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THE CARATHEODORY NUMBER FOR THE k-CORE

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The k-core of the set $S \subset \mathbf{R}^n$ is the intersection of the convex hull of all sets $A \subsetneq S$ with $|S \setminus A| \leq k$. The Caratheodory number of the k-core is the smallest integer $f(d,k)$ with the property that $x \in \text{core}_k S$, $S \subset \mathbf{R}^n$ implies the existence of a subset $T \subsetneq S$ such that $x \in \text{core}_k T$ and $|T| \leq f(d,k)$. In this paper various properties of $f(d,k)$ are established.

1. Definitions and results

The k-core of a set $S \subseteq \mathbf{R}^d$ is the intersection of the convex hulls of all sets $A \subseteq S$ with $|S \setminus A| \leq k$, i.e.,

$$\text{core}_k S = \cap\{\text{conv}A : A \subseteq S,\ |S \setminus A| \leq k\}.$$

Here and in what follows we assume S is a finite multiset in $\mathbf{R}^d$. This means that the points in S have "multiplicity". Strictly speaking, a multiset is a map $S : F \to \mathbf{R}^d$ and in our case F is finite. Then $\text{core}_k S = \cap\{\text{conv}S(E) :\ E \subseteq F,\ |F \setminus E| \leq k\}$. From now on we do not say explicitly that the sets in question are multisets. This will make the notation simpler and will not cause confusion.

Alternatively, we can define

$$\text{core}_k S = \cap\{H^+ :\ H^+ \text{ is a closed halfspace with } |H^+ \cap S| \geq |S| - k\}.$$

So the case $k = 0$ is the usual convex hull. Several properties of the k-core are known, c.f. [5] (or [2] Theorem 2.8). In [1], Boros and Füredi extend the definition of the k-core to every real number $k \geq 0$.

We define the Caratheodory number of the k-core as the smallest integer $f(d,k)$ with the property that $x \in \text{core}_k S$, $S \subset \mathbf{R}^d$ implies the existence of a subset $T \subseteq S$ such that $x \in \text{core}_k T$ and $|T| \leq f(d,k)$. By Caratheodory's theorem, $f(d,0) = d+1$. At the 1982 Oberwolfach conference on convex bodies, Micha Perles posed the problem of determining $f(d,k)$. In this paper various properties of

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$f(d,k)$ are established. We determine, for instance, $f(2,k)$ and $f(d,1)$ exactly. We also establish the order of magnitude of $f(d,k)$ when $d \to \infty$ and k is fixed.

Our first theorem shows that $f(d,k)$ is finite for every $d \geq 1$ and $k \geq 0$.

Theorem 1. *If $f(d,k-1)$ is finite then so is $f(d,k)$ and*

$$f(d,k) \leq \max\left((k+1)(d+1), d(1+f(d,k-1)\right)$$

It follows from here that $f(d,k) \leq d^{k+1} + 2d^k + d^{k-1} + \ldots + 1$ and this can be improved to $f(d,k) \leq d^{k+1}$ for all $k \geq 1$ and $d \geq 1$ except $(d,k) = (2,1)$ and $(2,2)$.

A simple lower bound on the Caratheodory number is this:

$$f(d,k) \geq (k+1)(d+1). \tag{1}$$

To see this take for S the set of vertices of a d-dimensional simplex $(k+1)$-times. Then the center of the simplex lies in $\mathrm{core}_k S$ but it does not lie in $\mathrm{core}_k T$ if T is a proper subset of S.

It is readily seen that for $d = 1$ equality holds in (1). This is the case, too, when $d = 2$.

Theorem 2. $f(2,k) = 3(k+1)$.

One might expect that equality holds in (1) for all d and k. That this is not the case is shown by

Theorem 3. *For any n with $d > n > k > 0$ we have*

$$f(d,k) \geq k + d + \binom{n}{k}(d-n). \tag{2}$$

When k is fixed and $d \to \infty$ we get from here and Theorem 1

$$\frac{1}{e^k(k+1)!}d^{k+1} \leq f(d,k) \leq d^{k+1},$$

thus establishing the order of magnitude of $f(d,k)$ for fixed k.

Theorem 4. $f(d,1) = \max(2(d+1), 1 + d + \lfloor d^2/4 \rfloor)$.

Our last theorem shows that $f(d,k)$ grows quite fast when $d \geq 5$ is fixed and k tends to infinity.

Theorem 5. *If $k > d$ and $d \geq 5$, then* $f(d,k) \geq k + d + \frac{1}{2}\binom{k}{\lfloor (d-1)/2 \rfloor}$.

2. Proof of Theorem 1

We start with an observation that will be basic (explicitly or implicitly) for most of the proofs to follow.

Lemma 6. *Assume $S \subset \mathbf{R}^d$ and $|S| > (d+1)(k+1)$ and $y \in \text{core}_k S$ but $y \notin \text{core}_k T$ for any proper subset $T \subset S$. Then there is a point x in* relbd $\text{core}_k S$ *such that $x \notin \text{core}_k T$ for any proper subset $T \subset S$.*

Proof of Lemma 6. As $|S| \geq (k+1)(d+1)+1$, it follows from Tverberg's theorem [6] that there are pairwise disjoint subsets $S_1, \dots, S_{k+1}$ of S whose conver hulls have a point, say $u \in \mathbf{R}^d$, in common. Then $u \in \text{core}_{k+1} S$ and, consequently, $u \in \text{core}_k T$ for every $T \subset S$ with $|T| = |S| - 1$. Let x be the last point in $\text{core}_k S$ on the halfline stemming from u and passing through y (clearly $u \neq y$). If $x \in \text{core}_k T$ for some subset $T \subset S$ with $|T| = |S| - 1$, then $u \in \text{core}_k T$ implies $y \in \text{core}_k T$, a contradiction. ∎

Now we prove Theorem 1. Let $S \subset \mathbf{R}^d$ and assume $|S| > (k+1)(d+1)$ and $x \in \text{core}_k S$ but $x \notin \text{core}_k T$ for any proper subset $T \subset S$. Applying Lemma 6 we may assume $x \notin \text{relbd}\,\text{core}_k S$. Then $x \in \text{bd}\,\text{conv}(S \setminus A)$ for some $A \subset S$, $|A| \leq k$ (as otherwise $x \in \text{int conv}(S \setminus A)$ for all A with $|A| \leq k$. Then by Caratheodory's theorem, applied in $\text{bd}\,\text{conv}(S \setminus A)$, there are points $z_1, \dots, z_d \in S$ with $x \in \text{conv}\{z_1, \dots, z_d\}$. Clearly $x \in \text{core}_{k-1}(S \setminus z_i)$ for each $i = 1, \dots, d$. So (by the induction hypothesis) there exists a subset $T_i \subseteq S \setminus z_i$ with $|T_i| \leq f(d, k-1)$ and $x \in \text{core}_{k-1} T_i$. Define $T = \{z_1, \dots, z_d\} \cup T_1 \cup \dots \cup T_d$. We claim that $x \in \text{core}_k T$. Let $K \subset T$ with $|K| = k$. We have to prove that $x \in \text{conv}(T \setminus K)$. This is obvious if K does not contain any one of the points $z_1, \dots, z_d$. So assume $z_i \in K$. Then $|K \cap T_i| \leq k-1$, consequently $x \in \text{core}_{k-1} T_i \subseteq \text{conv}(T \setminus K)$ as claimed.

This shows that $|T| \leq d + df(d, k-1)$ and proves the theorem. ∎

Remark 7. Using the fact that in Caratheodory's theorem one of the points out of the $d+1$ can be chosen arbitrarily (and some other arguments) we can give a slightly better estimate of $f(d,k)$. We can prove, for instance, that for all $d, k \geq 1$ except $(d,k) = (2,1)$ and $(2,2)$ $f(d,k) \leq d^{k+1}$. This can be further improved by using Theorem 4 as the starting step of the introduction. We omit the details.

Remark 8. This proof works in any abstract convexity space (see [2]) as well.

3. Proof of Theorem 2

In view of (1) we have to show that $f(2,k) \leq 3(k+1)$. We prove this by induction on k. The case $k = 0$ is trivial. So we assume the statement holds for $k-1$ and we prove it for k ($k \geq 1$). Let $S \subset \mathbf{R}^2$ and $|S| \geq 3(k+1)$ and $x \in \text{core}_k S$. We distinguish two cases.

Case 1. $x \in S$. Then clearly $x \in \text{core}_{k-1}(S \setminus x)$, and so, by induction, there is a subset $T \subseteq S \setminus x$, $|T| \leq 3k$ such that $x \in \text{core}_{k-1} T$. Then $|T \cup \{x\}| \leq 3k+1 < 3k+3$ and $x \in \text{core}_k(T \cup \{x\})$.

Case 2. $x \notin S$. Then we assume, without loss of generality, that the points of S are on a unit circle with center x, their clockwise order on this circle is $z_1, z_2, \dots, z_n$

where $n = |S|$. Observe that $x \in \text{core}_k S$ if and only if every $k+1$ consecutive points in the circle span an arc$\leq \pi$.

Suppose S is minimal with respect to $x \in \text{core}_k S$, i.e., $x \notin \text{core}_k T$ for any proper subset $T \subset S$. Then there are $k+3$ consecutive point, $z_1, \ldots, z_{k+3}$ say, spanning an arc larger than π, as otherwise $x \in \text{core}_k(S \setminus \{z_i\})$ for every i. Consider now the point z_{2k+4}. (If $|S| < 2k+4$ then we are finished at once.) As $x \notin \text{core}_k(S \subset \{z_{2k+4}\})$ there are $k+2$ points $z_i, \ldots, z_{2k+3}, z_{2k+5}, \ldots, z_{i+k+2}$ spanning an arc larger than π. Here $i+k+2$ is meant mod $|S|$. We have $i \geq k+3$ as $i+k+2$ cannot be less that $2k+5$. Then $i+k+2 \geq |S|+2$, for otherwise the two arcs of size larger than π wouldn't overlap. But $i \leq 2k+3$ and so $|S| \leq i+k \leq 3k+3$. ∎

4. The example proving Theorem 3

Let $e_1, \ldots, e_d$ be an orthonormal basis $\mathbf{R}^d$. Define $e = \sum_{i=1}^{d} e_i$ and $e_0 = (1/d)e$ and

$$f_i = e_0 + p(e_0 - e_i) \qquad (i = 1, \ldots, d),$$

where $p \geq 0$ will be specified later. Define $[t] = \{1, \ldots, t\}$ when $t > 0$ is an integer. Let $k+1 \leq n < d$. Set $F = \{K : K \subseteq [n], |K| = k\}$ and $D = [d] \setminus [n]$. For $(K, j) \in F \times D$ define

$$e(K,j) = \sum_{i \in [n] \setminus K} e_i + \Big(e_j - (d-n)^{-1} \sum_{i \in D} e_i\Big).$$

Set, finally

$$S = \{0, \ldots, 0, f_1, \ldots, f_d\} \cup \{e(K,j) : (K,j) \in F \times D\}$$

where 0 is taken k times.

This is the set that will prove the estimate in Theorem 3. To see this we first need a lemma.

Lemma 9. *The linear system with variables $\alpha_0, \alpha_i, \alpha(K,j)$*

$$e_0 = \sum_{i=k+1}^{d} \alpha_i f_i + \Sigma^* \alpha(K,j) e(K,j)$$

$$\alpha_0 + \sum_{k+1}^{d} \alpha_i \Sigma^* \alpha(K,j) = 1$$

$$\alpha_0, \alpha_i, \alpha(K,j) \geq 0$$

has a unique solution. Here Σ^* denotes summation over all pairs in $F \times D$.

Proof of Lemma 9. The Lemma means that e_0 is in the relative interior of a sim plex which is a face of the polytope $P = \text{conv}(\{0, f_{k+1}, \ldots, f_n\} \cup \{e(K,j) : (K,j)$ $F \times D\})$. Consider the vector $w = (n-d)(e_1 + \ldots + e_k) + k(e_{n+1} + \ldots + e_{\text{(}}$

and the hyperplane $H = \{x \in \mathbb{R}^d : w \cdot x = 0\}$. One can easily check that $w \cdot e_0 = w \cdot 0 = w \cdot f_{k+1} = \ldots = w \cdot f_n = 0$ while $w \cdot f_{n+1} = \ldots = w \cdot f_d = -pk$ and $w \cdot e(K,j) = (n-d)|[k] \setminus K| \leq 0$, with equality only if $K = [k]$. This means that H supports P in the face with vertices $0, f_{k+1}, \ldots, f_n, e([k], n+1), \ldots, e([k], d)$ with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < 1$ provided $p > 0$ is large enough. This representation is unique for the points $0, f_{k+1}, \ldots, f_n, e([k], n+1), \ldots, e([k], d)$ are affinely independent. The proof of the last statement is left to the reader. ∎

The Lemma shows that

(3) $\qquad e_0 \in \operatorname{conv}(S \setminus \{f_i : i \in K\})$ for every $K \in F$, and

(4) $\qquad e_0 \notin \operatorname{conv}\Big(S \setminus (\{f_i : i \in K\} \cup \{e(K,j)\})\Big)$ for any $(K,j) \in F \times D$.

Claim 10. *If $z \in S$, then $e_0 \notin \operatorname{core}_k(S \setminus z)$.*

Proof. When $z = e(K,j)$, this is exactly (4). If $z = f_i$ for some $i \in [d]$, then let $A = \{0, \ldots, 0\}$ k times. If $z = 0$, then let $A = \{f_1\} \cup \{0, \ldots, 0\}$ 0 taken $k-1$ times. In both cases, $e_0 \notin \operatorname{conv}(S \setminus A)$. This proves the claim. ∎

Claim 11. $e_0 \in \operatorname{core}_k S$.

Proof. We have to show that $e_0 \in \operatorname{conv}(S \setminus A)$ if $A \subseteq S$ and $|A| = k$. If $0 \in A$ or if $A \cap \{f_1, \ldots, f_n\} = \emptyset$, then this follows immediately. If $A \subset \{f_1, \ldots, f_n\}$, then this is just (3).

So assume $|A \cap \{f_1, \ldots, f_t\}| = t < k$ ($t = 0$ is possible), say $A \cap \{f_1, \ldots, f_n\} = \{f_1, \ldots, f_t\}$. Then we look for a set $K \in F$ with $[t] \subset K$ such that $e(K,j) \in S \setminus A$ for every $j \in D$ (when $t = 0$ let $[t] = \emptyset$). Call such a set "good". As there are at most $(k-t)$ vectors $e(K,j)$ in A, the number of "bad" K-s is at most $(k-t)$. There are altogether $\binom{n-t}{k-t}$ ways to choose K so the number of "good" K-s is at least

$$\binom{n-t}{k-t} - (k-t) \geq (n-t) - (k-t) = n-k \geq 1.$$

Now fix a "good" K. We will show now that

(5) $\qquad e_0 \in \operatorname{conv}(\{0, f_{t+1}, \ldots, f_n\} \cup \{e(K,j) : j \in D\}).$

As all of these points belong to $S \setminus A$ this will prove the Claim.

Lemma 9 implies $e_0 \in \operatorname{conv}(\{0\} \cup \{f_i : i \in [n] \setminus K\} \cup \{e(K,j) : j \in D\})$. We have $[t] \subset K$ so $[n] \setminus K \subset [n] \setminus [t]$ proving (5). ∎

5. Proof of Theorem 4

Set $g(d) = \max(2(d+1), 1 + d + \lfloor d^2/4 \rfloor)$. We have to prove that $f(d,1) = g(d)$. Inequalities (1) and (2) show that $f(d,1) \geq g(d)$.

So we have to prove that $f(d,1) \leq g(d)$. Assume this is false and take a counterexample $S \subset \mathbb{R}^d$ with minimal d. Then $d \geq 3$ and $|S| > g(d)$, $\dim S = d$ and $x \in \operatorname{core}_1 S$ for some x but $x \notin \operatorname{core}_1 T$ for any proper subset $T \subset S$. As

$|S| > 2(d+1)$ Lemma 6 applies and so we may assume that $x \in \operatorname{relbd} \operatorname{core}_1 S$. Then the alternative definition of the 1-core gives a closed halfspace H^+ with bounding hyperplane H such that $x \in H$ and $|H^+ \cap S| \geq |S| - 1$. If $|H^+ \cap S| = |S|$ were the case here, then $x \in \operatorname{core}_1(S \cap H)$ clearly and this is a contradiction: with $T = S \cap H$ $|T| < |S|$ and $x \in \operatorname{core}_1 T$. So there is a unique point $z_0 \in S \setminus H^+$.

Now $x \in \operatorname{bd} \operatorname{conv}(S \setminus z_0)$, so there are affinely independent vectors $z_1, \dots, z_r \in S$ $(r \leq d)$ with $x \in$relint $\operatorname{conv}\{z_1, \dots, z_r\}$. To simplify notation we set $x = 0$.

Define $C = S \setminus \{z_1, \dots, z_r\}$ and $E = \{0, 1, \dots, r\}$. For a subset $B \subset E$ write $z(B) = \{z_i : i \in B\}$. The condition $0 \in \operatorname{core}_1 S$ is the same, by definition, as $0 \in \operatorname{conv}(S \setminus z)$ for all $z \in S$, but $0 \in \operatorname{conv}(S \setminus z)$ is trivially satisfied unless $z = z_i$ for some $i \in [r]$. So $0 \in \operatorname{core}_1 S$ is equivalent to

$$0 \in \operatorname{conv}(z(E \setminus i) \cup C) \text{ for all } i \in [r].$$

Define $\mathscr{B}$ as the collection of sets $B \subset E$ with $0 \in B$ that are minimal with respect to the property

$$0 \in \operatorname{conv}(z(B) \cup C).$$

This means that $0 \in \operatorname{conv}(z(B) \cup C)$ but $0 \notin \operatorname{conv}(z(B') \cup C)$ for any proper subset B' (with $0 \in B'$) of B. If $\{0\} \in \mathscr{B}$, i.e., $0 \in \operatorname{conv}(\{z_0\} \cup C)$, then (by Caratheodory's theorem) we find a subset $C_0 \subset C$, $|C_0| \leq d$ with $0 \in \operatorname{conv}(\{z_0\} \cup c_0)$. Then for $T = z(E) \cup C_0 \subset S$ we have $0 \in \operatorname{core}_1 T$ and $|T| \leq d + 1 + r \leq 2d + 1 < |S|$, a contradiction. So $\{0\} \notin \mathscr{B}$. Clearly, for each $i \in [r]$ there is a $B \in \mathscr{B}$ with $i \notin B$, i.e., $\cap \mathscr{B} = \{0\}$. Let now $\mathscr{B}' \subseteq \mathscr{B}$ be a subfamily minimal with respect to the property $\cap \mathscr{B} = \{0\}$. Set, finally, $\mathscr{B}' = \{B_1, \dots, B_n\}$.

Claim 12. $\sum_{j=1}^{n} (d - |B_j|) \leq \lfloor d^2/4 \rfloor$.

Proof. By the minimality of $\mathscr{B}'$ for every $j \in [n]$ there is an element $i(j)$ missed by B_j only, i.e., $i(j) \in B_k$ iff $k \neq j$. This element is nonzero and is different for different j-s. Then $|B_j \setminus 0| \geq n - 1$ and so $|B_j| \geq n$ and

$$\sum_{j=1}^{n} (d - |B_j|) \leq n(d-n) \leq \lfloor (n + (d-n))^2/4 \rfloor = \lfloor d^2/4 \rfloor. \qquad \blacksquare$$

Now for each $j \in [n]$ choose a subset $C_j \subseteq C$ minimal with respect to the property

$$0 \in \operatorname{conv}(z(B_j) \cup C_j). \tag{6}$$

Set $T = z(E) \cup C_1, \cup \dots \cup C_n$. Let us prove now $0 \in \operatorname{core}_1 T$: We have to show that $0 \in \operatorname{conv}(T \setminus z)$ for all $z \in T$. If $z \neq z_i$ for some $i \in [r]$, then $0 \in \operatorname{conv}\{z_1, \dots, z_r\}$ and if $z = z_i$ for some $i \in [r]$, then defining j by $i \notin B_j$ we have $0 \in \operatorname{conv}(z(B_j) \cup C_j)$.

We would like to show that $|T| \leq g(d)$ as this would prove the theorem. At this point we have only

$$|T| \leq |z(E)| + \sum_{j=1}^{n} |C_j|$$

$$\leq (r+1) + \sum_{j=1}^{n} (d + 1 - |B_j|) \leq d + 1 + n + \lfloor d^2/4 \rfloor,$$

which is not good enough. With the Insertion Lemma (see below) one can get an element common to every C_j giving $|T| \le d + 2 + \lfloor d^2/4 \rfloor$ which is $g(d) + 1$ instead of $g(d)$. To get rid of the plus one we will have to consider a few cases. We need the following fact well-known from linear programming or from the proof of Caratheodory's theorem [2]:

Insertion Lemma. *Assume $X \subset \mathbb{R}^d$ consists of affinely independent points and $0 \in \operatorname{conv} X$. Assume, further, that $y \in \operatorname{aff} X$. Then there is a $z \in X$ such that $0 \in \operatorname{conv}((X \setminus z) \cup \{y\})$.*

Proof. (Which is well-known and we give it rather for further reference.) We have $\sum_{x \in X} \lambda(x)x = 0$ with a convex combination, i.e., $\lambda(x) \ge 0$ and $\Sigma\lambda(x) = 1$. There is an affine dependence $y + \sum_{x \in X} \gamma(x)x = 0$. Multiplying it by t and adding it to the convex combination we get

$$ty + \sum_{x \in X} (\lambda(x) + t\gamma(x))x = 0.$$

Set $t_0 = \max\{t \ge 0 : \lambda(x) + t\gamma(x) \ge 0\}$; such a t_0 clearly exists. Let $z \in X$ be defined by $\lambda(z) + t_0\gamma(z) = 0$. Then $t_0 y + \sum_{x \in X \setminus z} (\lambda(x) + t_0\gamma(x)) = 0$ is a convex combination again. So $0 \in \operatorname{conv}((X \setminus z) \cup \{y\})$ indeed. ∎

We say that y pushes z out from X when inserted. The pushed-out element is not uniquely determined but we think of it as fixed.

We return now to the proof of Theorem 4. Condition (6) can be written as

$$\sum_{i \in B_j} \lambda_i z_i + \sum_{c \in C_j} \lambda(c)c = 0$$

with a suitable convex combination. It is easy to see that $\lambda_0 = 0$ if and only if $C_j \subset H$. It follows from the minimality of B_j and C_j that

$$|B_j| + |C_j| = \dim \operatorname{aff}(z(B_j) \cup C_j) + 1, \text{ if } C_j \not\subset H, \text{ and} \tag{7}$$

$$|B_j \setminus 0| + |C_j| = \dim \operatorname{aff}(z(B_j \setminus 0) \cup C_j) + 1, \text{ if } C_j \subset H. \tag{8}$$

There are a few cases to consider now. We introduce the sets

$$J_1 = \{j \in [n] : \dim \operatorname{aff}(z(B_j) \cup C_j) = d \text{ and } C_j \not\subset H\},$$
$$J_2 = \{j \in [n] : \dim \operatorname{aff}(z(B_j) \cup C_j) = d - 1 \text{ and } C_j \not\subset H\},$$
$$J_3 = \{j \in [n] : \dim \operatorname{aff}(z(B_j) \cup C_j) \le d - 2 \text{ and } C_j \not\subset H\},$$
$$J_4 = \{j \in [n] : \operatorname{aff}(z(B_j \setminus 0) \cup C_j) = H\}.$$
$$J_5 = \{j \in [n] : \dim \operatorname{aff}(z(B_j \setminus 0) \cup C_j) \le d - 2 \text{ and } C_j \subset H\}.$$

We claim that the sets J_1, J_2, J_3, J_4, J_5 form a partition of $[n]$. Indeed, these sets are pairwise disjoint and if $j \notin J_1 \cup J_2 \cup J_3 \cup J_5$, then $C_j \subset H$ and $\dim \operatorname{aff}(z(B_j \setminus 0) \cup C_j) = d - 1$ implying $\operatorname{aff}(z(B_j \setminus 0) \cup C_j) = H$.

Observe now that if $C_j \subset H$ for all $j \in [n]$, then $T = S \setminus z_0$ is a proper subset of S with $0 \in \text{core}_1 T$, a contradiction. So there is $c_1 \in C_j$ for some $i \in J_1 \cup J_2 \cup J_3$ with $c_1 \notin H$ and $c_1 \in H^+$. The line segment connecting z_0 and c_1 intersects H at the point c_0. Insert now c_0 into every $X_j = z(B_j \setminus 0) \cup C_j$ with $j \in J_4$; let the pushed out element be x_j. Set $X'_j = (X_j \setminus x_j) \cup \{z_0, c_1\}$, then $0 \in \text{conv} X'_j$. Assume $x_j \in z(B_j)$. Then $0 \in \text{conv} X'_j = \text{conv}((z(B_j) \setminus x_j)) \cup C_j \cup \{c_1\})$ which contradicts the fact that B_j is minimal. Thus $x_j \in C_j$. Choose now a minimal $C'_j \subseteq (C_j \setminus x_j) \cup \{c_1\}$ with $0 \in \text{conv}(z(B_j) \cup C'_j)$. A straightforward checking shows that $c'_j \not\subset H$. Replacing now every old $z(B_j) \cup C_j$ by the new $z(B_j) \cup C'_j$ for $j \in J_4$ we get a new system of B_j-s and C_j-s, and J_4 will be empty.

So we may assume $J_4 = \emptyset$ from now on. Suppose $J_1 \neq [n]$ and choose some $a \in C_j$ with $j \in J_2 \cup J_3 \cup J_5$. Insert a into every $z(B_j) \cup C_j$ with $j \in J_1$. We see again that the pushed-out element c_j must come from C_j. Write $C_j(a)$ for the set $(C_j \setminus c_j) \cup \{a\}$. We have a new system B_j, $C_j(a)$ for $j \in J_1$ and B_j, C_j for the rest. Set

$$T = z(E) \cup \bigcup_{j \in J_1} C_j(a) \cup \bigcup_{j \in [n] \setminus J_1} C_j,$$

Again we have $0 \in \text{core}_1 T$. Moreover,

$$\Big| \bigcup_{j \in J_1} C_j(a) \cup \bigcup_{j \in [n] \setminus J_1} C_j \Big| \le \Big| \bigcup_{j \in J_1} (C_j(a) \setminus a) \Big| + \sum_{j \in [n] \setminus J_1} |C_j| \le$$

$$\le \sum_{j \in J_1} (d - |B_j|) + \sum_{j \in [n] \setminus J_1} (d - 1 - |B_j \setminus 0|) =$$

$$= \sum_{j \in [n]} (d - |B_j|) \le \lfloor d^2/4 \rfloor$$

according to Claim 12. Here we used (7) and (8) as well. Then $|T| \le 1 + r + \lfloor d^2/4 \rfloor \le g(d) < |S|$, a contradiction.

We have, finally, $J_1 = [n]$; $n = 0$ or 1 is clearly impossible. So $n \ge 2$. Consider $a \in C_1$ and insert a into every other $z(B_j) \cup C_j$. Again, a pushes out some element from C_j. Write $C_1(a) = C_1$. Setting again $T = z(E) \cup \sum_{j=1}^{n} C_j(a)$ we have $0 \in \text{core}_1 T$. The above estimation gives now $|T| \le 1 + r + \lfloor d^2/4 \rfloor + 1$. We can get one less if $|C_i(a) \cap C_j(a)| \ge 2$ for some $i \neq j$. So assume the sets $C_j(a) \setminus a$ are pairwise disjoint. Observe, further, that the proof of Claim 12 gives $\lfloor d^2/4 \rfloor - 1$ unless $|B_j| = n$ for all $j \in [n]$ and n equals $\lfloor d/2 \rfloor$ or $\lfloor (d+1)/2 \rfloor$. Thus $|C_j| = d + 1 - n \ge 2$. Then there is $b \in C_2(a)$ $b \neq a$. Try to insert b into $z(B_1) \cup C_1$. Then $|C_2(a) \cap C_1(b)| \ge 2$ unless b pushes out a from C_1. Similarly, take $a' \in C_1$, $a' \neq a$ and insert a' into $z(B_2) \cup C_2(a)$ and into every $z(B_j) \cup C_j$ $(j = 3, \dots, n)$. If a' pushes out a from $C_2(a)$, then $C_1(b)$, $C_2(a)(a')$, $C_3(a'), \dots, C_n(a')$ is a good system together with the unchanged B_j-s, because $C_1(b)$ and $C_2(a)(a')$ have two elements, a' and b in common. Similarly, if a' pushes out b from $C_2(a)$, then C_1, $C_2(a)(a')$, $C_3(a'), \dots, C_n(a')$ is a good system for the first two sets share two elements a and a'. ∎

6. The example proving Theorem 5

First set $n = \lfloor (d-1)/2 \rfloor$. Take a hyperplane $H \subset \mathbb{R}^d$ not passing through the origin and let $P \subset H$ be the set of vertices of an n-neighbourly polytope with $|P| = k$. This means that the convex hull of any n-subset of vertices is a face of the polytope. It is well-known that such polytopes exist, see for instance [3] or [4]. Set $Q = -\text{conv}P$, this is a $(d-1)$-dimensional n-neighbourly polytope lying in the hyperplane $-H$. For an n-subset A of P define $s(A)$ as the center of gravity of the pointset $-A$. Let v be a unit vector parallel with H and in general position relative to the faces of Q. Then at least one of the halflines $\{s(A) + tv : t \geq 0\}$ and $\{s(A) + tv : t \leq 0\}$ has a single point, $s(A)$, in common with Q. Then either $\{s(A) + tv : t \geq 0\} \cap Q = s(A)$ or $\{s(A) + tv : t \leq 0\} \cap Q = s(A)$ for at least half of the n-subsets A in P. We assume, without loss of generality, that $\{s(A) + tv : t \geq 0\} \cap Q = s(A)$ for at least half of the n-subsets. Denote the set of these n-subsets by $\mathcal{A}$; then $|\mathcal{A}| \geq \frac{1}{2}\binom{k}{n}$. For any vector u close enough to v we will have

$$\{s(A) + tu : t \geq 0\} \cap Q = s(A) \text{ for all } a \in \mathcal{A}. \tag{9}$$

Choose now points $z_1, \dots, z_n$ parallel to H and close to v and points $z_{n+1}, \dots, z_d$ parallel to H and close to $-v$ with $0 \in \text{conv}\{z_1, \dots, z_d\}$.

Let L be a hyperplane orthogonal to v and such that $Q + \{tv : t \geq 0\}$ is in one of the open halfspaces determined by L. Define $y(p,i) = L \cap \{-p - tz_i : t \geq 0\}$ for all $p \in P$ and $i \in [n]$. Clearly, $y(p,i)$ is a single point. Set now $Y = \{y(p,i) : p \in P,\ i \in [n]\}$, $Z = \{z_1, \dots, z_d\}$, $X = \{s(A) : A \in \mathcal{A}\}$ and $S = P \cup Z \cup X \cup Y$.

Claim 13. $0 \in \text{core}_k S$.

Proof. Take away a set K of k points from S; we have to prove that $0 \in \text{conv}(S \setminus K)$. Set $P' = P \cap K$, $Z' = Z \cap K$, $X' = X \cap K$ and $Y' = Y \cap K$. Let $|P'| = k - \alpha$, $|Z'| = \beta$, then $|X' \cup Y'| = \alpha - \beta$. We can assume that $\beta \geq 1$ as otherwise $0 \in \text{conv}(S \setminus K)$ trivially. Clearly $\alpha \geq \beta$. There are three cases to consider.

Case 1. When $\alpha > n$. Then $|P \setminus P'| = \alpha > n$ so there are at least $\binom{\alpha}{n}$ n-subsets in $P \setminus P'$ and $\binom{\alpha}{n} \geq \alpha > \alpha - \beta$, so there is an n-subset $A \subset P \setminus P'$ with $s(A) \notin X'$. Then

$$0 \in \text{conv}(s(A) \cup A) \subset \text{conv}(S \setminus K).$$

Case 2. When $\alpha \leq n$ and $\beta < n$. If there is a pair $p, i \in P \times [n]$ with $p \notin P'$, $z_i \notin Z'$ and $y(p,i) \notin Y'$, then $0 \in \text{conv}\{p, z_i, y(p,i)\} \subset \text{conv}(S \setminus K)$. The number of pairs with $p \notin tP'$ and $z_i \notin Z'$ ($i \in [n]$) is at least $|P \setminus P'|(n - \beta) = \alpha(n - \beta)$ and $|Y'| \leq \alpha - \beta$. So there is a pair $(p,i) \in P \times [n]$ with $p \notin P'$, $z_i \notin Z'$ and $y(p,i) \notin Y'$ unless $\alpha(n-\beta) \leq \alpha - \beta$, Now $\alpha - \beta \leq n - \beta$, so $\alpha = 1$ must hold. Then $\alpha(n - \beta) \leq \alpha - \beta$ implies $n = 1$ but we have $d \geq 5$.

Case 3. When $\alpha = \beta = n$. Then $P \setminus P'$ is an n-subset of P and $S(P \setminus P')$ is not deleted, so

$$0 \in \text{conv}(s(P \setminus P') \cup (P \setminus P')) \subset \text{conv}(S \setminus K). \qquad ∎$$

Claim 14. *For any* $A \in \mathcal{A}$, $0 \notin \text{core}_k(S \setminus s(A))$.

Sketch of the proof. Set $Z' = \{z_1, \dots, z_n\}$, $P' = P \setminus A$ and $K = Z' \cup P'$. Take a hyperplane H_1 passing through the origin, containing A and containing Q in one of the closed halfspaces determined by H_1. Such a hyperplane exists for Q is an n-neighbourly polytope in H. H_1 has a single point, $s(A)$, in common with $\text{conv}(X \cup Y)$. It is not difficult to see from this and from (9) that 0 does not lie in the convex hull of $(S \setminus s(A)) \setminus K$. We omit the details. ∎

By the claim, if $T \subset S$ and $0 \in \text{core}_k T$, then $s(A) \in T$ must hold for all $A \in \mathcal{A}$. This implies $|T| \geq |\mathcal{A}|$. One can see also that T must contain P and Z as well. This proves the theorem. ∎

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