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Intrinsic volumes and f-vectors of random polytopes

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1. Introduction

Let $K \subset R^d$ be a convex body (a convex compact set with nonempty interior) and choose points $x_1, \ldots, x_n \in K$ randomly, independently and according to the uniform distribution on K. Then $K_n = \text{conv}\{x_1, \ldots, x_n\}$ is a *random polytope*. It is clear that, with high probability, K_n gets nearer and nearer to K as n tends to infinity. There has been a lot of research to determine how well K_n approximates K in various measures of approximation. These measures usually are the expectation of $\varphi(K) - \varphi(K_n)$ where φ is some functional defined on the set of convex bodies, for instance volume, surface area, mean width, etc. Most of the research concentrated on the case $d = 2$ and on smooth convex bodies and polytopes.

Now let $\varepsilon > 0$ and define

$$K[\varepsilon] = \{x \in K: \text{vol}(K \cap H) \geqq \varepsilon \text{ for every halfspace } H \text{ with } x \in H\}.$$

This is a convex body again if ε is small enough. The main result of [BL] says that K_n is close to $K[1/n]$ in the following sense:

$$E\,\text{vol}(K \backslash K_n) \sim \text{vol}\left(K \backslash K\left[\frac{1}{n}\right]\right) \tag{1.1}$$

where E denotes expectation and the notation $f(n) \sim g(n)$ means that $\underline{\lim}\, f(n)/g(n) > 0$ and $\underline{\lim}\, g(n)/f(n) > 0$. That is, there are constants c_1 and c_2 such that for n large enough

$$c_1\,\text{vol}\left(K \backslash K\left[\frac{1}{n}\right]\right) < E\,\text{vol}(K \backslash K_n) < c_2\,\text{vol}\left(K \backslash K\left[\frac{1}{n}\right]\right).$$

This result shows that K_n and $K[1/n]$ approximate K in the same order and suggests that $K \backslash K_n$ is close to $K \backslash K[1/n]$ in some strong sense.

The aim of this paper is to further exploit the connection between $K \backslash K_n$ and

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$K \setminus K[1/n]$. The main results are: (1) the expectation of $V_s(K) - V_s(K_n)$ is about $V_s(K) - V_s(K[1/n])$ where V_s denotes the s-th intrinsic volume, $s = 1, 2, \ldots, d$, (2) the expectation of the number of s-dimensional faces of K_n is about $n \operatorname{vol}(K \setminus K[1/n])/\operatorname{vol} K$ ($s = 0, 1, \ldots, d-1$), (3) for a smooth convex body K the expectation of the Haussdorff distance between K and K_n is about $(\log n/n)^{2/(d+1)}$.

The paper is organized as follows. The second section introduces the necessary notation and terminology. The third contains the results. The basic auxiliary lemmata are given in the fourth section. Their proofs are postposed to the last section. The proofs of the results are in Sects. 5, 6, 7 and 8.

2. Notation

In this section we introduce some basic notation.

The set of all convex bodies in R^d is denoted by $\mathscr{K}^d$. $\mathscr{K}_1^d = \{K \in \mathscr{K}^d : \operatorname{vol} K = 1\}$. $\mathscr{K}^d(r, R)$ consists of all $K \in \mathscr{K}^d$ that contain a ball of radius r and are contained in a ball of radius R. We write $\mathscr{K}_1^d(r, R) = \mathscr{K}_1^d \cap \mathscr{K}^d(r, R)$.

For a set $X \subset R^d$ $\operatorname{conv} X$, $\operatorname{aff} X$ denotes its convex and affine hull. $\operatorname{dist}(X, Y)$ is the distance between $X, Y \subset R^d$, and $X + Y$ is their Minkowski sum. The Euclidean distance of two points $x, y \in R^d$ is denoted by $|x - y|$, their scalar product by $x \cdot y$. B^d stands for the Euclidean unit ball of R^d, S^{d-1} is its boundary. We write $\omega_d = \operatorname{vol} B^d$.

For a set $K \in \mathscr{K}^d$ $\operatorname{bd} K$ and $\operatorname{int} K$ denotes its boundary and interior, $h(a) = h_K(a)$ is its support function, i.e., $h(a) = \sup\{a \cdot x : x \in K\}$. For $a \in S^{d-1}$, $H(a, t)$ is the halfspace $\{x \in R^d : a \cdot x \geqq h(a) - t\}$. So $H(a, t) = H_K(a, t)$ depends on the underlying convex body $K \in \mathscr{K}^d$ but we will usually suppress this dependence. The bounding hyperplane of $H(a, t)$ is denoted by $H(a = t)$.

For $K \in \mathscr{K}^d$, $K[\varepsilon]$ was defined in the previous section. We let $K(\varepsilon)$ to be the closure of $K \setminus K[\varepsilon]$:

$$K(\varepsilon) = \{x \in K : \operatorname{vol}(K \cap H) \leqq \varepsilon \text{ for some halfspace } H \text{ with } x \in H\}$$

$K(\varepsilon)$ is a kind of "inner parallel layer" to K.

When P is a polytope $f_s(P)$ will denote the number of s-dimensional faces of $P, s = 0, 1, \ldots$. For $K \in \mathscr{K}^d$ $V_s(K)$ is the s-th intrinsic volume of $K (s = 1, 2, \ldots, d)$. For the definition see Sect. 3 or (6.2). We write $E(K, s, n)$ as a shorthand for $E(V_s(K) - V_s(K_n))$ when $K \in \mathscr{K}^d$.

In what follows $c_1, c_2, \ldots c_1(d), \ldots, c_1(K), \ldots, \text{const}(d, r, R)$ will denote various constants. The reader is warned that the constants $c_i(d)$ appearing in different sections do not coincide.

3. Results

We first give the results concerning the expected number of s-faces of the polytope K_n. As a non-degenerate affine transformation does not influence $f_s(K_n)$ we may consider $K \in \mathscr{K}_1^d$. An identity due to Efron [Ef] says that for $K \in \mathscr{K}_1^d$

$$E f_0(K_n) = n E \operatorname{vol}(K \setminus K_{n-1}). \tag{3.1}$$

Thus by (1.1) we have

$$Ef_0(K_n) \sim n \operatorname{vol}\left(K \backslash K\left[\frac{1}{n}\right]\right) = n \operatorname{vol} K\left(\frac{1}{n}\right).$$

We extend this to every f_s, $s = 0, 1, \ldots, d-1$:

Theorem 1. *Assume $K \in \mathscr{K}_1^d$ and $s \in \{0, 1, \ldots, d-1\}$. Then*

$$Ef_s(K_n) \sim n \operatorname{vol} K\left(\frac{1}{n}\right). \tag{3.2}$$

The implied constants depend only on d.

This theorem says that $Ef_s(K_n)$ is essentially the same for all $s = 0, 1, \ldots, d-1$. This is not so much surprising when one thinks of the boundary of K_n as locally R^{d-1} and the faces of K_n as a "random triangulation" on a piece of R^{d-1}. In a random triangulation of R^{d-1} one would expect the average degree bounded by a constant depending only on d, and so the average number of s-faces equal to the average number of vertices (up to a constant multiplier).

As $\operatorname{vol} K(1/n)$ is known for smooth convex bodies and polytopes (see [BL] and also [L]) Theorem 1 has the following immediate consequences.

Corollary 1. *For a polytope $P \in \mathscr{K}^d$ and $s \in \{0, 1, \ldots, d-1\}$*

$$Ef_s(P_n) \sim (\log n)^{d-1}. \tag{3.3}$$

Corollary 2. *For a $\mathscr{C}^2$ convex body $K \in \mathscr{K}^d$ and $s \in \{0, 1, \ldots, d-1\}$*

$$Ef_s(K_n) \sim n^{(d-1)/(d+1)}. \tag{3.4}$$

The case $s = d-1$ of Corollary 2 was proved by Wieacker [W] in asymptotic form, i.e.,

$$Ef_s(K_n) \approx c(K) n^{(d-1)/(d+1)}$$

with explicitly given constant $c(K)$ where the notation $f(n) \approx g(n)$ means that $\lim f(n)/g(n) = 1$. The case $s = d-1$ of Corollary 1 was proved by Dwyer [Dw] and by van Wel (see [S 2]) independently, when the polytope is simple.

The next corollary follows from Theorem 1 via Theorem 5 of [BL]:

Corollary 3. *If $K \in \mathscr{K}_1^d$, then for all $s \in \{0, 1, \ldots, d-1\}$*

$$c_1(d)(\log n)^{d-1} < Ef_s(K_n) < c_2(d) n^{(d-1)/(d+1)}. \tag{3.5}$$

Moreover, for any functions $\Omega(n) \to \infty$ and $\omega(n) \to 0$ and for most (in the Baire category sense) convex bodies $K \in \mathscr{K}_1^d$

$$\Omega(n)(\log n)^{d-1} > Ef_s(K_n) \tag{3.6}$$

for infinitely many n and

$$\omega(n) n^{(d-1)/(d+1)} < Ef_s(K_n) \tag{3.7}$$

for infinitely many n.

In other words inequality (3.5) in best possible apart from the constants $c_1(d)$ and $c_2(d)$.

Now we consider the intrinsic volume, $V_s(K)$, of a convex body $K \in \mathscr{K}^d$ which is defined (see [Mc; BF]) for $s = 0, 1, \ldots, d$ as

$$V_s(K) = \omega_{d-s}^{-1} \binom{d}{s} V(K, \ldots, K, B^d, \ldots, B^d)$$

where $V(K, \ldots, K, B^d, \ldots, B^d)$ is the mixed volume of K taken s times and B^d taken $d - s$ times. It is well-known [Mc; BF] that $V_d(K) = \operatorname{vol} K$, $V_{d-1}(K)$ equals the surface area of K and $V_1(K)$ is a constant multiple of the mean width of K. It turns out that the intrinsic volume of K_n is close to that of $K[1/n]$. More precisely we have

Theorem 2. *Assume $K \in \mathscr{K}^d(r, R)$ and $s \in \{1, \ldots, d\}$. Then*

$$E(V_s(K) - V_s(K_n)) \sim V_s(K) - V_s\left(K\left[\frac{1}{n}\right]\right). \tag{3.8}$$

with the implied constants depending only on d, r, R.

We will use the notation $E(K, s, n) = E(V_s(K) - V_s(K_n))$ and $V_s(K(1/n)) = V_s(K) - V_s(K[1/n])$. Using Theorem 2 one can compute $E(K, s, n)$ for different classes of convex bodies, namely, for smooth convex bodies and for polytopes.

Theorem 3. *If $K \in \mathscr{K}^d$ is a $\mathscr{C}^2$ convex body with positive Gaussian curvature, then for $s = 1, 2, .., d$*

$$E(K, s, n) \sim n^{-2/(d+1)}. \tag{3.9}$$

Theorem 4. *If $P \in \mathscr{K}^d$ is a polytope, then for $s = 1, 2, \ldots, d - 1$,*

$$E(P, s, n) \sim n^{-1/(d-s+1)}. \tag{3.10}$$

In the last two theorems the implied constants depend on the convex body (K and P) itself.

In the case when $s = d$ (i.e., when V_s is the usual volume) $E(P, d, n) \sim n^{-1}(\log n)^{d-1}$ according to Theorems 2 and 3 of [BL].

In some special cases Theorems 3 and 4 have been proved earlier and in stronger form. For instance, Rényi and Sulanke [RS] show that for a smooth enough convex body $K \in \mathscr{K}^2$

$$E(K, 1, n) \approx c(K) n^{-2/3}$$

with explicitly given $c(K)$. This was later extended to $d > 2$ by Schneider and Wieacker [SW]:

$$E(K, 1, n) \approx c(K) n^{-2/(d+1)}$$

with explicitly given $c(K)$, again. For polytopes Buchta [Bu 1] ($d = 2$) and Schneider [S1] ($d > 2$) proved

$$E(P, 1, n) \approx c(P) n^{-1/d}.$$

Schneider [S 1] showed further that for all $K \in \mathscr{K}^d$

$$c_1(K)n^{-2/(d+1)} < E(K,1,n) < c_2(K)n^{-1/d} \tag{3.11}$$

and that (3.11) is best possible apart from the constants $c_1(K)$ and $c_2(K)$. It would be interesting to have the analogous result for $E(K,s,n)$. One would expect the extreme classes to be the polytopes and smooth convex bodies. Thus the obvious guess would be this: for $1 \leqq s \leqq \frac{d+1}{2}$ and $K \in \mathscr{K}^d$

$$c_1(K)n^{-2/(d+1)} < E(K,s,n) < c_2(K)n^{-1/(d-s+1)},$$

and for $\frac{d+1}{2} \leqq s \leqq d-1$ and $K \in \mathscr{K}^d$

$$c_1(K)n^{-1/(d-s+1)} < E(K,s,n) < c_2(K)n^{-2/(d+1)}.$$

If true this would imply, for instance, that for all $K \in \mathscr{K}^3$ $E(K,2,n)$, the surface area of K minus the expectation of the surface area of K_n is about $n^{-1/2}$. I find this quite remarkable. Of course this is equivalent to

$$V_2(K) - V_2(K[\varepsilon]) \sim \varepsilon^{1/2} \quad \text{for all} \quad K \in \mathscr{K}^3.$$

For other results and questions on random polytopes see the excellent survey papers by Schneider [S 2] and Buchta [Bu 2].

The proof of Theorem 4 is not quite simple and we will need a strengthening of Theorem 3 of [BL] which we no describe. Let $P \in \mathscr{K}^d$ be a polytope and define $m(P)$ as the minimal number of simplices needed to triangulate P.

Theorem 5. *If $P \in \mathscr{K}^d$ is a polytope and $0 < \varepsilon < \frac{d^{-d}}{4} \operatorname{vol} P$, then*

$$\operatorname{vol} K(\varepsilon) \leqq c(d)m(P)\varepsilon\left(\log \frac{\operatorname{vol} P}{\varepsilon}\right)^{d-1}, \tag{3.12}$$

where the constant $c(d)$ depends only on d.

Our next result is about the Haussdorff distance of K and K_n. This is defined for $K, L \in \mathscr{K}^d$ as

$$\delta(K,L) = \inf\{h : K \subset L + hB^d, L \subset K + hB^d\}.$$

It is almost trivial that $E\delta(P,P_n) \sim n^{-1/d}$ for a polytope $P \in \mathscr{K}^d$.

Theorem 6. *Assume $K \in \mathscr{K}^d$ is a $\mathscr{C}^2$ convex body with positive Gaussian curvature. Then*

$$E\delta(K,K_n) \sim \left(\frac{\log n}{n}\right)^{2/(d+1)} \tag{3.13}$$

with the implied constants depending on K.

It is easy to see that $\delta(K, K|\varepsilon|) \sim \varepsilon^{2/(d+1)}$ for smooth enough convex bodies. So the similarity between K_n and $K[1/n]$ seems to break down here. This can be explained in the following way. K_n is close to $K[1/n]$, but K_n is random while $K[1/n]$ is not—K_n is a "random perturbation" of $K[1/n]$. This occurs at the boundary of $K[1/n]$ which is at distance $n^{-2/(d+1)}$ from that of K. The random fluctuation of the boundary of K_n around $\text{bd}\, K[1/n]$ is what makes this distance larger by a factor of $(\log n)^{2/(d+1)}$.

4. Definitions and auxiliary lemmata

Let $K \in \mathscr{K}^d$. A *cap* C of K is a set $C = K \cap H$ where H is a closed halfspace with $K \cap H \neq \emptyset$. Then $H = \{x \in R^d : a \cdot x \geqq \alpha\}$ for some $a \in S^{d-1}$ and $\alpha \in R^1$. Here $a \cdot x$ denotes the scalar product of a and x. It will be convenient to write $H = H(a, t)$ with $t = h(a) - \alpha$ where

$$h(a) = \max\{a \cdot x : x \in K\}$$

is the *support function* of K (see [BF]). With this notation t is the width of the cap C in direction a which we call the *depth* of the cap. We will also write $H(a = t)$ for the bounding hyperplane of $H(a, t)$.

For a cap $C = K \cap H(a, t)$ a point $t \in C$ is called the *centre* of C if $a \cdot z = h(a)$. A cap may have several centres but we think of a cap as having a fixed centre, say the centre of gravity of all centres. For a cap C with centre z define (when $\lambda > 0$)

$$C^\lambda = z + \lambda(C - z). \tag{4.1}$$

Obviously $C = C^1$. It is clear that for $\lambda \geqq 1$

$$C^\lambda \supset K \cap H(a, \lambda t). \tag{4.2}$$

Now we define a function $v: K \to R^1$ by

$$v(x) = \inf\{\text{vol}(K \cap H) : x \in H, H \text{ is a halfspace}\}.$$

Clearly, the set $K(v \geqq \varepsilon) = \{x \in K : v(x) \geqq \varepsilon\}$ coincides with $K[\varepsilon]$. Also, $K(\varepsilon) = K(v \leqq \varepsilon) = \{x \in K : v(x) \leqq \varepsilon\}$.

When $x \in K$, a *minimal cap* of K at x is defined as a cap $C(x)$ with $x \in C(x)$ and $\text{vol}\, C(x) = v(x)$. It is evident that for each $x \in K$ a minimal cap exists. The minimal cap $C(x)$ is, in general, not unique. See for instance when K is a triangle. A standard variational argument shows that for a minimal cap $C(x) = K \cap H(a, t)$ the point x is the centre of gravity of the section $K \cap H(a = t)$.

For $x \in K$ and $\lambda > 0$ we call the set

$$M(x, \lambda) = M_K(x, \lambda) = x + \lambda\{(K - x) \cap (x - K)\} \tag{4.3}$$

a *Macbeath region*. Such region were studied by Macbeath [Ma] and by Ewald et al. [ELR]. A Macbeath region is obviously convex and centrally symmetric with centre x. We define another map $u: K \to R^1$ by

$$u(x) = \text{vol}\, M(x, 1). \tag{4.4}$$

Macbeath [Ma] proved that the set $K(u \geqq \varepsilon) = \{x \in K : u(x) \geqq \varepsilon\}$ is convex. It is proved and extensively used in [BL] that u and v are very close to each other near the boundary K. This fact will be crucial for this paper as well.

It follows form the existence of the Löwner John ellipsoid [DGK] that

$$\max_{x\in K} v(x) \geqq \frac{1}{2d^d} \operatorname{vol} K,$$

and

$$\max_{x\in K} u(x) \geqq \frac{1}{d^d} \operatorname{vol} K. \tag{45}$$

Now we list some of the facts needed later. Most of them are proved in [ELR] or in [BL]. From now on we assume that $K \in \mathscr{K}^d(r, R)$ and that the centre of the concentric inscribed and circumscribed balls (of radius r and R, respectively) is the origin. Define

$$\varepsilon_0 = \varepsilon_0(d, r, R) = \frac{\omega_{d-1}}{4^d d} \left(\frac{r}{R}\right)^d r^d$$

Lemma A. *If* $M(x, 1/2) \cap M(y, 1/2) \neq \emptyset$, *then*

$$M(y, 1) \subset M(x, 5).$$

Lemma B. $u(x) \leqq 2v(x)$ *for all* $x \in K$.

Lemma C. *If* $x \in K$ *and* $v(x) \leqq \varepsilon_0$, *then*

$$C(x) \subseteq M(x, 3d)$$

for every minimal cap $C(x)$.

Lemma D. *If* $x \in K$ *and* $v(x) \leqq \varepsilon_0$, *then* $v(x) \leqq (3d)^d u(x)$.

Lemma E. *If* $x \in K$ *and* $u(x) \leqq (3d)^{-d} \varepsilon_0$ *then* $v(x) \leqq (3d)^d u(x)$.

Lemma F. $K[\varepsilon]$ *contains no line segment on its boundary provided* $\varepsilon > 0$.

Lemma G. *Assume* C *is a cap such the* $C \cap K[\varepsilon] = \{x\}$, *a single point. If* $\varepsilon < \varepsilon_0$, *then* $C \subset M(x, 3d)$. *If* $\operatorname{int} K[\varepsilon] \neq \emptyset$, *then* $\operatorname{vol} C \leqq d\varepsilon$.

Lemma H. (*Economic cap covering*) *Assume* $\varepsilon < \varepsilon_0$. *Then there are caps* $K_1, \ldots, K_m$ *and pairwise disjoint sets* $K'_1, \ldots, K'_m$ *with* $K'_i \subset K_i$ $(i = 1, \ldots, m)$ *such that*

(i) $\bigcup_{i=1}^m K'_i \subset K(\varepsilon) \subset \bigcup_{i=1}^m K_i$,

(ii) $\operatorname{vol} K'_i \geqq (6d)^{-d} \varepsilon$, $\operatorname{vol} K_i \leqq 6^d \varepsilon$.

Lemma I. *If* $0 < \varepsilon \leqq \varepsilon_0$ *and* $\lambda \geqq 1$, *then*

$$\operatorname{vol} K(v < \varepsilon) > c(d) \lambda^{-d} \operatorname{vol} K(v \leqq \lambda \varepsilon)$$

where the constant $c(d)$ *depends only on* d.

Lemma J. *Let $K \in \mathscr{K}_1^d$ and $x \in K$. Then*

(i) $(1 - v(x))^n \leqq \text{Prob}(x \notin K_n)$

(ii) $\text{Prob}(x \notin K_n) \leqq 2 \sum_{i=0}^{d-1} \binom{n}{i} \left(\frac{u(x)}{2}\right)^i \left(1 - \frac{u(x)}{2}\right)^{n-i}$.

5. Proof of Theorem 1

As $Ef_s(K_n)$ is affinely invariant we may assume that $K \in \mathscr{K}_1^d(r, R)$ with $dr \geqq R$. Define

$$\varepsilon_0(d) = (2d)^{-2d}$$

as in [BL]. We will need a strengthening of the economic cap covering theorem (Lemma H):

Theorem 7. *Assume $K \in \mathscr{K}_1^d(r, R)$ with $dr \geqq R$. Let $0 < \varepsilon \leqq \varepsilon_0(d)$. Then there are caps $K_1, \ldots, K_m$ and pairwise disjoint subsets $K'_1, \ldots, K'_m$ with $K'_i \subset K_i$ $i = 1, \ldots, m$ such that*

(i) $\bigcup_1^m K'_i \subset K(\varepsilon) \subset \bigcup_1^m K_i$,

(ii) $\text{vol}\, K_i \leqq (15d + 1)^d \varepsilon$, $\quad \text{vol}\, K'_i \geqq \frac{1}{2}(6d)^{-d} \varepsilon$.

(iii) *for every cap C with* $\text{vol}\, C \leqq \varepsilon$ *there exists an* $i \in \{1, \ldots, m\}$ *with* $C \subset K_i$.

Proof. (Using the proof of the economic cap covering theorem from [BL]). Choose a maximal system of points $x_1, \ldots, x_m$ from bd $K[\varepsilon]$ subject to the condition that for $i \neq j$

$$M(x_i, \tfrac{1}{2}) \cap M(x_j, \tfrac{1}{2}) = \emptyset.$$

This system is finite because the sets $M(x_i, 1/2)$ are pairwise disjoint, all of them lie in K and $\text{vol}\, M(x_i, 1/2) = 2^{-d} u(x_i) \geqq (6d)^{-d} v(x_i) = (6d)^{-d} \varepsilon$ according to Lemma D.

Claim 1. *For a cap C with* $\text{vol}\, C \leqq \varepsilon$ *there is an* $i \in \{1, \ldots, m\}$ *with* $C \subset M(x_i, 15d)$.

Proof. Set $C = K \cap H(a, t_0)$ and define

$$t_1 = \sup\{t > 0 : H(a, t) \cap K[\varepsilon] = \emptyset\}$$

Then $t_1 \geqq t_0$ and $C_1 = K \cap H(a, t_1) \supset C_0$. Clearly $C_1 \cap K[\varepsilon] \neq \emptyset$ but $(\text{int}\, C_1) \cap K[\varepsilon] = \emptyset$. By Lemma F $C_1 \cap K[\varepsilon] = \{x\}$, a single point. Then, by Lemma G

$$C_1 \subset M(x, 3d). \tag{5.1}$$

On the other hand the system $x_1, \ldots, x_m$ is maximal so

$$M(x, \tfrac{1}{2}) \cap M(x_i, \tfrac{1}{2}) \neq \emptyset$$

for some $i \in \{1, \ldots, m\}$. Then by Lemma A

$$M(x, 1) \subset M(x_i, 5).$$

We show now that

$$M(x, 3d) \subset M(x_i, 15d). \tag{5.2}$$

This will follow from a more general statement:

Fact. Assume A and B are centrally symmetric convex bodies with centre a and b respectively. Assume $B \subset A$. Then, for $\lambda \geqq 1$,

$$b + \lambda(B - b) \subset a + \lambda(A - a).$$

Proof. We may assume $a = 0$. Let $c \in B$, we have to prove $b + \lambda(c - b) \in \lambda A$. B is symmetric so $2b - c \in B \subset A$, and A is symmetric so $c - 2b \in A$. But A is convex and $c \in B \subset A$ so $(1/2)(c + (c - 2b)) = c - b \in A$. Then $c \in A$ and $c - b \in A$ so $\lambda c \in \lambda A$ and $\lambda(c - b) \in \lambda A$. But $b + \lambda(c - b)$ lies on the line segment connecting λc and $\lambda(c - b)$:

$$b + \lambda(c - b) = \frac{1}{\lambda}(\lambda c) + \left(1 - \frac{1}{\lambda}\right)\lambda(c - b) \in A,$$

proving the fact. □

(5.2) follows from this by choosing $A = M(x_i, 5)$, $B = M(x, 1)$ and $\lambda = 3d$.
Now we have by (5.1) and (5.2)

$$C \subset C_1 \subset M(x, 3d) \subset M(x_i, 15d).$$

Next we define the caps $K_1, \ldots, K_m$ and the sets $K'_1, \ldots, K'_m$. Let $C(x_i) = K \cap H(a_i, t_i)$ be a minimal cap at x_i. Then $\operatorname{vol} C(x_i) = \varepsilon$. Define

$$K_i = K \cap H(a_i, (15d + 1)t_i),$$

$$K'_i = M(x_i, \tfrac{1}{2}) \cap H(a_i, t_i).$$

It is clear that $K'_i \subset H(a_i, t_i) \cap K \subset K(v \leqq \varepsilon)$. We have seen already that $\operatorname{vol} M(x_i, 1/2) \geqq (6d)^{-d}\varepsilon$, $\operatorname{vol} K'_i = 1/2 \operatorname{vol} M(x_i, 1/2)$. The other part of condition (ii) follows from (4.2). Condition (iii) follows from Claim 1 and the obvious fact that $M(x_i, 15d) \subset K_i$. Finally condition (iii) clearly implies $K(v \leqq \varepsilon) \subset \bigcup_1^m K_i$. □

Now let $x_1, \ldots, x_s \in K$, $s \in \{1, \ldots, d\}$. Define $A = \operatorname{aff}\{x_1, \ldots, x_s\}$ and $v(A) = \max\{v(x): x \in A\}$. This maximum attained for v is continuous. We write K^s for the space of all ordered s-tuples $(x_1, \ldots, x_s)$ with $x_1, \ldots, x_s \in K$. The direct product of the Lebesgue measure on K defines a probability measure ν on K^s. We need one more theorem before we get to the proof of Theorem 1.

Theorem 8. *If* $0 < \varepsilon < \varepsilon_0(d)$, *then*

$$\nu(\{(x_1, \ldots, x_s) \in K^s: v(A) < \varepsilon\}) \sim \varepsilon^{s-1} \operatorname{vol} K(v \leqq \varepsilon). \tag{5.3}$$

Proof. We only prove that $\nu(\{(x_1, \ldots, x_s): v(A \leqq \varepsilon)\}) \leqq c(d)\varepsilon^{s-1} \operatorname{vol} K(v \leqq \varepsilon)$. The other inequality is also true, its proof is more or less straightforward, but we will not need it in the sequel.

A simple separation argument shows that if $v(A) \leqq \varepsilon$ then there is a cap C with $A \cap K \subset C$ and $\operatorname{vol} C \leqq \varepsilon$. Thus

$$\{(x_1, \dots, x_s): v(A) \leqq \varepsilon\} \subset \cup \{(C, \dots, C): C \text{ is a cap with } \operatorname{vol} C \leqq \varepsilon\}$$
$$\subset \bigcup_{i=1}^{m} (K_i, \dots, K_i),$$

where $K_1, \dots, K_m$ come from the previous cap covering theorem. Then

$$\begin{aligned} v(\{(x_1, \dots, x_s): v(A) \leqq \varepsilon\}) &\leqq v\left(\bigcup_{i=1}^{m} (K_i, \dots, K_i) \right) \\ &\leqq \sum_1^m v(K_i, \dots, K_i) \leqq m(15d+1)^{ds} \varepsilon^s \\ &\leqq (15d+1)^{ds} 2(6d)^d \varepsilon^{s-1} \sum_1^m \operatorname{vol} K_i' \\ &\leqq (15d+1)^{d^2} 2(6d)^d \varepsilon^{s-1} \operatorname{vol} K(v \leqq \varepsilon). \quad \square \end{aligned}$$

Now we prove Theorem 1. K_n is a simplicial polytope with probability 1. Double counting the pairs (F_i, F_j) where F_i and F_j are faces of dimension i and j of K_n with $F_i \subset F_j$(and $i<j$) we get

$$f_i(K_n) = \sum_{F_i} 1 \leqq \sum_{(F_i,F_j)} 1 \leqq \binom{j+1}{i+1} \sum_{F_j} 1 = \binom{j+1}{i+1} f_j(K_n).$$

So we see that we have to prove the inequalities

$$Ef_0(K_n) \geqq c_1 n \operatorname{vol} K\left(\frac{1}{n}\right), \tag{5.4}$$

and

$$Ef_{d-1}(K_n) \leqq c_2 n \operatorname{vol} K\left(\frac{1}{n}\right). \tag{5.5}$$

The first inequality follows from (3.1) and (1.1). Yet for further reference we give its simple proof here:

$$\begin{aligned} Ef_0(K_n) &= nE \operatorname{vol}(K \backslash K_{n-1}) \\ &= n \int_{x \in K} \operatorname{Prob}(x \notin K_{n-1})\, dx \\ &\geqq n \int_{x \in K} (1 - v(x))^n\, dx \\ &\geqq n \int_{v(x) \leqq 1/n} (1 - v(x))^n\, dx \\ &\geqq n\left(1 - \frac{1}{n}\right)^n \operatorname{vol} K\left(v \leqq \frac{1}{n}\right) \end{aligned}$$

$$> \tfrac{1}{4} n \operatorname{vol} K\left(\frac{1}{n}\right),$$

if $n \geqq 4$, say. (We used Lemma J (i) here.)

The proof of the second inequality is more involved. It follows that of Theorem 1 in [BL]. Some notation is needed. Write $A = \operatorname{aff}\{x_1, \dots, x_d\}$, $u(A) = \max\{u(x): x \in A\}$. Then $u(z) = u(A)$ for some $z \in A$ which we denote by z_A. Clearly

$$E f_{d-1}(K_n) = \binom{n}{d} \int \cdots \int \begin{Bmatrix} 1 & \text{if} \quad \operatorname{conv}\{x_1 \dots, x_d\} \text{ is a face of } K_n \\ 0 & \text{otherwise} \end{Bmatrix} dx_1 \cdots dx_n$$

$$= \binom{n}{d} \int \cdots \int \operatorname{Prob}(A \cap \operatorname{conv}\{x_{d+1}, \dots, x_n\} = \varnothing)\, dx_1 \cdots dx_d \tag{5.6}$$

where Prob is meant with A fixed and $x_{d+1}, \dots, x_n$ chosen randomly, independently, and uniformly from K. Now by Lemma J

$$\int \cdots \int \operatorname{Prob}(A \cap \operatorname{conv}\{x_{d+1}, \dots, x_n\} = \varnothing)\, dx_1 \cdots dx_d$$

$$\leqq \int \cdots \int \operatorname{Prob}(z_A \notin \operatorname{conv}\{x_{d+1}, \dots, x_n\})\, dx_1 \cdots dx_d$$

$$\leqq \int \cdots \int 2 \sum_{i=0}^{d-1} \binom{n-d}{i} \left(\frac{u(z_A)}{2}\right)^i \left(1 - \frac{u(z_A)}{2}\right)^{n-d-i} dx_1 \cdots dx_d$$

$$\leqq 2 \sum_{\lambda=1}^{n} \sum_{i=0}^{d-1} \binom{n-d}{i} \int_{(\lambda-1)/n \leqq u(A) \leqq \lambda/n} \cdots \int \left(\frac{u(A)}{2}\right)^i \left(1 - \frac{u(A)}{2}\right)^{n-d-i} dx_1 \cdots dx_d$$

$$\leqq 2 \sum_{\lambda=1}^{n} \sum_{i=0}^{d-1} \binom{n-d}{i} \left(\frac{\lambda}{2n}\right)^i \left(1 - \frac{\lambda - 1}{2n}\right)^{n-d-i} \operatorname{Prob}\left(\mathrm{u}(A) \leqq \frac{\lambda}{n}\right). \tag{5.7}$$

Here $\operatorname{Prob}(u(A) \leqq \lambda/n)$ denotes the probability content (in our case, volume,) of the set of those d-tuples $x_1, \dots, x_d \in K^d$ for which $u(A) \leqq \dfrac{\lambda}{n}$. Then

$$\sum_{i=0}^{d-1} \binom{n-d}{i} \left(\frac{\lambda}{2n}\right)^i \left(1 - \frac{\lambda-1}{2n}\right)^{n-d-i}$$

$$\leqq \sum_{i=0}^{d-1} \frac{(n-d)^i}{i!} \frac{\lambda^i}{2^i n^i} \exp\left\{-\frac{(\lambda-1)(n-d-i)}{2n}\right\}$$

$$\leqq \sum_{i=0}^{d-1} \frac{\lambda^{d-1}}{i! 2^i} \exp\left\{-\frac{\lambda}{2}\right\} \exp\left\{\frac{(\lambda-1)(d+i)}{2n} + \frac{1}{2}\right\}$$

$$\leqq d \lambda^{d-1} e^{-\lambda/2} e^{d+1/2}. \tag{5.8}$$

Define now $n_0 = [(3d)^{-d}(2d)^{-2d} n]$. Then for $\lambda \leqq n_0$

$$\operatorname{Prob}\left(u(A) \leqq \frac{\lambda}{n}\right) = \operatorname{Prob}\left(A \cap K\left(u \geqq \frac{\lambda}{n}\right) = \varnothing\right)$$

$$\leqq \operatorname{Prob}\left(A \cap K\left(v \geqq (3d)^d \frac{\lambda}{n}\right) = \varnothing\right)$$

for $K(u \geqq \lambda/n) \supset K(v \geqq (3d)^d \lambda/n)$ if $\frac{\lambda}{n} \leqq (3d)^{-d}\varepsilon_0 = (3d)^{-d}(2d)^{-2d}$ according to Lemma E (or Lemma 2 of [BL]). Then, for $\lambda \leqq n_0$, Theorem 8 implies

$$\begin{aligned}
\operatorname{Prob}\left(u(A) \leqq \frac{\lambda}{n}\right) &\leqq \operatorname{Prob}\left(v(A) \leqq (3d)^d \frac{\lambda}{n}\right) \\
&\leqq c_1(d)\left((3d)^d \frac{\lambda}{n}\right)^{d-1} \operatorname{vol} K\left(v \leqq (3d)^d \frac{\lambda}{n}\right) \\
&\leqq c_2(d)\left((3d)^d \frac{\lambda}{n}\right)^{d-1} ((3d)^d \lambda)^d \operatorname{vol} K\left(v \leqq \frac{1}{n}\right) \\
&\leqq c_3(d) \frac{\lambda^{2d-1}}{n^{d-1}} \operatorname{vol} K\left(v \leqq \frac{1}{n}\right),
\end{aligned}$$

where the last inequality is justified by Lemma I. Then by (5.8)

$$\begin{aligned}
&2 \sum_{\lambda=1}^{n_0} \sum_{i=0}^{d-1} \binom{n-d}{i}\left(\frac{\lambda}{2n}\right)^i \left(1 - \frac{\lambda-1}{2n}\right)^{n-d-i} \operatorname{Prob}\left(u(A) \leqq \frac{\lambda}{n}\right) \\
&\leqq 2 \sum_{\lambda=1}^{n_0} c_3(d) \frac{\lambda^{3d-2}}{n^{d-1}} e^{-\lambda/2} \operatorname{vol} K\left(v \leqq \frac{1}{n}\right) \\
&\leqq c_4(d) \frac{1}{n^{d-1}} \operatorname{vol} K\left(v \leqq \frac{1}{n}\right). \qquad (5.9)
\end{aligned}$$

When $n > n_0$ we use the trivial inequalities $\operatorname{Prob}(u(A) \leqq \lambda/n) \leqq 1$ and $\operatorname{vol} K(v \leqq 1/n) \geqq 1/n$. Then for large enough n

$$\frac{1}{n^{d-1}} \operatorname{vol} K\left(v \leqq \frac{1}{n}\right) \geqq \frac{1}{n^d} \geqq \exp\{-\tfrac{1}{4}(3d)^{-d}(2d)^{-2d} n\}.$$

By (5.8)

$$\begin{aligned}
&2 \sum_{\lambda=n_0+1}^{n} \sum_{i=0}^{d-1} \binom{n-d}{i}\left(\frac{\lambda}{2n}\right)^i \left(1 - \frac{\lambda-1}{2n}\right)^{n-d-i} \operatorname{Prob}\left(u(A) \leqq \frac{\lambda}{n}\right) \\
&\leqq 2 \sum_{\lambda=n_0+1}^{2} d e^{d+1/2} \lambda^{d-1} e^{-\lambda/4} e^{-n_0/4} \\
&\leqq 2 \sum_{\lambda=n_0+1}^{n} d e^{d+1/2} \lambda^{d-1} e^{-\lambda/4} \frac{1}{n^{d-1}} \operatorname{vol} K\left(v \leqq \frac{1}{n}\right) \\
&\leqq c_5(d) \frac{1}{n^{d-1}} \operatorname{vol} K\left(v \leqq \frac{1}{n}\right). \qquad (5.10)
\end{aligned}$$

Now by (5.6), (5.7), (5.9) and (5.10)

$$\begin{aligned}
E f_{d-1}(K_n) &\leqq \binom{n}{d} \max(c_4(d), c_5(d)) \frac{1}{n^{d-1}} \operatorname{vol} K\left(v \leqq \frac{1}{n}\right) \\
&\leqq \operatorname{const}(d) n \operatorname{vol} K\left(v \leqq \frac{1}{n}\right). \quad \square
\end{aligned}$$

6. Proof of Theorem

Let $K \in \mathscr{K}^d(r, R)$ as in the theorem and consider F, an s-dimensional subspace of R^d. Let $\mathrm{pr} = \mathrm{pr}_F : R^d, F$ denote the orthogonal projection into F. We will drop the subscript F if is there is no ambiguity. Define $L = \mathrm{pr}\, K$ and $L_\varepsilon = \mathrm{pr}\, K[\varepsilon]$. We need a cap covering theorem for $L \backslash L_\varepsilon$ (cf. Lemma H).

Theorem 9. *There are caps $L_1, \ldots, L_m$ of L and pairwise disjoint subsets $L'_1, \ldots, L'_m$ with $L'_i \subset L_i$ $(i = 1, \ldots, m)$ such that*

(i) $\bigcup_1^m L'_i \subset L \backslash L_\varepsilon \subset \bigcup_1^m L_i$,

(ii) $\mathrm{vol}\, L'_i \geqq c(d) \,\mathrm{vol}\, L_i \quad (i = 1, \ldots, m)$

where the constant $c(d)$ depends only on d.

Proof. First we replace K by its symmetral K^* with respect to F. That is, for each $x \in L$ we compute the $(d-s)$-dimensional volume of $K \cap (x + F^\perp)$ and put a $(d-s)$-dimensional ball of this same volume and having centre x into the affine subspace $x + F^\perp$. The union of all such balls is K^*. It is known [BF] that $K^* \in \mathscr{K}^d(r, R)$ and $\mathrm{vol}\, K^* = \mathrm{vol}\, K$. Obviously $\mathrm{pr}\, K = \mathrm{pr}\, K^* = L$.

Now we prove

$$\mathrm{pr}\, K[\varepsilon] \subset \mathrm{pr}\, K[(3d)^{-2d}\varepsilon] \quad \text{if} \quad \varepsilon \leqq \varepsilon_0, \tag{6.1}$$

$$\mathrm{pr}\, K^*[d\varepsilon] \subset \mathrm{pr}\, K[\varepsilon] \quad \text{if} \quad \mathrm{int}\, K[\varepsilon] \neq \varnothing. \tag{6.2}$$

Let us see (6.1) first. Assume $z \in \mathrm{bd}\,\mathrm{pr}\, K[\varepsilon]$. Then there is $y \in \mathrm{pr}^{-1} z$ with $v(y) = \varepsilon$ and then $u(y) \geqq (3d)^{-d}\varepsilon$ by Lemma D. As it is well-known, $M(y, 1)$ contains an ellipsoid with volume at least

$$d^{-d/2} u(y) \geqq (3d)^{-d} d^{-d/2} \varepsilon.$$

The symmetral of this ellipsoid is contained in K^* so

$$v^*(y) := v_{K^*}(y) \geqq \tfrac{1}{2}(3d)^{-d} d^{-d/2} \varepsilon \geqq (3d)^{-2d}\varepsilon.$$

Now both sets $\mathrm{pr}\, K[\varepsilon]$ and $\mathrm{pr}\, K^*[(3d)^{-2d}\varepsilon]$ are convex and the latter contains all boundary points of the first. This proves (6.1).

To see (6.2) assume $z \in F$ but $z \notin \mathrm{pr}\, K[\varepsilon]$. Then there is a halfspace H with $H \cap K[\varepsilon] = \varnothing$ whose bounding hyperplane contains $z + F^\perp$. Let H' be the parallel translate of H such that $H' \cap K[\varepsilon]$ is a single point. (H' exists by Lemma F.) Applying Lemma G to the cap $C = H' \cap K$ we get

$$\mathrm{vol}\,(H \cap K) \leqq \mathrm{vol}\,(H' \cap K) \leqq d\varepsilon$$

which means that every point in $\mathrm{pr}^{-1}(z) \cap K$ can be cut off by the cap C that has volume $d\varepsilon$ at most. As the symmetral of C has the same volume we conclude that $z \notin \mathrm{pr}\, K^*[d\varepsilon]$, proving (6.2).

Set now $\eta = (3d)^{-2d}\varepsilon$ and $L^*_\eta = \mathrm{pr}\, K^*[\eta]$. It follows form the definition that

$$L^*_\eta = F \cap K^*[\eta]$$

and

$$M_L(x,\lambda)=M_{\mathrm{pr}K}(x,\lambda)=M_{F\cap K}(x,\lambda)=F\cap M_{K^*}(x,\lambda)$$

for all $x\in F$ and $\lambda>0$.

Choose now a maximal system of points $x_1,\dots,x_m$ from bd L^*_η (with bd meant in F) subject to the condition

$$M_{K^*}(x_i,\tfrac{1}{2})\cap M_{K^*}(x_j,\tfrac{1}{2})=\emptyset.$$

The argument from the proof of Theorem 7 shows that this system is finite. Then, in the same way as in [BL], Claim 1 we see that

$$L\backslash L^*_\eta\subset\bigcup_1^m M_{K^*}(x_i,5),$$

and so

$$L\backslash L^*_\eta\subset\bigcup_1^m M_L(x_i,5.)$$

Now let a_i be an outer unit normal to L^*_η at x_i with $a_i\in F$ $(i=1,\dots,m)$. Set $D_i=L\cap H_i$ where H_i is the halfspace in R^d whose bounding hyperplane contains x_i and has normal a_i. Let C_i be the "lifting" of D_i into K^*, i.e., $C_i=K^*\cap H_i$. Clearly $\mathrm{pr}\,C_i=D_i$. Then, by Lemma G

$$D_i\subset M_{K^*}(x_i,3d),$$

and consequently

$$D_i\subset M_L(x_i,3d).$$

On the other hand $H_i=H(a_i,t_i)$ with a suitable t_i. Here $H(a_i,t_i)$ can be regarded as defined through K, K^* or L. Set now

$$L_i'=M_L(x_i,\tfrac{1}{2})\cap H_i$$

and

$$L_i'=L\cap H(a_i,6t_i).$$

Then we see in the same way as in [BL] that the L_i'-s are pairwise disjoint, $L_i'\subset L_i''$ and $M_L(x_i,5)\subset L_i''$. So

$$\bigcup_1^m L_i'\subset L\backslash L^*_\eta\subset\bigcup_1^m L_i'',$$

$$\mathrm{vol}\,L_i''\leqq 6^s\,\mathrm{vol}\,D_i,$$

and

$$\mathrm{vol}\,L_i'=\tfrac{1}{2}\mathrm{vol}\,M_L(x_i,\tfrac{1}{2})=\tfrac{1}{2}(6d)^{-s}\,\mathrm{vol}\,M_L(x_i,3d)$$
$$\geqq\tfrac{1}{2}(6d)^{-s}\,\mathrm{vol}\,D_i.$$

So we have an economic cap covering for $L\backslash L^*_\eta$. But we need one for $L\backslash L_\varepsilon$.

Even more generally, we are going to produce an economic cap covering for $L\backslash L_{\lambda\varepsilon}$ with $\lambda\geqq 1$. Set $\mu=d^2(3d)^{2d}\lambda$ and

$$L_i=L\cap H(a_i,\mu t_i)\quad i=1,\dots,m.$$

Claim 2. $L\backslash L_{\lambda\varepsilon} \subset \bigcup_1^m L_i$ *if* $\operatorname{int} K[\lambda\varepsilon] \neq \emptyset$.

Proof. (Which will be similar to that of Theorem 7 from [BL].) Take a point $x \in L\backslash L_{\lambda\varepsilon}$. We are going to show that $x_i \in L_i$ for some $i \in \{1, \dots, m\}$, so we may assume that $x \notin L\backslash L_\eta^*$ as $L\backslash L_\eta^* \subset \bigcup_1^m L_i$ clearly.

Set $v^* = v_{K^*}$ and $v^*(x) = v$. Let $a \in F$ be the outer normal to $K^*[v]$ at x. Then the cap $C(x) = K^* \cap H(a, t)$ with $x \in H(a = t)$ has centre $z \in F$, say, and the line segment through x and z intersects $\operatorname{bd} L_\eta^*$ at the point y. Let t' be defined by $y \in H(a = t')$. As $v^*(y) = \eta$ we have

$$\begin{aligned}\eta = v^*(y) &\leqq \operatorname{vol} K^* \cap H(a, t) \\ &= \int_0^{t'} \operatorname{vol}_{d-1}[K^* \cap H(a = \tau)]\, d\tau \\ &\leqq t' \max\{\operatorname{vol}_{d-1}[K^* \cap H(a = \tau)]: 0 \leqq \tau \leqq t'\} \\ &\leqq t' \max\{\operatorname{vol}_{d-1}[K^* \cap H(a = \tau)]: 0 \leqq \tau \leqq t\}.\end{aligned}$$

On the other hand $x \in L\backslash L_{\lambda\varepsilon}$ and (6.2) implies $\operatorname{vol} C(x) \leqq d\lambda\varepsilon = d\lambda(3d)^{2d}\eta$. Thus

$$\begin{aligned}d\lambda(3d)^{2d}\eta &\geqq \operatorname{vol} C(x) = \operatorname{vol} K^* \cap H(a, t) \\ &= \int_0^{t} \operatorname{vol}_{d-1}[K^* \cap H(a = \tau)]\, d\tau \\ &\geqq \frac{t}{d} \max\{\operatorname{vol}_{d-1}[K^* \cap H(a = \tau)]: 0 \leqq \tau \leqq t\}.\end{aligned}$$

So we have

$$\frac{t}{t'} = \frac{|z - x|}{|z - y|} \leqq d^2(3d)^{2d}\lambda = \mu.$$

Consider now the cap L_i'' from the cap covering of $L\backslash L_\eta^*$ containing y. Let z_i be the centre of L_i'' and write y_i for the intersection of $H(a_i = 6t_i)$ with the line segment connecting x and z_i. The line through z and x intersects the hyperplanes $H(a_i = 0)$ and $H(a_i = 6t_i)$ in the points z' and y', respectively. It is easy to see that the points z', z, y, y', x are collinear and come in this order on their line. Then

$$\frac{|x - z_i|}{|y_i - z_i|} = \frac{|x - z'|}{|y' - z'|} \leqq \frac{|x - z'|}{|y - z'|} = \frac{|x - z| + |z - z'|}{|y - z| + |z - z'|} \leqq \frac{|x - z|}{|y - z|} \leqq \mu.$$

So $x \in L_i$ and the Claim is proved. □

Now in the case $\lambda = 1$ we have

$$\operatorname{vol} L_i = \operatorname{vol} L \cap H(a_i, \mu t_i) \leqq \mu^s \operatorname{vol} L \cap H(a_i, t_i) = \mu^s \operatorname{vol} D_i \leqq \operatorname{const}(d) \operatorname{vol} L_i'.$$

The proof of Theorem 9 is complete. □

When $\lambda > 1$ and $\operatorname{int} K[\lambda\varepsilon] \neq \emptyset$ we have, similarly,

$$\operatorname{vol} L_i \leqq c_1(d)\lambda^s \operatorname{vol} L_i' \text{ and so}$$

$$\operatorname{vol} \bigcup_{i=1}^{m} L_i \leqq c_1(d)\lambda^s \sum_1^m \operatorname{vol} L_i' \leqq c_1(d)\lambda^s \operatorname{vol}(L\backslash L_\eta^*)$$

$$\leqq c_1(d)\lambda^s \operatorname{vol}(L\backslash L_\varepsilon). \tag{6.3}$$

But (6.3) remains true (with another constant $c_1(d,r,R)$ instead of $c_1(d)$ even if $\operatorname{int} K[\lambda\varepsilon] = \emptyset$. To see this observe first that in this case $\lambda\varepsilon > 1/2\omega_d r^d$ and $\operatorname{vol}(L\backslash L_\varepsilon) \leqq \operatorname{vol} L \leqq \omega_s R^s$. Consider a cap C whose bounding hyperplane touches $K[\varepsilon]$. Then $\operatorname{vol} C \geqq \varepsilon$ and

$$\varepsilon \leqq \operatorname{vol} C = \int_{x \in \operatorname{pr} C} \operatorname{vol}_{d-s}(\operatorname{pr}^{-1}(x) \cap K)\, dx$$

$$\leqq \omega_{d-s} R^{d-s} \operatorname{vol}_s \operatorname{pr} C,$$

and so

$$\operatorname{vol}_s \operatorname{pr} C \geqq \varepsilon (R^{d-s}\omega_{d-s})^{-1}.$$

So the left hand side of (6.3) is at most $\operatorname{vol} L \leqq \omega_s R^s$ and the right hand side is at least

$$c_1(d)\left(\frac{r^d \omega_d}{2\varepsilon}\right)^s \varepsilon (R^{d-s}\omega_{d-s})^{-1}.$$

Then (6.3) holds for all $\lambda \geqq 1$ and $\varepsilon \leqq \varepsilon_0$ if the corresponding constant $c_1(d,r,R)$ is chosen large enough. We proved

Theorem 10. *If $0 < \varepsilon \leqq \varepsilon_0$ and $s \in \{0, 1, \ldots, d-1\}$, then*

$$\operatorname{vol}_s(L\backslash L_\varepsilon) \geqq \operatorname{const}(d,r,R)\lambda^{-s} \operatorname{vol}_s(L\backslash L_{\lambda\varepsilon}). \quad \square$$

This result is analogous to Lemma I.

Now we start with the proof of Theorem 2. We assume, without loss of generality, that $K \in \mathscr{K}_1^d(r,R)$. We need the following fact (see [H] or [BF]) that can be taken for the definition of intrinsic volume:

$$V_s(K) = A \int_{F \in \mathscr{G}} \operatorname{vol}_s(\operatorname{pr}_F K)\, d\omega(F) \tag{6.4}$$

where $\mathscr{G} = \mathscr{G}_{d,s}$ is the Grassmannian of the s-dimensional subspaces of R^d, $F \in \mathscr{G}$, $\omega(\cdot)$ is the Haar measure on $\mathscr{G}$ normalized by $\omega(\mathscr{G}) = 1$, and A is a constant depending on d and s. Thus

$$E(K,s,n) = AE \int_{\mathscr{G}} \operatorname{vol}_s(\operatorname{pr}_F(K)\backslash \operatorname{pr}_F(K_n))\, d\omega(F)$$

$$= A \int_{\mathscr{G}} [E \operatorname{vol}_s(\operatorname{pr}_F(K)\backslash \operatorname{pr}_F(K_n))]\, d\omega(F) \tag{6.5}$$

where the application of Fubini's theorem is easily justified.

We will drop the subscript F from pr_F while the subspace F is being kept fixed. As symmetrisation does not change the value of $E \operatorname{vol}_s(\operatorname{pr} K_n)$ we have

$$\begin{aligned} E\,\mathrm{vol}_s(\mathrm{pr}(K)\backslash\mathrm{pr}(K_n)) &= E(\mathrm{vol}_s(\mathrm{pr}(K)) - \mathrm{vol}_s(\mathrm{pr}(K_n))) \\ &= E(\mathrm{vol}_s(K^*)) - \mathrm{vol}_s(\mathrm{pr}\,(K_n^*)) \\ &= \int_{\mathrm{pr}K^*} \mathrm{Prob}\,(x \notin \mathrm{pr}\,K_n^*)\,dx \\ &= \int_{\mathrm{pr}K^*} \mathrm{Prob}\,(x \notin K_n^*)\,dx, \end{aligned} \tag{6.6}$$

where $\mathrm{Prob}\,(x\notin \mathrm{pr}\,K_n^*)$ is the probability that $x\notin\mathrm{pr}\,K_n^*$ for a fixed $x\in\mathrm{pr}\,K^*$. We write again v^* and u^* instead of v_{K^*} and u_{K^*}. Let $x\subset F$ with $v(x)=\eta$ and consider the minimal cap $D(x)$ of $\mathrm{pr}\,K^*$. Then its lifting, $C(x)=\mathrm{pr}^{-1}(D(x))\cap K^*$ is a cap touching $K^*(\eta)$. Then by Lemma G its volume is at most $d\eta$ and

$$\begin{aligned} &\int_{\mathrm{pr}K^*} \mathrm{Prob}\,(x\notin K^*)\,dx \\ &\geqq \int_{\mathrm{pr}K^*} (1-\mathrm{vol}\,C(x))^n\,dx \geqq \int_{\mathrm{pr}K^*\backslash \mathrm{pr}K^*[d/n]} (1-\mathrm{vol}\,C(x))^n\,dx \\ &\geqq \left(1-\frac{d^2}{n}\right)^n \mathrm{vol}_s\left(\mathrm{pr}\,(K^*)\backslash\mathrm{pr}\left(K^*\left[\frac{d}{n}\right]\right)\right) \\ &\geqq c_2(d)\,\mathrm{vol}_s\left(\mathrm{pr}\,(K)\backslash\mathrm{pr}\left(K\left[\frac{1}{n}\right]\right)\right). \end{aligned} \tag{6.7}$$

This shows, using (6.4), (6.5) and (6.6) that

$$\begin{aligned} E(K,s,n) &\geqq \mathrm{const}\,(d,s)\int_{\mathscr{G}} \mathrm{vol}_s\left(\mathrm{pr}_F(K)\backslash\mathrm{pr}_F\left(K\left[\frac{1}{n}\right]\right)\right)d\omega(F) \\ &\geqq \mathrm{const}\,(d,s)\left(V_s(K) - V_s\left(K\left[\frac{1}{n}\right]\right)\right) \end{aligned}$$

which is the first inequality to be proved in Theorem 2.

For the other inequality we observe that by Lemma J

$$\mathrm{Prob}\,(x\notin K_n^*) \leqq 2\sum_{i=0}^{d-1}\binom{n}{i}\left(\frac{u^*(x)}{2}\right)^i\left(1-\frac{u^*(x)}{2}\right)^{n-i}.$$

Continuing (6.6) and this we get

$$\begin{aligned} &E\,\mathrm{vol}_s(\mathrm{pr}\,(K^*)\backslash\mathrm{pr}\,(K_n^*)) \\ &\leqq \int_{\mathrm{pr}K^*} 2\sum_{i=0}^{d-1}\binom{n}{i}\left(\frac{u^*(x)}{2}\right)^i\left(1-\frac{u^*(x)}{2}\right)^{n-i} dx \\ &= \sum_{\lambda=1}^{n}\sum_{i=0}^{d-1} 2 \int_{(\lambda-1)/n \leqq u^*(x)\leqq \lambda/n} \binom{n}{i}\left(\frac{u^*(x)}{2}\right)^i\left(1-\frac{u^*(x)}{2}\right)^{n-i} dx \\ &\leqq \sum_{\lambda=1}^{n}\sum_{i=0}^{d-1} 2\binom{n}{i}\left(\frac{\lambda}{2n}\right)^i\left(1-\frac{\lambda-1}{2n}\right)^{n-i}\mathrm{vol}_s\left(\mathrm{pr}\,K^*\backslash\mathrm{pr}\,K^*\left(u^*\geqq\frac{\lambda}{n}\right)\right). \end{aligned}$$

Similarly as in (5.8) we have

$$\sum_{i=0}^{d-1} 2\binom{n}{i}\left(\frac{\lambda}{2n}\right)^i\left(1-\frac{\lambda-1}{2n}\right)^{n-i} \leqq e^2\lambda^{d-1}e^{-\lambda/2}. \tag{6.8}$$

By Lemma E

$$K^*\left(u^* \geqq \frac{\lambda}{n}\right) \supseteq K^*\left(v^* \geqq (3d)^d\frac{\lambda}{n}\right)$$

provided $\lambda/n \leqq (3d)^{-d}\varepsilon_0$. According to (6.2)

$$\operatorname{pr} K^*\left(v^* \geqq (3d)^d\frac{\lambda}{n}\right) \supseteq \operatorname{pr} K\left(v \geqq (3d)^{3d}\frac{\lambda}{n}\right)$$

provided $(3d)^d\lambda/n \leqq \varepsilon_0$. Then for $\lambda \leqq n_0 = [(3d^{-d}\varepsilon_0 n]$

$$\begin{aligned}
&\operatorname{vol}_s\left(\operatorname{pr}(K^*)\backslash\operatorname{pr} K^*\left(u^* \geqq \frac{\lambda}{n}\right)\right)\\
&\leqq \operatorname{vol}_s\left(\operatorname{pr} K\backslash\operatorname{pr} K\left(v \geqq (3d)^{3d}\frac{\lambda}{n}\right)\right)\\
&\leqq c_1(d,s,r,R)((3d)^{3d}\lambda)^s\operatorname{vol}_s\left(\operatorname{pr} K\backslash\operatorname{pr} K\left[\frac{1}{n}\right]\right)
\end{aligned} \tag{6.9}$$

where the last inequality follows from Theorem 10. Then splitting the last sum in (6.7) into two parts we get

$$\begin{aligned}
\sum_{\lambda=1}^{n_0}\cdots &\leqq \sum_{\lambda=1}^{n_0} e^2\lambda^{d-1}e^{-\lambda/2}c_1(d,s,r,R)\lambda^s\operatorname{vol}_s\left(\operatorname{pr} K\backslash\operatorname{pr} K\left[\frac{1}{n}\right]\right)\\
&\leqq c_2(d,s,r,R)\operatorname{vol}_s\left(\operatorname{pr} K\backslash\operatorname{pr} K\left[\frac{1}{n}\right]\right).
\end{aligned} \tag{6.10}$$

It is not difficult to see that

$$\operatorname{vol}_s\left(\operatorname{pr} K\backslash\operatorname{pr} K\left[\frac{1}{n}\right]\right) \geqq \exp\{-\tfrac{1}{4}n_0\}$$

if n is large enough. (We omit the details.) Then

$$\begin{aligned}
\sum_{\lambda=n_0+1}^{n}\cdots &\leqq \sum_{\lambda=n_0+1}^{n} e^2\lambda^{d-1}e^{-\lambda/2}\\
&\leqq c_3(d,s,R)\sum_{\lambda=n_0+1}^{n}\lambda^{d-1}e^{-\lambda/4}e^{-n_0/4}\\
&\leqq c_3(d,s,R)\sum_{\lambda=n_0+1}^{n}\lambda^{d-1}e^{-\lambda/4}\operatorname{vol}_s\left(\operatorname{pr} K\backslash\operatorname{pr} K\left[\frac{1}{n}\right]\right)\\
&\leqq c_4(d,s,R)\operatorname{vol}_s\left(\operatorname{pr} K\backslash\operatorname{pr} K\left[\frac{1}{n}\right]\right).
\end{aligned} \tag{6.11}$$

Then (6.6), (6.7), (6.10) and (6.11) imply

$$E(K,s,n) \leqq \mathrm{const}(d,s,r,R) \int_{\mathscr{G}} \mathrm{vol}_s\left(\mathrm{pr}(K)\backslash \mathrm{pr}\, K\left[\frac{1}{n}\right]\right) d\omega(F)$$

$$= \mathrm{const}(d,s,r,R)\left(V_s(K) - V_s\left(K\left[\frac{1}{n}\right]\right)\right). \quad \square$$

7. Proof of Theorems 3 and 4

We want to compute $V_s(K) - V_s(K[\varepsilon])$ when K is smooth and when K is a polytope. According to (6.4)

$$V_s(K) - V_s(K[\varepsilon]) = A \int_{\mathscr{G}} \mathrm{vol}_s(\mathrm{pr}_F(K)\backslash \mathrm{pr}_F(K[\varepsilon]))\, d\omega(F). \tag{7.1}$$

The integrand here is the s-volume of the union of all $\mathrm{pr}_F C$ where C is a cap of K with $C \cap K[\varepsilon] \neq \emptyset$ but $\mathrm{int}\, C \cap K[\varepsilon] = \emptyset$ and such that the normal of its bounding hyperplane lies in F.

Proof. (of Theorem 3 which is much simpler.) Let $K \in \mathscr{K}^d(r,R)$ be a $\mathscr{C}^2$ convex body with positive Gaussian curvature. As the curvature of K is bounded away from zero and infinity, C is very close to a cap of an ellipsoid if $\varepsilon > 0$ is small enough. One can estimate $\mathrm{vol}_s\, \mathrm{pr}_F C$ easily:

$$c_1(K)\varepsilon^{(s+1)/(d+1)} \leqq \mathrm{vol}_s\, \mathrm{pr}_F C \leqq c_2(K)\varepsilon^{(s+1)/(d+1)}.$$

Moreover, $\mathrm{pr}_F K$ satisfies the conditions of Theorem 4 in [BL]. So applying that theorem, Theorem 1 of [BL] and a result of Groemer [Gro] we get

$$\mathrm{vol}_s(\mathrm{pr}_F(K)\backslash \mathrm{pr}_F K[\varepsilon]) \sim \mathrm{vol}_s\, \mathrm{pr}_F(K)\left(v_{\mathrm{pr}_F(K)} \leqq c(K)\varepsilon^{(s+1)/(d+1)}\right) \sim \varepsilon^{2/(d+1)}$$

with the implied constants depending on K (and independent of F and ε). This proves Theorem 3. $\square$

Proof of Theorem 4. Let $P \in \mathscr{K}^d(r,R)$ be a polytope. We prove first that

$$V_s(P) - V_s(P[\varepsilon]) \geqq \mathrm{const}(P)\varepsilon^{1/(d-s+1)}.$$

Let $Q(a,\alpha)$ denote the circular cone with apex O and half-angle α ($0 < \alpha < \pi/2$), its axis having direction $a \in S^{d-1}$. Clearly, for almost every $F \in \mathscr{G}$ there is an $(s-1)$-dimensional face, L, of P and a circular cone $Q(a,\alpha)$ such that $P \subset L + Q(a,\alpha)$ and there is a hyperplane H with normal $a \in F$ supporting P with $H \cap P = L$. (According to our notation convention $H = H(-a, 0)$.) Moreover, L (and then H) can be chosen so that

$$\mathrm{vol}_{s-1} L \leqq c_1(P)\mathrm{vol}_{s-1}\, \mathrm{pr}_F L$$

holds. This can be seen in the following way. $\mathrm{pr}_F P$ is an s-dimensional polytope with surface area larger then that of rB^s and number of facets less than $f_{s-1}(P)$.

Further $\mathrm{vol}_{s-1}L \leqq \omega_{s-1}R^{s-1}$ for all $(s-1)$-faces of P. Then

$$\max\{\mathrm{vol}_{s-1}(\mathrm{pr}_F L)\colon \mathrm{pr}_F L \text{ is a facet of } \mathrm{pr}_F P\}$$
$$\geqq \frac{\text{surface area of } \mathrm{pr}_F P}{f_{s-1}(P)} \geqq \frac{(s-1)\omega_{s-1}r^{s-1}}{f_{s-1}(P)}.$$

So for almost every $F\in\mathscr{G}$ there is an $(s-1)$-face, L, of P such that

(i) $\mathrm{pr}_F L$ is a facet of $\mathrm{pr}_F P$,

(ii) $P\subset L+Q(a,\alpha)$ for some $a\in F$ and $\alpha<\frac{\pi}{2}$,

(iii) $\mathrm{vol}_{s-1}\,\mathrm{pr}_F L \geqq \frac{s-1}{f_{s-1}(P)}\left(\frac{r}{R}\right)^{s-1}\mathrm{vol}_{s-1}L.$

Then there exists an angle $\alpha_0<\pi/2$ and a set $\mathscr{F}\subset\mathscr{G}$ with $\omega(\mathscr{F})>1/2$ such that there exists an $(s-1)$-face L of P satisfying (i), (ii) and (iii) with $\alpha=\alpha_0$ in (ii). Here α_0 depends only on P. It is easy to see that for the cap $C(a,t)=P\cap H(-a,t)$ we have

$$\mathrm{vol}\,C(a,t)\sim t^{d-s+1}\,\mathrm{vol}_{s-1}L$$

when $t\leqq t_0$ is small enough (where t_0 and the constants implied by $\sim$ depend only on P). Moreover

$$\mathrm{vol}_s\,\mathrm{pr}_F C(a,t)\sim t\,\mathrm{vol}_{s-1}L.$$

Let us fix t so that $\mathrm{vol}\,C(a,t)=\varepsilon$. Then $\mathrm{pr}_F C(a,t)\subset \mathrm{pr}_F P\backslash \mathrm{pr}_F P[\varepsilon]$ and

$$\mathrm{vol}_s\,\mathrm{pr}_F C(a,t)\sim \varepsilon^{1/(d-s+1)}.$$

with the implied constants depending only on P. Then we get

$$\mathrm{vol}_s(\mathrm{pr}_F P\backslash \mathrm{pr}_F P[\varepsilon])\geqq c_2(P)\varepsilon^{1/(d-s+1)}$$

for all $F\in\mathscr{F}$. So by (7.1)

$$V_s(P)-V_s(P[\varepsilon]) = A\int_{\mathscr{G}} \mathrm{vol}_s(\mathrm{pr}_F P\backslash \mathrm{pr}_F P[\varepsilon])\,d\omega(F)$$
$$\geqq A\int_{\mathscr{F}} c_2(P)\varepsilon^{1/(d-s+1)}d\omega(F)$$
$$\geqq c_3(P)\varepsilon^{1/(d-s+1)}.$$

Quite similar arguments show that for all $F\in\mathscr{G}$ and for all caps $C=P\cap H(a,t)$ with $a\in F$ and $\mathrm{int}\,C\cap P[\varepsilon]=\emptyset$ one has

$$\mathrm{vol}_s\,\mathrm{pr}_F C\leqq c_4(P)\varepsilon^{1/(d-s+1)}.$$

Then by (1.1), Theorem 3 of [BL] and (7.1)

$$V_s(P)-V_s(P[\varepsilon])\leqq c_5(P)\varepsilon^{1/(d-s+1)}\left(\log\frac{1}{\varepsilon}\right)^{s-1}$$

which is only slightly weaker than the inequality we have to prove, namely:

$$V_s(P) - V_s(P[\varepsilon]) \leqq \text{const}(P)\varepsilon^{1/(d-s+1)} \tag{7.2}$$

for small enough ε. Set $p = d - s + 1$. We will prove this by showing that for all $F \in \mathscr{G}$

$$\text{vol}_s(\text{pr}_F P \backslash \text{pr}_F P[\varepsilon]) \leqq \text{const}(P)\varepsilon^{1/p}. \tag{7.3}$$

We drop the subscript F. By (6.2)

$$\text{pr}\, P[\varepsilon] \supset \text{pr}\, P^*[d\varepsilon]$$

and by Lemma B

$$P^*[d\varepsilon] = P^*(v^* \geqq d\varepsilon) \supset P^*(u^* \geqq 2d\varepsilon).$$

Clearly $\text{pr}\, P = \text{pr}\, P^* = P^* \cap F$ is a polytope Q in $F \simeq R^s$ and $Q \in \mathscr{K}^s(r, R)$. Define $K = \text{conv}(Q \cup rB^d)$. Then $K \in \mathscr{K}^d(r, R)$, $\text{pr}\, K = K \cap F = Q$ and $K = K^* \subset P^*$. Then $u_K(x) \leqq u^*(x) := u_{P^*}(x)$. Consequently

$$P^*(u^* \geqq 2d\varepsilon) \supset K(u_K \geqq 2d\varepsilon),$$

and we have $\text{pr}\, P[\varepsilon] \supset \text{pr}\, K(u_K \geqq 2d\varepsilon)$ and so

$$\begin{aligned} \text{pr}\, P \backslash \text{pr}\, P[\varepsilon] &\subset \text{pr}\, K \backslash \text{pr}\, K(u_K \geqq 2d\varepsilon) \\ &\subset F \cap K(u_K \leqq 2d\varepsilon). \end{aligned}$$

Set $\eta = 2d\varepsilon$. We will prove (7.3) by showing

$$\text{vol}_s F \cap K(u_K \leqq \eta) \leqq \text{const}(P)\eta^{1/p} \tag{7.4}$$

when $\eta \leqq \eta_0(d, r) = 2^{-d} r^d \omega_d$.

Let $x \in F \cap \text{int}\, K$, let $z \in \text{bd}\, K$ be such that x is on the line segment connecting 0 and z. Write $\tau = |z - x|$ and let L be the facet of Q containing z. Set $t = \text{dist}(x, \text{aff}\, L)$ and choose $\varrho > 0$ maximal with

$$x + \varrho B^d \subset K.$$

The facts $K \in \mathscr{K}^d(r, R)$, $Q \in \mathscr{K}^s(r, R)$ and some standard arguments show that

$$\begin{aligned} t \sim \tau &\sim \varrho \\ u_K(x) &\sim t^{d-s} u_Q(x) \\ u_Q(x) &\sim t u_{Q \cap H(t)}(x) \end{aligned} \tag{7.5}$$

where $H(t)$ is the hyperplane (in F) parallel with L and containing x (so $\text{dist}(H(t), \text{aff}\, L) = t$). Set

$$L^0 = \text{conv}(L \cup \{0\})$$
$$Q(t) = Q \cap H(t),$$

and

$$h = \text{dist}(0, \text{aff}\, L).$$

Clearly

$$\text{vol}_s F \cap K(u_K \leqq \eta) = \sum \text{vol}_s [L^0 \cap K(u_K \leqq \eta)] \tag{7.6}$$

where the summation is taken for all facets L of Q. We assume $s \geqq 2$ as case $s=1$ of Theorem 3 is proved in [S1]. Moreover

$$\mathrm{vol}_s[L^0 \cap K(u_K \leqq \eta)] = \int_{t=0}^{h/2} \mathrm{vol}_{s-1}[L^0 \cap H(t) \cap K(u_K \leqq \eta)]\, dt \tag{7.7}$$

where the upper bound $h/2$ in the integration is explained in the following way: If $t > h/2 \geqq r/2$, then

$$M_K(x) \supset x + \frac{r}{2} B^d,$$

consequently $u_K(x) \geqq 2^{-d} r^d \omega_d = \eta_0(r,d)$. We continue (7.7):

$$\begin{aligned} &\int_{t=0}^{h/2} \mathrm{vol}_{s-1}[L^0 \cap H(t) \cap K(u_K \leqq \eta)]\, dt \\ &\leqq \int_{t=0}^{h/2} \mathrm{vol}_{s-1}\left[L^0 \cap Q(t)\left(u_{Q(t)} \leqq c_6 \frac{\eta}{t^p}\right)\right] dt \\ &\leqq \int_{t=0}^{t_0} \mathrm{vol}_{s-1}(L^0 \cap H(t))\, dt + \int_{t_0}^{h/2} \mathrm{vol}_{s-1} Q(t)\left(u_{Q(t)} \leqq c_6 \frac{\eta}{t^p}\right) dt \end{aligned} \tag{7.8}$$

where the first inequality and constant $c_6 = c_6(d,r,R)$ come from (7.5) and t_0 is defined as

$$t_0 = \left(\frac{c_6 \cdot 4 \cdot 2^{s-1} d^d \eta}{\mathrm{vol}\, L}\right)^{1/p}$$

if this is less than $h/2$, and $t_0 = h/2$ otherwise. We estimate the first integral in the right hand side of (7.8):

$$\begin{aligned} \int_0^{t_0} \mathrm{vol}_{s-1}[L^0 \cap H(t)]\, dt &\leqq \int_0^{t_0} \mathrm{vol}_{s-1} L\, dt = t_0 \mathrm{vol}_{s-1} L \\ &\leqq (c_6 2^{s+1} d^d)^{1/p} \eta^{1/p} (\mathrm{vol}_{s-1} L)^{1-1/p} \\ &\leqq (c_6 2^{s+1} d^d)^{1/p} \eta^{1/p} (R^{s-1} \omega_{s-1})^{1-1/p} \leqq c_7(P) \eta^{1/p}. \end{aligned} \tag{7.9}$$

Using the definition of t_0 (when $t_0 < h/2$) we have for $t \geqq t_0$

$$c_6 \frac{\eta}{t^p} \leqq \frac{\mathrm{vol}_{s-1} L}{2^{s-1} \cdot 4 d^d} \leqq \frac{\mathrm{vol}_{s-1} Q(t)}{4 d^d},$$

because $\mathrm{vol}_{s-1} Q(t) \geqq \mathrm{vol}_{s-1} Q(t) \cap L^0 \geqq \mathrm{vol}_{s-1} Q\left(\frac{h}{2}\right) \cap L^0 = 2^{-(s-1)} \mathrm{vol}_{s-1} L$. So we may apply Theorem 5 to $Q(t)$.

$$\mathrm{vol}_{s-1} Q(t)\left(u_{Q(t)} \leqq c_6 \frac{\eta}{t^p}\right) \leqq C(s-1) m(Q(t)) c_6 \frac{\eta}{t^p}\left(\log \frac{t^p \mathrm{vol}_{s-1} Q(t)}{c_1 \eta}\right)^{s-2}$$

where $C(s-1)$ is the constant in Theorem 5 and $m(Q(t))$ is the minimal number of simplices needed to triangulate the polytope $Q(t)$. Clearly $m(Q(t) \leqq c_8(P)$ for a suitable constant depending only on P. So the second integral in the right hand side of (7.8) can be estimated as follows:

$$\int_{t_0}^{h/2} \operatorname{vol}_{s-1} Q(t)\left(u_{Q(t)} \leqq c_6 \frac{\eta}{t^p}\right) dt$$

$$\leqq \int_{t_0}^{h/2} C(s-1) c_8(P) c_6 \frac{\eta}{t^p}\left(\log \frac{t^p \operatorname{vol}_{s-1} Q(t)}{c_6 \eta}\right)^{s-2} dt$$

$$\leqq C(s-1) c_8(P) \int_{t_0}^{h/2} \frac{c_6 \eta}{t^p}\left(\log \frac{t^p \omega_{s-1} R^{s-1}}{c_6 \eta}\right)^{s-2} dt. \tag{7.10}$$

When $s=2$, we can integrate simply, and the definition of t_0 shows that this is less than $\operatorname{const}(P)\eta^{1/p}$. When $s>2$ we substitute

$$y=\frac{t^p \omega_{s-1} R^{s-1}}{c_6 \eta},$$

and

$$y_0=\frac{t_0^p \omega_{s-1} R^{s-1}}{c_6 \eta}=\frac{2^{s+1} d^d \omega_{s-1} R^{s-1}}{\operatorname{vol} L} \geqq 1.$$

We continue (7.10):

$$\leqq C(s-1) c_8(P) \frac{1}{\omega_{s-1} R^{s-1}} \int_{y_0}^{\infty} \frac{1}{p}\left(\frac{c_6 \eta}{\omega_{s-1} R^{s-1}}\right)^{1/p} y^{1/p-1} \frac{(\log y)^{s-2}}{y} dy$$

$$\leqq \frac{C(s-1) c_8(P)}{p \omega_{s-1} R^{s-1}}\left(\frac{c_6 \eta}{\omega_{s-1} R^{s-1}}\right)^{1/p} \int_1^{\infty} \frac{(\log y)^{s-2}}{y^{2-1/p}} dy$$

$$\leqq c_9(P, s) \eta^{1/p}.$$

So we get from this and (7.9) that

$$\operatorname{vol}_s(L^0 \cap K(u_K \leqq \eta)) \leqq \operatorname{const}(P, s) \eta^{1/p}.$$

The number of terms of the sum in (7.6) is bounded by a constant depending on P and independent of F. So we proved (7.4). □

8. Proof of Theorem 6

We may assume $K \in \mathscr{K}_1^d$, i.e., $\operatorname{vol} K=1$. First we prove that

$$E\delta(K, K_n) \geqq \operatorname{const}(K)\left(\frac{\log n}{n}\right)^{2/(d+1)}.$$

A certain $\varepsilon \in(0,1)$ will be fixed later. Take a maximal system of pairwise disjoint caps $C_1, \ldots, C_m$ with $\operatorname{vol} C_i=\varepsilon$. We show that

$$c_1(K) \varepsilon^{-(d-1)/(d+1)} \leqq m \leqq c_2(K) \varepsilon^{-(d-1)/(d+1)} \tag{8.1}$$

for small enough ε. According to Theorem 8 of [BL]

$$\operatorname{vol} K(\varepsilon) \sim \varepsilon^{2/(d+1)}$$

with the implied constants depending on K. As

$$\bigcup_1^m C_i \subset K(\varepsilon)$$

the right hand side inequality of (8.1) follows. To see the other inequality we claim that

$$\bigcup_1^m C_i^5 \supset K(\varepsilon).$$

(For the definition of C_i^5 see (4.1).) Consider $y \in K(\varepsilon)$ and a minimal cap $C(y)$ with centre z. Let C_0 be the cap "parallel" with $C(y)$ and such that $C_0 \bigcap K[\varepsilon] = \{x\}$, a single point. We will prove the existence of $i \in \{1,\ldots,m\}$ with $C_0 \subset C_i^5$ provided ε is small enough. As K is a convex body with positive Gaussian curvature, K is very close to an ellipsoid E in a small neighbournood, N, of z. Let $C_i = K \cap H_i$ $(i = 0, 1, \ldots, m)$. Then the caps C_i that lie in N are very close to the caps $D_i = E \cap H_i$ of E. The maximality of the system $C_1, \ldots, C_m$ implies $C_0 \cap C_i \neq \emptyset$ for some $i \in \{1,\ldots,m\}$. Then $D_0^2 \cap D_i^2 \neq \emptyset$ can be seen easily. This shows (by a routine argument) that $D_0^2 \subset D_i^5$ which, in turn, implies $C_0 \subset C_i^5$. So indeed $\bigcup_1^m C_i^5 \supset K(\varepsilon)$ and (8.1) is proved.

It is clear that, for small enough ε, the depth of the cap C_i, $h(C_i)$ satisfies

$$c_3(K)\varepsilon^{2/(d+1)} \leqq h(C_i) \leqq c_4(K)\varepsilon^{2/(d+1)}. \tag{8.2}$$

Choose now $\varepsilon \in (0, 1)$ so that

$$\varepsilon^{-(d-1)/(d+1)}(1-\varepsilon)^n = \frac{1}{c_2(K)}. \tag{8.3}$$

This is possible for the function on the left hand side is continuous and decreasing in $(0, 1)$. It is 0 at $\varepsilon = 1$ and tends to infinity as $\varepsilon \to 0$. It is easily seen that the solution to (8.3) satisfies

$$\frac{d-2}{d+1}\frac{\log n}{n} < \varepsilon < \frac{d-1}{d+1}\frac{\log n}{n} \tag{8.4}$$

(at least for n large enough). Now

$$\begin{aligned}
&\operatorname{Prob}(\delta(K, K_n) > c_4(K)\varepsilon^{2/(d+1)}) \\
&\geqq \operatorname{Prob}(\exists i \in \{1,\ldots,m\}\colon C_i \cap K_n = \emptyset) \\
&= \sum_{k=1}^m (-1)^{k+1} \sum_{i_1 \cdots i_k} \operatorname{Prob}((C_{i_1} \cup \cdots \cup C_{i_k}) \cap K_n = \emptyset) \\
&= \sum_{k=1}^m (-1)^{k+1} \binom{m}{k} (1-k\varepsilon)^n \\
&= \sum_{\substack{k=1 \\ k \text{ odd}}}^m \binom{m}{k}(1-k\varepsilon)^n \left[1 - \frac{m-k}{k+1}\left(1 - \frac{\varepsilon}{1-k\varepsilon}\right)^n\right].
\end{aligned} \tag{8.5}$$

The expression $\frac{m-k}{k+1}\left(1-\frac{\varepsilon}{1-k\varepsilon}\right)^n$ is decreasing in k and for $k=1$

$$\frac{m-1}{2}\left(1-\frac{\varepsilon}{1-\varepsilon}\right)^n<\frac{m}{2}(1-\varepsilon)^n \leqq \tfrac{1}{2}c_2(K)\varepsilon^{-(d-1)/(d+1)}(1-\varepsilon)^n=\tfrac{1}{2}.$$

We continue (8.5):

$$\geqq \binom{m}{1}(1-\varepsilon)^n\left[1-\frac{m-1}{2}\left(1-\frac{\varepsilon}{1-\varepsilon}\right)^n\right] \geqq c_1(K)\varepsilon^{(d-1)/(d+1)}(1-\varepsilon)^n(1-\tfrac{1}{2})\frac{c_1(K)}{2c_2(K)}.$$

Then

$$\begin{aligned} E\delta(K,K_n) &\geqq c_4(K)\varepsilon^{2/(d+1)}\operatorname{Prob}(\delta(K,K_n)>c_4(K)\varepsilon^{2/(d+1)}) \\ &\geqq \frac{c_4c_1}{2c_2}\left(\frac{d-2}{d+1}\cdot\frac{\log n}{n}\right)^{2/(d+1)} \\ &\geqq c_5(K)\left(\frac{\log n}{n}\right)^{2/(d+1)} \end{aligned}$$

indeed.

Next we show

$$E\delta(K,K_n) \leqq \operatorname{const}(K)\left(\frac{\log n}{n}\right)^{2/(d+1)}. \tag{8.6}$$

We write K^t for the inner paarallel body of K with distance t. Using the fact that K is close to an ellipsoid at any point of its boundary it can be seen that

$$\operatorname{vol}(C\cap K^t)\geqq c_6(K)t^{(d+1)/2}=: f(t)$$

for every cap C of depth $2t$. This implies

$$\operatorname{Prob}(\delta(K,K_n)>2t)\leqq \operatorname{Prob}(\operatorname{vol}(K^t\backslash K_n)\geqq f(t)).$$

Then by Markov's inequality (see [R])

$$\operatorname{Prob}(\delta(K,K_n)>2t)\leqq\frac{E\operatorname{vol}(K^t\backslash K_n)}{f(t)}.$$

We choose

$$t=c_7(K)\left(\frac{\log n}{n}\right)^{2/(d+1)}$$

and show that

$$E\operatorname{vol}(K^t\backslash K_n)<tf(t)=c_6c_7\left(\frac{\log n}{n}\right)^{(d+3)/(d+1)}. \tag{8.7}$$

This will prove (8.6) because

$$\begin{aligned} E\delta(K,K_n) &\leqq \operatorname{diam}K\cdot\operatorname{Prob}(\delta(K,K_n)>2t)+2t\operatorname{Prob}(\delta(K,K_n)\leqq 2t) \\ &\leqq t\operatorname{diam}K+2t\leqq\operatorname{const}(K)\left(\frac{\log n}{n}\right)^{2/(d+1)}. \end{aligned}$$

To prove (8.7) one checks first that $x \in K^t$ implies $u(x) \geqq 5\frac{d+3}{d+1}\frac{\log n}{n}$ if $c_7(K)$ is chosen large enough. Then, setting $p = 5\frac{d+3}{d+1}\log n$

$$
\begin{aligned}
E \operatorname{vol}(K^t \backslash K_n) &\leqq \int_{u(x) \geqq p/n} \operatorname{Prob}(x \notin K_n) dx \\
&\leqq \int_{u(x) \geqq p/n} 2 \sum_{i=0}^{d-1} \binom{n}{i} \left(\frac{u(x)}{2}\right)^i \left(1 - \frac{u(x)}{2}\right)^{n-i} dx \\
&\leqq \sum_{\lambda=[p]}^{n} 2 \sum_{i=0}^{d-1} \binom{n}{i} \left(\frac{\lambda}{2n}\right)^i \left(1 - \frac{\lambda - 1}{2n}\right)^{n-i} \\
&\leqq \sum_{\lambda=[p]}^{n} e^{d+1/2} \lambda^{d-1} e^{-\lambda/2} \\
&\leqq e^{d+1/2} e^{-(5/4)d+3/d+1} \log n \sum_{\lambda=[p]}^{n} \lambda^{d-1} e^{-\lambda/4} \\
&\leqq \operatorname{const}(d) n^{-(5/4)d+3/d+1}.
\end{aligned}
$$

This proves (3.7) when n is large enough. □

9. Proof of Theorem 5 and the auxiliary lemmata

Most of these lemmata are proved in [BL]. Lemma A comes from [ELR]. Lemma B which is quite easy is proved in [BL]. Lemma C, D and E follow in the same way as Lemma 2 in [BL] except that, at the end, r and R must not be eliminated.

Proof. (of Lemma F). Let $x, y \in \operatorname{bd} K[\varepsilon]$ and assume $z = \frac{1}{2}(x+y) \in K[\varepsilon]$ as well. Then there is a minimal cap $C(z)$ with volume ε. $C(z)$ cannot contain x (or y) in its interior since otherwise a smaller "parallel" cap would contain x (or y). Then $C(z)$ must contain both x and y on its bounding hyperplane H. Then $C(z)$ is a minimal cap for both x and y. But the centre of gravity of $K \cap H$ cannot be both x and y at the same time unless $x = y$. □

Proof. (of Lemma G.) Denote the set of outer normals to $K[\varepsilon]$ at z by $N(z)$. If $\operatorname{int} K[\varepsilon] \neq \emptyset$, then $K[\varepsilon]$ is a convex body again. It is well-known (see, e.g., [Ro]) that $N(z)$ is a closed pointed cone and it coincides with the convex hull of its extreme rays:

$$N(z) = \operatorname{conv} \operatorname{ext} N(z).$$

For $b \in S^{d-1}$ define C^b as the unique cap $C^b = K \cap H(b,t)$ such that $C^b \cap K[\varepsilon] \neq \emptyset$ but $\operatorname{int} C^b \cap K[\varepsilon] = \emptyset$.

Our first aim is to show that $\operatorname{vol} C^b = \varepsilon$ if $b \in S^{d-1}$ is the direction of an extreme ray of $N(z)$. To prove this we use a classical result of Alexandrov (see c.f. [S3]) stating that at almost every point z on the boundary of a convex body the set of outer normals at z is a halfline (which is the same as the supporting hyperplane at z is unique). If the convex body is $K[\varepsilon]$ and $z \in \mathrm{bd}\, K[\varepsilon]$ is such a point then we write $b(z) = N(z) \cap S^{d-1}$.

Notice first that $N(z)$ is the polar of the minimal cone whose apex is z and which contains $K[\varepsilon]$ (see [Ro]). So there is a vector $u \in S^{d-1}$ such that $u \cdot b = 0$ and $u \cdot x < 0$ for all $x \in N(z), x \neq \lambda b (\lambda > 0)$ and such that there are points $z(t) \in \mathrm{bd}\; K[\varepsilon]$ (for $t > 0$ small enough) with

$$|(z(t) - z) - tu| = o(t) \quad \text{as} \quad t \to 0.$$

Choose now a sequence $z_k \in \mathrm{bd}\, K[\varepsilon]$ (using Alexandrov's theorem) very close to $z(t = 1/k)$, i.e.,

$$|(z_k - z) - \frac{1}{k} u| = o\left(\frac{1}{k}\right) \quad \text{as} \quad k \to \infty,$$

and such that $b(z_k)$ exists for all $k = 1, 2, \ldots$. We may assume that $\lim b(z_k) = b_0$ exists for S^{d-1} is compact. It is easy to see and actually well-known that $b_0 \in N(z)$. Assume $b_0 \neq b$. Then, as $b(z_k) \in N(z_k)$

$$0 \geqq b(z_k) \cdot (y - z_k)$$

for every $y \in K[\varepsilon]$. In particular, for $y = z$ we get

$$0 \geqq b(z_k) \cdot (z - z_k) = -\frac{1}{k} b(z_k) \cdot u + o\left(\frac{1}{k}\right)$$

$$> \frac{1}{2k} b_0 \cdot u + o\left(\frac{1}{k}\right) > 0$$

for k large enough. A contradiction. So $b_0 = b$. Then the continuity of the map $b \to \mathrm{vol}\, C^b$ implies $\mathrm{vol}\, C^b = \varepsilon$.

Now let a be the outer unit normal of the bounding hyperplane of the cap C (from the statement of the lemma). Then $a \in N(x)$ and so $a \in \mathrm{conv}\,\mathrm{ext}\, N(x)$. This implies by Caratheodory's theorem the existence of vectors $b_1, \ldots, b_d \in S^{d-1}$ such that each of them represents and extreme ray of $N(x)$ and such that a is in the cone hull of $b_1, \ldots, b_d$. Then $C^a = C$ is contained in

$$\bigcup_{i=1}^{d} C^{b_i}.$$

This proves that $\mathrm{vol}\, C \leqq d \,\mathrm{vol}\, C^{b_i} = d\varepsilon$.

This shows finally that every C^b with $\mathrm{vol}\, C^b = \varepsilon$ is a minimal cap. Then we have $C^b \subset M(x, 3d)$ from Lemma C provided $\varepsilon \leqq \varepsilon_0$. So $C^{b_i} \subset M(x, 3d)$ for $i = 1, \ldots, d$. Consequently

$$C \subset \bigcup_{i=1}^{d} C^{b_i} \subset M(x, 3d). \quad \square$$

Lemma H is Theorem 6 and Lemma I is Theorem 7 in [BL]. Finally, Lemma J (i) is simple and its proof is given in the beginning of the proof of Theorem 1 in [BL] and Lemma J (ii) is formula (3.6) from [BL].

Proof of Theorem 5
When $P = S$ is a simplex with $\operatorname{vol} S = 1$ then a simple checking of the proof of Theorem 3 of [BL] shows that

$$\operatorname{vol} P(v \leqq \varepsilon) \leqq c_1(d)\varepsilon\left(\log\frac{1}{\varepsilon}\right)^{d-1}$$

with $c_1(d) = 2^d$ or anything larger, when $0 < \varepsilon < \frac{1}{4d^d}$. When $P = S$ is a simplex with arbitrary volume then we use the fact that

$$\frac{\operatorname{vol} K(v \leqq \varepsilon \operatorname{vol} K)}{\operatorname{vol} K}$$

does not change when a (non-degenerate) affinity is applied to K. So

$$\begin{aligned}\operatorname{vol} S(v \leqq \varepsilon) &= \left(\frac{\operatorname{vol} S\left(v \leqq \frac{\varepsilon}{\operatorname{vol} S}\cdot \operatorname{vol} S\right)}{\operatorname{vol} S}\right)\cdot \operatorname{vol} S \\ &\leqq c_1(d)\frac{\varepsilon}{\operatorname{vol} S}\left(\log\frac{\operatorname{vol} S}{\varepsilon}\right)^{d-1} \quad \operatorname{vol} S \\ &= c_1(d)\varepsilon\left(\log\frac{\operatorname{vol} S}{\varepsilon}\right)^{d-1}\end{aligned}$$

as claimed. Now when $P = \bigcup_1^m S_i$ is a triangulation of P with $m = m(P)$, then clearly

$$\operatorname{vol} P(v \leqq \varepsilon) \leqq \sum_{i=1}^m \operatorname{vol} S_i(v_i \leqq \varepsilon)$$

where $v_i = v_{S_i}$. If $\varepsilon \leqq \frac{1}{4d^d}\operatorname{vol} S_i$, then we apply the previous step. And if $\varepsilon > \frac{1}{4d^d}\operatorname{vol} S_i$, then

$$\operatorname{vol} S_i(v_i \leqq \varepsilon) \leqq \operatorname{vol} S_i \leqq 4d^d\varepsilon \leqq c_2(d)\varepsilon\left(\log\frac{\operatorname{vol} P}{\varepsilon}\right)^{d-1}$$

if $c_2(d)$ is chosen large enough. Then

$$\begin{aligned}\operatorname{vol} P(v \leqq \varepsilon) &\leqq \sum_{i=1}^m \operatorname{vol} S_i(v_i \leqq \varepsilon) \\ &\leqq \max(c_1(d), c_2(d))m(P)\varepsilon\left(\log\frac{\operatorname{vol} P}{\varepsilon}\right)^{d-1} \\ &= c(d)m(P)\varepsilon\left(\log\frac{\operatorname{vol} P}{\varepsilon}\right)^{d-1}. \quad \square\end{aligned}$$

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