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A Combinatorial Result About Points and Balls in Euclidean Space

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Abstract. For each $n \geq 1$ there is $c_n > 0$ such that for any finite set $X \subseteq \mathbb{R}^n$ there is $A \subseteq X$, $|A| \leq \frac{1}{2}(n+3)$, having the following property: if $B \supseteq A$ is an n-ball, then $|B \cap X| \geq c_n |X|$. This generalizes a theorem of Neumann-Lara and Urrutia which states that $c_2 \geq \frac{1}{60}$.

A theorem of Neumann-Lara and Urrutia [3] is generalized from the plane to arbitrary n-dimensional Euclidean space $\mathbb{R}^n$, solving Problem 2 of [3]. By an *n-ball* we mean a set of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : (x_1 - a_1)^2 + (x_2 - a_2)^2 + \ldots + (x_n - a_n)^2 \leq r\},$$

where $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $r > 0$.

Theorem 1. *For each $n \geq 1$ there is $c_n > 0$ such that for any finite set $X \subseteq \mathbb{R}^n$ there is $A \subseteq X$, $|A| \leq [\frac{1}{2}(n+3)]$, having the following property: if $B \supseteq A$ is an n-ball, then $|B \cap X| \geq c_n |X|$.*

The bound $[\frac{1}{2}(n+3)]$ in the theorem will be shown in Theorem 6 to be optimal in quite a strong way. For now, let X be any finite set of points on the *moment curve* $\alpha(t) = (t, t^2, t^3, \ldots, t^n)$, $|X| = m \geq n+1$. Then X is the set of vertices of a convex polyhedron (known as *the cyclic n-polytope with m vertices*) and every $[n/2]$-element subset $A \subseteq X$ is the set of vertices of one of its faces. (See Sections 4.7 and 7.4 of [2].) Clearly then, for each such A there is an n-ball B such that $B \cap X = A$.

The following notation will be used. For a set S, $\mathcal{P}_n(S)$ is the set of n-element subsets of S. If $A \subseteq \mathbb{R}^n$, then conv A is the convex hull of A.

Lemma 2. *Let $Y \in \mathcal{P}_{n+3}(\mathbb{R}^n)$. Then there is $A \subseteq Y$, $|A| = [\frac{1}{2}(n+3)]$, such that for any n-ball $B \supseteq A$, $(Y \backslash A) \cap B \neq \varnothing$.*

Proof. There exist disjoint $A_1, A_2 \subseteq Y$ such that $|A_1| = |A_2| = [\frac{1}{2}(n+3)]$ and conv $A_1 \cap$ conv $A_2 \neq \varnothing$. The argument for obtaining A_1 and A_2 is essentially in [1] and [4]. Let $Y = \{y_1, y_2, \ldots, y_{n+3}\}$, and then let $\bar{Y} = \{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n+3}\} \subseteq \mathbb{R}^2$ be its Gale transform. (Here we are assuming, without loss of generality, that $\mathbb{R}^n$ is the affine span of Y.) For some $y_i \in Y$ the line l in $\mathbb{R}^2$ through $\bar{y}_i$ and the origin divides $\mathbb{R}^2$ into two open half-planes P_1, P_2 such that $|P_1 \cap \bar{Y}|, |P_2 \cap \bar{Y}| \leq [\frac{1}{2}(n+3)]$. Let C_1, C_2, $Z \subseteq Y$ be such that $\bar{C}_1 = P_1 \cap \bar{Y}$, $\bar{C}_2 = P_2 \cap \bar{Y}$, and $\bar{Z} = l \cap \bar{Y}$. By Lemma 1 of [4], conv $(C_1 \cup Z_1) \cap$ conv$(C_2 \cup Z_2) \neq \varnothing$ whenever $Z_1 \cup Z_2 = Z$. But this implies conv $C_1 \cap$ conv $C_2 \neq \varnothing$. So just let $A_1, A_2 \subseteq Y$ be disjoint sets such that $C_1 \subseteq A_1$, $C_2 \subseteq A_2$, and $|A_1| = |A_2| = [\frac{1}{2}(n+3)]$.

We now claim that either $A = A_1$ works or $A = A_2$ works.

In order to derive a contradiction, let $a \in$ conv $A_1 \cap$ conv A_2, and let B_1, B_2 be n-balls for which $A_1 \subseteq B_1$, $A_2 \subseteq B_2$, and $B_1 \cap A_2 = \varnothing = B_2 \cap A_1$. Clearly, $B_1 \cap B_2 \neq \varnothing$ since $a \in B_1 \cap B_2$, and also $B_1 \backslash B_2 \neq \varnothing \neq B_2 \backslash B_1$. Therefore, there is a unique hyperplane h such that $h \cap \partial B_1 = h \cap \partial B_2 = \partial B_1 \cap \partial B_2$ (where ∂B_i denotes the boundary of B_i). Let H_1, H_2 be the closed half-spaces such that $H_1 \cap H_2 = h$, $B_1 \backslash B_2 \subseteq H_1$, and $B_2 \backslash B_1 \subseteq H_2$. Then $a \in H_1 \cap H_2 = h$, so there must be some $b \in A_1 \cap h$. But then $b \in B_2$, which is a contradiction. □

A simple counting argument allows us to deduce Theorem 1 from Lemma 2. This is abstracted in the next lemma.

Lemma 3. *Let S be a set, $\mathcal{B}$ a collection of subsets of S, and $r > m$ positive integers. Suppose that for each $Y \in \mathcal{P}_r(S)$ there is $A \in \mathcal{P}_m(Y)$ such that whenever $A \subseteq B \in \mathcal{B}$, then $(Y \backslash A) \cap B \neq \varnothing$. Let $c = (m!(r-m-1)!)/r!$. Then for any finite $X \subseteq S$ with $|X| \geq r$ there is $A \in \mathcal{P}_m(X)$ such that whenever $A \subseteq B \in \mathcal{B}$, it follows that $|B \cap X| > c|X|$.*

Proof. Let $X \in \mathcal{P}_t(S)$ where $t \geq r$. There are sets $A \in \mathcal{P}_m(X)$ and $\mathcal{R} \subseteq \mathcal{P}_r(X)$ such that $|\mathcal{R}| \geq \binom{t}{r} \Big/ \binom{t}{m}$ and for each $Y \in \mathcal{R}$, A is as in the hypothesis of the lemma. We claim that this is the desired A.

Suppose that $A \subseteq B \in \mathcal{B}$ and $|B \cap X| = k$. The number of sets $Y \in \mathcal{P}_r(X)$ for which $A \subseteq Y$ and $(Y \backslash A) \cap B \neq \varnothing$ is clearly at most $(k-m)\binom{t-m-1}{r-m-1}$, so this is an upper bound for $|\mathcal{R}|$. Therefore, $(k-m) \geq c(t-m)$, so $k > ct$. □

Lemmas 2 and 3 show that $c_2 \geq \frac{1}{30}$, improving the constant in [3]. Refinements of this proof show that $c_2 > \frac{1}{20}$.

The proof of Lemma 3 shows that its conclusion is true on the average, in the sense that as A ranges over $\mathcal{P}_m(X)$, the average value of $f(A) = \min\{|B \cap X| : A \subseteq B \in \mathcal{B}\}$ is greater than $c|X|$.

Theorem 1 has several generalizations. We mention just one of them.

Theorem 4. *For each $m \geq [\frac{1}{2}(n+3)]$ there is $c_{n,m} > 0$ such that for any finite $X \subseteq \mathbb{R}^n$, $|X| \geq m$, there is $A \in \mathcal{P}_m(X)$ having the following property: if B is an n-ball and $|A \cap B| \geq [\frac{1}{2}(n+3)]$, then $|B \cap X| \geq c_{n,m}|X|$.*

This theorem is a consequence of Lemma 5 below (which is the analogue of Lemma 2) and a version of Lemma 3 whose statement and proof can easily be supplied.

Let $R_s(t)$ be the Ramsey number defined as follows: $R_s(t)$ is the least r such that whenever $|Y| \geq r$ and $\mathcal{P}_s(Y) = P_1 \cup P_2$, then there is $W \in \mathcal{P}_t(Y)$ such that either $\mathcal{P}_s(W) \subseteq P_1$ or $\mathcal{P}_s(W) \subseteq P_2$.

Lemma 5. *Let $m \geq s = [\frac{1}{2}(n+3)]$, let $t > m/c_n$ be an integer (c_n is from Theorem 1), and let $r = R_s(t)$. Suppose $Y \in \mathcal{P}_r(\mathbb{R}^n)$. Then there is $A \in \mathcal{P}_m(Y)$ such that if B is an n-ball and $|B \cap A| \geq s$, then $(Y \backslash A) \cap B \neq \varnothing$.*

Proof. Let

$$P = \{Z \in \mathcal{P}_s(Y)\text{: for each } n\text{-ball } B \supseteq Z, |B \cap Y| \geq c_n t\}.$$

By Ramsey's theorem there is $W \in \mathcal{P}_t(Y)$ such that $\mathcal{P}_s(W) \subseteq P$ or $\mathcal{P}_s(W) \cap P = \varnothing$. By Theorem 1, $\mathcal{P}_s(W) \cap P \neq \varnothing$; hence, $\mathcal{P}_s(W) \subseteq P$. Any $A \in \mathcal{P}_m(W)$ will do, for $|B \cap A| \geq s$ implies $|B \cap Y| \geq c_n t > m = |A|$. □

The next theorem shows that the bound on the dimension in Theorem 1 (as well as Lemma 2 and Theorem 4) is sharp.

Theorem 6. *There is an infinite subset $X \subseteq \mathbb{R}^n$ such that whenever $A \subseteq X$ and $|A| < [\frac{1}{2}(n+3)]$, then there is some n-ball B for which $A = B \cap X$.*

Proof. Clearly we can assume that $n \geq 3$ and n is odd, so let $n = 2k-1$ where $k \geq 2$. Consider the moment curve $\alpha(t) = (t, t^2, t^3, \ldots, t^n)$. We will obtain X as $\{\alpha(t): 0 < t < \varepsilon\}$ for some appropriately small $\varepsilon > 0$.

Suppose $\varepsilon > t_1 \geq t_2 \geq \cdots \geq t_k > 0$. We can find parameters a_{2n}, a_{2n-1}, $a_{2n-2}, \ldots, a_1, a_0$ such that

$$(t-a_1)^2 + (t^2-a_2)^2 + \ldots + (t^n - a_n)^2 = a_0 + [(t-t_1)(t-t_2)\ldots(t-t_k)]^2 p(t),$$

where $p(t) = a_{n+1} + a_{n+2}t + a_{n+3}t^2 + \cdots + a_{2n}t^{2k-2}$. These parameters are uniquely determined, and can be found in the order listed, a_j being determined by the coefficients of t^j in the above equation. It is easy to see that:

$$a_j = 1 + O(\varepsilon) \quad \text{if } j \geq 2k \text{ and } j \text{ is even;}$$

$$a_j = 0 + O(\varepsilon) \quad \text{if } j \text{ is odd;}$$

$$a_j = \tfrac{1}{2} + O(\varepsilon) \quad \text{if } 1 \leq j \leq 2k-1 \text{ and } j \text{ is even;}$$

and, finally,

$$a_0 = \frac{k-1}{4} + O(\varepsilon).$$

Now just pick $\varepsilon > 0$ small enough so that $a_0 > 0$ and $p(t) > 0$ whenever $0 < t < \varepsilon$. Then if $\varepsilon > t_1 \geq t_2 \geq \cdots \geq t_k > 0$, let B be the n-ball with center $(a_1, a_2, \cdots, a_n)$ and radius $\sqrt{a_0}$. Clearly, $X \cap B = \{\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_k)\}$. □

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Note added in proof. Ryan Hayward has shown that $C_2 \geq \frac{5}{84}$.