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Probability Theory and Related Fields

On the Shape of the Convex Hull of Random Points

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Summary. Denote by E_n the convex hull of n points chosen uniformly and independently from the d-dimensional ball. Let $\mathrm{Prob}(d, n)$ denote the probability that E_n has exactly n vertices. It is proved here that $\mathrm{Prob}(d, 2^{d/2} d^{-\varepsilon}) \to 1$ and $\mathrm{Prob}(d, 2^{d/2} d^{(3/4)+\varepsilon}) \to 0$ for every fixed $\varepsilon > 0$ when $d \to \infty$. The question whether E_n is a k-neighbourly polytope is also investigated.

1. Random Points in the Ball

Let us denote by $\mathrm{Prob}(d, n)$ the probability that choosing n points uniformly and independently from the d-dimensional unit ball B_d, none of the points is contained in the convex hull of the others, i.e., their convex hull has exactly n vertices. Clearly, $\mathrm{Prob}(d, n) = 1$ for $n \leqq d+1$. For $n = d+2$ Blaschke ([2], pp. 55–60) and Hostinsky ([10], pp. 22–26) determined $\mathrm{Prob}(2, 4)$ and $\mathrm{Prob}(3, 5)$, respectively. Decades later Kingman [11] calculated the probability $\alpha(d)$ that the convex hull of $d+2$ points in B_d is a simplex:

$$\alpha(d) = 1 - \mathrm{Prob}(d, d+2) = \frac{d+2}{2^d} (b_{d+1})^{d+1} (b_{(d+1)^2})^{-1} \tag{1}$$

where $b_i = \Gamma(i+1)\,\Gamma\left(\frac{i}{2}+1\right)^{-2}$. Using the well-known asymptotic formula $\Gamma(x+1) = x^x \sqrt{2\pi x}\, e^{-x} e^{1/(12x+\theta)}$ where $0 \leqq \theta < 1$ for $x \geqq 1$ (see, e.g., Rényi [14], p. 138) (1) implies

$$\alpha(d) = (2\pi d)^{-d/2} d^{3/2} e^{-3/4} (1 + O(d^{-1})).$$

Hence $\mathrm{Prob}(d, d+2) \to 1$ as $d \to \infty$. Miles ([13], pp. 369–374) proved that $\lim_{d \to \infty} \mathrm{Prob}(d, d+3) = 1$ holds. He conjectured and Buchta [3] proved that

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$\operatorname{Prob}(d, d+m) \to 1$ for all fixed m. Actually Buchta's method gives that $\lim_{d\to\infty} \operatorname{Prob}(d, (\frac{3}{2}-\varepsilon)d) = 1$ holds for every fixed $\varepsilon > 0$.

We call the set $\{Y_1, \ldots, Y_k\} \subset R^d$ 1-convex if its convex hull has k vertices. It is easy to compute the expected value of the number of non-1-convex $(d+2)$-sets of $\{X_1, \ldots, X_n\}$, where all $X_i \in B_d$:

$$E(\text{number of non-1-convex } (d+2)\text{-sets}) = \binom{n}{d+2} \alpha(d) > \left(\frac{n}{2\pi d^{3/2}}\right)^d.$$

Here the right hand side is larger than exponential in d for $n > d^2$ (say). So it might be surprising that $\operatorname{Prob}(d, n) = 1 - o(1)$ holds for exponentially large n. More explicitly we have

Theorem 1.1. *Let c be a positive real. Then*

$$\operatorname{Prob}(d, c2^{d/2}) > 1 - c^2.$$

Theorem 1.2. *Let $c \geqq 20$. Then, for $d \geqq 100$*

$$\operatorname{Prob}(d, c d^{3/4} 2^{d/2}) < 2e^{-c/2}.$$

It seems likely that the threshold function of the 1-convexity of random points is about $d^{1/2} 2^{d/2}$. Other properties of random point-sets can be found e.g. in Rényi and Sulanke [15] (they consider the case $d = 2$). For any extensive bibliography see Buchta and Müller [4].

2. *k*-Convexity of Random Pointsets

Let k be a positive integer. A finite set $E \subset R^d$ is said to be k-convex if for every subset $A \subset E$ with $|A| \leqq k$ the set conv A is a face of the polytope conv E. For instance, if E is the vertex set of conv E, then E is 1-convex and this coincides with the definition in the previous section. The vertex set of the d-dimensional simplex is d-convex and the vertex set of the cyclic polytope is $[d/2]$-convex. Denote by $\operatorname{Prob}_k(d, n)$ the probability that choosing n points uniformly and independently from the d-dimensional unit ball B_d we get a k-convex set. Clearly, $\operatorname{Prob}_1(d, n) = \operatorname{Prob}(d, n)$. The next two theorems are about the threshold function of k-convexity.

Theorem 2.1. *For k fixed,*

$$\operatorname{Prob}_k\left(d, \left(1 + \frac{1}{4k}\right)^d\right) = 1 - o(1) \tag{2}$$

$$\operatorname{Prob}_k\left(d, \left(1 + \frac{1}{2k-4}\right)^d\right) = o(1) \quad \textit{when } k \geqq 3 \tag{3}$$

$$\operatorname{Prob}_2(d, 1.4^d) = o(1)$$

when $d \to \infty$.

Theorem 2.2. *When* $k=\left[\frac{d}{2A\log d}\right]$, *then*

$$\mathrm{Prob}_k(d, d^{A/6})=1-o(1) \tag{4}$$

$$\mathrm{Prob}_k(d, d^{A+1+\varepsilon})=o(1) \tag{5}$$

as $d\to\infty$.

3. The Largest Polytopes in the Ball

The volume of the d-dimensional unit ball B_d is $\gamma_d=\pi^{d/2}\,\Gamma\left(\frac{d}{2}+1\right)^{-1}$. Let us denote by $V(d, n)$ how well an inscribed polytope with n vertices can fill the ball, i.e.,

$$V(d, n)=\sup\left\{\frac{\mathrm{Vol}(\mathrm{conv}\{P_1, \ldots, P_n\})}{\gamma_d}: P_1, \ldots, P_n\in B_d\right\}.$$

Clearly $V(d, n)=0$ for $n\leqq d$.

$$V(d, d+1)=\sqrt{d+1}\left(\frac{d+1}{d}\right)^{d/2}(d!\,\gamma_d)^{-1}\approx\left(\frac{e}{2\pi d}\right)^{d/2}\sqrt{\frac{de}{2}}.$$

The asymptotic behaviour of $V(d, n)$ when d is fixed and n tends to infinity is fairly well-known, see e.g., Gruber [9]. Recently Elekes [6] (see also in Lovász [12], p. 55) proved that

$$V(d, n)\leqq n2^{-d}. \tag{6}$$

This bound is not the best (for $n>2^d$ it gives even more than 1), but it is valid for every value of n and d and its proof is very nice. The proofs of Theorems 1.1 and 1.2 give as a byproduct that the Elekes bound is not as bad as it seems at first sight, at least when n is about $2^{d/2}$.

Theorem 3.1. *For* $d\geqq 100$ *and* $n=20d^{3/4}2^{d/2}$

$$V(d, n)>\frac{n}{500d^{3/2}2^d}.$$

Elekes's result and Theorem 3.1 give quite good estimates for $V(d, n)$ when $n=20d^{3/4}2^{d/2}$:

$$0.04d^{-3/4}2^{-d/2}\leqq V(d, n)\leqq 20d^{3/4}2^{-d/2}.$$

In a forthcoming paper [1] we are going to return to the determination of $V(d, n)$ when n is relatively small, polynomial in d, say, and d tends to infinity.

4. Proof of Theorem 1.1

Set $E=\{X_1, \ldots, X_n\}$. Clearly,

$$\operatorname{Prob}(E \text{ is 1-convex}) \geqq 1-\sum_{i=1}^{n} \operatorname{Prob}(X_i \in \operatorname{conv}(E \setminus \{X_i\}).$$

By definition

$$\operatorname{Prob}(X_i \in \operatorname{conv}(E \setminus \{X_i\})) = \frac{1}{\gamma_d} \operatorname{Vol} \operatorname{conv}(E \setminus \{X_i\}) \leqq V(d, n-1),$$

and this gives

$$\operatorname{Prob}(d, n) \geqq 1 - n V(d, n-1). \tag{7}$$

Now we use (6) to show that $\operatorname{Prob}(d, n) > 1 - \frac{n^2}{2^d} = 1 - c^2$. $\square$

5. Proof of Theorem 1.2

We will choose the random points in two (independent) steps. Let first $E = \{X_1, \ldots, X_{[n/2]}\}$ and $F = \{X_{[n/2]+1}, \ldots, X_n\}$. Denote by 0 the center of B_d and let A_d be the ball with center 0 and radius $r = 2^{-1/2} d^{-3/4d} \approx 2^{-1/2}\left(1 - \frac{3 \log d}{4d}\right)$. Then we have

$$\operatorname{Prob}(A_d \cap E = \emptyset) < e^{-c/2}. \tag{8}$$

Indeed, $\operatorname{Prob}(X_i \in A_d) = \frac{1}{\gamma_d} \operatorname{Vol}(A_d) = r^d$. Hence

$$\operatorname{Prob}(A_d \cap E = \emptyset) = (1 - r^d)^{n/2} < \exp(-r^d [n/2]) \leqq \exp(-c/2).$$

From now on we assume that $A_d \cap E \neq \emptyset$, let $Y \in A_d \cap E$, say. Define O' so that Y is the midpoint of the line segment OO'. Let B'_d be the ball with center O' and radius 1. Set, further, $H = B_d \cap B'_d$. Now we show that for $d \geqq 100$

$$\frac{\operatorname{Vol}(H)}{\gamma_d} \geqq 0.46 d^{1/4} 2^{-d/2}. \tag{9}$$

Indeed, $\|OY\| \leqq r$ and so

$$\begin{aligned}
\frac{1}{\gamma_d} \operatorname{Vol}(H) &\geqq \frac{2}{\gamma_d} \cdot \gamma_{d-1} \int_{x=r}^{1} (1-x^2)^{\frac{d-1}{2}} dx \\
&\geqq \frac{2\gamma_{d-1}}{\gamma_d} \int_{r}^{1} x(1-x^2)^{\frac{d-1}{2}} dx \\
&= \frac{\gamma_{d-1}}{\gamma_d} \cdot \frac{2}{d+1} (1-r^2)^{\frac{d+1}{2}} \\
&= \frac{\gamma_{d-1}}{\gamma_d} \cdot \frac{2}{d+1} (1 - \tfrac{1}{2} d^{-\frac{3}{2d}})^{\frac{d+1}{2}}.
\end{aligned}$$

Taking into account the asymptotic formula $\frac{\gamma_{d-1}}{\gamma_d} \approx \sqrt{\frac{d}{2\pi}}$ one can easily see that

$$\lim_{d\to\infty} \frac{\gamma_{d-1}}{\gamma_d} \cdot \frac{2}{d+1} \cdot \frac{(1-\frac{1}{2}d^{3/2d})^{\frac{d+1}{2}}}{d^{1/4}2^{-d/2}} = \frac{1}{\sqrt{\pi}} = 0.564\ldots.$$

This shows that (9) is true for d large enough, for instance when $d \geqq 100$.

Now define k as the number of elements of $F \cap H$. The expected value of k, $E(k)$ equals $\left[\frac{n}{2}\right](\gamma_d)^{-1}\operatorname{Vol}(H) \geqq 0.23cd$. Now we show that

$$\operatorname{Prob}(k<0.1cd)<e^{-0.03cd} \tag{10}$$

using the large deviation theorem of Chernoff [5] applied to the binomial distribution (see Spencer [16]). This says that for $a \leqq pm$, $0<p<1$

$$\sum_{i<a} \binom{m}{i} p^i(1-p)^{m-i} < \exp(-(mp-a)^2/2mp). \tag{11}$$

In our case $p = \frac{1}{\gamma_d}\operatorname{Vol}(H) > 0.46\, d^{1/4}\, 2^{-d/2}$ (if $d \geqq 100$) and $m = \left[\frac{cd}{2}\right]^{3/4} 2^{d/2}$. Set $a = 0.1cd$, then (11) yields (10).

Finally, we use a result of Wendel [17] (see also in [8]).

Theorem [17]. Let K be a centrally symmetric bounded domain in R^d with center 0, $\operatorname{Vol}(K)>0$. Choose the points $Y_1, \ldots, Y_k$ randomly, uniformly and independently from K. Then

$$\operatorname{Prob}(0 \in \operatorname{conv}\{Y_1, \ldots, Y_k\}) = 1 - \frac{1}{2^{k-1}} \sum_{i \leqq d-1} \binom{k-1}{i}. \tag{12}$$

Now (12) implies that

$$\operatorname{Prob}(Y \notin \operatorname{conv} F \mid k \geqq 0.1cd) \leqq 2^{\frac{1}{0.1cd-1}} \sum_{i \leqq d-1} \binom{0.1cd-1}{i} < e^{-0.01cd} \tag{13}$$

when $c \geqq 20$.

Denote the event that $E \cup F$ is 1-convex by U and the events in the left hand sides of (8), (10) and (13) by V, W and Z, respectively. Clearly $U \subset V+W+Z+U\overline{VWZ}$, hence $\operatorname{Prob}(U) \leqq \operatorname{Prob}(V)+\operatorname{Prob}(W)+\operatorname{Prob}(Z)+\operatorname{Prob}(U\overline{VWZ})$. But $\operatorname{Prob}(U\overline{VWZ})=0$ because if $E \cup F$ is 1-convex then $Y \in \operatorname{conv}(F)$ is impossible. Using (8), (10) and (13) we get

$$\begin{aligned}\operatorname{Prob}(d,n) &= \operatorname{Prob}(E \cup F \text{ is 1-convex}) \\ &= \operatorname{Prob}(U) < e^{-c/2} + e^{-0.03cd} + e^{-0.01cd} < 2e^{-c/2}\end{aligned}$$

for $d \geqq 100$, $c \geqq 20$.

6. Proof of Theorem 3.1

Let us apply Theorem 1.2 and (7) when $n=20\cdot 2^{d/2} d^{3/4}$. We get

$$2e^{-10} > \operatorname{Prob}(d, n) > 1 - nV(d, n-1)$$

which shows that

$$V(d, n) > V(d, n-1) > \frac{1-2e^{-10}}{n} > \frac{n}{500\, 2^d d^{3/2}}. \quad \square$$

7. Proof of the Lower Bounds in Theorems 2.1 and 2.2

We will use a result from Bárány and Füredi [13]. To formulate it we need some preparation.

Given a convex set $C \subset R^d$ with $L=\operatorname{aff}(C)$, define $L^\perp$ as the maximal subspace of R^d orthogonal to L. Further, for $\rho>0$ let

$$C^\rho = C + (L^\perp \cap \rho B_d).$$

In other words, C^ρ is a cylinder above C, i.e., C^ρ is the set of points $x \in R^d$ such that if x' is the nearest point to x in C, then $\|x-x'\| \leqq \rho$ and $x-x'$ is orthogonal to L. Define $\rho(d, 1)=1$, $\rho(d, d)=d^{-1}$ and for $1<k<d$

$$\rho(d, k) = \left(\frac{d-k+1}{d(k-1)}\right)^{1/2}.$$

Lemma 7.1. *Given a simplex F in B_d and $k \in \{1, 2, \ldots, d\}$ and a point $x \in F$, there is a $(k-1)$-face F_k of F with $x \in F_k^{\rho(d,k)}$.*

Now we prove (3). Fix $k \geqq 3$ and take the n points in two steps again: in the first step $3d$ points E_{3d} and in the second $n-3d$ points E_{n-3d}. (The case $k=2$ can be dealt with in a slightly different way.)

Observe that by Wendel's theorem $O \in \operatorname{conv} E_{3d}$ with probability $1-o(1)$. Then by Caratheodory's theorem $O \in \operatorname{conv} E_{d+1}$ for some $E_{d+1} \subset E_{3d}$ with $|E_{d+1}|=d+1$. Then by Lemma 7.1 $O \in (\operatorname{conv} E_k)^\rho$ for some $E_k \subset E_{d+1}$, $|E_k|=k$ and $\rho=\rho(\mathrm{d}, \mathrm{k})$. Denoting the nearest point to the origin in $\operatorname{conv} E_k$ by Y we have

$$\operatorname{dist}(O, Y) \leqq \rho(d, k) = \sqrt{\frac{d-k+1}{d(k-1)}} < \frac{1}{\sqrt{k-1}}.$$

From now on we work in the same way as in the proof of Theorem 1.2. Define O' so that Y is the midpoint of OO' and set $H=B_d \cap B'_d$ where B'_d is the unit ball around O'. Then

$$E(|E_{n-3d} \cap H|) = \frac{1}{\gamma_d} \operatorname{Vol}(H)(n-3d) > \frac{1}{2\sqrt{d}}\left(1-\frac{1}{k-1}\right)^{d/2}(n-3d).$$

so if we take $n \geqq 3d + 2d^2 \left(\frac{k-1}{k-2}\right)^{d/2}$, say, then with probability $1-o(1)$

$$|E_{n-3d} \cap H| > 3d$$

and we apply Wendel's theorem again. So from here we get that (3) holds for every fixed $k \geqq 3$ when $d \to \infty$.

In exactly the same way one can prove (5) as well. This method works when k is linear in d, $k = \alpha d$, (say) and gives

$$\mathrm{Prob}_{k=\alpha d}(d, 3(1+e^{(1-\alpha)/2\alpha}\sqrt{\pi/\alpha})\, d) = o(1).$$

Remark. We think that

$$\mathrm{Prob}_{k=\alpha d}(d, f(\alpha)\, d) = 1 - o(1)$$

when $\alpha \in (0, 1/2)$, where $f(\alpha)$ depends on α only, probably

$$f(\alpha) > \exp(c(\tfrac{1}{2} - \alpha)/\alpha).$$

8. Proof of the Upper Bounds in Theorems 2.1 and 2.2

For the proof of (2) and (4) we need some more notation and facts. Denote by $\mathrm{aff}(X_1, \ldots, X_k)$ the affine hull of the points $X_1, \ldots, X_k$. For $a, b > 0$ and $0 \leqq t \leqq 1$ set

$$B(a, b, t) = \int_{x=0}^{t} (1-x^2)^a x^{2b}\, dx,$$

and

$$B(a, b) = B(a, b, 1).$$

The following proposition is proved in Miles [13].

Proposition 8.1. *Choosing the points $X_1, \ldots, X_k$ uniformly and independently from B_d $(k \leqq d)$ we have*

$$\mathrm{Prob}(\mathrm{dist}(O, \mathrm{aff}(X_1, \ldots, X_k)) \leqq t) = \frac{B(\frac{1}{2}(d+1)(k-1), \frac{1}{2}(d-k), t)}{B(\frac{1}{2}(d+1)(k-1), \frac{1}{2}(d-k))}. \quad \square$$

Proposition 8.2.

$$\frac{a^a b^b}{(a+b)^{a+b+1}} \leqq B(a, b) \leqq \frac{a^a b^b}{(a+b)^{a+b}}$$

and for $t \leqq \sqrt{b/(a+b)}$ we have

$$B(a, b, t) \leqq (1-t^2)^a t^{2b}.$$

To prove this proposition one can use standard techniques, taking into account that the maximum of $(1-x^2)^a x^{2b}$ is attained at $\sqrt{b/(a+b)}$. $\square$

Now let E_n be the random n-set from B_d. We are going to estimate Prob(E_n is not k-convex) from above. If E_n is not k-convex, then there is $A_k \subset E_n$ with $|A_k|=k$ and $\operatorname{conv} A_k \cap \operatorname{conv}(E_n \setminus A_k) \neq \emptyset$. Then, by Caratheodory's theorem and by the cylinder lemma there exists $A_s \subset E_n \setminus A_k$. $|A_s|=s$ with $\operatorname{conv} A_k \cap (\operatorname{conv} A_s)^\rho \neq \emptyset$ where $\rho=\rho(d,s)$. We will choose $s\in\{1,\dots,d-1\}$ later. Thus

$$\begin{aligned}\operatorname{Prob}_k(d,n) &= \operatorname{Prob}(E_n \text{ is not } k\text{-convex})\\ &\leqq \binom{n}{k}\binom{n-k}{s}\operatorname{Prob}(\operatorname{dist}(\operatorname{conv} A_k, \operatorname{conv} A_s) \leqq \rho) \qquad (14)\end{aligned}$$

where A_k and A_s are sets of k, s points respectively chosen uniformly and independently from B_d.

Recall that for an affine subspace $M\subset R^d$, $M^\perp$ is the subspace of R^d orthogonal to M with $\dim M+\dim M^\perp=d$. Let $L_1=(\operatorname{lin} A_s)^\perp$, $L_2=(\operatorname{aff} A_k)^\perp$ and $L_0=L_1\cap L_2$. Let π_i denote the orthogonal projections from R^d to L_i and $B(L_i)$ the unit ball of the subspace L_i, i.e., $B(L_i)=L_i\cap B_d$ $(i=0,1,2)$. Finally write π^* for the orthogonal projection from L_2 to L_0. Now

$$\begin{aligned}\operatorname{Prob}(\operatorname{dist}(\operatorname{conv} A_k, \operatorname{conv} A_s)\leqq\rho) &\leqq \operatorname{Prob}(\operatorname{dist}(\operatorname{aff} A_k, \operatorname{lin} A_s)\leqq\rho)\\ &= \operatorname{Prob}(\pi_0(\operatorname{aff} A_k)\in\rho B(L_0)). \qquad (15)\end{aligned}$$

Now $\pi_0=\pi^*\circ\pi_2$ and so if $\pi_2(\operatorname{aff} A_k)\in\rho B(L_2)$, then $\pi_0(\operatorname{aff} A_k)\in\rho B(L_0)$ trivially. On the other hand, $\pi_2(\operatorname{aff} A_k)$ is just a point $Z\in L_2$. We continue (15):

$$\begin{aligned}&\leqq \operatorname{Prob}(\pi_2(\operatorname{aff} A_k)\in\rho B(L_2))\\ &+\int_{t=\rho}^{1}\operatorname{Prob}(\pi^*(Z)\in\rho B(L_0)\,|\,Z\in B(L_2),\ \|Z\|=t)\\ &\cdot d_t\operatorname{Prob}(\pi_2(\operatorname{aff} A_k)\in tB(L_2)). \qquad (16)\end{aligned}$$

But $\pi_2(\operatorname{aff} A_k)\in tB(L_2)$ is equivalent to $\operatorname{dist}(O,\operatorname{aff} A_k)\leqq t$ so from Proposition 8.1 we infer

$$\operatorname{Prob}(\pi_2(\operatorname{aff} A_k)\in tB(L_2))=\frac{B(\frac{1}{2}(d+1)(k-1),\frac{1}{2}(d-k),t)}{B(\frac{1}{2}(d+1)(k-1),\frac{1}{2}(d-k))}=: B_1(t).$$

Moreover

$$\begin{aligned}&\operatorname{Prob}(\pi^*(Z)\in\rho B(L_0)\,|\,Z\in B(L_2),\ \|Z\|=t)\\ &=\operatorname{Prob}\left(\pi^*(Z)\in\frac{\rho}{t}B(L_0)\,\middle|\,Z\in B(L_2),\ \|Z\|=1\right)\\ &=\operatorname{Prob}\left(\pi^*(Z)\in\frac{\rho}{t}B(L_0)\right)\end{aligned}$$

where one chooses Z from the unit sphere of $L_2\cong R^{d-k+1}$ uniformly. This probability can be computed easily

$$\operatorname{Prob}\left(\pi^*(Z)\in\frac{\rho}{t}B(L_0)\right)=\frac{B\left(\frac{1}{2}s,\frac{1}{2}(d-k-s),\frac{\rho}{t}\right)}{B(\frac{1}{2}s,\frac{1}{2}(d-k-s))}=: B_2\left(\frac{\rho}{t}\right).$$

A real number t_0 with $\rho < t_0 < 1$ will be specified later. We continue (16) using the fact that $B_1(t)$ and $B_2(\rho/t)$ are monotone increasing and decreasing, respectively; and both of them are less than or equal to 1:

$$\begin{aligned}
&\operatorname{Prob}(\operatorname{dist}(\operatorname{conv} A_k, \operatorname{conv} A_s) \leqq \rho)\\
&\leqq B_1(\rho) + \int_{t=\rho}^{1} B_2(\rho/t)\, B_1'(t)\, dt\\
&= B_1(\rho) + \int_{t=\rho}^{t_0} B_2(\rho/t)\, B_1'(t)\, dt + \int_{t=t_0}^{1} B_2\ (\rho/t)\, B_1'(t)\, dt\\
&\leqq B_1(\rho) + \int_{t=\rho}^{t_0} B_1'(t)\, dt + \max_{t_0 \leqq t \leqq 1} B_2(\rho/t) \int_{t=t_0}^{1} B_1'(t)\, dt\\
&= B_1(\rho) + [B_1(t_0) - B_1(\rho)] + B_2(\rho/t_0)[1 - B_1(t_0)]\\
&\quad < B_1(t_0) + B_2(\rho/t_0). \qquad (17)
\end{aligned}$$

Proposition 8.2 shows that $B_1(t_0)$ is very close to zero when

$$t_0 < t_1 = \sqrt{\frac{d-k}{(d+1)(k-1)+d-k}} \approx \frac{1}{\sqrt{k}},$$

and $B_2(\rho/t_0)$ is very close to zero when

$$t_0 > t_2 = \sqrt{\frac{(d-s+1)(d-k-1)}{d(s-1)(d-k-s)}} \approx \frac{1}{\sqrt{s}}.$$

Our plan to prove (2) and (4) is to find s and t_0 such that

$$t_2 < t_0 < t_1, \qquad (18)$$

$$\binom{n}{k}\binom{n-k}{s} B_1(t_0) = o(1), \qquad (19)$$

$$\binom{n}{k}\binom{n-k}{s} B_2\left(\frac{\rho}{t_0}\right) = o(1). \qquad (20)$$

If this can be done then from (17) and (14) we get that $\operatorname{Prob}_k(d, n) = o(1)$.

When k is constant, $s = 1.2k$ and $t_0 = (1.1k)^{-1/2}$ is a good choice, (18), (19) and (20) will hold with $n = (1 + (4k)^{-1})^d$. This can be seen using Proposition 8.2. Similarly, when $k = d(2A \log d)^{-1}$ then setting

$$s = d(1.2A \log d)^{-1}$$

and

$$t_0 = [1.99A\,(\log d)/d]^{-1/2}$$

and

$$n = d^{A/6}$$

one can check that (18), (19) and (20) hold. This implies (4).

We mention finally that it is possible to prove somewhat, but not significantly, better estimates than those in (2) and (4), if one projects onto $(\operatorname{aff} A_k)^\perp$, $(\operatorname{aff} A_s)^\perp$ and their intersection. Then the computations become more involved.

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References

1. Bárány, I., Füredi, Z.: Approximation of the sphere by polytopes having few vertices. To appear in Proc. Am. Math. Soc.
2. Blaschke, W.: Vorlesungen über Differentialgeometrie II. Affine Differentialgeometrie. Berlin: Springer 1923
3. Buchta, C.: On a conjecture of R.E. Miles about the convex hull of random points. Monatsh. Math. **102**, 91–102 (1986)
4. Buchta, C., Müller, J.: Random polytopes in a ball. J. Appl. Probab. **21**, 753–762 (1984)
5. Chernoff, H.: A measure of asymptotic efficiency for tests of an hypothesis based on the sum of observations. Ann. Math. Statist. **23**, 493–507 (1952)
6. Elekes, G.: A geometric inequality and the complexity of computing volume. Discrete. Comput. Geom. **1**, 289–292 (1986)
7. Erdös, P., Spencer, J.: Probabilistic methods in combinatorics. Budapest: Akadémiai Kiadó 1974
8. Füredi, Z.: Random polytopes in the *d*-dimensional cube. Discrete Comput. Geom. **1**, 315–319 (1986)
9. Gruber, P.M.: Approximation of convex bodies by polytopes. In: Gruber, P.M., Wills, J.M. (eds.). Convexity and its applications, pp. 131–162. Basel: Birkhauser 1983
10. Hostinsky, B.: Sur les probabilités géométriques. Publ. Fac. Sci. Univ. Masaryk. Brno. 1925
11. Kingman, J.F.C.: Random secants of a convex body. J. Appl. Probab. **6**, 660–672 (1969)
12. Lovász, L.: An algorithmic theory of numbers, graphs and convexity. Preprint, Report No. 85368-OR, University of Bonn 1985
13. Miles, R.E.: Isotropic random simplices. Adv. Appl. Probab. **3**, 353–382 (1971)
14. Rényi, A.: Probability theory. Amsterdam: North-Holland 1970
15. Rényi, A., Sulanke, R.: Über die convexe Hülle von *n* zufällig gewählten Punkten I-II. Z. Wahrscheinlichkeitstheor. Verw. Geb. **2**, 75–84 (1963); **3**, 138–147 (1964)
16. Spencer, F.: Sequences with small discrepancy relative to *n* events. Compositio Math. **47**, 365–392 (1982)
17. Wendel, J.G.: A problem in geometric probability. Math. Scand. **11**, 109–111 (1962)

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