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ON AFFINELY EMBEDDABLE SETS IN THE PROJECTIVE PLANE

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In this note we prove a conjecture of Bisztriczky and Schaer [1] about convex sets in the real projective plane P^2. It will be simpler to formulate the result for convex cones in R^3 and then show that it implies the conjecture. A cone $C \subset R^3$ is called pointed if it contains no line, i.e., when $x \in C$ and $-x \in C$ imply $x=0$. Here is the result:

THEOREM 1. *Assume $n \geqq 3$ and $C_1, \ldots, C_n \subset R^3$ are closed, pointed, convex cones with common apex the origin O. Assume that for $i \neq j$ $(i, j=1, 2, \ldots, n)$ there is an $e(i,j) \in \{-1, +1\}$ such that for all $k=1, \ldots, n$, $k \neq i, j$ and for both $e=1, -1$*

$$(i, j; k, e) \qquad (eC_k) \cap (C_1 + e(i,j)C_j) = \{O\}.$$

Then there is a plane P through O such that for all $i=1, \ldots, n$, $P \cap C_i = \{O\}$.

We will now translate this theorem from R^3 to P^2. For a convex pointed cone $C \subset R^3$ set $S(C) = S^2 \cap C$ where S^2 is the unit sphere of R^3. P^2 is obtained from S^2 by identifying antipodal points. With this identification the points of $S(C)$ and $-S(C) = S(-C)$ give rise to a set $P(C) \subset P^2$. Clearly, $P(C) = P(-C)$.

A set $A \subset P^2$ is called convex if there exists a line L in P^2 disjoint from A and A is convex in the affine plane $P^2 \setminus L$ (cf. [2] or [1]). A convex set A in P^2 gives rise to two connected subsets $S^+(A)$ and $S^-(A) = -S^+(A)$ of S^2, whose cone hulls are $C^+(A)$ and $C^-(A)$, respectively. Evidently, $C^+(A) = -C^-(A)$. In this way one can see that $A \subset P^2$ is convex if and only if $A = P(C)$ for some pointed convex cone $C \subset R^3$.

Now let $A_1, A_2 \subset P^2$ be convex. We want to define the convex hull of their union. Then $A_j = P(C_j)$ for some pointed convex cone $C_j \subset R^3$ and also $A_j = P(-C_j)$ $(j=1, 2)$. So the union of A_1 and A_2 will have, in general, two convex hulls: $H_1(A_1, A_2) = P(\text{conv}\,(C_1, C_2))$ and $H_2(A_1, A_2) = P(\text{conv}\,(C_1, -C_2))$. Of course, H_1 and H_2 will be convex only if $C_1 - C_2 = \text{conv}\,(C_1, -C_2)$ and $C_1 + C_2 = \text{conv}(C_1, C_2)$ are pointed cones.

We can now formulate Theorem 1 in P^2.

THEOREM 2. *Let $A_1, \ldots, A_n$ be closed convex sets in P^2 $(n \geqq 3)$. Assume that for $i \neq j$ $(i, j=1, \ldots, n)$ either $A_k \cap H_1(A_i, A_j) = \emptyset$ for all $k \neq i, j$ or $A_k \cap H_2(A_i, A_j) = \emptyset$ for all $k \neq i, j$. Then there is a line $L \subset P^2$ disjoint from each A_i.*

In [1], the collection of the sets $A_1, \ldots, A_n$ is called affinely embeddable when the conclusion of Theorem 2 holds.

In the proof of Theorem 1 we will use standard techniques from the theory of convex cones in finite dimensional spaces (cf. [3], [4] or [5]).

When proving Theorem 1 we will obtain its dual form which seems to be worth mentioning:

THEOREM 3. *Assume* $D_1, \ldots, D_n \subset R^3$ $(n \geqq 3)$ *are closed, pointed, convex cones with common apex the origin. Suppose that for* $i \neq j$ $(i, j = 1, \ldots, n)$ *there is an* $e(i, j) \in \{-1, +1\}$ *such that for all* $k = 1, \ldots, n$, $k \neq i, j$ *and for both* $e = 1$ *and* -1 $(eD_k) \cap D_i \cap (e(i, j) D_j) \neq \{O\}$. *Then there are signs* $e_1, \ldots, e_n$ $(e_i = +1$ or $-1)$ *and a vector* $p \in R^3 \setminus \{O\}$ *such that* $p \in e_i D_i$ *for all* $i = 1, \ldots, n$.

PROOF OF THEOREM 1. Assume the theorem is false and take a counterexample $C_1, \ldots, C_n \subset R^3$ of closed, convex, pointed cones satisfying condition $(i, j; k, e)$ such that for all planes P through the origin there is an $i \in \{1, \ldots, n\}$ with $P \cap C_i \neq \{O\}$.

We will modify this counterexample. We *claim* first that for $i \neq j$ both $C_i + C_j$ and $C_i - C_j$ are pointed and closed convex cones. We prove this for $C_i + C_j$, the proof for $C_i - C_j$ is identical. By condition $(i, k; j, -1)$

$$(-C_j) \cap C_i \subset (-C_j) \cap (C_i + e(i, k) C_k) = \{O\},$$

so C_i and $(-C_j)$ can be separated (strictly, because they are closed), i.e,. there exists $v \in R^3$ such that $v \cdot x < 0$ for all $x \in C_i \setminus \{O\}$ and $v \cdot y > 0$ for all $y \in (-C_j) \setminus \{O\}$. (Here $v \cdot x$ denotes the scalar product of $v, x \in R^3$.) Then $v \cdot z < 0$ for all $z \in (C_i + C_j) \setminus \{O\}$ proving that $(C_i + C_j)$ is pointed.

Now we prove that $C_i + C_j$ is closed. Assume it is not, then there are elements $x_m \in C_i$ and $y_m \in C_j$ with $x_m, y_m \in S^2$ and positive numbers α_m, β_m such that $z_m = \alpha_m x_m + \beta_m y_m$ is in $(C_i + C_j) \cap S^2$ but $z = \lim z_m$ is not. By the compactness of S^2 we may assume that $x = \lim x_m$ and $y = \lim y_m$ exists. Then α_m and β_m must tend to infinity and so $z_m \in S^2$ is possible only if $x + y = 0$. This implies that $C_i + C_j$ contains the line through x and $-x = y$ which is impossible because it is a pointed cone.

We define, for a closed pointed cone $C \subset R^3$ and for $\alpha > 0$ the set

$$C^\alpha = \{x \in R^3: \text{ there is } y \in C \text{ with } \sphericalangle xOy \leqq \alpha\},$$

where $\sphericalangle xOy$ denotes the angle of the triangle xOy at vertex O. C^α is clearly a convex, pointed cone with nonempty interior provided α is small enough.

Condition $(i, j; k, e)$ says that the two closed and pointed cones $C_i + e(i, j) C_j$ and eC_k are disjoint (except for the common apex). Then there is $\alpha(i, j; k, e) > 0$ such that for $0 < \alpha < \alpha(i, j; k, e)$

$$(eC_k^\alpha) \cap (C_i^\alpha + e(i, j) C_j^\alpha) = \{O\};$$

and C_i^α, C_j^α, C_k^α, $C_i^\alpha + e(i, j) C_j^\alpha$ are all pointed, convex, closed cones. Set $\beta = \min \alpha(i, j; k, e)$ and take a closed polyhedral cone B_i with nonempty interior satisfying

$$C_i \subset B_i \subset C_i^\beta \quad \text{for} \quad i = 1, \ldots, n.$$

We may choose the finitely many halflines generating the cones B_i to be in general position. We will clarify later what is meant by general position here.

This is what we have now: The cones B_i are convex, closed, pointed and polyhedral with nonempty interior, and they satisfy condition $(i, j; k, e)$. Moreover, for each plane P through the origin $P \cap \operatorname{int} B_i \neq \{O\}$ for some $i = 1, \ldots, n$.

Consider now the polars $D_i = B_i^*$ of B_i defined as

$$D_i = \{x \in R^3 \colon\ x \cdot y \leqq 0 \ \text{for}\ y \in B_i\}.$$

The D_i's are convex, closed, pointed, polyhedral cones in R^3 with nonempty interior. We *claim* now that condition $(i, j; k, e)$ implies the following condition:

$$(i, j;\ k, e)^* \qquad (-eD_k) \cap D_i \cap (e(i, j) D_j) \neq \{O\},$$

and the last condition in the theorem implies this one: For each $p \in R^3 \setminus \{O\}$ there is an $i \in \{1, \ldots, n\}$ such that

$$(*) \qquad\qquad p \notin D_i \quad \text{and} \quad p \notin -D_i.$$

We prove this claim using standard techniques from the theory of convex polyhedral cones (cf. [4] or [5]). Condition $(i, j; k, e)$ for the cones B_i is of the form $B_k \cap (B_i + B_j) = \{O\}$ (here we dropped the signs) that has polar form $D_k + (D_i \cap D_j) = R^3$. Assume now that $(-D_k) \cap (D_i \cap D_j) = \{O\}$, then the cones $-D_k$ and $(D_i \cap D_j)$ can be separated, i.e., there is $v \in R^3 \setminus \{O\}$ such that $v \cdot x \leqq 0$ for all $x \in -D_k$ and $v \cdot y \geqq 0$ for all $y \in D_i \cap D_j$. But then $v \cdot z \geqq 0$ for all $z \in D_k + (D_i \cap D_j)$, a contradiction. Let us see now the last condition:

$$P \cap \operatorname{int} B_i \neq \{O\},$$

and consider $q \in P \cap \operatorname{int} B_i$ with $q \neq O$. Write p for a normal of the plane P. Then $q \cdot p = 0$ and $q \cdot x < 0$ for all $x \in B_i^* \setminus \{O\} = D_i \setminus \{O\}$, so indeed, $\pm p \notin D_i$.

(As a matter of fact, from now on we will give the proof of Theorem 3 in the case when the sets D_i are polyhedral cones in R^3 with nonempty interior. The general case follows by a standard continuity argument.)

Choose a point $d_i \in \operatorname{int} D_i$ now for $i = 1, \ldots, n$ and shrink each set D_i to the point d_i linearly and simultaneously with a parameter $t \in [0, 1]$, so that the shrinking set $D_i(t)$ equals D_i when $t = 1$ and d_i when $t = 0$. Write I for the set of indices i, j, k, e_i, e_j, e_k and set

$$D_I(t) = (e_i D_i(t)) \cap (e_j D_j(t)) \cap (e_k D_k(t))$$

when $t \in [0, 1]$. We assume that the cones B_i and the points d_i are in general position to ensure that $D_I(1) \neq \{O\}$ implies that $\operatorname{int} D_I(1)$ is nonempty. Moreover, as the cones $D_i(t)$ shrink, the cones $D_I(t)$ shrink as well and $D_I(t) = \{O\}$ for $t < t_0(I)$ where $t_0(I)$ is the smallest t for which $D_I(t)$ is different from $\{O\}$. (If, for some, $D_I(1) = \{O\}$ already, then $t_0(I)$ is not defined.) We assume that the cones B_i and the points d_i are in general position to ensure that $D_I(t)$ is a halfline when $t = t_0(I)$ and that $\operatorname{int} D_I(t) \neq \emptyset$ for $t > t_0(I)$.

As t decreases, condition $(*)$ remains true because the cones D_i get smaller and smaller. But conditions $(i, j; k, e)^*$ will fail for each $(i, j; k, e)$ for some t because $D_I(0) = \{O\}$ for all I. The condition $(i, j; k, e)^*$ holds for all $t > t(i, j; k, e)$ and fails for all $t \leqq t(i, j; k, e)$ where $t(i, j; k, e)$ is uniquely determined. Write t_0 for the largest $t(i, j; k, e)$, then $t_0 = t(i, j; k, e)$ for some $(i, j; k, e)$. We may assume without loss of generality that $i=1$, $j=2$, $k=3$ and $e(1,2)=1$ and $e=-1$. So condition $(1,2;3,-1)^*$ fails, i.e.,

$$D_1(t_0)\cap D_2(t_0)\cap D_3(t_0) = K$$

where K is a halfline of the form $\{\alpha v\colon \alpha \geqq 0\}$ with $v\in R^3\setminus\{O\}$. We know that $D_1(t)\cap \cap D_2(t)\cap D_3(t)$ is $\{O\}$ for $t<t_0$ and has nonempty interior for $t>t_0$. We *claim* now that for each $j=1,2,\ldots,n$, $v\in D_j(t_0)$ or $v\in -D_j(t_0)$. This will contradict condition $(*)$ and so prove the theorem.

The claim is evident when $j=1,2$ and 3. We are going to prove it with notation $j=4$. There are two cases to consider.

1st case. When the intersection of two of the cones $D_j(t_0)$ $(j=1,2,3)$ is equal to K, $D_1(t_0)\cap D_2(t_0)=K$, say. From condition $(2,4;1,e=-1)$ we get for $t=t_0$ that

$$D_1(t_0)\cap D_2(t_0)\cap\big(e(2,4)D_4(t_0)\big) \neq \{O\}.$$

But $K=D_1(t_0)\cap D_2(t_0)$ and so $v\in K\subset e(2,4)D_4(t_0)$ indeed.

2nd case. When the intersection of any two cones $D_j(t_0)$ have nonempty interior $(j=1,2,3)$. Then, by a wellknown theorem (see [3], for instance), there are vectors $a_j\in R^3$ such that $a_j\cdot x\leqq 0$ for all $x\in D_j(t_0)$ $(j=1,2,3)$ and O is in the convex hull of a_1, a_2 and a_3. The case when some a_j is parallel with some other a_i has been dealt with in the first case. So we assume that every a_j is nonzero and $0=\alpha_1a_1+\alpha_2a_2+\alpha_3a_3$ and every $\alpha_j>0$. Then $a_j\cdot x\leqq 0$ $(j=1,2,3)$ implies that $x=\beta v$ for some real number β. Moreover, $a_j\cdot v=0$ for $j=1,2,3$.

Assume now that $\pm v\notin D_4(t_0)$. Then L, the line through v and $-v$ can be separated from $D_4(t_0)$, i.e., there exists a nonzero $a_4\in R^3$ such that $a_4\cdot x<0$ when $x\in D_4(t_0)\setminus\{O\}$ and $a_4\cdot x=0$ when $x\in L$. This shows that the vectors a_i $(i=1,2,3,4)$ are all orthogonal to v and so $a_4=\beta_1a_1+\beta_2a_2$ for some real numbers β_1 and β_2. We show now that β_1 and β_2 are both different from zero. Assume that $\beta_2=0$, say. Then a_1 and a_4 are parallel and, then $D_1(t_0)$ is separated either from $D_4(t_0)$ or from $-D_4(t_0)$, contradicting condition $(1,j;4,\pm 1)^*$.

Consider now condition $(1,2;4,e)^*$: there exists an $x\in R^3\setminus L$ such that

$$x\in\big(-eD_4(t_0)\big)\cap D_1(t_0)\cap D_2(t_0).$$

Then $-ea_4\cdot x<0$, $a_1\cdot x\leqq 0$ and $a_2\cdot x\leqq 0$. This implies that β_1 and β_2 cannot be of the same sign. We may assume that $\beta_1>0$ and $\beta_2<0$.

Suppose now that $e(3,4)=1$ and consider condition $(3,4;2,-1)^*$. In the same way as above this implies the existence of an $x\in R^3\setminus L$ with $a_3\cdot x\leqq 0$, $a_4\cdot x<0$ and $a_2\cdot x\leqq 0$. Now a_1 is a positive linear combination of a_2 and a_4, so $a_1\cdot x<0$. But $a_1\cdot x<0$, $a_2\cdot x\leqq 0$, $a_3\cdot x\leqq 0$ is impossible. Assume now that $e(3,4)=-1$ and consider condition $(3,4;1,-1)^*$. Again, this implies the existence of an $x\in R^3\setminus L$ with $a_3\cdot x\leqq 0$, $a_4\cdot x>0$ and $a_1\cdot x\leqq 0$. Now a_2 is a positive linear combination of a_1 and $-a_4$, so $a_2\cdot x<0$. But $a_1\cdot x\leqq 0$, $a_2\cdot x<0$, $a_3\cdot x\leqq 0$ is impossible.

We mention finally that it is possible to extend these results to higher dimensional spaces but, unfortunately, the conditions in the theorems become rather unintelligible.

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