RANDOM POLYTOPES IN SMOOTH CONVEX BODIES

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Abstract. Let $K \subset R^d$ be a convex body and choose points $x_1, x_2, \ldots, x_n$ randomly, independently, and uniformly from K. Then $K_n = \operatorname{conv}\{x_1, \ldots, x_n\}$ is a random polytope that approximates K (as $n \to \infty$) with high probability. Answering a question of Rolf Schneider we determine, up to first order precision, the expectation of $\operatorname{vol} K - \operatorname{vol} K_n$ when K is a smooth convex body. Moreover, this result is extended to quermassintegrals (instead of volume).

§1. *Introduction and the theorems.* Assume $K \subset R^d$ is a convex body (a convex compact set with nonempty interior) and let $x_1, \ldots, x_n$ be points chosen randomly, independently, and uniformly from K. Set $X_n = \{x_1, \ldots, x_n\}$ and call $K_n = \operatorname{conv} X_n$ a random polytope. In this paper we determine $E(K, n)$, the expectation of $\operatorname{vol}(K \backslash K_n)$ for smooth convex bodies up to first order precision when n tends to infinity.

The asymptotic behaviour of $E(K, n)$ has been known for different classes of convex bodies. Rényi and Sulanke [**8** and **9**] proved that for a polygon $P \subset R^2$

$$E(P, n) = \tfrac{2}{3}(\#\text{ vert } P)(\text{Area } P)\frac{\log n}{n} + O\left(\frac{1}{n}\right),$$

and for smooth convex bodies $K \subset R^2$ (with positive curvature κ)

$$E(K, n) = (\tfrac{2}{3})^{1/3}\Gamma(\tfrac{5}{3}) \int_{\partial K} \kappa^{1/3} dz \left(\frac{n}{\text{Area } K}\right)^{-2/3} + O\left(\frac{1}{n}\right). \tag{1.1}$$

Wieacker [**13**] determined the asymptotic behaviour of $E(B^d, n)$ where B^d is the Euclidean unit ball of R^d:

$$E(B^d, n) = c_0(d) n^{-2/(d+1)}(1 + o(1)),$$

where $c_0(d)$ is an explicit constant. This can be improved using a recent result of Affentranger [**1**] to

$$E(B^d, n) = c_0(d) n^{-2/(d+1)} + O(n^{-4/(d+1)}) \tag{1.2}$$

(for $d \geq 3$ only, for $d = 2$ one has (1.1)). This improvement is implicit in [**1**] and, in fact, if the computations there are carried out to second order precision then one obtains (1.2).

The asymptotic behaviour of $E(P, n)$ was determined by van Wel [**14**] and also by Affentranger and Wieacker [**2**] for simple polytopes $P \subset R^d$ and by Bárány and Buchta [**4**] for general ones.

In this paper we answer a question of Schneider and Wieacker [**12**, p. 69] which is repeated as a conjecture in Schneider [**11**, p. 222].

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THEOREM 1. *If $K \subset R^d$ is a $\mathscr{C}^3$ convex body with positive Gauss Kronecker curvature κ, then*

$$E(K, n) = c(d) \int_{\partial K} \kappa^{1/(d+1)} dz \left(\frac{n}{\text{vol } K}\right)^{-2/(d+1)} + O(n^{-3/(d+1)} \log^2 n),$$

where $c(d)$ is a constant.

This has been known for $d=2$ and also for $d=3$ [**13**]. The idea behind the proof of the theorem is that K is very close to an ellipsoid E near $z \in \partial K$ and so K_n is close to E_m, the random polytope in E with m points for some suitable m. The main difficulty in the proof is to show that K_n near $z \in \partial K$ does not depend on the x_i's far from z. This means that the shape of K_n near z is "independent" of the shape of K far from z—a fact that has been successfully applied to determine $E(P, n)$ where P is a polytope (see [**4**]).

The same idea works when one wants to determine, up to first order precision, the expectation of the difference between the quermassintegrals of K and K_n. Denote by $W_i^{(d)}(K) = W_i(K)$ the i-th quermassintegral of K $(i = 0, 1, \ldots, d-1)$, see [**5**] or [**10**] for a definition. This includes and generalizes the cases of volume, surface area, and mean width that are constant multiples of W_0, W_1 and W_{d-1}. We introduce the notation

$$E(K, i, n) = E(W_i(K) - W_i(K_n)).$$

Affentranger [**1**] determined $E(B^d, i, n)$ up to first order precision:

$$E(B^d, i, n) = c_0(d, i) n^{-2/(d+1)}(1 + o(1)). \tag{1.3}$$

Here and in the next theorem, $o(1)$ could be replaced by $O(n^{-1/(d+1)} \log^2 n)$, again.

THEOREM 2. *If $K \subset R^d$ is a C^3 convex body with positive Gauss Kronecker curvature κ, then*

$$E(K, i, n) = c(d, i) \int_{\partial K} \kappa^{[i/(d-1)]+[1/(d+1)]} dz \left(\frac{n}{\text{vol } K}\right)^{-2/(d+1)} (1 + o(1)). \tag{1.4}$$

This has been known for $i = d-1$, *i.e.*, for the mean width [**12**]. The case $i = 0$ is Theorem 1 and the proof will be very similar so I will give only a sketch.

Let $f_i(P)$ denote the number of i-dimensional faces of the polytope $P \subset R^d$. It is clear (at least for the author) that there will be a similar theorem for the expectation of $f_i(K_n)$ and that this theorem will have the following form

$$Ef_i(K_n) = b(d, i) \int_{\partial K} \kappa^{1/(d+1)} dz \left(\frac{n}{\text{vol } K}\right)^{(d-1)/(d+1)} (1 + o(1)), \tag{1.5}$$

for some constant $b(d, i)$ provided K satisfies the conditions of Theorem 1 (*cf.* [**3**]). (1.5) is known to be true for $i = d-1$ [**13**] and $i = d-2$ since $f_{d-2} = \frac{1}{2} d f_{d-1}$ for a simplicial d-polytope, and for $i = 0$ it follows from Theorem

1 and from an identity due to Efron [7] saying

$$Ef_0(K_n) = \frac{n}{\text{vol } K} E(K, n-1).$$

As K_n is a simplicial polytope with probability one the numbers $Ef_i(K_n)$ satisfy the Dehn-Sommerville equations. This shows that conjecture (1.5) is true for $i = 0, 1, \ldots, d-1$ when $d = 3, 4, 5$. When $d > 5$ the conjecture is open even for $K = B^d$, the unit ball.

§2. *Preliminaries.* We will need several properties of smooth convex bodies. So assume $K \subset R^d$ is a $\mathscr{C}^3$ convex body with positive curvature $\kappa = \kappa(z)$ at every $z \in \partial K$. Then there is a constant $t_0 > 0$ such that $x \in K$, dist $(x, \partial K) = t \leq t_0$ implies x can be written uniquely as

$$x = z - ta, \tag{2.1}$$

where $z \in \partial K$ and a is the outer unit normal to K at z. Here z, a, t all depend on x but we will usually not denote this dependence. The constant t_0 depends on K only. This will be true for all the constants Δ_0, $b_1, b_2, \ldots, c_1, c_2, \ldots$, to come (unless stated otherwise).

Assume now that the principal radii of K at $z \in \partial K$ are all equal, $R = R_1 = R_2 = \ldots = R_{d-1} = \kappa^{-1/(d-1)}$. Let H^t be the halfspace

$$H^t = \{y \in R^d : (y - (z - at)) \,.\, a \geq 0\}, \tag{2.2}$$

with the notation of (2.1). Also, write $B(y, r)$ for the ball with centre y and radius r. Then, for $t \leq \Delta$

$$H^t \cap B(z - (R - \Delta)a, R - \Delta) \subset H^t \cap K \subset H^t \cap B(z - (R + \Delta)a, R + \Delta), \tag{2.3}$$

provided $\Delta \leq \Delta_0$ for some constant $\Delta_0 > 0$. Consequently

$$b_1(R - \Delta)^{(d-1)/2} t^{(d+1)/2} \leq \text{vol}\,(H^t \cap K) \leq b_1(R + \Delta)^{(d-1)/2} t^{(d+1)/2}, \tag{2.4}$$

where the constant b_1 depends only on d. Write $D = B(z - (R + \Delta)a, R + \Delta)$. We can estimate vol $(H^\Delta \cap K)$ with small error

$$|\text{vol}\,(H^\Delta \cap K) - \text{vol}\,(H^\Delta \cap D)| \leq b_2 \Delta^{(d+2)/2}. \tag{2.5}$$

Define now $u : K \to R$ by

$$u(x) = \text{vol}\,(K \cap (x - K)).$$

The region $K \cap (x - K)$ is centrally symmetric with respect to x. Moreover, if x is close to ∂K then $K \cap (x - K)$ is close to $(H^t \cap K) \cup (x - (H^t \cap K))$ with t and H^t coming from (2.1) and (2.2). More precisely, for $x \in K$ and $t = \text{dist}\,(x, \partial K) < t_0$

$$b_3 t^{(d+1)/2} \leq u(x) \leq b_4 t^{(d+1)/2}. \tag{2.6}$$

This follows from (2.4) if K is "circular" around z (*i.e.*, $R_1 = R_2 = \ldots = R_{d-1}$).

Otherwise (2.4) changes to

$$b_1\left(\prod_{i=1}^{d-1}(R_i-\Delta)\right)^{1/2} t^{(d+1)/2} \leq \operatorname{vol}(H^t \cap K) \leq b_1\left(\prod_{i=1}^{d-1}(R_i+\Delta)\right)^{1/2} t^{(d+1)/2},$$

and $\prod_{i=1}^{d-1} R_i = \kappa$ so (2.6) follows again.

A proof of the above facts can be found in [12] pages 71-72 of Schneider and Wieacker.

We will often use the following inequality (see [6] or [3] for a proof)

$$\operatorname{Prob}(x \notin K_n) \leq 2\sum_{i=0}^{d-1}\binom{n}{i}\left(\frac{u(x)}{2\operatorname{vol} K}\right)^i\left(1-\frac{u(x)}{2\operatorname{vol} K}\right)^{n-i}. \tag{2.7}$$

Here $\operatorname{Prob}(x \notin K_n)$ is meant with x fixed and K_n, the random polytope in K varying.

§3. *Proof of Theorem* 1. We assume $d \geq 3$ (for $d = 2$ see [8] or (1.1)) and also that $\operatorname{vol} K = 1$. The proof is split into several steps.

Step 1. The theorem is true for the ball rB^d.

Proof. If $K = rB^d$ then $\kappa = r^{-(d-1)}$ and (1.4) says

$$E(rB^d, n) = c(d)\int_{\partial(rB^d)} \kappa^{1/(d+1)} dz\left(\frac{n}{\omega_d r^d}\right)^{-2/(d+1)} + O(n^{-3/(d+1)}\log^2 n)$$

$$= c(d)d\omega_d^{(d+3)/(d+1)} r^d n^{-2/(d+1)} + O(n^{-3/(d+1)}\log^2 n), \tag{3.1}$$

which is correct according to (1.2).

Step 2. There is a constant c_1 such that with

$$t_1 = t_1(n) = c_1((\log n)/n)^{2/(d+1)},$$

$$E(K, n) = \int_{x: t \leq t_1} \operatorname{Prob}(x \notin K_n)dx + O(n^{-1}), \tag{3.2}$$

where the integration is taken over all $x \in K$ with $t = \operatorname{dist}(x, \partial K) \leq t_1$.

Proof. Clearly $E(K, n) = \int_K \operatorname{Prob}(x \notin K_n)dx$. According to (2.6) $t \geq t_1$ implies $u(x) \geq 3((\log n)/n)$ with $t_1 = c_1((\log n)/n)^{2/(d+1)}$. Then

$$I = \int_{x: t \geq t_1} \operatorname{Prob}(x \notin K_n)dx \leq \int_{u(x) \geq 3(\log n)/n} \operatorname{Prob}(x \notin K_n)dx.$$

Set $\lambda_0 = \lceil 3 \log n \rceil$ and apply (2.7). Then, the same way as in [6],

$$I \leqslant \int_{u(x) \geqslant 3(\log n)/n} 2 \sum_{i=0}^{d-1} \binom{n}{i} \left(\frac{u(x)}{2}\right)^i \left(1 - \frac{u(x)}{2}\right)^{n-i} dx$$

$$\leqslant \sum_{\lambda=\lambda_0}^{n} 2 \int_{(\lambda-1)/n \leqslant u(x) \leqslant \lambda/n} \sum_{0}^{d-1} \binom{n}{i} \left(\frac{u(x)}{2}\right)^i \left(1 - \frac{u(x)}{2}\right)^{n-i} dx$$

$$\leqslant \sum_{\lambda=\lambda_0}^{n} 2 \sum_{i=0}^{d-1} \binom{n}{i} \left(\frac{\lambda}{2n}\right)^i \left(1 - \frac{\lambda - 1}{2n}\right)^{n-i}$$

$$\leqslant \sum_{\lambda=\lambda_0}^{n} 2 \sum_{i=0}^{d-1} \frac{\lambda^i}{2^i i!} \exp\left(-\frac{\lambda}{2} + \frac{d}{2}\right) \leqslant O(n^{-1}).$$

Let $x \in K$ with $x = z - ta$ as in (2.1). Assume $0 \leqslant t \leqslant t_1$. It is easy to see that

$$dx = \left(1 + O\left(\left(\frac{\log n}{n}\right)^{2/(d+1)}\right)\right) dt dz.$$

Using this in (3.2) we get

$$\int_{x : t \leqslant t_1} \operatorname{Prob}(x \notin K_n) dx$$

$$= \int_{z \in \partial K} \int_{\substack{x = z - at \\ 0 \leqslant t \leqslant t_1}} \operatorname{Prob}(x \notin K_n) dt dz \left(1 + O\left(\left(\frac{\log n}{n}\right)^{2/(d+1)}\right)\right). \tag{3.3}$$

Near the point z, K looks like an ellipsoid $E(z)$. If one applies an affine transformation $T = T(z)$ of determinant 1 that leaves every point of the line $z - ta$ unchanged and the tangent plane to K at z invariant, then $\operatorname{Prob}(x \notin K_n) = \operatorname{Prob}(x \notin (TK)_n)$ identically for every $x = z - ta$, $t \geqslant 0$. Choose such a T that carries $E(z)$ to a ball $B(z)$. Clearly $B(z) = B(z - ra, r)$ where $r = \kappa^{-1/(d-1)}$ with $\kappa = \kappa(z)$. From now on we assume that $T(z)$ has been applied at z. Set $m = \lfloor n \omega_d r^d \rfloor$. The basic idea of this proof is that K_n is similar to $B(z)_m$ near the point z. More precisely our aim is to prove

$$\left| \int_{x : t \leqslant t_1} \operatorname{Prob}(x \notin K_n) dx - \int_{x : t \leqslant t_1} \operatorname{Prob}(x \notin B(z)_m) dx \right|$$

$$= O(n^{-3/(d+1)} \log^2 n). \tag{3.4}$$

Here in the second integral $x : t \leqslant t_1$ means $\{x \in B(z) : \operatorname{dist}(x, \partial B(z)) \leqslant t_1\}$. Notice that $t_1 = c_1'((\log m)/m)^{2/(d+1)}$ for some constant c_1'.

Step 3. (3.4) implies the theorem.

Proof. According to (3.1)

$$E(B(z), m) = c(d) d \omega_d^{(d+3)/(d+1)} r^d m^{-2/(d+1)} + O(m^{-3/(d+1)} \log^2 m).$$

Step 2, applied to $K=B(z)$ and $n=m$ and (3.3) give

$$E(B(z),m)=\left[\int_{\bar z\in\partial B(z)}\int_{\substack{x:t\le t_1\\ x=\bar x-at}}\operatorname{Prob}(x\notin B(z)_m)\,dt\right]d\bar z+O(m^{-1}).$$

The expression in the brackets does not depend on $\bar z$ so we get from the two representations of $E(B(z),m)$ that, with the particular choice $\bar z=z$,

$$\int_{\substack{x:t\le t_1\\ x=z-at}}\operatorname{Prob}(x\notin B(z)_m)\,dt=c(d)\omega_d^{2/(d+1)}rm^{-2/(d+1)}+O(m^{-3/(d+1)}\log^2 m)$$

$$=c(d)\kappa^{1/(d+1)}n^{-2/(d+1)}+O(n^{-3/(d+1)}\log^2 n). \tag{3.5}$$

Then (3.2), (3.3), (3.4) and (3.5) prove that, indeed

$$E(K,n)=c(d)\int_{\partial K}\kappa^{1/(d+1)}dz\,n^{-2/(d+1)}+O(n^{-3/(d+1)}\log^2 n). \tag{3.6}$$

From now on I will drop z from the notation if there is no ambiguity. So $B=B(z)$, $a=a(z)$, *etc.* Set $\Delta=c_2((\log n)/n)^{2/(d+1)}$ where c_2 will be fixed later and will be much larger than c_1. Recall notation (2.2),

$$H^t=\{y\in R^d:(y-(z-at))\,.\,a\ge 0\}.$$

Write $D=B(z-(r+\Delta)a,\,r+\Delta)$ and

$$D^\Delta=D\cap H^\Delta,\qquad B^\Delta=B\cap H^\Delta,\qquad K^\Delta=K\cap H^\Delta.$$

Then, by (2.3), (2.4) and (2.5) we get

$$D^\Delta\supset K^\Delta,\qquad D^\Delta\supset B^\Delta,$$
$$\operatorname{vol}(D^\Delta\backslash K^\Delta)\le b_2\Delta^{(d+2)/2},$$
$$\operatorname{vol}(D^\Delta\backslash B^\Delta)\le b_2\Delta^{(d+2)/2},$$
$$\operatorname{vol}D^\Delta\approx b_1(r+\Delta)^{(d-1)/2}\Delta^{(d+1)/2},$$

provided Δ is small enough (*i.e.*, n is large enough).

Set $p=\lfloor n\omega_d(r+\Delta)^d\rfloor$. We mention at once that

$$\left|\int_{x:t\le t_1}\operatorname{Prob}(x\notin D_p)\,dx-\int_{x:t\le t_1}\operatorname{Prob}(x\notin B_m)\,dx\right|=O(n^{-3(d+1)}\log^2 n).$$

This follows from (3.5) immediately. (Notice that $t_1=c_1''((\log p)/p)^{2/(d+1)}$ and $t_1=c_1'((\log m)/m)^{2/(d+1)}$ with suitable constants c_1', c_1''.) This shows, in turn, that it is enough to prove (3.4) with D and p instead of B and m, *i.e.*,

$$\left|\int_{x:t\le t_1}\operatorname{Prob}(x\notin K_n)\,dx-\int_{x:t\le t_1}\operatorname{Prob}(x\notin D_p)\,dx\right|$$

$$=O(n^{-3/(d+1)}\log^2 n). \tag{3.7}$$

Next I show that $K_n \cap H^\Delta$ is essentially independent of the x_i's not lying in H^Δ.

Step 4. If $\operatorname{dist}(x, \partial K) \leq t_1$, then

$$|\operatorname{Prob}(x \notin \operatorname{conv}(X_n \cap K^\Delta)) - \operatorname{Prob}(x \notin \operatorname{conv} X_n)| = O(n^{-1}), \tag{3.8}$$

where $X_n = \{x_1, \ldots, x_n\}$ is the random n-set from K.

We need the following lemma (see [4] for a similar statement with a similar, if more involved, application). We write $\operatorname{ray}(x, y) = \{x + \tau y : \tau > 0\}$.

LEMMA. *If $x, x_1, \ldots, x_n$ are in general position and $x \in K^\Delta \cap \operatorname{conv} X_n$ but $x \notin \operatorname{conv}(X_n \cap K^\Delta)$, then there is an $x_i \in X_n \backslash K^\Delta$ with*

$$\operatorname{ray}(x, x_i) \cap \operatorname{conv}(X_n \backslash (K' \cup \{x_i\})) = \varnothing.$$

Proof of the lemma. Identify x with the origin for this proof. Then the conditions say that $\operatorname{cone} X_n = R^d$ but $\operatorname{cone}(X_n \cap K^\Delta) \neq R^d$ and $\operatorname{cone}(X_n \backslash K') \neq R^d$. But $\operatorname{cone} X_n = \operatorname{cone}(X_n \cap K^\Delta) + \operatorname{cone}(X_n \backslash K')$ so $\operatorname{cone}(X_n \backslash K')$ must contain an extreme ray, defined by some $x_i \in X_n \backslash K'$ which is not in $\operatorname{cone}(X_n \cap K^\Delta)$. Then $x_i \notin X_n \cap K^\Delta$ as well and the ray does not meet $\operatorname{conv}(X_n \backslash (K' \cap \{x_i\}))$ as required.

Proof of Step 4. Clearly

$$\begin{aligned} 0 &\leq \operatorname{Prob}(x \notin \operatorname{conv}(X_n \cap K^\Delta)) - \operatorname{Prob}(x \notin \operatorname{conv} X_n) \\ &= \operatorname{Prob}(x \notin \operatorname{conv}(X_n \cap K^\Delta) \text{ but } x \in \operatorname{conv} X_n) \\ &\leq \operatorname{Prob}(\exists x_i \in X_n \backslash K^\Delta : \operatorname{ray}(x, x_i) \cap \operatorname{conv}(X_n \backslash (K' \cup \{x_i\})) = \varnothing) \\ &\qquad\qquad \text{(by the Lemma)} \\ &\leq n \int_{y \in K \backslash K^\Delta} \operatorname{Prob}(\operatorname{ray}(x, y) \cap \operatorname{conv}(X_{n-1} \backslash K') = \varnothing)\, dy. \end{aligned}$$

Fix $t \in [0, t_1]$ and write u_t for the u-function of the convex set $K \backslash K^t$. Let $y_0 \in \operatorname{ray}(x, y)$ be the point maximizing u_t on $\operatorname{ray}(x, y)$. We claim that if c_2 is large enough then

$$u_t(y_0) \geq 4d(\log n)/n.$$

Indeed,

$$u_t(y_0) \geq u(y_0) - 2u(x).$$

By (2.6) $u(x) \leq b_4 t_1^{(d+1)/2}$ and $u(y_0) \geq b_3[\operatorname{dist}(y_0, \partial K)]^{(d+1)/2}$. As K is very close to a ball near z, $\operatorname{dist}(y_0, \partial K) \geq \Delta/4$ follows quite easily. So

$$\begin{aligned} u_t(y_0) &\geq b_3\left(\frac{\Delta}{4}\right)^{(d+1)/2} - 2b_4 t_1^{(d+1)/2} = \left(b_3\left(\frac{c_2}{4}\right)^{(d+1)/2} - 2b_4 c_1^{(d+1)/2}\right)\frac{\log n}{n} \\ &\geq 4d\frac{\log n}{n}, \end{aligned}$$

as claimed if c_2 is large enough. Evidently $u_t(y_0) \leq 1$.

Now with $\alpha = \text{vol } K^t$ we get

$$\begin{aligned}
&\text{Prob}\,(\text{ray}\,(x, y) \cap \text{conv}\,(X_{n-1} \backslash K^t) = \emptyset) \\
&\quad = \sum_{j=0}^{n-1} \text{Prob}\,(\text{ray}\,(x, y) \cap \text{conv}\,(X_{n-1} \backslash K^t) = \emptyset \mid |X_{n-1} \cap K^t| = j) \\
&\qquad \times \binom{n-1}{j} \alpha^j (1-\alpha)^{n-1-j} \\
&\quad = \sum_{j=0}^{n-1} \text{Prob}\,(\text{ray}\,(x, y) \cap (K \backslash K^t)_{n-j-1} = \emptyset) \binom{n-1}{j} \alpha^j (1-\alpha)^{n-1-j} \\
&\quad \leq \sum_{j=0}^{n-1} \text{Prob}\,(y_0 \notin (K \backslash K^t)_{n-j-1}) \binom{n-1}{j} \alpha^j (1-\alpha)^{n-1-j} \quad (\text{by } (2.7)) \\
&\quad \leq \sum_{j=0}^{n-1} 2 \sum_{i=0}^{d-1} \binom{n-j-1}{i} \left(\frac{u_t(y_0)}{2(1-\alpha)} \right)^i \left(1 - \frac{u_t(y_0)}{2(1-\alpha)} \right)^{n-1-j-i} \\
&\qquad \times \binom{n-1}{j} \alpha^j (1-\alpha)^{n-1-j} \\
&\quad = 2 \sum_{i=0}^{d-1} \binom{n-1}{i} \left(\frac{u_t(y_0)}{2} \right)^i \left(1 - \frac{u_t(y_0)}{2} \right)^{n-1-i} \\
&\quad \leq 2dn^{d-1} \left(1 - \frac{4d \log n}{2n} \right)^{n-1} = O(n^{-2}).
\end{aligned}$$

Now we apply Step 4 with $K = D$ and $n = p$ to get

$$|\text{Prob}\,(x \notin \text{conv}\,(Y_p \cap D^\Delta)) - \text{Prob}\,(x \notin \text{conv } Y_p)| = O(n^{-1}), \tag{3.9}$$

where $Y_p = \{y_1, \ldots, y_p\}$ is a random p-set in D.

Write now $\beta = \text{vol } K^\Delta$ and $\gamma = \text{vol } D^\Delta / \text{vol } D$. Clearly

$$\begin{aligned}
&\text{Prob}\,(x \notin \text{conv}\,(X_n \cap K^\Delta)) \\
&\quad = \sum_{k=0}^{n} \text{Prob}\,(x \notin \text{conv}\,(X_n \cap K^\Delta) \mid |X_n \cap K^\Delta| = k) \binom{n}{k} \beta^k (1-\beta)^{n-k} \\
&\quad = \sum_{k=0}^{n} \text{Prob}\,(x \notin K_k^\Delta) \binom{n}{k} \beta^k (1-\beta)^{n-k},
\end{aligned} \tag{3.10}$$

and analogously

$$\text{Prob}\,(x \notin \text{conv}\,(Y_p \cap D^\Delta)) = \sum_{k=0}^{p} \text{Prob}\,(x \notin D_k^\Delta) \binom{p}{k} \gamma^k (1-\gamma)^{p-k}. \tag{3.11}$$

We know that $\beta \approx \gamma (r+\Delta)^d \omega_d \approx \text{const}\,(\log n)/n$ and $p \approx n(r+\Delta)^d \omega_d$. The next two steps follow easily from the properties of the binomial distribution and the choice of γ and p (we omit the proofs).

Step 5. There are numbers $k_1 = \lfloor c_3 \log n \rfloor$ and $k_2 = \lceil c_4 \log n \rceil$ with $k_1 < k_2$ such that the contribution of the terms with $k < k_1$ and $k > k_2$ in both (3.10) and (3.11) is less than $O(n^{-1})$.

Step 6. For $k = k_1, k_1 + 1, \ldots, k_2$

$$\left|\binom{n}{k}\beta^k(1-\beta)^{n-k} - \binom{p}{k}\gamma^k(1-\gamma)^{p-k}\right| = \binom{n}{k}\beta^k(1-\beta)^{n-k}\left(1 + O\left(n\left(\frac{\log n}{n}\right)^{(d+2)/(d+1)}\right)\right).$$

Step 7. For $t \in [0, t_1]$ and for $k = k_1, \ldots, k_2$

$$|\text{Prob}\,(x \notin K_k^\Delta) - \text{Prob}\,(x \notin D_k^\Delta)| = O\left(\left(\frac{\log n}{n}\right)^{1/(d+1)} \log n\right).$$

Proof. Let $Z_k = \{z_1, \ldots, z_k\}$ denote the random k-set in D^Δ. Set $\delta = \text{vol}\,(D^\Delta \backslash K^\Delta)/\text{vol}\, D^\Delta$. Then $\delta < b_5 \Delta^{1/2}$ by (2.5) and (2.6). Define

$$P_1 = |\text{Prob}\,(x \notin \text{conv}\,(Z_k \cap K^\Delta)) - \text{Prob}\,(x \notin K_k^\Delta)|,$$
$$P_2 = |\text{Prob}\,(x \notin \text{conv}(Z_k \cap K^\Delta)) - \text{Prob}\,(x \notin D_k^\Delta)|.$$

Clearly $|\text{Prob}\,(x \notin K_k^\Delta) - \text{Prob}\,(x \notin D_k^\Delta)| \le P_1 + P_2$. Moreover,

$$\begin{aligned} P_1 &\le \sum_{j=0}^{k} |\text{Prob}\,(x \notin \text{conv}\,(Z_k \cap K^\Delta) \mid |Z_k \cap K^\Delta| = k-j) \\ &\qquad - \text{Prob}\,(x \notin K_k^\Delta)|\binom{k}{j}\delta^j(1-\delta)^{k-j} \\ &= \sum_{j=0}^{k} |\text{Prob}\,(x \notin K_{k-j}^\Delta) - \text{Prob}\,(x \notin K_k^\Delta)|\binom{k}{j}\delta^j(1-\delta)^{k-j} \\ &\le \sum_{j=1}^{k}\binom{k}{j}\delta^j(1-\delta)^{k-j} < \sum_{j=1}^{k}(k\delta)^j < \frac{k\delta}{1-k\delta} = O\left(\left(\frac{\log n}{n}\right)^{1/(d+1)} \log n\right). \end{aligned}$$

Quite similarly,

$$\begin{aligned} P_2 &\le \sum_{j=0}^{k} |\text{Prob}\,(x \notin \text{conv}\,(Z_k \cap K^\Delta) \mid |Z_k \cap K^\Delta| = k-j) \\ &\qquad - \text{Prob}\,(x \notin \text{conv}\, Z_k \mid |Z_k \cap K^\Delta| = k-j)|\binom{k}{j}\delta^j(1-\delta)^{k-j} \\ &= \sum_{j=1}^{k} |\text{Prob}\,(\cdots) - \text{Prob}\,(\cdots)|\binom{k}{j}\delta^j(1-\delta)^{k-j} = O\left(\left(\frac{\log n}{n}\right)^{1/(d+1)} \log n\right). \end{aligned}$$

Finally, we prove (3.7)

$$
\begin{aligned}
&|\mathrm{Prob}\,(x\notin K_n)-\mathrm{Prob}\,(x\notin D_p)|\\
&\quad\le |\mathrm{Prob}\,(x\notin \mathrm{conv}\,(X_n\cap K^\Delta))-\mathrm{Prob}\,(x\notin \mathrm{conv}\,(Y_p\cap D^\Delta))|+O(n^{-1})\\
&\quad\le \sum_{k=k_1}^{k_2}\Bigg|\mathrm{Prob}\,(x\notin K_k^\Delta)\binom{n}{k}\beta^k(1-\beta)^{n-k}-\mathrm{Prob}\,(x\notin D_k^\Delta)\\
&\qquad\qquad\times\binom{p}{k}\gamma^k(1-\gamma)^{p-k}\Bigg|+O(n^{-1})\\
&\quad\le \sum_{k=k_1}^{k_2}|\mathrm{Prob}(x\notin K_k^\Delta)-\mathrm{Prob}\,(x\notin D_k^\Delta)|\binom{n}{k}\beta^k(1-\beta)^{n-k}\\
&\qquad\qquad+O\left(n\left(\frac{\log n}{n}\right)^{(d+2)/(d+1)}\right)\\
&\quad\le \sum_{k=k_1}^{k_2}c_5\left(\frac{\log n}{n}\right)^{1/(d+1)}\log n\binom{n}{k}\beta^k(1-\beta)^{n-k}+O\left(n\left(\frac{\log n}{n}\right)^{(d+2)/(d+1)}\right)\\
&\quad=O\left(\left(\frac{\log n}{n}\right)^{1/(d+1)}\log n\right).
\end{aligned}
$$

Integrating this on $[0, t_1]$ with $t_1=c_1((\log n)/n)^{2/(d+1)}$ we get (3.7).

§4. *Sketch of the proof of Theorem* 2. Recall first **[5, 10]** that

$$
W_i^{(d)}(K)=\int_{F\in G}\mathrm{vol}_{d-i}\,(\mathrm{proj}_F\,(K))\,d\omega(F),
$$

where $G=G(d,d-i)$ is the Grassmannian of the $(d-i)$-dimensional subspaces of R^d, ω is the (unique) rotation-invariant measure on G normalized suitably, and $\mathrm{proj}_F: R^d\to F$ denotes orthogonal projection onto $F\in G$. Then

$$
\begin{aligned}
E(K,i,n)&=E(W_i^{(d)}(K)-W_i^{(d)}(K_n))\\
&=E\int_G \mathrm{vol}_{d-i}\,(\mathrm{proj}_F\,(K)\backslash\mathrm{proj}_F\,(K_n))\,d\omega(F)\\
&=\int_G E\,\mathrm{vol}_{d-i}\,(\mathrm{proj}_F\,(K)\backslash\mathrm{proj}_F\,(K_n))\,d\omega(F)\\
&=\int_G\int_{\bar{x}\in F}\mathrm{Prob}\,(\bar{x}\notin\mathrm{proj}_F\,(K_n))\,d\bar{x}\,d\omega(F). \qquad (4.1)
\end{aligned}
$$

Now Step 1 follows as before using (1.3). Also, Step 2 goes the same way because $\mathrm{Prob}\,(\bar{x}\notin\mathrm{proj}_F\,(K_n))$ is very small when $\bar{x}$ is far from the boundary of $\mathrm{proj}_F\,(K)$.

Write $t = \text{dist}(\bar{x}, \partial \,\text{proj}_F(K))$, $\bar{x} = \bar{z} - \bar{a}t$ with $\bar{z} \in \partial \,\text{proj}_F(K)$ and $\bar{a}$ the outer unit normal to $\text{proj}_F(K)$ at $\bar{z}$. Clearly $\bar{z} = \text{proj}_F z$ for a unique $z \in \partial K$ where $a = \bar{a}$ is the outer normal to K at z.

So we get from Step 2 and (4.1) with $t_1 \leq c_1((\log n)/n)^{2/(d+1)}$

$$E(K, i, n) = \int_G \int_{\substack{\bar{x} \in F \\ t \leq t_1}} \text{Prob}\,(\bar{x} \notin \text{proj}_F(K_n))\, d\bar{x}\, d\omega(F) + O(n^{-1})$$

$$= \int_{F \in G} \int_{\substack{z \in \partial K \\ a \in F}} \int_{\substack{\bar{x} = \bar{z} - ta \\ t \leq t_1}} \text{Prob}\,(\bar{x} \notin \text{proj}_F(K_n))\, dt\, dz\, d\omega(F) + O(n^{-1})$$

$$= \int_{z \in \partial K} \int_{\substack{x = z - ta \\ t \leq t_1}} \int_{\substack{F \in G \\ a \in F}} \text{Prob}\,((x + F^{\perp}) \cap K_n = \varnothing)\, d\omega(F)\, dt\, dz + O(n^{-1}), \tag{4.2}$$

because $\bar{x} \notin \text{proj}_F(K_n)$, if, and only if, $(x + F^{\perp}) \cap K_n = \varnothing$ where $x = z - ta$, $a \in F$. ($F^{\perp}$ denotes the orthogonal complementary subspace of F.)

Apply now the same affine transformation T as in the previous proof. Then, for $x = z - ta$ and $a \in F$,

$$\text{Prob}\,((x + F^{\perp}) \cap K_n = \varnothing) = \text{Prob}\,((x + (TF)^{\perp}) \cap (TK)_n = \varnothing)$$

identically in x. Our aim is to prove (with the same notation as earlier) that

$$\int_{t \leq t_1} \int_{\substack{F \in G \\ a \in F}} (\text{Prob}\,((x + F^{\perp}) \cap K_n = \varnothing) - \text{Prob}\,((x + F^{\perp}) \cap B_m = \varnothing))\, d\omega(F)\, dt$$
$$= o(n^{-2/(d+1)}). \tag{4.3}$$

Now Step 3 says that (4.3) implies the theorem and the proof is analogous.

Set $A = x + F^{\perp}$. This is an i-dimensional affine subspace. For $y \in A \cap K$ $\text{dist}(y, \partial K) \leq t$ where t comes from (2.1), *i.e.*, from $x = z - ta$. Moreover,

$$\max\,\{\text{dist}\,(y, \partial K) \colon y \in A \cap K\} = t.$$

Letting $\Delta = c_2((\log n)/n)^{2/(d+1)}$ again, Step 4 says that

$$|\text{Prob}\,(A \cap \text{conv}\,(X_n \cap K^{\Delta}) = \varnothing) - \text{Prob}\,(A \cap \text{conv}\, X_n = \varnothing)| = o(1).$$

The proof of this follows the same lines. The auxiliary lemma we need here is

LEMMA. *If $A, x_1, x_2, \ldots, x_n$ are in general position and $A \cap K \subset K^{\Delta}$, $A \cap \text{conv}\, X_n \neq \varnothing$ but $A \cap \text{conv}\,(X_n \cap K^{\Delta}) = \varnothing$, then there exists an $x_i \subset X_n \backslash K^{\Delta}$ with*

$$(A + \text{ray}\,(0, x_i - x)) \cap \text{conv}\,(X_n \backslash (K^t \cup d\{x_i\})) = \varnothing.$$

The rest of the proof is the same.

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