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A Stability Property of the Densest Circle Packing

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Abstract. A new stability property of the densest circle packing in the plane is proved. This property is related to a conjecture of L. FEJES TÓTH.

1. The Result

Let $\mathscr{C} = \{C_0, C_1, C_2, \ldots\}$ be a densest packing of congruent circles (of unit diameter, say) in the plane. It is well-known [2, 4] that the centres of the circles in $\mathscr{C}$ form a regular triangular lattice. Let $\mathscr{C}_0$ be the packing obtained from $\mathscr{C}$ by removing one circle, C_0 say, from $\mathscr{C}$. L. FEJES TÓTH [3] conjectures that $\mathscr{C}$ has the following very interesting stability property: Assume $(n + 1)$ circles, $C_0, C_1, \ldots, C_n$ are removed from $\mathscr{C}$ and n circles, $C'_1, \ldots, C'_n$, each congruent with C_0 are put back in such a way that the family $\mathscr{C}' = \{C'_1, \ldots, C'_n, C_{n+1}, \ldots\}$ is again a packing. Then, according to the conjecture, $\mathscr{C}'$ is congruent to $\mathscr{C}_0$.

This conjecture is open and the solution seems to be difficult. In this paper we prove the following related result.

Theorem. *Assume the circles $\{C'_1, \ldots, C'_n, C_{n+1}, \ldots\}$ form a packing and the centre of C'_i is within ε distance of the centre of C_i $(i = 1, \ldots, n)$. Then $C'_i = C_i$ $(i = 1, \ldots, n)$ provided $\varepsilon < \frac{1}{40}$.*

Inspired by this result A. BEZDEK and R. CONNELLY [1] call a packing $\mathscr{P}$ of congruent circles in the plane *uniformly stable* if there is $\varepsilon > 0$ such that no finite subset of the circles can be rearranged such that each circle is moved with a distance less than ε and the rearranged circles together with the rest form a packing not congruent to the original one. In this terminology our theorem says that $\mathscr{C}_0$ is uniformly stable. BEZDEK and CONNELLY establish in [1] that certain

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circle packings are uniformly stable and some others are not. They show, for instance, that the packing where the centres of the circles form the integer lattice in R^2 is not uniformly stable.

The Theorem says that $\mathscr{C}_0$ is uniformly stable with $\varepsilon \leqslant \varepsilon_0 = \frac{1}{40}$. We have not tried to get the largest possible ε_0 here, though some considerations show that our proof works at $\varepsilon_0 = \frac{1}{16}$ and does not work at $\varepsilon_0 = \frac{1}{12}$.

2. Two Lemmas and One Proof

The proof of the Theorem is based on two lemmas. The first is quite simple:

Lemma 1. *If in a triangle T the length of every edge is at least* 1 *and an angle is $\frac{\pi}{3} + \delta$ with $|\delta| \leqslant \frac{1}{10}$, then*

$$\operatorname{Area} T \geqslant \frac{\sqrt{3}}{4} + 0.229\,|\delta|\,. \tag{1}$$

In order to state the second lemma some preparation is needed. Let $p_1, p_2, \ldots, p_6$ be the vertices of a regular hexagon in this order along the perimeter. Choose points $q_1, \ldots, q_6$ with

$$|q_i - p_i| \leqslant \tfrac{1}{40},\ i = 1, \ldots, 6\,.$$

Then $q_1, \ldots, q_6$ are the vertices of a hexagon H "close" to the regular. Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ be the angle of H at vertex q_2, q_4 and q_6, respectively, and define α, β, γ by setting

$$\bar{\alpha} = \frac{2\pi}{3} + \alpha,\quad \bar{\beta} = \frac{2\pi}{3} + \beta,\quad \bar{\gamma} = \frac{2\pi}{3} + \gamma\,.$$

Lemma 2. *If the length of every edge of H is at least one, then*

$$\operatorname{Area} H \geqslant \frac{\sqrt{27}}{2} - 0.565\,(\alpha^2 + \beta^2 + \gamma^2)\,. \tag{2}$$

Proof of the Theorem using Lemma 1 and 2. Consider a packing $\mathscr{C}' = \{C'_1, \ldots, C'_n, C_{n+1}, \ldots\}$ with the centres $q_1, q_2, \ldots$ of the circles as in Fig. 1.

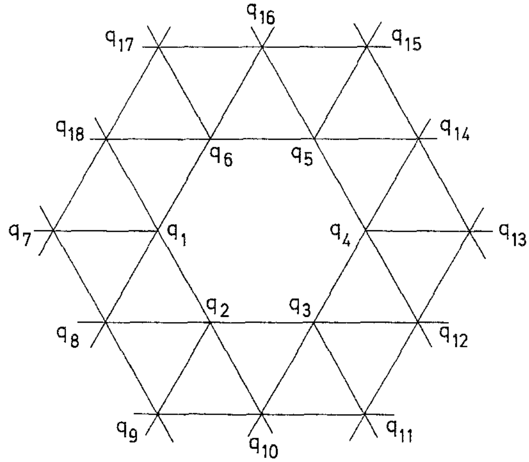

Figure 1

By our assumption, every q_i is within $\varepsilon \leqslant \frac{1}{40}$ distance of the corresponding lattice point p_i (when $i = 1, \ldots, n$) and coincides with p_i when $i > n$. The centre of the missing circle is $p_0 = q_0$.

The centres q_i, q_j, q_k are said to form the triangle $q_i q_j q_k$ if the circles C_i, C_j and C_k pairwise touch each other. As the edges of a triangle T are of length at least one and the triangle is "close" to the regular,

$$\text{Area } T \geqslant \frac{\sqrt{3}}{4} . \tag{3}$$

Denote by K the large hexagon with vertices $q_7, q_9, q_{11}, q_{13}, q_{15}, q_{17}$. All triangles outside K have area $\frac{\sqrt{3}}{4}$ at least and for q_i far enough from p_0, we have $q_i = p_i$. This implies

$$\text{Area } K \leqslant 24 \frac{\sqrt{3}}{4} = 6\sqrt{3} . \tag{4}$$

Now write $\frac{\pi}{3} + \delta_1, \frac{\pi}{3} + \delta_2, \frac{\pi}{3} + \delta_3, \frac{\pi}{3} + \delta_4$ for the angle at q_2 of the triangle $q_1 q_8 q_2$, $q_8 q_9 q_2$, $q_9 q_{10} q_2$ and $q_{10} q_3 q_2$, respectively. Let H be the hexagon with vertices $q_1, q_2, \ldots, q_6$. Then, with the notation of Lemma 2, $\delta_1 + \delta_2 + \delta_3 + \delta_4 + \alpha = 0$ and so

$$|\delta_1| + |\delta_2| + |\delta_3| + |\delta_4| \geqslant |\alpha| .$$

A simple computation shows that the conditions $|q_i - p_i| \leqslant \frac{1}{40}$ imply

$$|\delta_i| \leqslant \tfrac{1}{10},\ i = 1, \ldots, 4,\ |\alpha| \leqslant \tfrac{1}{10} .$$

So Lemma 1 applies: the area of the four triangles at q_2 is at least

$$4\frac{\sqrt{3}}{4} + 0.229\,(|\delta_1| + |\delta_2| + |\delta_3| + |\delta_4|) \geqslant \sqrt{3} + 0.229\,|\alpha|\,.$$

Estimating the area of the four triangles around q_4 (and q_6) in the same way we get

$$\text{Area}\,(K \setminus H) \geqslant 6\frac{\sqrt{3}}{4} + 3\sqrt{3} + 0.229\,(|\alpha| + |\beta| + |\gamma|)\,. \qquad (5)$$

Furthermore, Lemma 2 applies to the hexagon H:

$$\text{Area}\,H \geqslant \frac{3\sqrt{3}}{2} - 0.565\,(\alpha^2 + \beta^2 + \gamma^2)\,. \qquad (6)$$

Combining (4), (5) and (6) we have

$$6\sqrt{3} \geqslant \text{Area}\,K = \text{Area}\,(K \setminus H) + \text{Area}\,H \geqslant$$
$$\geqslant 6\sqrt{3} - 0.565\,(\alpha^2 + \beta^2 + \gamma^2) + 0.229\,(|\alpha| + |\beta| + |\gamma|)\,,$$

which implies that

$$0.229(|\alpha| + |\beta| + |\gamma|) \leqslant 0.565(\alpha^2 + \beta^2 + \gamma^2) \leqslant 0.565(|\alpha| + |\beta| + |\gamma|)^2\,.$$

So either $\alpha = \beta = \gamma = 0$ or $|\alpha| + |\beta| + |\gamma| \geqslant \dfrac{0.229}{0.565} > 0.4$. The latter case contradicts to $|\alpha|, |\beta|, |\gamma| \leqslant 0.1$.

So $\alpha = \beta = \gamma = 0$. Repeating the same argument with vertices q_1, q_3, q_5 (instead of q_2, q_4, q_6) we get that all angles of H are equal to $\dfrac{2\pi}{3}$. Then $\text{Area}\,H > \dfrac{3\sqrt{3}}{2}$ unless all edges of H are equal to one. Similarly, $\text{Area}\,(K \setminus H) > \dfrac{9\sqrt{3}}{2}$ unless all triangles in $K \setminus H$ are regular. So we get $q_i = p_i$, $i = 1, 2, \ldots$. □

3. Proof of Lemma 1

Let x and y be the lengths of the edges of the triangle, incident to the angle $\dfrac{\pi}{3} + \delta$. Assume first that $\delta \geqslant 0$. Then

$$\text{Area}\,T = \frac{1}{2}xy\sin\left(\frac{\pi}{3} + \delta\right) \geqslant \frac{1}{2}\sin\left(\frac{\pi}{3} + \delta\right).$$

We are going to use Taylor expansion quite often so we describe its first use in more detail. The Taylor expansion of $\sin t$ around $t = \frac{\pi}{3}$ is this:

$$\sin\left(\frac{\pi}{3} + \delta\right) = \sin\frac{\pi}{3} + \left(\cos\frac{\pi}{3}\right)\delta + \frac{1}{2}\left(-\sin\left(\frac{\pi}{3} + \bar{\delta}\right)\right)\delta^2$$

where $\bar{\delta}$ is between 0 and δ. Thus

$$\sin\left(\frac{\pi}{3} + \bar{\delta}\right) = \sin\frac{\pi}{3}\cos\bar{\delta} + \cos\frac{\pi}{3}\sin\bar{\delta} =$$

$$= \frac{\sqrt{3}}{2}\cos\bar{\delta} + \frac{1}{2}\sin\bar{\delta} \leqslant \frac{\sqrt{3}}{2} + \frac{1}{2}|\bar{\delta}| \leqslant \frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{1}{10}.$$

Then

$$\text{Area } T \geqslant \frac{1}{2}\sin\left(\frac{\pi}{3} + \delta\right) \geqslant \frac{\sqrt{3}}{4} + \frac{1}{4}\delta - \frac{1}{4}\left(\frac{\sqrt{3}}{2} + \frac{1}{20}\right)\delta^2 \geqslant$$

$$\geqslant \frac{\sqrt{3}}{4} + \frac{1}{4}\delta - 0.2041\,\delta^2 \geqslant \frac{\sqrt{3}}{4} + 0.229\,\delta$$

when $0 < \delta < 0.1$, indeed.

Consider now the case $\delta < 0$. It is an elementary exercise to see that Area T will be the smallest when the triangle is isosceles with two angles equal to $\frac{\pi}{3} + \delta$. The third is $\frac{\pi}{3} - 2\delta$, then. Using Taylor expansion again

$$\text{Area } T \geqslant \frac{1}{2}\sin\left(\frac{\pi}{3} - 2\delta\right) \geqslant \frac{\sqrt{3}}{4} - \frac{1}{2}\delta - \frac{\sqrt{3}}{2}\delta^2 \geqslant \frac{\sqrt{3}}{4} + 0.229\,|\delta|$$

when $-0.1 < \delta \leqslant 0$. □

4. Proof of Lemma 2

Denote by x_i the length of the edge $q_i q_{i+1}$ of H ($i = 1, \ldots, 6$, obviously $q_7 = q_1$ here). We want to estimate

$$h_0 = \min\{\text{Area } H: |q_i - p_i| \leqslant \tfrac{1}{40},\ x_i \geqslant 1,\ i = 1, \ldots, 6\}.$$

As we mentioned earlier these conditions imply $|\alpha|, |\beta|, |\gamma| \leqslant 0.1$. Then

$$h_0 \geqslant \min\{\text{Area}\, H: |\alpha|, |\beta|, |\gamma| \leqslant 0.1,\ x_i \geqslant 1,\ i = 1, \ldots, 6\}\ .$$

We claim that if the last minimum is attained then all $x_i = 1$. Indeed, assume $x_1 > 1$ (say), keep $\alpha, \beta, \gamma,\ x_2, x_3, \ldots, x_6$ fixed and decrease x_1 to 1. Then

$$\text{Area}\, H = \text{Area}\, q_1 q_2 q_3 + \text{Area}\, q_3 q_4 q_5 + \text{Area}\, q_5 q_6 q_1 + \text{Area}\, q_1 q_3 q_5 \quad (7)$$

and in this formula, Area $q_1 q_2 q_3$ decreases and the triangles $q_3 q_4 q_5$ and $q_5 q_6 q_1$ do not change. In the triangle $q_1 q_3 q_5$ the edges $q_1 q_5$ and $q_3 q_5$ do not change and $q_1 q_3$ decreases. Then the angle at q_5 decreases and so does Area $q_1 q_3 q_5$. Thus

$$h_0 \geqslant h_1 = \min\{\text{Area}\, H: |\alpha|, |\beta|, |\gamma| \leqslant 0.1,\ x_i = 1,\ i = 1, \ldots, 6\}\ .$$

We estimate h_1 using formula (7). Write $\max(|\alpha|, |\beta|, |\gamma|) = \Delta$. With Taylor expansion

$$\begin{aligned} \text{Area}\, q_1 q_2 q_3 &= \frac{1}{2} x_1 x_2 \sin\left(\frac{2\pi}{3} + \alpha\right) = \frac{1}{2} \sin\left(\frac{2\pi}{3} + \alpha\right) \geqslant \\ &\geqslant \frac{\sqrt{3}}{4} - \frac{1}{4}\alpha - \frac{1}{4}\left(\frac{\sqrt{3}}{2} + \Delta\right)\alpha^2. \end{aligned} \quad (8)$$

To compute the area of the triangle $q_1 q_3 q_5$ is more involved. Set $|q_3 - q_5|^2 = a$, $|q_5 - q_1|^2 = b$, $|q_1 - q_3|^2 = c$. Clearly $a = x_1^2 + x_2^2 - 2 x_1 x_2 \cos\bar{\alpha} = 2 - 2\cos\bar{\alpha}$, $b = 2 - 2\cos\bar{\beta}$, $c = 2 - 2\cos\bar{\gamma}$. Then, by Heron's formula

$$\begin{aligned} \text{Area}\, q_1 q_3 q_5 &= \tfrac{1}{4}(-a^2 - b^2 - c^2 + 2ab + 2bc + 2ca)^{\frac{1}{2}} = \\ &= \tfrac{1}{2}[2 - 2(\cos^2\bar{\alpha} + \cos^2\bar{\beta} + \cos^2\bar{\gamma}) + (\cos\bar{\alpha} + \cos\bar{\beta} + \cos\bar{\gamma} - 1)^2]^{\frac{1}{2}}\,. \end{aligned}$$

Using Taylor expansion again

$$\cos\bar{\alpha} = \cos\left(\frac{2\pi}{3} + \alpha\right) \leqslant -\frac{1}{2} - \frac{\sqrt{3}}{2}\alpha + \frac{1}{2}\left(\frac{1}{2} + \Delta\right)\alpha^2$$

and

$$(\cos\bar{\alpha} + \cos\bar{\beta} + \cos\bar{\gamma} - 1)^2 \geqslant$$

$$\geqslant \left[-\frac{5}{2} - \frac{\sqrt{3}}{2}(\alpha + \beta + \gamma) + \frac{1}{2}\left(\frac{1}{2} + \Delta\right)(\alpha^2 + \beta^2 + \gamma^2)\right]^2 \geqslant$$

$$\geqslant \frac{25}{4} + \frac{5\sqrt{3}}{2}(\alpha + \beta + \gamma) + \frac{3}{4}(\alpha + \beta + \gamma)^2 -$$

$$- \left(\frac{5}{4} + 4.06\,\Delta\right)(\alpha^2 + \beta^2 + \gamma^2)\,.$$

Here the term $4.06\,\Delta$ comes from a routine estimation using $\Delta \leqslant 0.1$.

With another Taylor expansion

$$\cos^2\bar{\alpha} = \cos^2\left(\frac{2\pi}{3} + \alpha\right) \leqslant \frac{1}{4} + \frac{\sqrt{3}}{2}\alpha + \frac{1}{2}(1 + 4\,\Delta)\,\alpha^2$$

and so

$$\cos^2\bar{\alpha} + \cos^2\bar{\beta} + \cos^2\bar{\gamma} \leqslant$$

$$\leqslant \frac{3}{4} + \frac{\sqrt{3}}{2}(\alpha + \beta + \gamma) + \frac{1}{2}(1 + 4\,\Delta)(\alpha^2 + \beta^2 + \gamma^2)\,.$$

Then

$$\text{Area}\, q_1 q_3 q_5 \geqslant \frac{\sqrt{27}}{2}\left\{1 + \left[\frac{2\sqrt{3}}{9}(\alpha + \beta + \gamma) + \frac{1}{9}(\alpha + \beta + \gamma)^2 - \right.\right.$$

$$\left.\left. - \left(\frac{1}{3} + 1.2\,\Delta\right)(\alpha^2 + \beta^2 + \gamma^2)\right]\right\}^{\frac{1}{2}}\,. \qquad (9)$$

Write t for the expression in square brackets here. We estimate t using $\Delta \leqslant \frac{1}{10}$ again

$$|\,t\,| \leqslant \frac{2\sqrt{3}}{9}3\,\Delta + \frac{1}{9}(3\,\Delta)^2 + \left(\frac{1}{3} + 1.2\,\Delta\right)3\,\Delta^2 < 1.4\,\Delta\,.$$

The Taylor expansion for $(1 + t)^{\frac{1}{2}}$ around $t = 0$ with $|\,t\,| \leqslant 1.4\,\Delta$ gives

$$(1 + t)^{\frac{1}{2}} \geqslant 1 + \tfrac{1}{2}t - \tfrac{1}{8}(1 + 3.4\,\Delta)\,t^2\,.$$

Also,

$$t^2 \leqslant (\tfrac{4}{27} + 0.27\,\Delta)(\alpha + \beta + \gamma)^2 + 1.55\,\Delta\,(\alpha^2 + \beta^2 + \gamma^2)\,.$$

Then we get

$$(1+t)^{\frac{1}{2}} \geqslant 1 + \frac{1}{\sqrt{27}}(\alpha+\beta+\gamma) + \left(\frac{1}{27} - 0.12\,\Delta\right)(\alpha+\beta+\gamma)^2 - \\ - \left(\frac{1}{6} + 0.9\,\Delta\right)(\alpha^2+\beta^2+\gamma^2)\,. \tag{10}$$

Substituting (8), (9) and (10) into (7) we get

$$h_1 \geqslant \frac{3\sqrt{3}}{2} - \left(\frac{\sqrt{3}}{4} + 1.3\,\Delta\right)(\alpha^2+\beta^2+\gamma^2) + \\ + \left(\frac{1}{12\sqrt{3}} - 0.2\Delta\right)(\alpha+\beta+\gamma)^2 \geqslant \\ \geqslant \frac{\sqrt{27}}{2} - 0.565(\alpha^2+\beta^2+\gamma^2).\ \square$$

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