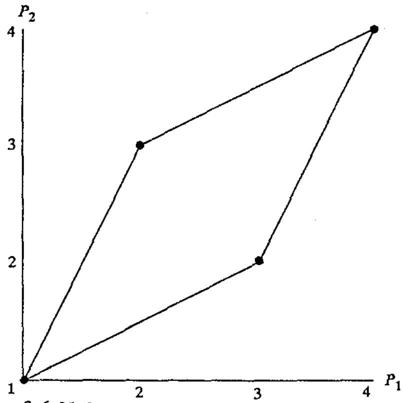
$$\begin{bmatrix} 3,4 & 4,3 \\ 2,2 & 1,1 \end{bmatrix} \begin{bmatrix} 3,4 & 4,3 \\ 1,1 & 2,2 \end{bmatrix} \begin{bmatrix} 2,2 & 3,4 \\ 4,3 & 1,1 \end{bmatrix} \begin{bmatrix} 2,2 & 4,3 \\ 3,4 & 1,1 \end{bmatrix}$$



20, 50, 68, 69

Fig. 2

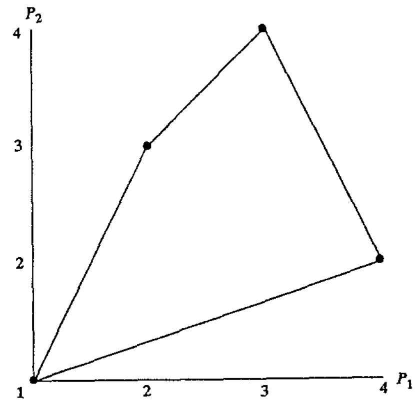
$$\begin{bmatrix} 4,4 & 3,2 \\ 2,3 & 1,1 \end{bmatrix} \begin{bmatrix} 4,4 & 2,3 \\ 3,2 & 1,1 \end{bmatrix} \begin{bmatrix} 4,4 & 3,2 \\ 1,1 & 2,3 \end{bmatrix} \begin{bmatrix} 4,4 & 2,3 \\ 1,1 & 3,2 \end{bmatrix}$$



3, 6, 25, 58

Fig. 3

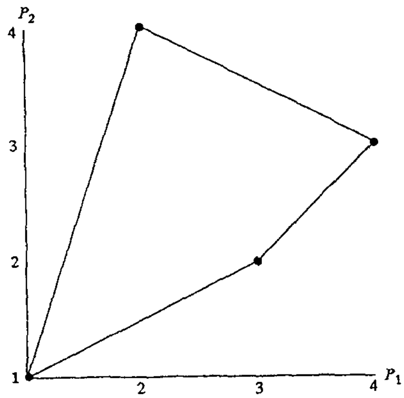
$$\begin{bmatrix} 3,4 & 4,2 \\ 2,3 & 1,1 \end{bmatrix} \begin{bmatrix} 3,4 & 4,2 \\ 1,1 & 2,3 \end{bmatrix} \begin{bmatrix} 2,3 & 4,2 \\ 1,1 & 3,4 \end{bmatrix} \begin{bmatrix} 2,3 & 3,4 \\ 4,2 & 1,1 \end{bmatrix} \begin{bmatrix} 3,4 & 1,1 \\ 2,3 & 4,2 \end{bmatrix}$$



13, 52, 57, 67, 34

Fig. 4

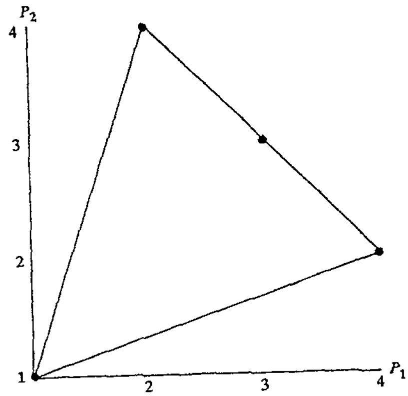
$$\begin{bmatrix} 2,4 & 4,3 \\ 1,1 & 3,2 \end{bmatrix}$$



55

Fig. 5

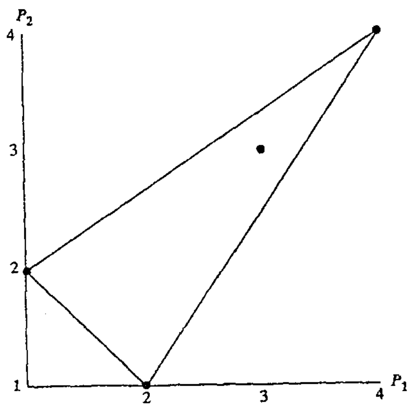
$$\begin{bmatrix} 3,3 & 4,2 \\ 2,4 & 1,1 \end{bmatrix}\begin{bmatrix} 3,3 & 2,4 \\ 4,2 & 1,1 \end{bmatrix}\begin{bmatrix} 3,3 & 4,2 \\ 1,1 & 2,4 \end{bmatrix}\begin{bmatrix} 2,4 & 4,2 \\ 1,1 & 3,3 \end{bmatrix}$$



7, 66, 54, 56

Fig. 6

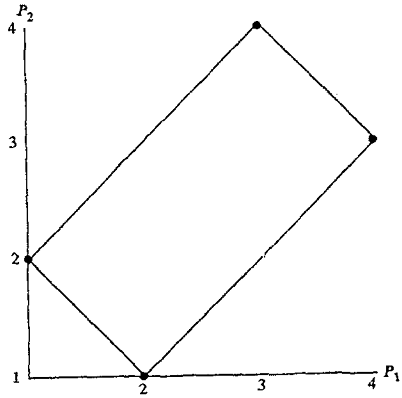
$$\begin{bmatrix} 4,4 & 3,3 \\ 1,2 & 2,1 \end{bmatrix}\begin{bmatrix} 4,4 & 3,3 \\ 2,1 & 1,2 \end{bmatrix}\begin{bmatrix} 4,4 & 1,2 \\ 2,1 & 3,3 \end{bmatrix}\begin{bmatrix} 4,4 & 1,2 \\ 2,1 & 3,3 \end{bmatrix}$$



2, 22, 60, 3

Fig. 7

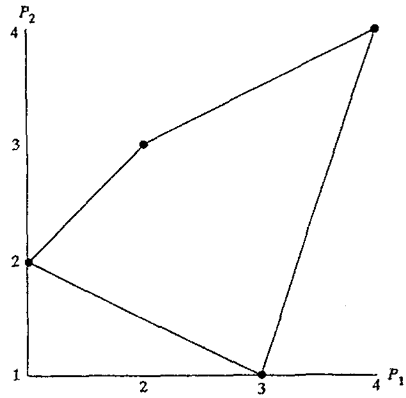
$$\begin{bmatrix} 3,4 & 4,3 \\ 1,2 & 2,1 \end{bmatrix}\begin{bmatrix} 3,4 & 4,3 \\ 2,1 & 1,2 \end{bmatrix}\begin{bmatrix} 3,4 & 2,1 \\ 1,2 & 4,3 \end{bmatrix}$$



19, 49, 64

Fig. 8

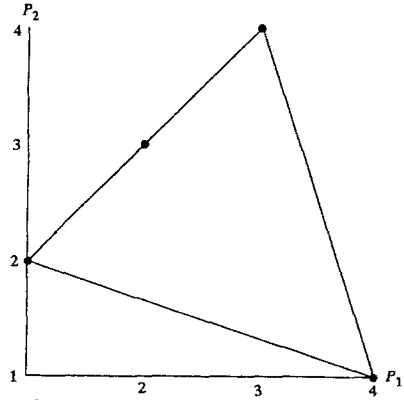
$$\begin{bmatrix} 4,4 & 1,2 \\ 3,1 & 2,3 \end{bmatrix} \begin{bmatrix} 4,4 & 3,1 \\ 1,2 & 2,3 \end{bmatrix} \begin{bmatrix} 4,4 & 2,3 \\ 3,1 & 1,2 \end{bmatrix}$$



62, 29, 26

Fig. 9

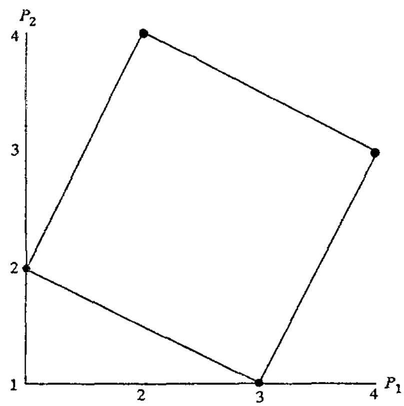
$$\begin{bmatrix} 3,4 & 4,1 \\ 2,3 & 1,2 \end{bmatrix} \begin{bmatrix} 3,4 & 1,2 \\ 2,3 & 4,1 \end{bmatrix} \begin{bmatrix} 3,4 & 4,1 \\ 1,2 & 2,3 \end{bmatrix} \begin{bmatrix} 3,4 & 2,3 \\ 1,2 & 4,1 \end{bmatrix} \begin{bmatrix} 2,3 & 4,1 \\ 1,2 & 3,4 \end{bmatrix}$$



15, 33, 41, 37, 47

Fig. 10

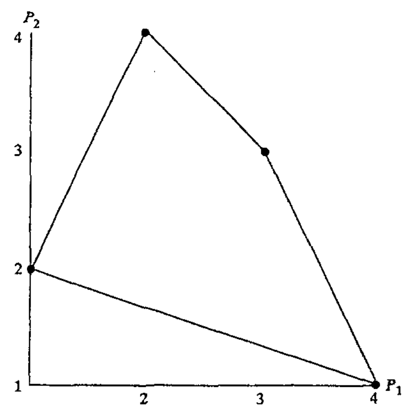
$$\begin{bmatrix} 2,4 & 4,3 \\ 1,2 & 3,1 \end{bmatrix} \begin{bmatrix} 2,4 & 3,1 \\ 1,2 & 4,3 \end{bmatrix}$$



21, 65

Fig. 11

$$\begin{bmatrix} 3,3 & 4,1 \\ 1,2 & 2,4 \end{bmatrix} \begin{bmatrix} 2,4 & 4,1 \\ 1,2 & 3,3 \end{bmatrix} \begin{bmatrix} 2,4 & 3,3 \\ 1,2 & 4,1 \end{bmatrix}$$



43, 44, 39

Fig. 12

$$\begin{bmatrix} 4,4 & 3,2 \\ 1,3 & 2,1 \end{bmatrix} \begin{bmatrix} 4,4 & 3,2 \\ 2,1 & 1,3 \end{bmatrix} \begin{bmatrix} 4,4 & 2,1 \\ 3,2 & 1,3 \end{bmatrix} \begin{bmatrix} 4,4 & 2,1 \\ 1,3 & 3,2 \end{bmatrix}$$

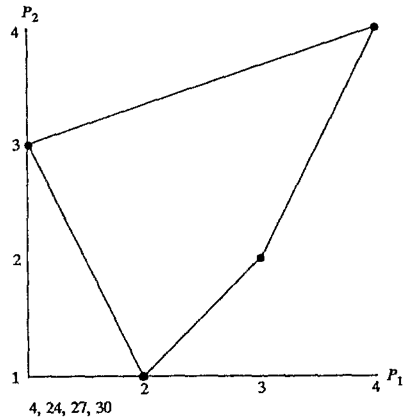


Fig. 13

$$\begin{bmatrix} 3,4 & 4,2 \\ 1,3 & 2,1 \end{bmatrix} \begin{bmatrix} 3,4 & 2,1 \\ 1,3 & 4,2 \end{bmatrix} \begin{bmatrix} 3,4 & 4,2 \\ 2,1 & 1,3 \end{bmatrix} \begin{bmatrix} 3,4 & 2,1 \\ 4,2 & 1,3 \end{bmatrix}$$

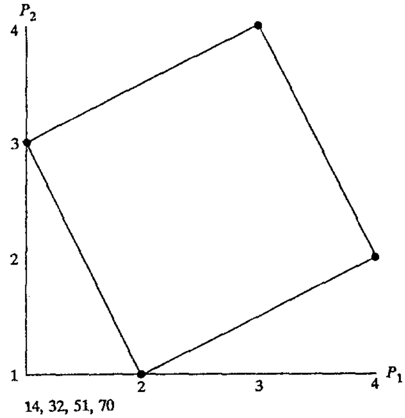


Fig. 14

$$\begin{bmatrix} 4,4 & 3,1 \\ 1,3 & 2,2 \end{bmatrix} \begin{bmatrix} 4,4 & 3,1 \\ 2,2 & 1,3 \end{bmatrix} \begin{bmatrix} 4,4 & 1,3 \\ 3,1 & 2,2 \end{bmatrix}$$

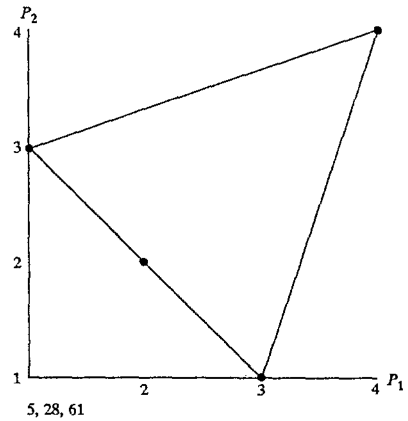


Fig. 15

$$\begin{bmatrix} 3,4 & 4,1 \\ 1,3 & 2,2 \end{bmatrix} \begin{bmatrix} 3,4 & 2,2 \\ 1,3 & 4,1 \end{bmatrix} \begin{bmatrix} 3,4 & 1,3 \\ 2,2 & 4,1 \end{bmatrix} \begin{bmatrix} 3,4 & 2,2 \\ 2,2 & 1,3 \end{bmatrix} \begin{bmatrix} 2,2 & 4,1 \\ 1,3 & 3,4 \end{bmatrix}$$

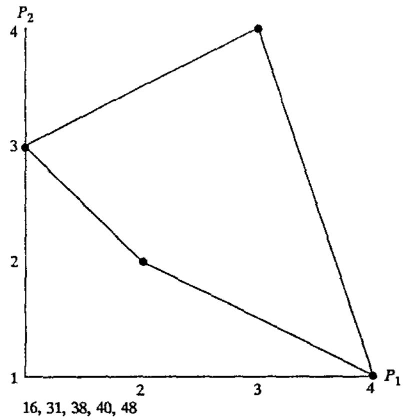
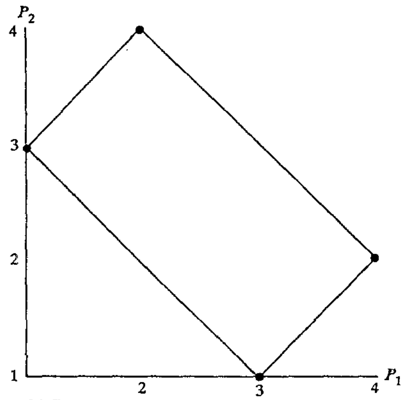


Fig. 16

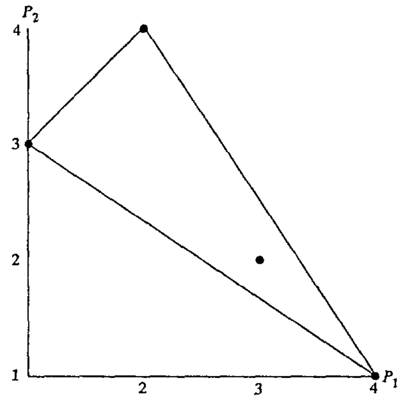
$$\begin{bmatrix} 2,4 & 3,1 \\ 1,3 & 4,2 \end{bmatrix} \overset{M}{\begin{bmatrix} 2,4 & 3,1 \\ 4,2 & 1,3 \end{bmatrix}} \begin{bmatrix} 2,4 & 4,3 \\ 1,3 & 3,1 \end{bmatrix}$$



36, 74, 17

Fig. 17

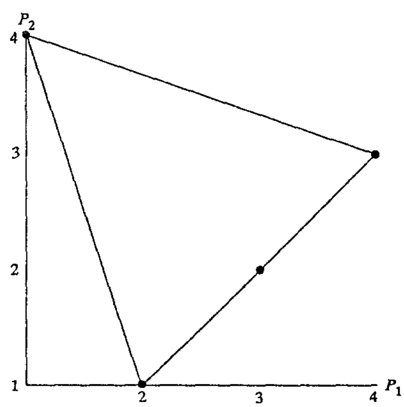
$$\begin{bmatrix} 2,4 & 4,1 \\ 1,3 & 3,2 \end{bmatrix} \begin{bmatrix} 2,4 & 3,2 \\ 1,3 & 4,1 \end{bmatrix} \overset{M}{\begin{bmatrix} 2,4 & 4,1 \\ 3,2 & 1,3 \end{bmatrix}} \begin{bmatrix} 3,2 & 4,1 \\ 1,3 & 2,4 \end{bmatrix}$$



18, 35, 73, 46

Fig. 18

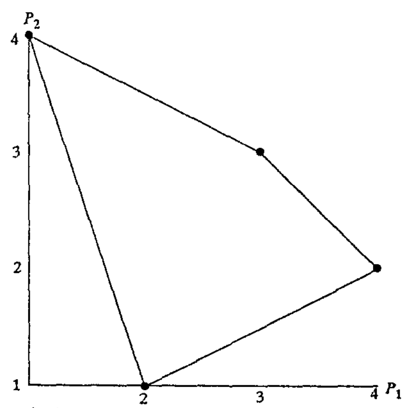
$$\overset{M}{\begin{bmatrix} 3,2 & 2,1 \\ 4,3 & 1,4 \end{bmatrix}}$$



72

Fig. 19

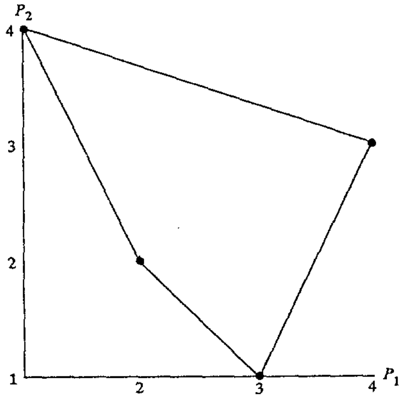
$$\begin{bmatrix} 3,3 & 4,2 \\ 1,4 & 2,1 \end{bmatrix} \begin{bmatrix} 3,3 & 4,2 \\ 2,1 & 1,4 \end{bmatrix} \overset{M}{\begin{bmatrix} 3,3 & 2,1 \\ 4,2 & 1,4 \end{bmatrix}}$$



8, 53, 71

Fig. 20

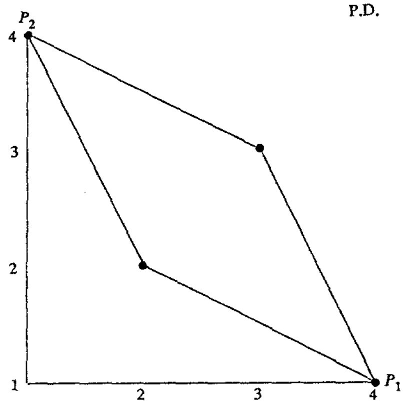
$$\begin{array}{c} M \\ \begin{bmatrix} 2,2 & 3,1 \\ 4,3 & 1,4 \end{bmatrix} \end{array}$$



78

Fig. 21

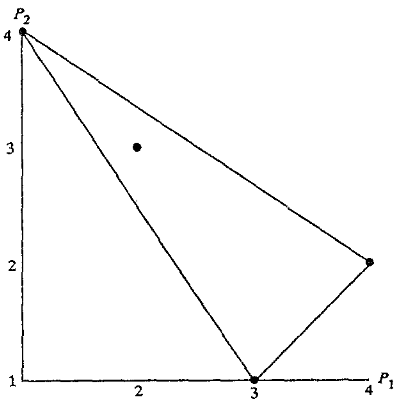
$$\begin{bmatrix} 3,3 & 4,1 \\ 1,4 & 2,2 \end{bmatrix} \begin{bmatrix} 3,3 & 4,1 \\ 2,2 & 1,4 \end{bmatrix} \begin{array}{c} M \\ \begin{bmatrix} 2,2 & 4,1 \\ 3,3 & 1,4 \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{bmatrix} 2,2 & 4,1 \\ 1,4 & 3,3 \end{bmatrix} \\ \text{P.D.} \end{array}$$



9, 42, 77, 12

Fig. 22

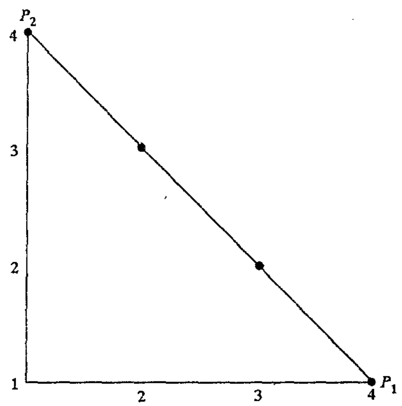
$$\begin{bmatrix} 2,3 & 4,2 \\ 1,4 & 3,1 \end{bmatrix} \begin{array}{c} M \\ \begin{bmatrix} 2,3 & 3,1 \\ 4,2 & 1,4 \end{bmatrix} \end{array}$$



10, 76

Fig. 23

$$\begin{bmatrix} 2,3 & 4,1 \\ 1,4 & 3,2 \end{bmatrix} \begin{bmatrix} 3,2 & 4,1 \\ 2,3 & 1,4 \end{bmatrix} \begin{array}{c} M \\ \begin{bmatrix} 2,3 & 4,1 \\ 3,2 & 1,4 \end{bmatrix} \end{array}$$



11, 45, 75

Fig. 24

We note that of the 24 diagrams there are only seven distinct shapes, all others can be obtained from these seven by rotation or reflextion. The seven basically different shapes are indicated by Figures 1, 2, 3, 4, 6, 8, and 11, of which we observe that all but Figures 4 and 11 are symmetric by interchanging the players (i.e., for any point (x, y) in the set, (y, x) is also in the set). Table 1 groups the figures together in terms of the baxic seven, with the others obtained by reflection and/or rotation. In the table, figure numbers are separated by a slash if the associated figures are related by reflection about the line $x=y$ (i.e., interchange of players).

Table 1

1, 24
2, 7, 18/23
3, 22
4/5, 9/13, 12/20, 16/21
6, 10/19, 15
8, 17
11/14

We caution the reader that if games G_1 and G_2 are related by interchange of players, then although they are equivalent in the sense of Rapoport et al., they may be associated with different figures (1-24). Specifically, if G_1 and G_2 are related in the sense of rapoport et al., and G_1 and G_2 are associated with Figures F_1 and F_2, respectively, then either $F_1=F_2$ or F_1 and F_2 are related by reflection about the line $x=y$.

Noncooperative Equilibria. Surveying all 78 games, the distribution of pure strategy noncooperative equilibrium points is shown in Table 2. There are a total of $(0\cdot 9)+(1\cdot 58)+(2\cdot 11)=80$ noncooperative quilibria, and we indicate the number that are and are not Pareto optimal.

Table 2

	Number of Games	Pareto optimal	Not optimal
0 NE	9	0	0
1 NE	58	57	1
2 NE	11	16	6

In essnce the concern with the noncooperative equilibrium (abbreviated as NE) is for decentralized decision making. Hopefully a unique Pareto- optimal NE might be reached by simple sequential best response, this is true for the 2×2 matrix, but it is not generally true for a 3×3 or larger matrix. the 3×3 below provides a counterexample. If the players do not start in either row or column 2, they will cycle on the four corners.

$$\begin{bmatrix} 5,6 & 4,4 & 8,5 \\ 3,3 & 9,9 & 2,2 \\ 6,7 & 1,1 & 7,8 \end{bmatrix}$$

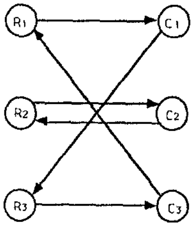


Fig. III

The Expected Ranking at an NE. We may consider three forms of simplistic best response. A point is selected randomly, then (1) Player *I* moves first and selects his myopic immediate best response, (2) Player *II* moves first, and (3) Players *I* and *II* move simultaneously each selecting his myopic bestz response.

If we apply these three methods to the 3×3 matrix above, we obtain the following for either of the sequential moves there is a probability of 5/9 of attaining (9,9) and a probability of 4/9 of going into a 4-cycle on the corners. For the simultaneous move, there is a probability of 1/9 of attaining (9,9) and a probability of 4/9 each for a 4-cycle on the corners or on the four middle outside cells.

Individual rationality. Although we have calculated the Pareto set, some of the points in this set may not be individually rational. In order to calculate the individually rational part of the Pareto set, we must calculate the pure strategy maxmin for each player. The maxmin will either be at 2 or 3 for each player. As we should expect in Figure 24 (games of pure opposition), if the game has a saddlepoint, it coincides with the NE and the Pareto set is reduced to a single point.

Table 3 shows, for each figure, the number of Pareto optimal outcomes (both when we consider the payoffs to be ordinal and cardinal) and the minimum payoff to player *I* among his individually rational payoffs that are Pareto optimal. We have also included the area of each convex hull as it gives a measure of the size of the payoff set assuming that the payoffs are cardinal and that correlated mixed strategies are permitted.

The $n \times n$ Grid. The representation of the payoff matrix gives rise to a question concerning the shapes of the convex hull of the feasible set of payoffs. However, we can ask this question more generally about an $n \times n$ grid where not all grid sizes are related to games. For example, we may ask how many shapes can be obtained with 5 points placed on a 5×5 grid with only one point occupying any row or column. There are 23 different shapes for this "$\sqrt{5} \times \sqrt{5}$ game". In this "game" and all smaller there is a 1:1 relationship between the convex hull and the payoffs of the game. The first time this does not occur is in the 2×3 game (see Figure IV).

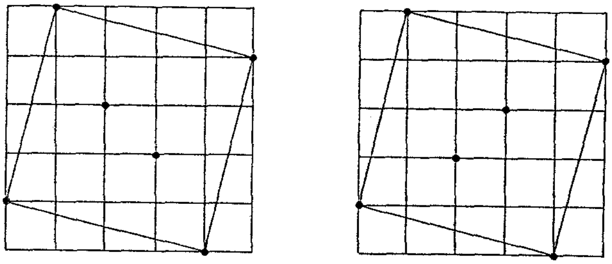


Fig. IV

On the Interpretation of Limiting Results. In the text we have concentrated on limiting results as the matrices become large. A *caveat* should be stressed. There are special games, such as those of pure opposition (where payoff matrix $A = n+1-B$) and pure coordination (where payoff matrix $A = B$) which fast become an insignificant percentage of all games. But the applications of game theory to human activity frequently depend on special structure, hence a class of games which in some general sense may tend towards a set of measure zero, may in actuality be of considerable significance. In particular, this is true of games involving payoffs with ties.

Table 3

Figure	Ordinal PO	Cardinal PO	min IR PÜO	Area
1	1	1	1	0
2	2	2	2	2.5
3	1	1	1	3
4	2	2	2	4
5	2	2	2	4
6	3	3	1	4
7	1	1	1	2.5
8	2	2	2	4
9	1	1	1	4
10	2	2	1	4
11	2	2	2	5
12	3	3	2	4
13	1	1	1	4
14	2	2	2	5
15	1	1	1	4
16	2	2	1	4
17	2	2	1	4
18	3	2	1	2.5
19	2	2	1	4
20	3	3	2	4
21	2	2	1	4
22	3	3	1	3
23	3	2	1	2.5
24	4	4	1	0
Averages:	48/24	46/24	31/24	82/24

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Received July 1991
Revised version July 1992